VOLUME
23/2015
No. 2

ISSN 1804-1388 (Print)
ISSN 2336-1298 (Online)

COMMUNICATIONS IN MATHEMATICS

Available online at
http://cm.osu.cz

UNIVERSITAS OSTRAVIENSIS

Subscriptions
This journal is available for exchange. Information on exchange and/or subscription is available at: Library of the University of Ostrava, Bráfova 7, 701 03 Ostrava, Czech Republic
Website: http://cm.osu.cz

The journal is reviewed in Zentralblatt für Mathematik and Mathematical Reviews.

ISSN 1804-1388 (Print), ISSN 2336-1298 (Online)

Communications in Mathematics 23 (2015) 101–112

The Morse-Sard-Brown Theorem for Functionals on Bounded Fréchet-Finsler Manifolds

Kaveh Eftekharinasab

Abstract. In this paper we study Lipschitz-Fredholm vector fields on bounded Fréchet-Finsler manifolds. In this context we generalize the Morse-Sard-Brown theorem, asserting that if M is a connected smooth bounded Fréchet-Finsler manifold endowed with a connection $\mathcal{K}$ and if ξ is a smooth Lipschitz-Fredholm vector field on M with respect to $\mathcal{K}$ which satisfies condition (WCV), then, for any smooth functional l on M which is associated to ξ, the set of the critical values of l is of first category in $\mathbb{R}$. Therefore, the set of the regular values of l is a residual Baire subset of $\mathbb{R}$.

1 Introduction

The notion of a Fredholm vector field on a Banach manifold B with respect to a connection on B was introduced by Tromba [13]. Such vector fields arise naturally in non-linear analysis from variational problems. There are geometrical objects such as harmonic maps, geodesics and minimal surfaces which arise as the zeros of a Fredholm vector field. Therefore, it would be valuable to study the critical points of functionals which are associated to Fredholm vector fields. In [12], Tromba proved the Morse-Sard-Brown theorem for this type of functionals in the case of Banach manifolds. Such a theorem would have applications to problems in the calculus of variations in the large such as Morse theory [11] and index theory [13].

The purpose of this paper is to extend the theorem of Tromba [12, Theorem 1 (MSB)] to a new class of generalized Fréchet manifolds, the class of the so-called bounded Fréchet manifolds, which was introduced in [8]. Such spaces arise in geometry and physical field theory and have many desirable properties. For instance, the space of all smooth sections of a fibre bundle (over closed or noncompact manifolds), which is the foremost example of infinite dimensional manifolds, has the structure of a bounded Fréchet manifold, see [8, Theorem 3.34]. The idea to introduce this category of manifolds was to overcome some permanent difficulties (i.e.,

2010 MSC: 58K05, 58B20, 58B15
Key words: Fréchet manifolds, condition (CV), Finsler structures, Fredholm vector fields

problems of intrinsic nature) in the theory of Fréchet spaces. For example, the lack of a non-trivial topological group structure on the general linear group of a Fréchet space. As for the importance of bounded Fréchet manifolds, we refer to [3], [4] and [8].

Essentially, to define the index of Fredholm vector fields we need the stability of Fredholm operators under small perturbations, but this is unobtainable in the case of proper Fréchet spaces (non-normable spaces) in general, see [3]. Also, we need a subtle notion of a connection via a connection map, but (because of the aforementioned problem) such a connection can not be constructed for Fréchet manifolds in general (cf. [2]). However, in the case of bounded Fréchet manifolds under the global Lipschitz assumption on Fredholm operators, the stability of Lipschitz--Fredholm operators was established in [3]. In addition, the notion of a connection via a connection map was defined in [4]. By using these results, we introduce the notion of a Lipschitz-Fredholm vector field in Section 3. With regard to a kind of compactness assumption (condition (WCV)), which one needs to impose on vector fields, we will be interested in manifolds which admit a Finsler structure. We then define Finsler structures for bounded Fréchet manifolds in Section 4. Finally, after we explained all subsequent portions for proving the Morse-Sard-Brown theorem, we formulate the theorem in the setting of Finsler manifolds in Section 5. A key point in the proof of the theorem is Proposition 2 which in its simplest form says that a Lipschitz-Fredholm vector field ξ near *origin* locally has a representation of the form $\xi(u,v) = (u, \eta(u,v))$, where η is a smooth map. Indeed, this is a consequence of the inverse function theorem (Theorem 3). One of the most important advantage of the category of bounded Fréchet manifold is the availability of the inverse function theorem of Nash and Moser (see [8]).

Morse theory and index theories for Fréchet manifolds have not been developed. Nevertheless, our approach provides some essential tools (such as connection maps, covariant derivatives, Finsler structures) which would create a proper framework for these theories.

2 Preliminaries

In this section we summarize all the necessary preliminary material that we need for a self-contained presentation of the paper. We shall work in the category of smooth manifolds and bundles. We refer to [4] for the basic geometry of bounded Fréchet manifolds.

A Fréchet space (F, d) is a complete metrizable locally convex space whose topology is defined by a complete translation-invariant metric d. A metric with absolutely convex balls will be called a standard metric. Note that every Fréchet space admits a standard metric which defines its topology: If (α_n) is an arbitrary sequence of positive real numbers converging to *zero* and if (ρ_n) is any sequence of continuous seminorms defining the topology of F, then

$$d_{\alpha,\rho}(e,f) := \sup_{n\in\mathbb{N}} \alpha_n \frac{\rho_n(e-f)}{1+\rho_n(e-f)} \tag{1}$$

is a metric on F with the desired properties. We shall always define the topology of Fréchet spaces with this type of metrics. Let (E, g) be another Fréchet space and let $\mathcal{L}_{g,d}(E,F)$ be the set of all linear maps $L : E \to F$ such that

$$\operatorname{Lip}(L)_{g,d} := \sup_{x\in E\setminus\{0\}} \frac{d(L(x),0)}{g(x,0)} < \infty.$$

We abbreviate $\mathcal{L}_g(E) := \mathcal{L}_{g,g}(E,E)$ and write $\operatorname{Lip}(L)_g = \operatorname{Lip}(L)_{g,g}$ for $L \in \mathcal{L}_g(E)$. The metric $D_{g,d}$ defined by

$$D_{g,d} : \mathcal{L}_{g,d}(E,F) \times \mathcal{L}_{g,d}(E,F) \longrightarrow [0,\infty), \qquad (L,H) \mapsto \operatorname{Lip}(L-H)_{g,d}\,, \tag{2}$$

is a translation-invariant metric on $\mathcal{L}_{d,g}(E,F)$ turning it into an Abelian topological group (see [6, Remark 1.9]). The latter is not a topological vector space in general, but a locally convex vector group with absolutely convex balls. The topology on $\mathcal{L}_{d,g}(E,F)$ will always be defined by the metric $D_{g,d}$. We shall always equip the product of any finite number k of Fréchet spaces $(F_i,d_i), 1 \leq i \leq k$, with the maximum metric

$$d_{\max}\big((x_1,\ldots,x_k),(y_1,\ldots,y_k)\big) := \max_{1\leq i\leq k} d_i(x_i,y_i).$$

Let E,F be Fréchet spaces, U an open subset of E and $P : U \to F$ a continuous map. Let $CL(E,F)$ be the space of all continuous linear maps from E to F topologized by the compact-open topology. We say P is differentiable at a point $p \in U$ if there exists a linear map $\mathrm{d}P(p) : E \to F$ such that

$$\mathrm{d}P(p)h = \lim_{t\to 0} \frac{P(p+th)-P(p)}{t},$$

for all $h \in E$. If P is differentiable at all points $p \in U$, if $\mathrm{d}P(p) : U \to CL(E,F)$ is continuous for all $p \in U$ and if the induced map

$$P' : U \times E \to F, \qquad (u,h) \mapsto \mathrm{d}P(u)h$$

is continuous in the product topology, then we say that P is Keller-differentiable. We define $P^{(k+1)} : U \times E^{k+1} \to F$ inductively by

$$P^{(k+1)}(u,f_1,...,f_{k+1}) = \lim_{t\to 0} \frac{P^{(k)}(u+tf_{k+1})(f_1,...,f_k) - P^{(k)}(u)(f_1,...,f_k)}{t}.$$

If P is Keller-differentiable, $\mathrm{d}P(p) \in \mathcal{L}_{d,g}(E,F)$ for all $p \in U$, and the induced map $\mathrm{d}P(p) : U \to \mathcal{L}_{d,g}(E,F)$ is continuous, then P is called b-differentiable. We say P is MC^0 and write $P^0 = P$ if it is continuous. We say P is an MC^1 and write $P^{(1)} = P'$ if it is b-differentiable. Let $\mathcal{L}_{d,g}(E,F)_0$ be the connected component of $\mathcal{L}_{d,g}(E,F)$ containing the zero map. If P is b-differentiable and if $V \subseteq U$ is a connected open neighbourhood of $x_0 \in U$, then $P'(V)$ is connected and hence contained in the connected component $P'(x_0)+\mathcal{L}_{d,g}(E,F)_0$ of $P'(x_0)$ in $\mathcal{L}_{d,g}(E,F)$. Thus,

$$P'|_V - P'(x_0) : V \to \mathcal{L}_{d,g}(E,F)_0$$

is again a map between subsets of Fréchet spaces. This makes possible a recursive definition: If P is MC^1 and V can be chosen for each $x_0 \in U$ such that

$$P'|_V - P'(x_0) : V \to \mathcal{L}_{d,g}(E,F)_0$$

is MC^{k-1}, then P is called an MC^k-map. We make a piecewise definition of $P^{(k)}$ by $P^{(k)}|_V := (P'|_V - P'(x_0))^{(k-1)}$ for x_0 and V as before. The map P is MC^∞ if it is MC^k for all $k \in \mathbb{N}_0$. We shall denote the derivative of P at p by $\mathrm{D}\,P(p)$.

A bounded Fréchet manifold is a second countable Hausdorff space with an atlas of coordinate charts taking their values in Fréchet spaces such that the coordinate transition functions are all MC^∞-maps.

3 Lipschitz-Fredholm vector fields

Throughout the paper we assume that (F,d) is a Fréchet space and M is a bounded Fréchet manifold modelled on F. Let $(U_\alpha, \varphi_\alpha)_{\alpha\in\mathcal{A}}$ be a compatible atlas for M. The latter gives rise to a trivializing atlas $(\pi_M^{-1}(U_\alpha), \psi_\alpha)_{\alpha\in\mathcal{A}}$ on the tangent bundle $\pi_M : TM \to M$, with

$$\psi_\alpha : \pi_M^{-1}(U_\alpha) \to \varphi_\alpha(U_\alpha) \times F, \qquad j_p^1(f) \mapsto \big(\varphi_\alpha(p), (\varphi_\alpha \circ f)'(0)\big),$$

where $j_p^1(f)$ stands for the 1-jet of an MC^∞-mapping $f : \mathbb{R} \to M$ that sends *zero* to $p \in M$. Let N be another bounded Fréchet manifold and $h : M \to N$ an MC^k-map. The tangent map $Th : TM \to TN$ is defined by $Th(j_p^1(f)) = j_{h(p)}^1(h \circ f)$. Let $\Pi_{TM} : T(TM) \to TM$ be an ordinary tangent bundle over TM with the corresponding trivializing atlas $\big(\Pi_{TM}^{-1}(\pi_M^{-1}(U_\alpha)), \widetilde{\psi}_\alpha\big)_{\alpha\in\mathcal{A}}$. A connection map on the tangent bundle TM (possible also for general vector bundles) was defined in [4]. It is a smooth bundle morphism

$$\mathcal{K} : T(TM) \to TM$$

such that the maps $\tau_\alpha : \varphi_\alpha(U_\alpha) \times F \to \mathcal{L}_d(F)$ defined by the local forms

$$\mathcal{K}_\alpha := \psi_\alpha \circ \mathcal{K} \circ (\widetilde{\psi}_\alpha)^{-1} : \varphi_\alpha(U_\alpha) \times F \times F \times F \to \varphi_\alpha(U_\alpha) \times F, \quad \alpha \in \mathcal{A} \tag{3}$$

of $\mathcal{K}$ by the rule

$$\mathcal{K}_\alpha(f,g,h,k) = (f, k + \tau_\alpha(f,g)\cdot h),$$

are smooth. A connection on M is a connection map on its tangent bundle $\pi_M : TM \to M$. A connection $\mathcal{K}$ is linear if and only if it is linear on the fibres of the tangent map. Locally $T\pi$ is the map $U_\alpha \times F \times F \times F \to U_\alpha \times F$ defined by $T\pi(f,\xi,h,\gamma) = (f,h)$, hence locally its fibres are the spaces $\{f\} \times F \times \{h\} \times F$. Therefore, $\mathcal{K}$ is linear on these fibres if and only if the maps $(g,k) \mapsto k + \tau_\alpha(f,g)h$ are linear, and this means that the mappings τ_α need to be linear with respect to their second variables.

A linear connection $\mathcal{K}$ is determined by the family $(\Gamma_\alpha)_{\alpha\in\mathcal{A}}$ of its Christoffel symbols consisting of smooth mappings

$$\Gamma_\alpha : \varphi_\alpha(U_\alpha) \to \mathcal{L}(F \times F; F), \quad p \mapsto \Gamma_\alpha(p)$$

defined by $\Gamma_\alpha(p)(g,h) = \tau_\alpha(p,g)h$.

Remark 1. If $\varphi : U \subset M \to F$ is a local coordinate chart for M, then a vector field ξ on M induces a vector field ξ on F called the local representative of ξ by the formula $\xi(x) = T\varphi \cdot \xi(\varphi^{-1}(x))$. Here and in what follows we use ξ itself to denote this local representation.

In the following we adopt Elliason's definition of a covariant derivative [5].

Definition 1. Let $\pi_M : TM \to M$ be the tangent bundle of M. Let N be a bounded Fréchet manifold modelled on F, $\lambda : N \to M$ a Fréchet vector bundle with fibre F, and K_λ a connection map on TN. If $\xi : M \to N$ is a smooth section of λ, we define the covariant derivative of ξ at $p \in M$ to be the bundle map $\nabla\xi : TM \to N$ given by

$$\nabla\xi(p) = K_\lambda \circ T_p\xi, \quad T_p\xi = T\xi|_{T_pM}.$$

In a local coordinate chart (U, Φ) we have

$$\nabla\xi(x) \cdot y = \mathrm{D}\,\xi(x) \cdot y + \Gamma_\Phi(x) \cdot (y, \xi(x)),$$

where Γ_Φ is the Christoffel symbol for K_λ with respect to the chart (U, Φ).

The covariant derivative $\nabla\xi(p)$ is a linear map from the tangent space T_pM to $F_p := \lambda^{-1}(p)$. This is because it is the combination of the tangent map $T_p\xi$ that maps T_pM linearly into $T_{\xi(p)}N$ with K_λ which is a linear map from $T_{\xi(p)}N$ to F_p.

Definition 2. ([3], Definition 3.2) Let (F, d) and (E, g) be Fréchet spaces. A map φ in $\mathcal{L}_{g,d}(E, F)$ is called a Lipschitz-Fredholm operator if it satisfies the following conditions:

1. The image of φ is closed.
2. The dimension of the kernel of φ is finite.
3. The co-dimension of the image of φ is finite.

We denote by $\mathcal{LF}(E, F)$ the set of all Lipschitz-Fredholm operators from E into F. For $\varphi \in \mathcal{LF}(E, F)$ we define the index of φ as follows:

$$\operatorname{Ind}\varphi = \dim\ker\varphi - \operatorname{codim}\operatorname{Img}\varphi.$$

Theorem 1. *([3], Theorem 3.2) $\mathcal{LF}(E, F)$ is open in $\mathcal{L}_{g,d}(E, F)$ with respect to the topology defined by the metric* (2). *Furthermore, the function $T \to \operatorname{Ind} T$ is continuous on $\mathcal{LF}(E, F)$, and hence it is constant on the connected components of $\mathcal{LF}(E, F)$.*

Now we define a Lipschitz-Fredholm vector field on M with respect to a connection on M.

Definition 3. A smooth vector field $\xi : M \to TM$ is called Lipschitz-Fredholm with respect to a connection $\mathcal{K} : T(TM) \to TM$ if for each $p \in M$, $\nabla\xi(p) : T_pM \to T_pM$

is a linear Lipschitz-Fredholm operator. The index of ξ at p is defined to be the index of $\nabla\xi(p)$, that is

$$\operatorname{Ind}\nabla\xi(p) = \dim\ker\nabla\xi(p) - \operatorname{codim}\operatorname{Img}\nabla\xi(p).$$

By Theorem 1, if M is connected, then the index is independent of the choice of p, and the common value is called the index of ξ. If M is not connected, then the index is constant on its components, and we shall require it to be the same on all these components.

Remark 2. Note that the notion of a Lipschitz-Fredholm vector field depends on the choice of the connection $\mathcal{K}$. If p is a *zero* of ξ, $\xi(p) = 0$, then by Definition 1 we have $\nabla\xi(p) = \mathrm{D}\,\xi(p)$, and hence the covariant derivative at p does not depend on $\mathcal{K}$. In this case, the derivative of ξ at p, $\mathrm{D}\,\xi(p)$, can be viewed as a linear endomorphism from T_pM into itself.

4 Finsler structures

A Finsler structure on a bounded Fréchet manifold M is defined in the same way as in the case of Fréchet manifolds (see [1] for the definition of Fréchet-Finsler manifolds). However, we need a countable family of seminorms on its Fréchet model space F which defines the topology of F. As mentioned in the Preliminaries, we always define the topology of a Fréchet space by a metric with absolutely convex balls. One reason for this consideration is that a metric with this property can give us back the original seminorms. More precisely:

Remark 3. ([8], Theorem 3.4) Assume that (E, g) is a Fréchet space and g is a metric with absolutely convex balls. Let $B^g_{\frac{1}{i}}(0) := \{y \in E \mid g(y, 0) < \frac{1}{i}\}$, and suppose that $(U_i)_{i\in\mathbb{N}}$ is a family of convex subsets of $B^g_{\frac{1}{i}}(0)$. Define the Minkowski functionals

$$\|v\|_i := \inf\left\{\epsilon > 0 \;\middle|\; \epsilon \in \mathbb{R},\ \frac{1}{\epsilon}\cdot v \in U_i\right\}.$$

These Minkowski functionals are continuous seminorms on E. A collection $(\|v\|_i)_{i\in\mathbb{N}}$ of these seminorms gives the topology of E.

Definition 4. Let F be as before. Let X be a topological space and $V = X \times F$ the trivial bundle with fibre F over X. A Finsler structure for V is a collection of functions $\|\cdot\|^n : V \to \mathbb{R}^+$, $n \in \mathbb{N}$, such that

1. For any fixed $b \in X$, $\|(b, x)\|^n = \|x\|^n_b$ is a collection of seminorms on F which gives the topology of F.
2. Given $K > 1$ and $x_0 \in X$, there exits a neighborhood $\mathcal{U}$ of x_0 such that

$$\frac{1}{K}\|f\|^n_{x_0} \leqq \|f\|^n_x \leqq K\|f\|^n_{x_0} \tag{4}$$

for all $x \in \mathcal{U}$, $n \in \mathbb{N}$, $f \in F$.

Let $\pi_M : TM \to M$ be the tangent bundle of M and let $\|\cdot\|^n : TM \to \mathbb{R}^+$, $n \in \mathbb{N}$, be a family of functions. We say that $(\|\cdot\|^n)_{n\in\mathbb{N}}$ is a *Finsler structure* for TM if for a given $m_0 \in M$ and any open neighborhood U of m_0 which trivializes the tangent bundle TM, i.e., there exists a diffeomorphism

$$\psi : \pi_M^{-1}(U) \approx U \times \big(F_{m_0} := \pi_M^{-1}(m_0)\big),$$

the family $(\|\cdot\|^n \circ \psi^{-1})_{n\in\mathbb{N}}$ is a Finsler structure for $U \times F_{m_0}$.

Definition 5. A bounded Fréchet-Finsler manifold is a bounded Fréchet manifold together with a Finsler structure on its tangent bundle.

Proposition 1. *Let N be a paracompact bounded Fréchet manifold modelled on a Fréchet space (E, g). If all seminorms $\|\cdot\|_i$, $i \in \mathbb{N}$, (which are defined as in Remark 3) are smooth maps on $E \setminus \{0\}$, then N admits a partition of unity. Moreover, N admits a Finsler structure.*

Proof. See [1], Propositions 3 and 4. □

If $(\|\cdot\|^n)_{n\in\mathbb{N}}$ is a Finsler structure for M then eventually we can obtain a graded Finsler structure, denoted again by $(\|\cdot\|^n)_{n\in\mathbb{N}}$, for M (see [1]). Let $(\|\cdot\|^n)_{n\in\mathbb{N}}$ be a graded Finsler structure for M. We define the length of piecewise MC^1-curve $\gamma : [a, b] \to M$ by

$$L^n(\gamma) = \int_a^b \|\gamma'(t)\|^n_{\gamma(t)}\, \mathrm{d}t.$$

On each connected component of M, the distance is defined by

$$\rho^n(x, y) = \inf_\gamma L^n(\gamma),$$

where the infimum is taken over all continuous piecewise MC^1-curve connecting x to y. Thus, we obtain an increasing sequence of pseudometrics $\rho^n(x, y)$ and define the distance ρ by

$$\rho(x, y) = \sum_{n=1}^{n=\infty} \frac{1}{2^n} \cdot \frac{\rho^n(x, y)}{1 + \rho^n(x, y)}. \tag{5}$$

Lemma 1. *([1], Lemma 2) A family $(\sigma^i)_{i\in\mathbb{N}}$ of pseudometrics on F defines a unique topology $\mathcal{T}$ such that for every sequence $(x_n)_{n\in\mathbb{N}} \subset F$, we have $x_n \to x$ in the topology $\mathcal{T}$ if and only if $\sigma^i(x_n, x) \to 0$, for all $i \in \mathbb{N}$. The topology is Hausdorff if and only if $x = y$ when all $\sigma^i(x, y) = 0$. In addition,*

$$\sigma(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\sigma^n(x, y)}{1 + \sigma^n(x, y)}$$

is a pseudometric on F, which defines the same topology.

With the aid of this lemma, the proof of the following theorem is quite similar to the proof given for Banach manifolds (cf. [10]).

Theorem 2. *Suppose M is a connected manifold endowed with a Finsler structure $(\|\cdot\|^n)_{n\in\mathbb{N}}$. Then the distance ρ defined by (5) is a metric for M. Furthermore, the topology induced by this metric coincides with the original topology of M.*

Proof. The distance ρ is pseudometric by Lemma 1. We prove that $\rho(x_0, y_0) > 0$ if $x_0 \neq y_0$. Let $(\|\cdot\|_n)_{n\in\mathbb{N}}$ be the family of all seminorms on F (which are defined as in Remark 3). Given $x_0 \in M$, let $\varphi : U \to F$ be a chart for M with $x_0 \in U$ and $\varphi(x_0) = u_0$. Let $y_0 \in M$, and let $\gamma : [a, b] \to M$ be an MC^1-curve connecting x_0 to y_0. Let $B_r(u_0)$ be a ball with center u_0 and radius $r > 0$. Choose r small enough so that $\mathcal{U} := \varphi^{-1}(B_r(u_0)) \subset U$ and for a given $K > 1$,

$$\frac{1}{K}\|f\|^n_{x_0} \leqq \|f\|^n_x \leqq K\|f\|^n_{x_0},$$

for all $x \in \mathcal{U}$, $n \in \mathbb{N}$, $f \in F$. Let $I = [a, b]$ and $\mu(t) := \varphi \circ \gamma(t)$. If $\gamma(I) \subset \mathcal{U}$, then let $\beta = b$. Otherwise, let β be the first $t > 0$ such that $\|\mu(t) - u_0\|_n = r$ for all $n \in \mathbb{N}$. Then, since for every $x \in U$ the map $\phi(x) : T_xM \to F$ given by $j^1_x \mapsto \varphi(x)$ is a homeomorphism, it follows that for all $n \in \mathbb{N}$ we have

$$\begin{aligned}\int_a^b \|\gamma'(t)\|^n_{\gamma(t)}\,\mathrm{d}t &\geq \frac{1}{K}\int_a^\beta \|\phi^{-1}(x) \circ \mu'(t)\|^n_{x_0}\mathrm{d}t \geq k_1 \int_a^\beta \|\mu'(t))\|_n \mathrm{d}t \\ &\geq k_1\|\int_a^\beta \mu'(t)\mathrm{d}t\|_n = k_1\|\mu(\beta) - \mu(a)\|_n \quad \text{for some } k_1 > 0.\end{aligned}$$

(The last inequality follows from [7, Theorem 2.1.1].) Thereby, if $x_0 \neq y_0$ then $\rho^n(x_0, y_0) > 0$ and hence $\rho(x_0, y_0) > 0$. Now we prove that the topology induced by ρ coincides with the topology of M. By virtue of Lemma 1, we only need to show that $(\rho^n)_{n\in\mathbb{N}}$ induces the topology which is consistent with the topology of M. If $x_i \to x_0$ in M then eventually $x_i \in \mathcal{U}$. Define $\lambda_i : [0, 1] \to \mathcal{U}$, an MC^1-curve connecting x_0 to x_i, by $\lambda_i(t) := t\varphi(x_i)$. Then, for all $n \in \mathbb{N}$

$$\begin{aligned}\rho^n(x_i, x_0) \leq L^n(\lambda_i) &= \int_0^1 \|\lambda'_i\|^n_{\lambda_i(t)}\mathrm{d}t = \int_0^1 \|\varphi(x_i)\|^n_{t\varphi(x_i)}\mathrm{d}t \\ &\leq K\int_0^1 \|\varphi(x_i)\|^n_{x_0}\mathrm{d}t = K\|\varphi(x_i)\|_n.\end{aligned}$$

But $\varphi(x_i) \to 0$ as $x_i \to x_0$, thereby $\rho^n(x_i, x_0) \to 0$ for all $n \in \mathbb{N}$. Conversely, if for all $n \in \mathbb{N}$, $\rho^n(x_i, x_0) \to 0$ then eventually we can choose r small enough so that $x_i \in \mathcal{U}$. Then, for all $n \in \mathbb{N}$ we have $\|\varphi(x_i)\|^n_{x_0} \leq K\rho^n(x_i, x_0)$ so $\|\varphi(x_i)\|^n_{x_0} \to 0$ in $T_{x_0}M$, whence $\varphi(x_i) \to 0$. Therefore, $x_i \to x_0$ in $\mathcal{U}$ and hence in M. □

The metric ρ is called the *Finsler metric* for M.

5 Morse-Sard-Brown Theorem

In this section we prove the Morse-Sard-Brown theorem for functionals on bounded Fréchet-Finsler manifolds. The proof relies on the following inverse function theorem.

Theorem 3 (Inverse Function Theorem for MC^k-maps). *([6], Proposition 7.1) Let (E, g) be a Fréchet space with standard metric g. Let $U \subset E$ be open, $x_0 \in U$ and $f\colon U \subset E \to E$ an MC^k-map, $k \geq 1$. If $f'(x_0) \in \operatorname{Aut}(E)$, then there exists an open neighbourhood $V \subseteq U$ of x_0 such that $f(V)$ is open in E and $f|_V : V \to f(V)$ is an MC^k-diffeomorphism.*

The following consequence of this theorem is an important technical tool.

Proposition 2 (Local representation). *Let F_1, F_2 be Fréchet spaces and U an open subset of $F_1 \times F_2$ with $(0,0) \in U$. Let E_2 be another Fréchet space and $\phi : U \to F_1 \times E_2$ an MC^∞-map with $\phi(0,0) = (0,0)$. Assume that the partial derivative $\mathrm{D}_1\, \phi(0,0) : F_1 \to F_1$ is linear isomorphism. Then there exists a local MC^∞-diffeomorphism ψ from an open neighbourhood $V_1 \times V_2 \subseteq F_1 \times F_2$ of $(0,0)$ onto an open neighbourhood of $(0,0)$ contained in U such that*

$$\phi \circ \psi(u, v) = (u, \mu(u, v)),$$

where $\mu : V_1 \times V_2 \to E_2$ is an MC^∞-mapping.

Proof. Let $\phi = \phi_1 \times \phi_2$, where $\phi_1 : U \to F_1$ and $\phi_2 : U \to E_2$. By assumption we have $\mathrm{D}_1\, \phi_1(0,0) = \mathrm{D}_1\, \phi(0,0)|_{F_1} \in \operatorname{Iso}(F_1, F_1)$. Define the map

$$g : U \subset F_1 \times F_2 \to F_1 \times E_2, \quad g(u_1, u_2) := \big(\phi_1(u_1, u_2), u_2\big)$$

locally at $(0,0)$. Then, for all $u = (u_1, u_2) \in U$, $f_1 \in F_1$, $f_2 \in F_2$ we have

$$\mathrm{D}\, g(u) \cdot (f_1, f_2) = \begin{pmatrix} \mathrm{D}_1\, \phi_1(u) & \mathrm{D}_2\, \phi_1(u) \\ 0 & \mathrm{Id}_{E_2} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

and hence $\mathrm{D}\, g(u)$ is a linear isomorphism at $(0,0)$. By the inverse function theorem, there are open sets U' and $V = V_1 \times V_2$ and an MC^∞-diffeomorphism $\Psi : V \to U'$ such that $(0,0) \in U' \subset U$, $g(0,0) \in V \subset F_1 \times E_2$, and $\Psi^{-1} = g|_{U'}$. Hence if $(u, v) \in V$, then

$$(u, v) = (g \circ \Psi)(u, v) = g\big(\Psi_1(u, v), \Psi_2(u, v)\big) = \big(\phi_1 \circ \Psi_1(u, v), \Psi_2(u, v)\big),$$

where $\Psi = \Psi_1 \times \Psi_2$. This shows that $\Psi_2(v, v) = v$ and $(\phi_1 \circ \Psi)(u, v) = u$. If $\eta = \phi_2 \circ \Psi$, then

$$(\phi \circ \Psi)(u, v) = \big(\phi_1 \circ \Psi(u, v), \phi_2 \circ \Psi(u, v)\big) = (u, \eta(u, v)).$$

This completes the proof. □

In the sequel, we assume that M is connected and it is endowed with a Finsler structure $(\|\cdot\|^n)_{n \in \mathbb{N}}$ and the induced Finsler metric ρ.

Definition 6. Let $l : M \to \mathbb{R}$ be an MC^∞-functional and $\xi M \to TM$ a smooth vector field. By saying that l and ξ are associated we mean $\mathrm{D}\, l(p) = 0$ if and only if $\xi(p) = 0$. A point $p \in M$ is called a critical point for l if $\mathrm{D}\, l(p) = 0$. The corresponding value $l(p)$ is called a critical value. Values other than critical are called regular values. The set of all critical points of l is denoted by Crit_l.

The following is our version of the compactness condition due to Tromba [11].

Condition 1 (CV). *Let $(m_i)_{i\in\mathbb{N}}$ be a bounded sequence in M. We say that a vector field $\xi : M \to TM$ satisfies condition* (CV) *if $\|\xi(m_i)\|^n \to 0$ for all $n \in \mathbb{N}$ implies that $(m_i)_{i\in\mathbb{N}}$ has a convergent subsequence.*

If ξ satisfies condition (CV) then the set of its zeros in any closed bounded set is compact (see [11, Proposition 1, p. 55]). This property turns out to be important. We then say ξ satisfies condition (WCV) if the set of its zeros in any closed bounded set is compact.

A subset G of a Fréchet space E is called topologically complemented or split in E if there is another subspace H of E such that E is homeomorphic to the topological direct sum $G \oplus H$. In this case we call H a topological complement of G in F.

We need the following facts:

Theorem 4. *([8], Theorem 3.14) Let E be a Fréchet space. Then*

1. *Every finite-dimensional subspace of E is closed.*
2. *Every closed subspace $G \subset E$ with* $\operatorname{codim}(G) = \dim(E/G) < \infty$ *is topologically complemented in E.*
3. *Every finite-dimensional subspace of E is topologically complemented.*
4. *Every linear isomorphism $G \oplus H \to E$ between the direct sum of two closed subspaces and E, is a homeomorphism.*

The proof of the Morse-Sard-Brown theorem requires Proposition 2 and Theorem 4. Except the arguments which involve these results and the Finslerian nature of manifolds, the rest of arguments are similar to that of Banach manifolds case, see [12, Theorem 1].

Theorem 5 (Morse-Sard-Brown Theorem). *Assume that (M, ρ) is endowed with a connection $\mathcal{K}$. Let ξ be a smooth Lipschitz-Fredholm vector field on M with respect to $\mathcal{K}$ which satisfies condition* (WCV). *Then, for any MC^∞-functional l on M which is associated to ξ, the set of its critical values $l(\mathrm{Crit}_l)$ is of first category in $\mathbb{R}$. Therefore, the set of the regular values of l is a residual Baire subset of $\mathbb{R}$.*

Proof. We can assume $M = \bigcup_{i\in\mathbb{N}} M_i$, where all the M_i's are closed bounded balls of radius i about some fixed point $m_0 \in M$. The boundedness and the radii of balls are relative to the Finsler metric ρ. Thus to conclude the proof it suffices to show that the image $l(C_B)$ of the set C_B of the zeros of ξ in some bounded set B is compact without interior. If, in addition, B is closed, then C_B is compact because ξ satisfies condition (WCV).

Let B be a closed bounded set and let C_B as before. If $p \in C_B$ then eventually $\xi(p) = 0$. Since C_B is compact we only need to show that for a bounded neighbourhood U of p, $l(C_B \cap \overline{U})$ is compact without interior. In other words, we can work locally. Therefore, we may assume without loss of generality that $p = 0 \in F$

and ξ, l are defined locally on an open neighbourhood of p. An endomorphism $\mathrm{D}\,\xi(p) : F \to F$ is a Lipschitz-Fredholm operator because ξ is a Lipschitz-Fredholm vector field (see Remark 2). Thereby, in the light of Theorem 4 it has a split image F_1 with a topological complement F_2 and a split kernel E_2 with a topological complement E_1. Moreover, $\mathrm{D}\,\xi(p)$ maps E_1 isomorphically onto F_1 so we can identify F_1 with E_1. Then, by Proposition 2, there is an open neighborhood $U \subset E_1 \times E_2$ of p such that $\xi(u,v) = (u, \eta(u,v))$ for all $(u,v) \in U$, where $\eta : U \to F_2$ is an MC^∞-map. Thus, if $\xi(u,v) = 0 = (u, \eta(u,v))$ then $u = 0$. Therefore, in this local representation, the zeros of ξ (and hence the critical points of l) in $\overline{U}$ are in $\overline{U}_1 := \overline{U} \cap (\{0\} \times E_2)$. The restriction of l, $l_{\overline{U}_1} : \overline{U}_1 \to \mathbb{R}$, is again MC^∞ and $C_B \cap \overline{U} = C_B \cap \overline{U}_1$ so $l(C_B \cap \overline{U}) = l(C_B \cap \overline{U}_1)$.

We have for some constant $k \in \mathbb{N}$, $\dim \overline{U}_1 = \dim E_2 = k$ because $\xi(p)$ is a Lipschitz-Fredholm operator and E_2 is its kernel. Thus, by the classical Sard theorem, $l(C_B \cap \overline{U}_1)$ has measure zero (note that MC^k-differentiability implies the usual C^k-differentiability for maps of finite dimensional manifolds). Therefore, since $C_B \cap \overline{U}_1$ is compact it follows that $l(C_B \cap \overline{U}_1)$ is compact without interior and hence $l(C_B \cap \overline{U})$ is compact without interior. □

Remark 4. From the preceding proof we see that $\dim F_2 = m$, where $m \in \mathbb{N}$ is constant. Thus, the index of ξ is the $\operatorname{Ind} \xi = \dim E_2 - \dim F_2 = k - m$.

References

[1] C.L. Bejan: Finsler structures on Fréchet bundles. In: *Proc. 3-rd Seminar on Finsler spaces, Univ. Braşov 1984.* Societatea de ştiinţe Matematice Romania, Bucharest (1984) 49–54.

[2] C.T.J. Dodson: Some recent work in Fréchet geometry. *Balkan J. Geometry and Its Applications* 17 (2) (2012) 6–21.

[3] K. Eftekharinasab: Sard's theorem for mappings between Fréchet manifolds. *Ukrainian Math. J.* 62 (12) (2011) 1896–1905.

[4] K. Eftekharinasab: Geometry of Bounded Fréchet Manifolds. *Rocky Mountain J. Math., to appear*().

[5] H. Eliasson: Geometry of manifolds of maps. *J. Differential Geometry* 1 (1967) 169–194.

[6] H. Glöckner: Implicit functions from topological vector spaces in the presence of metric estimates. *preprint, Arxiv:math/6612673*(2006).

[7] R.S. Hamilton: The inverse function theorem of Nash and Moser. *Bulletin of the AMS* 7 (1982) 65–222.

[8] O. Müller: A metric approach to Fréchet geometry. *Journal of Geometry and Physics* 58 (11) (2008) 1477–1500.

[9] R.S. Palais: Lusternik-Schnirelman theory on Banach manifolds. *Topology* 5 (2) (1966) 115–132.

[10] R.S. Palais: Critical point theory and the minimax principle. *Proc. Symp. Pur. Math.* 15 (1970) 185–212.

[11] A.J. Tromba: A general approach to Morse theory. *J. Differential Geometry* 12 (1) (1977) 47–85.

[12] A.J. Tromba: The Morse-Sard-Brown theorem for functionals and the problem of Plateau. *Amer. J. Math.* 99 (1977) 1251–1256.

[13] A.J. Tromba: The Euler characteristic of vector fields on Banach manifolds and a globalization of Leray-Schauder degree. *Advances in Mathematics* 28 (2) (1978) 148–173.

Author's address:
Kaveh Eftekharinasab, Topology dept., Institute of Mathematics of NAS of Ukraine, Tereshchenkivs'ka str. 3, 01601 Kyiv, Ukraine

E-mail: kaveh@imath.kiev.ua

Received: 25 August, 2014
Accepted for publication: 24 August, 2015
Communicated by: Olga Rossi

Communications in Mathematics 23 (2015) 113–117

On $X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8$ and $Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4$

Susil Kumar Jena

Abstract. The two related Diophantine equations: $X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8$ and $Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4$, have infinitely many nontrivial, primitive integral solutions. We give two parametric solutions, one for each of these equations.

1 Introduction

In this note, we study the two related Diophantine equations

$$X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8 \tag{1}$$

and

$$Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4. \tag{2}$$

It seems that no parametric solutions are known for (1). Choudhry [1] has found parametric solutions of a similar equation

$$A^4 + 4B^4 = C^4 + 4D^4$$

involving only fourth powers. Though, the parametric solution of (2) is already known which is based on the identity

$$(p^4 + 2q^4)^4 = (p^4 - 2q^4)^4 + (2p^3q)^4 + 4(2pq^3)^4,$$

we give a new parametric solution of (2). For a historical background and references of these equations, and similar Diophantine problems on fourth powers, we refer to Guy ([3], pp. 215–218) and Dickson ([2], pp. 647–648). The parameterisations of (1) and (2) are based on a result from our paper [4] in which we proved the following theorem:

2010 MSC: 11D41, 11D72

Key words: Diophantine equation $A^4 + nB^4 = C^2$, Diophantine equation $A^4 - nB^4 = C^2$, Diophantine equation $X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8$, Diophantine equation $Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4$

Theorem 1. *(Jena, [4]) For any integer n, if (A_t, B_t, C_t) is a solution of the Diophantine equation*

$$A^4 + nB^4 = C^2 \tag{3}$$

with A, B, C as integers, then $(A_{t+1}, B_{t+1}, C_{t+1})$ is also the solution of the same equation such that

$$(A_{t+1}, B_{t+1}, C_{t+1}) = \{(A_t^4 - nB_t^4), (2A_tB_tC_t), (A_t^8 + 6nA_t^4B_t^4 + n^2B_t^8)\} \tag{4}$$

and if A_t, nB_t, C_t are pairwise coprime and A_t, nB_t are of opposite parity, then $A_{t+1}, nB_{t+1}, C_{t+1}$ will also be pairwise coprime and A_{t+1}, nB_{t+1} will be of opposite parity with $A_{t+1}, B_{t+1}, C_{t+1}$ as an odd, even, odd integer respectively.

Changing n to $-n$ at appropriate places in Theorem 1, we get its equivalent theorem:

Theorem 2. *For any integer n, if (A_t, B_t, C_t) is a solution of the Diophantine equation*

$$A^4 - nB^4 = C^2 \tag{5}$$

with A, B, C as integers, then $(A_{t+1}, B_{t+1}, C_{t+1})$ is also the solution of the same equation such that

$$(A_{t+1}, B_{t+1}, C_{t+1}) = \{(A_t^4 + nB_t^4), (2A_tB_tC_t), (A_t^8 - 6nA_t^4B_t^4 + n^2B_t^8)\} \tag{6}$$

and if A_t, nB_t, C_t are pairwise coprime and A_t, nB_t are of opposite parity, then $A_{t+1}, nB_{t+1}, C_{t+1}$ will also be pairwise coprime and A_{t+1}, nB_{t+1} will be of opposite parity with $A_{t+1}, B_{t+1}, C_{t+1}$ as an odd, even, odd integer respectively.

Theorem 1 and Theorem 2 are based on two equivalent polynomial identities

$$(a-b)^4 + 16ab(a+b)^2 = (a^2 + 6ab + b^2)^2;$$

and

$$(a+b)^4 - 16ab(a-b)^2 = (a^2 - 6ab + b^2)^2,$$

which can be used to parameterise (3) and (5) respectively.

2 Core Results

The following lemma will be used for obtaining the main results of this paper.

Lemma 1. *The Diophantine equation*

$$c^4 - 2d^4 = t^2 \tag{7}$$

has infinitely many non-zero, coprime integral solutions for (c, d, t).

Proof. The integral solutions of (7) are generated by using Theorem 2. If we take the initial solution of (7) as $(c_1, d_1, t_1) = (3, 2, 7)$, then from (6) we get the next solution

$$(c_2, d_2, t_2) = \{(c_1^4 + 2d_1^4), (2c_1d_1t_1), (c_1^8 - 6 \times 2c_1^4d_1^4 + 2^2d_1^8)\} = (113, 84, -7967).$$

We take $(c_2, d_2, t_2) = (113, 84, 7967)$ as c, d and t are raised to even powers in (7). Note that $(c_1, 2d_1, t_1) = (3, 4, 7)$ are pairwise coprime, $c_1 = 3$ is odd, and $2d_1 = 4$ is even. So, according to Theorem 2 we expect $(c_2, 2d_2, t_2)$ to be pairwise coprime, and c_2 and $2d_2$ to be of opposite parity. In fact, our expectation is true as $(c_2, 2d_2, t_2) = (113, 168, 7967)$ are pairwise coprime, $c_2 = 113$ is odd, and $2d_2 = 168$ is even. Thus, (7) has infinitely many non-zero and coprime integral solutions. □

It is easy to verify the two polynomial identities

$$(a + b)^4 - (a - b)^4 = 8ab(a^2 + b^2); \tag{8}$$

and

$$(c^4 - 2d^4)^2 + 4(cd)^4 = c^8 + 4d^8 \tag{9}$$

by direct computation.

Put $c^4 - 2d^4 = t^2$ from (7) in (9) to get

$$t^4 + 4(cd)^4 = c^8 + 4d^8. \tag{10}$$

Putting $a = c^4$ and $b = 2d^4$ in (8) we get

$$\begin{aligned}
&(c^4 + 2d^4)^4 - (c^4 - 2d^4)^4 = 16c^4d^4(c^8 + 4d^8);\\
\Rightarrow &(c^4 + 2d^4)^4 = (c^4 - 2d^4)^4 + (2cd)^4\{t^4 + 4(cd)^4\}; \quad \text{[from (10)]}\\
\Rightarrow &(c^4 + 2d^4)^4 = (c^4 - 2d^4)^4 + (2cdt)^4 + 4(2c^2d^2)^4. \qquad (11)
\end{aligned}$$

2.1 Diophantine equation $X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8$

Theorem 3. *The Diophantine equation*

$$X_1^4 + 4X_2^4 = X_3^8 + 4X_4^8 \tag{12}$$

has infinitely many nontrivial, primitive integral solutions for (X_1, X_2, X_3, X_4). *To get the primitive solutions of* (12), *we assume that* $\gcd(X_1, X_2, X_3, X_4) = 1$.

Proof. In accordance with Lemma 1, we have infinitely many integral values of c, d, t with $c^4 - 2d^4 = t^2$ and $\gcd(c, 2d) = 1$ for which (10) has solutions. Comparing (12) with (10) we get $(X_1, X_2, X_3, X_4) = (t, cd, c, d)$. Since $X_3 = c$, $X_4 = d$ and $\gcd(c, 2d) = 1$, we get $\gcd(X_3, X_4) = 1$, and hence, $\gcd(X_1, X_2, X_3, X_4) = 1$. So, (12) has infinitely many nontrivial, primitive integral solutions for (X_1, X_2, X_3, X_4). □

Example 1.

$$\begin{aligned}(c_1,d_1,t_1) &= (3,2,7) : (X_{1_1},X_{2_1},X_{3_1},X_{4_1}) = (t_1,c_1d_1,c_1,d_1) = (7,6,3,2);\\ &\Rightarrow 7^4 + 4\times 6^4 = 3^8 + 4\times 2^8.\\ (c_2,d_2,t_2) &= (113,84,7967) : (X_{1_2},X_{2_2},X_{3_2},X_{4_2})\\ &= (t_2,c_2d_2,c_2,d_2) = (7967,9492,113,84);\\ &\Rightarrow 7967^4 + 4\times 9492^4 = 113^8 + 4\times 84^8.\end{aligned}$$

2.2 Diophantine equation $Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4$

Theorem 4. *The Diophantine equation*

$$Y_1^4 = Y_2^4 + Y_3^4 + 4Y_4^4 \tag{13}$$

has infinitely many nontrivial, primitive integral solutions for (Y_1,Y_2,Y_3,Y_4)*. To get the primitive solutions of* (13)*, we assume that* $(Y_1,Y_2,Y_3,Y_4)=1$.

Proof. Using Lemma 1, we get infinitely many integral values of c,d,t such that $c^4-2d^4=t^2$ and $\gcd(c,2d)=1$ for which (11) is satisfied. Comparing (13) with (11) we get

$$(Y_1,Y_2,Y_3,Y_4) = \big\{(c^4+2d^4),(c^4-2d^4),2cdt,2c^2d^2\big\}.$$

Since $\gcd(c,2d)=1$, we have

$$\gcd\big((c^4+2d^4),(c^4-2d^4)\big) = 1.$$

Thus, $\gcd(Y_1,Y_2)=1$; or, $\gcd(Y_1,Y_2,Y_3,Y_4)=1$. So, (13) has infinitely many nontrivial, primitive integral solutions for (Y_1,Y_2,Y_3,Y_4). □

Example 2.

$$\begin{aligned}(c_1,d_1,t_1) &= (3,2,7) :\\ (Y_{1_1},Y_{2_1},Y_{3_1},Y_{4_1}) &= \big\{(c_1^4+2d_1^4),(c_1^4-2d_1^4),2c_1d_1t_1,2c_1^2d_1^2\big\} = (113,49,84,72);\\ &= \big\{(c_1^4+2d_1^4),t_1^2,2c_1d_1t_1,2c_1^2d_1^2\big\} = (113,7^2,84,72).\\ \Rightarrow 113^4 &= 49^4+84^4+4\times 72^4 = 7^8+84^4+4\times 72^4.\end{aligned}$$

$$\begin{aligned}(c_2,d_2,t_2) &= (113,84,7967) :\\ (Y_{1_2},Y_{2_2},Y_{3_2},Y_{4_2}) &= \big\{(c_2^4+2d_2^4),(c_2^4-2d_2^4),2c_2d_2t_2,2c_2^2d_2^2\big\}\\ &= (262621633,63473089,151245528,180196128);\\ &= \big\{(c_2^4+2d_2^4),t_2^2,2c_2d_2t_2,2c_2^2d_2^2\big\}\\ &= (262621633,7967^2,151245528,180196128).\\ \Rightarrow 262621633^4 &= 63473089^4+151245528^4+4\times 180196128^4;\\ &= 7967^8+151245528^4+4\times 180196128^4.\end{aligned}$$

3 Conclusion

We make no attempt of giving the complete parametric solutions to the two Diophantine equations of the title. There might exist some singular solutions. It is expected that the prospective scholars will continue further exploration to find the complete solutions of these two equations.

Acknowledgment

During the failing moments of my life they appeared from nowhere to fill my soul with the nectar of hope. Noble angels, I acknowledge your grace in this brief note!

References

[1] A. Choudhry: The Diophantine equation $A^4 + 4B^4 = C^4 + 4D^4$. *Indian J. Pure Appl. Math.* 29 (1998) 1127–1128.

[2] L. E. Dickson: *History of the Theory of Numbers.* Chelsea Publishing Company, New York (1952).

[3] R. K. Guy: *Unsolved Problems in Number Theory.* Springer Science+Business Media Inc., New York (2004). Third Edition.

[4] S. K. Jena: Beyond the Method of Infinite Descent. *J. Comb. Inf. Syst. Sci.* 35 (2010) 501–511.

Author's address:
(Former Professor of Electronics and Telecommunication Engineering)
Plot No-170, Sector-5, Niladri Vihar, Chandrasekharpur, Bhubaneswar-751021, Odisha, India

E-mail: susil_kumar@yahoo.co.uk

Received: 18 September, 2014
Accepted for publication: 16 June, 2015
Communicated by: Attila Bérczes

Communications in Mathematics 23 (2015) 119–129

On the equivalence of control systems on Lie groups

Rory Biggs, Claudiu C. Remsing

Abstract. We consider state space equivalence and feedback equivalence in the context of (full-rank) left-invariant control systems on Lie groups. We prove that two systems are state space equivalent (resp. detached feedback equivalent) if and only if there exists a Lie group isomorphism relating their parametrization maps (resp. traces). Local analogues of these results, in terms of Lie algebra isomorphisms, are also found. Three illustrative examples are provided.

1 Introduction

Geometric control theory began in the late 1960s with the study of (nonlinear) control systems by using concepts and methods from differential geometry (cf. [14], [21]). In the spirit of Klein's *Erlanger Programm*, a way of understanding the structure of a class of (geometric) objects is to define equivalence relations (or group actions) and then to study their invariants. In order to understand the local geometry of general control systems one needs to introduce natural equivalence relations in the class of such systems or in various distinguished subclasses. We will consider (smooth) control systems of the form

$$\dot{x} = \Xi(x, u), \qquad x \in \mathsf{M},\ u \in U \tag{1}$$

where the state space M and the space of control parameters (shortly the input space) U are smooth manifolds, and the map $\Xi : \mathsf{M} \times U \to T\mathsf{M}$ is smooth. (Ξ defines a family of smooth vector fields on M, smoothly parametrized by the controls.) The class $\mathcal{U}$ of admissible controls is contained in the space of all U-valued measurable maps defined on intervals of the real line $\mathbb{R}$ (see, e.g., [2], [14], [21]). We shall denote a control system (1) by (M, Ξ) (cf. [3]). Let $\mathcal{X} = (\Xi_u = \Xi(\cdot, u))_{u \in U}$ be the associated family of vector fields (on M). The control system $\Sigma = (\mathsf{M}, \Xi)$

2010 MSC: 93B27, 22E60
Key words: left-invariant control system, state space equivalence, detached feedback equivalence

satisfies the Lie algebra rank condition (LARC) at $x_0 \in \mathsf{M}$ provided the Lie algebra (of vector fields on M) generated by $\mathcal{X}$ spans the whole tangent space $T_{x_0}\mathsf{M}$.

The most natural equivalence relation for such control systems is equivalence up to coordinate changes in the state space. This is called *state space equivalence*. Two control systems (M, Ξ) and (M', Ξ') are called state space equivalent (shortly S-equivalent) if there exists a diffeomorphism $\phi : \mathsf{M} \to \mathsf{M}'$ which transforms Σ to Σ'; this amounts to saying that the diffeomorphism ϕ conjugates the families $\mathcal{X}$ and $\mathcal{X}'$ (see [11]). S-equivalence is well understood. It establishes a one-to-one correspondence between the trajectories of the equivalent systems. However, this equivalence relation is very strong. We recall the following result due to Krener [16] and Sussmann [20] (see also [2], [21]).

Proposition 1. *Let Σ and Σ' be two analytic control systems having the same input space $U = U'$ and satisfying the LARC at x_0 and x_0', respectively. Then they are (locally) S-equivalent at x_0 and x_0', respectively, if and only if there exists a linear isomorphism $\psi : T_{x_0}\mathsf{M} \to T_{x_0'}\mathsf{M}'$ such that the equality*

$$\psi\big[\cdots[\Xi_{u_1}, \Xi_{u_2}], \ldots, \Xi_{u_k}\big](x_0) = \big[\cdots[\Xi'_{u_1}, \Xi'_{u_2}], \ldots, \Xi'_{u_k}\big](x_0')$$

holds for any $k \geq 1$ and any $u_1, \ldots, u_k \in U$. Furthermore, if in addition M and M' are simply connected and the vector fields Ξ_u and Ξ'_u are complete, then local state space equivalence implies global state space equivalence.

Therefore, there are so many S-equivalence classes that any general classification appears to be very difficult if not impossible. However, there is a chance for some reasonable classification in low dimensions.

Another fundamental equivalence relation for control systems is that of *feedback equivalence*. We say that two control systems (M, Ξ) and (M', Ξ') are feedback equivalent (shortly F-equivalent) if there exists a diffeomorphism $\Phi : \mathsf{M} \times U \to \mathsf{M}' \times U'$ of the form

$$\Phi(x, u) = \big(\phi(x), \varphi(x, u)\big)$$

which transforms the first system to the second. Note that the map ϕ plays the role of a change of coordinates (in the state space), while the *feedback transformation* φ changes coordinates in the input space in a way which is state dependent. Two feedback equivalent control systems have the same set of trajectories (up to a diffeomorphism in the state space) which are parametrized differently by admissible controls. F-equivalence has been extensively studied in the last few decades (see [18] and the references therein). There are a few basic methods used in the study of F-equivalence. These methods are based either on (studying invariant properties of) associated distributions or on Cartan's method of equivalence [9] or inspired by the Hamiltonian formalism [12]; also, another fruitful approach is closely related to Poincaré's technique for linearization of dynamical systems. Feedback transformations play a crucial role in control theory, particularly in the important problem of *feedback linearization* [13]. The study of F-equivalence of general control systems can be reduced to the case of control affine systems (cf. [11]). For a thorough study of the equivalence and classification of (general) control affine systems, see [8].

In the context of left-invariant control systems, state space equivalence and feedback equivalence have not yet been considered in a general and systematic manner; we do so in this paper. Characterizations of state space equivalence and (detached) feedback equivalence are obtained: globally, in terms of Lie group isomorphisms (Theorems 1 and 3, respectively) and locally, in terms of Lie algebra isomorphisms (Theorems 2 and 4, respectively). A few examples exhibiting the use of (local) equivalences are provided.

2 Left-invariant control systems

Invariant control systems on Lie groups were first considered in 1972 by Brockett [7] and by Jurdjevic and Sussmann [15]. A *left-invariant control system* $\Sigma = (\mathsf{G}, \Xi)$ is a control system evolving on some (real, finite-dimensional) Lie group G, whose dynamics $\Xi : \mathsf{G} \times U \to T\mathsf{G}$ are invariant under left translations, i.e., the push-forward $(L_g)_*\Xi_u$ equals Ξ_u for all $g \in \mathsf{G}$ and $u \in U$. (The tangent bundle $T\mathsf{G}$ is identified with $\mathsf{G} \times \mathfrak{g}$, where $\mathfrak{g} = T_{\mathbf{1}}\mathsf{G}$ denotes the associated Lie algebra.) Such a control system is described as follows (cf. [14], [2], [19], [17])

$$\dot{g} = \Xi(g, u), \qquad g \in \mathsf{G},\ u \in U$$

where $\Xi(g, u) = g\Xi(\mathbf{1}, u) \in T_g\mathsf{G}$. (The notation $g\Xi(\mathbf{1}, u)$ stands for the image of the element $\Xi(\mathbf{1}, u) \in \mathfrak{g}$ under the tangent map of the left translation $dL_g = T_{\mathbf{1}}L_g : \mathfrak{g} \to T_g\mathsf{G}$.) The input space U is a smooth manifold and admissible controls are piecewise continuous U-valued maps, defined on compact intervals $[0, T]$. The family $\mathcal{X} = (\Xi_u = \Xi(\cdot, u))_{u \in U}$ consists of left-invariant vector fields on G. We further assume that the parametrization map $\Xi(\mathbf{1}, \cdot) : U \to \mathfrak{g}$ is an embedding. This means that the image set $\Gamma = \operatorname{im}\Xi(\mathbf{1}, \cdot)$, called the *trace* of Σ, is a submanifold of $\mathfrak{g}$. By identifying (the left-invariant vector field) $\Xi(\cdot, u)$ with $\Xi(\mathbf{1}, u) \in \mathfrak{g}$, we have that $\Gamma = \{\Xi_u : u \in U\}$. We say that Σ has *full rank* if its trace generates the Lie algebra $\mathfrak{g}$ (i.e., $\mathsf{Lie}(\Gamma) = \mathfrak{g}$). We note that Σ satisfies the LARC at identity (and hence everywhere) if and only if Σ has full rank.

A *trajectory* for an admissible control $u(\cdot) : [0, T] \to U$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\Xi(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$. We say that a system Σ is *controllable* if for any $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(T) = g_1$. Necessary conditions for controllability are that the group G is connected and the system has full rank. Henceforth, we shall assume that all the systems under consideration have full rank and that all Lie groups under consideration are connected.

Left-invariant control *affine* systems are those systems for which the parametrization map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is affine. When the state space G is fixed, we specify such a system Σ by its parametrization map and simply write

$$\Sigma : \quad A + u_1B_1 + \cdots + u_\ell B_\ell.$$

Σ is said to be *homogeneous* if $A = \Xi(\mathbf{1}, 0) \in \operatorname{span}(B_1, \ldots, B_\ell)$, i.e., Γ is a linear subspace of $\mathfrak{g}$; otherwise Σ is *inhomogeneous*.

3 State space equivalence

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be left-invariant control systems with the same input space U. Then Σ and Σ' are called *locally state space equivalent* (shortly $\mathrm{S_{loc}}$-equivalent) at points $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if there exist open neighbourhoods N and N' of a and a', respectively, and a diffeomorphism $\phi : N \to N'$ (mapping a to a') such that $T_g\phi \cdot \Xi(g,u) = \Xi'(\phi(g),u)$ for $g \in N$ and $u \in U$. Systems Σ and Σ' are called *state space equivalent* (shortly S-equivalent) if this happens globally (i.e., $N = \mathsf{G}$ and $N' = \mathsf{G}'$).

Firstly, we characterize (global) S-equivalence.

Lemma 1. *Let $\phi : \mathsf{G} \to \mathsf{G}'$ be a diffeomorphism. The push-forward $\phi_* X$ of any left-invariant vector field X on G is left invariant if and only if ϕ is the composition of a Lie group isomorphism $\bar{\phi} : \mathsf{G} \to \mathsf{G}'$ and a left translation $L_{\phi(\mathbf{1})}$ on G', i.e., $\phi = L_{\phi(\mathbf{1})} \circ \bar{\phi}$.*

Proof. Suppose the push-forward $\phi_* X$ of any left-invariant vector field X on G is left invariant. By composition with an appropriate left translation, we may assume $\phi(\mathbf{1}) = \mathbf{1}$. Let $A \in \mathfrak{g}$ and $X(g) = gA$ be the corresponding left-invariant vector field. As $\phi_* X$ is left invariant, there exists $A' \in \mathfrak{g}'$ such that

$$(\phi_* X)(\phi(g)) = \phi(g)A'.$$

Thus, as ϕ maps the flow of X to the flow of $\phi_* X$, we have that

$$\phi(g \exp(tA)) = \phi(g) \exp(tA')$$

for all $g \in \mathsf{G}$. Consequently, we find that

$$\phi(g \exp(A)) = \phi(g)\phi(\exp(A))$$

for all $g \in \mathsf{G}$, $A \in \mathfrak{g}$. As any element $g \in \mathsf{G}$ can be written as a finite product

$$g = \exp(A_1)\exp(A_2)\cdots\exp(A_k)$$

where $A_1, \ldots, A_k \in \mathfrak{g}$, it follows that ϕ is a Lie group homomorphism (and hence an isomorphism). The converse is trivial. □

Theorem 1. *Σ and Σ' are S-equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1},u) = \Xi'(\mathbf{1},u)$ for all $u \in U$.*

Proof. Suppose that Σ and Σ' are S-equivalent. There exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $\phi_* \Xi_u = \Xi'_u$ for $u \in U$. Moreover,

$$\phi_*[\Xi_u, \Xi_{\bar{u}}] = [\phi_*\Xi_u, \phi_*\Xi_{\bar{u}}] = [\Xi'_u, \Xi'_{\bar{u}}]$$

for $u, \bar{u} \in U$. The same holds true for higher order brackets, i.e.,

$$\phi_*\big[\cdots[\Xi_{u_1}, \Xi_{u_2}], \ldots, \Xi_{u_k}\big] = \big[\cdots[\Xi'_{u_1}, \Xi'_{u_2}], \ldots, \Xi'_{u_k}\big].$$

As Σ has full rank, it follows that $\{\Xi_u : u \in U\}$ generates $\mathfrak{g}$. Hence, as the Lie bracket of any two left-invariant vector fields is left invariant, it follows that the push-forward $\phi_* X$ of any left-invariant vector field X on G is left invariant. By composition with an appropriate left-translation, we may assume that $\phi(\mathbf{1}) = \mathbf{1}$. Thus $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$ and, by Lemma 1, ϕ is a Lie group isomorphism.

Conversely, suppose that $\phi : \mathsf{G} \to \mathsf{G}'$ is a Lie group isomorphism as prescribed. Then $\phi \circ L_g = L_{\phi(g)} \circ \phi$ and so

$$T_g\phi \cdot \Xi(g, u) = T_{\mathbf{1}} L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\phi(g), u). \qquad \square$$

We now turn to $\mathrm{S}_{\mathrm{loc}}$-equivalence. Note that (by left translation) Σ and Σ' are $\mathrm{S}_{\mathrm{loc}}$-equivalent at $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if and only if they are $\mathrm{S}_{\mathrm{loc}}$-equivalent at $\mathbf{1} \in \mathsf{G}$ and $\mathbf{1} \in \mathsf{G}'$. We give a characterization of $\mathrm{S}_{\mathrm{loc}}$-equivalence, analogous to Theorem 1. The result may be proved by "localizing" the argument made in the proof of Theorem 1, or by considering the covering systems on the simply connected universal covering groups (cf. [3]) and applying Theorem 1; the result also follows as a fairly direct consequence of Proposition 1.

Theorem 2. *Σ and Σ' are S_{loc}-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$ for all $u \in U$.*

Remark 1. As left-invariant vector fields are complete, $\mathrm{S}_{\mathrm{loc}}$-equivalence implies S-equivalence when the state spaces are simply connected (Proposition 1). This fact can also be readily deduced from Theorems 1 and 2 by use of the following classic result: Let G and G' be connected and simply connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. If $\psi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra isomorphism, then there exists a unique Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi = \psi$ (see, e.g., [10]).

We conclude the section with an example of the classification, under $\mathrm{S}_{\mathrm{loc}}$-equivalence, of a class of systems on the three-dimensional Euclidean group.

Example 1. Any two-input inhomogeneous control affine system on the Euclidean group $\mathsf{SE}(2)$ is $\mathrm{S}_{\mathrm{loc}}$-equivalent to exactly one of the following systems

$$\begin{aligned}
\Sigma_{1,\alpha\beta\boldsymbol{\gamma}} &: \quad \alpha E_3 + u_1(E_1 + \gamma_1 E_2) + u_2(\beta E_2) \\
\Sigma_{2,\alpha\beta\boldsymbol{\gamma}} &: \quad \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2 \\
\Sigma_{3,\alpha\beta\boldsymbol{\gamma}} &: \quad \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3).
\end{aligned}$$

Here $\alpha > 0$, $\beta \neq 0$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives. (The standard basis elements E_1, E_2, E_3 of the Lie algebra $\mathfrak{se}(2)$ have commutator relations given by $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, and $[E_1, E_2] = 0$.) For a classification, under $\mathrm{S}_{\mathrm{loc}}$-equivalence, of full-rank left-invariant control affine systems on $\mathsf{SE}(2)$, see [1].

Indeed, the group of automorphisms of $\mathfrak{se}(2)$ is

$$\mathsf{Aut}(\mathfrak{se}(2)) = \left\{ \begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} : x, y, v, w \in \mathbb{R},\, x^2 + y^2 \neq 0,\, \varsigma = \pm 1 \right\}.$$

Let $\Sigma = (\mathsf{SE}(2), \Xi)$,

$$\Xi(\mathbf{1}, u) = \sum_{i=1}^{3} a_i E_i + u_1 \sum_{i=1}^{3} b_i E_i + u_2 \sum_{i=1}^{3} c_i E_i,$$

or in *matrix form*

$$\Sigma : \quad \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right].$$

It is then straightforward to show that there exists an automorphism $\psi \in \mathsf{Aut}(\mathfrak{se}(2))$ such that

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & \gamma_1 & \beta \\ \alpha & 0 & 0 \end{array}\right] \quad \text{if} \quad b_3 = 0,\, c_3 = 0$$

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} \beta & 0 & 0 \\ \gamma_1 & 0 & 1 \\ \gamma_2 & \alpha & 0 \end{array}\right] \quad \text{if} \quad b_3 \neq 0,\, c_3 = 0$$

or

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} \beta & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array}\right] \quad \text{if} \quad c_3 \neq 0.$$

Thus Σ is $\mathrm{S_{loc}}$-equivalent to $\Sigma_{1,\alpha\beta\boldsymbol{\gamma}}$, $\Sigma_{2,\alpha\beta\boldsymbol{\gamma}}$, or $\Sigma_{3,\alpha\beta\boldsymbol{\gamma}}$. It is a simple matter to verify that these class representatives are non-equivalent.

4 Detached feedback equivalence

We specialize feedback equivalence in the context of left-invariant control systems by requiring that the feedback transformations are G-invariant. Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be left-invariant control systems. Then Σ and Σ' are called *locally detached feedback equivalent* (shortly $\mathrm{DF_{loc}}$-equivalent) at points $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if there exist open neighbourhoods N and N' of a and a', respectively, and a diffeomorphism

$$\Phi : N \times U \to N' \times U', \quad (g, u) \mapsto (\phi(g), \bar{\varphi}(u))$$

such that $\phi(a) = a'$ and $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \bar{\varphi}(u))$ for $g \in N$ and $u \in U$. Systems Σ and Σ' are called *detached feedback equivalent* (shortly DF-equivalent) if this happens globally (i.e., $N = \mathsf{G}$ and $N' = \mathsf{G}'$).

We firstly characterize DF-equivalence. (The argument is very similar to the one used in the proof of Theorem 1.)

Theorem 3. *Σ and Σ' are DF-equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.*

Proof. Suppose that Σ and Σ' are DF-equivalent. There exists diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi : U \to U'$ such that $\phi_* \Xi_u = \Xi'_{\varphi(u)}$ for $u \in U$. Moreover,

$$\phi_*[\Xi_u, \Xi_{\bar{u}}] = [\phi_* \Xi_u, \phi_* \Xi_{\bar{u}}] = [\Xi'_{\varphi(u)}, \Xi'_{\varphi(\bar{u})}]$$

for $u, \bar{u} \in U$ and similarly for higher order brackets. Therefore it follows that the push-forward $\phi_* X$ of any left-invariant vector field X on G is left invariant. By composition with an appropriate left-translation, we may assume that $\phi(\mathbf{1}) = \mathbf{1}$. Thus

$$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$$

and so $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. Also, by Lemma 1, ϕ is a Lie group isomorphism.

Conversely, suppose that $\phi : \mathsf{G} \to \mathsf{G}'$ is a Lie group isomorphism as prescribed. As $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$, there exists a unique diffeomorphism $\phi : U \to U'$ such that

$$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u)).$$

Hence, as $\phi \circ L_g = L_{\phi(g)} \circ \phi$, it follows that

$$T_g\phi \cdot \Xi(g, u) = T_{\mathbf{1}}\phi(g) \cdot T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\phi(g), \varphi(u)). \qquad \square$$

Remark 2. Systems Σ and Σ' are F-equivalent if there exists a *diffeomorphism* $\phi : \mathsf{G} \to \mathsf{G}'$ such that (the push-forward) $\phi_* \mathcal{F} = \mathcal{F}'$. Here $g \mapsto \mathcal{F}(g) = g\Gamma$ is the field of admissible velocities. The specialization to DF-equivalence corresponds to the existence of a *Lie group isomorphism* ϕ such that $\phi_* \mathcal{F} = \mathcal{F}'$. Thus F-equivalence is weaker than DF-equivalence. For example, suppose $\Gamma = \mathfrak{g}$, $\Gamma' = \mathfrak{g}'$, and G is diffeomorphic to G'. Then Σ and Σ' are F-equivalent. However, Σ and Σ' will be DF-equivalent only if G and G' are, in addition, isomorphic as Lie groups.

We now proceed to $\mathrm{DF_{loc}}$-equivalence. We point out that systems Σ and Σ' are $\mathrm{DF_{loc}}$-equivalent at $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if and only if they are $\mathrm{DF_{loc}}$-equivalent at $\mathbf{1} \in \mathsf{G}$ and $\mathbf{1} \in \mathsf{G}'$. We give a characterization of $\mathrm{DF_{loc}}$-equivalence, analogous to Theorem 3. As with S-equivalence, the result may be proved by "localizing" the argument made in the proof of Theorem 3, or by considering the covering systems on the simply connected universal covering groups (cf. [3]) and applying Theorem 3. Alternatively, one can make use of the fact that any $\mathrm{DF_{loc}}$-equivalence (resp. DF-equivalence) transformation decomposes into a $\mathrm{S_{loc}}$-equivalence (resp. S-equivalence) transformation and a reparametrization (by which we mean a transformation of the form $\Xi'(g, u) = \Xi(g, \varphi(u))$).

Theorem 4. *Σ and Σ' are DF_{loc}-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$.*

Remark 3. As with S-equivalence, we have that $\mathrm{DF_{loc}}$-equivalence implies DF--equivalence when the state spaces are simply connected (cf. Remark 1).

We revisit the class of systems considered in Example 1 and, in contrast, now classify these systems up to $\mathrm{DF_{loc}}$-equivalence. We also give an example of the classification of a class of systems on another three-dimensional Lie group, namely the pseudo-orthogonal group.

Example 2. Any two-input inhomogeneous control affine system on $\mathsf{SE}(2)$ is $\mathrm{DF}_{\mathrm{loc}}$-equivalent to exactly one of the following systems (see [6])

$$\begin{aligned} \Sigma_1 : \quad & E_1 + u_1 E_2 + u_2 E_3 \\ \Sigma_{2,\alpha} : \quad & \alpha E_3 + u_1 E_1 + u_2 E_2. \end{aligned}$$

Here $\alpha > 0$ parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

Indeed, let $\Sigma = (\mathsf{SE}(2), \Xi)$ be an inhomogeneous system with trace

$$\Gamma = \sum_{i=1}^{3} a_i E_i + \left\langle \sum_{i=1}^{3} b_i E_i, \sum_{i=1}^{3} c_i E_i \right\rangle.$$

If $c_3 \neq 0$ or $b_3 \neq 0$, then

$$\Gamma = a_1' E_1 + a_2' E_2 + \left\langle b_1' E_1 + b_2' E_2, c_1' E_1 + c_2' E_2 + E_3 \right\rangle.$$

Now either $b_1' \neq 0$ or $b_2' \neq 0$, and so

$$\begin{bmatrix} b_2' & -b_1' \\ b_1' & b_2' \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_2' \\ a_1' \end{bmatrix}$$

has a unique solution (with $v_2 \neq 0$). Therefore

$$\psi = \begin{bmatrix} v_2 b_2' & v_2 b_1' & c_1' \\ -v_2 b_1' & v_2 b_2' & c_2' \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that

$$\psi \cdot \Gamma_1 = \psi \cdot (E_1 + \langle E_2, E_3 \rangle) = \Gamma.$$

Thus Σ is $\mathrm{DF}_{\mathrm{loc}}$-equivalent to Σ_1. On the other hand, suppose $b_3 = 0$ and $c_3 = 0$. Then $\Gamma = a_3 E_3 + \langle E_1, E_2 \rangle$. Hence $\psi = \mathrm{diag}(1, 1, \mathrm{sgn}(a_3))$ is an automorphism such that $\psi \cdot \Gamma = \alpha E_3 + \langle E_1, E_2 \rangle$ with $\alpha > 0$. Thus Σ is $\mathrm{DF}_{\mathrm{loc}}$-equivalent to $\Sigma_{2,\alpha}$. As the subspace $\langle E_1, E_2 \rangle$ is invariant (under automorphisms), Σ_1 and $\Sigma_{2,\alpha}$ cannot be $\mathrm{DF}_{\mathrm{loc}}$-equivalent. It is easy to show that $\Sigma_{2,\alpha}$ a $\Sigma_{2,\alpha'}$ are $\mathrm{DF}_{\mathrm{loc}}$-equivalent only if $\alpha = \alpha'$.

Example 3. Any two-input homogeneous control affine system on the pseudo-orthogonal group $\mathsf{SO}(2,1)$ is $\mathrm{DF}_{\mathrm{loc}}$-equivalent to exactly one of the following systems (see [4])

$$\begin{aligned} \Sigma_1 : \quad & \Xi_1(\mathbf{1}, u) = u_1 E_1 + u_2 E_2 \\ \Sigma_2 : \quad & \Xi_2(\mathbf{1}, u) = u_1 E_2 + u_2 E_3. \end{aligned}$$

(Here the commutator relations are given by $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, and $[E_1, E_2] = -E_3$.)

Indeed, the group of automorphisms of $\mathfrak{so}(2,1)$ is

$$\mathsf{Aut}(\mathfrak{so}(2,1)) = \mathsf{SO}(2,1) = \left\{ g \in \mathbb{R}^{3\times 3} : g^\top J g = J,\ \det g = 1 \right\}.$$

Here $J = \operatorname{diag}(1,1,-1)$ and each automorphism ψ is identified with its corresponding matrix g. The Lorentzian product $\odot$ on $\mathfrak{so}(2,1)$ is given by

$$A \odot B = a_1 b_1 + a_2 b_2 - a_3 b_3.$$

(Here $A = \sum_{i=1}^{3} a_i E_i$ and $B = \sum_{i=1}^{3} b_i E_i$.) Any automorphism ψ preserves $\odot$, i.e., $(\psi \cdot A) \odot (\psi \cdot B) = A \odot B$.

Let Σ be a system with trace $\Gamma = \langle A, B \rangle$. The *sign* $\sigma(\Gamma)$ of Γ is given by

$$\sigma(\Gamma) = \operatorname{sgn}\left(\begin{vmatrix} A \odot A & A \odot B \\ A \odot B & B \odot B \end{vmatrix} \right).$$

($\sigma(\Gamma)$ does not depend on the parametrization.) As $\odot$ is preserved by automorphisms, it follows that $\sigma(\psi \cdot \Gamma) = \sigma(\Gamma)$. A straightforward computation shows that if $\sigma(\Gamma) = 0$, then Σ does not have full rank.

Suppose $\sigma(\Gamma) = -1$. Then we may assume that $a_3 \neq 0$ or $b_3 \neq 0$. Hence

$$\Gamma = \left\langle a_1' E_1 + a_2' E_2 + E_3, r \sin\theta E_1 + r \cos\theta E_2 \right\rangle.$$

Thus

$$\psi = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle a_1'' E_1 + E_3, E_2 \rangle$. Now, as $\sigma(\psi \cdot \Gamma) = -1$, we have $(a_1'')^2 - 1 < 0$ and so $\psi \cdot \Gamma = \langle \sinh\vartheta E_1 + \cosh\vartheta E_3, E_2 \rangle$. Therefore

$$\psi' = \begin{bmatrix} \cosh\vartheta & 0 & -\sinh\vartheta \\ 0 & 1 & 0 \\ -\sinh\vartheta & 0 & \cosh\vartheta \end{bmatrix}$$

is an automorphism such that $\psi' \cdot \psi \cdot \Gamma = \langle E_3, E_2 \rangle$. Thus Σ is $\mathrm{DF}_{\mathrm{loc}}$-equivalent to Σ_1.

If $\sigma(\Gamma) = 1$, then a similar argument shows that there exists an automorphism ψ such that $\psi \cdot \Gamma = \langle E_1, E_2 \rangle$ (and so Σ is $\mathrm{DF}_{\mathrm{loc}}$-equivalent to Σ_2). Lastly, Σ_1 and Σ_2 are non-equivalent systems, as $\sigma(\Gamma_1) = 1$ and $\sigma(\Gamma_2) = -1$.

5 Conclusion

In recent decades, attention has been drawn to invariant control systems evolving on (matrix) Lie groups of low dimension. We believe that this paper facilitates the structuring and comparison of such systems. We summarize the results in a table (see the next page).

Equivalence	Characterization	
S-equiv	$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1},\cdot) = \Xi'(\mathbf{1},\cdot)$	$\phi : \mathsf{G} \to \mathsf{G}'$ Lie group isomorphism
DF-equiv	$T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$	
$\mathrm{S}_{\mathrm{loc}}$-equiv	$\psi \cdot \Xi(\mathbf{1},\cdot) = \Xi'(\mathbf{1},\cdot)$	$\psi : \mathfrak{g} \to \mathfrak{g}'$ Lie algebra isomorphism
$\mathrm{DF}_{\mathrm{loc}}$-equiv	$\psi \cdot \Gamma = \Gamma'$	

The (four) characterizations of equivalences provide efficient means to classify various distinguished subclasses of left-invariant control systems. For instance, if one considers the problem of classifying under $\mathrm{DF}_{\mathrm{loc}}$-equivalence, one may restrict to systems with a fixed Lie algebra $\mathfrak{g}$. Σ and Σ' are then $\mathrm{DF}_{\mathrm{loc}}$-equivalent if and only if their traces Γ and Γ' are equivalent under the relation

$$\Gamma \sim \Gamma' \iff \exists \psi \in \mathsf{Aut}(\mathsf{G}),\ \psi \cdot \Gamma = \Gamma'.$$

This reduces the problem of classifying control affine systems (under $\mathrm{DF}_{\mathrm{loc}}$-equivalence) to that of classifying affine subspaces of $\mathfrak{g}$. In the case of control affine systems evolving on three-dimensional Lie groups, a full classification under $\mathrm{DF}_{\mathrm{loc}}$-equivalence has been obtained in [4], [5], [6].

Acknowledgements

This research was supported in part by the European Union's Seventh Framework Programme (FP7/2007–2013, grant no. 317721). R. Biggs would also like to acknowledge the financial support of the Claude Leon Foundation.

References

[1] R.M. Adams, R. Biggs, C.C. Remsing: Equivalence of control systems on the Euclidean group SE(2). *Control Cybernet.* 41 (2012) 513–524.

[2] A.A. Agrachev, Y.L. Sachkov: *Control Theory from the Geometric Viewpoint.* Springer Science & Business Media (2004).

[3] R. Biggs, C.C. Remsing: A category of control systems. *An. Şt. Univ. Ovidius Constanţa* 20 (2012) 355–368.

[4] R. Biggs, C.C. Remsing: Control affine systems on semisimple three-dimensional Lie groups. *An. Şt. Univ. "A.I. Cuza" Iaşi. Ser. Mat.* 59 (2013) 399–414.

[5] R. Biggs, C.C. Remsing: Control affine systems on solvable three-dimensional Lie groups, I. *Arch. Math. (Brno)* 49 (2013) 187–197.

[6] R. Biggs, C.C. Remsing: Control affine systems on solvable three-dimensional Lie groups, II. *Note Mat.* 33 (2013) 19–31.

[7] R.W. Brockett: System theory on group manifolds and coset spaces. *SIAM J. Control* 10 (1972) 265–284.

[8] V.I. Elkin: Affine control systems: their equivalence, classification, quotient systems, and subsystems. *J. Math. Sci.* 88 (1998) 675–721.

[9] R.B. Gardner, W.F. Shadwick: Feedback equivalence of control systems. *Systems Control Lett.* 8 (1987) 463–465.

[10] V.V. Gorbatsevich, A.L. Onishchik, E.B. Vinberg: *Foundations of Lie Theory and Lie Transformation Groups*. Springer Science & Business Media (1997).

[11] B. Jakubczyk: Equivalence and invariants of nonlinear control systems. *Nonlinear Controllability and Optimal Control* (1990) 177–218. In: H.J. Sussmann (ed.)

[12] B. Jakubczyk: Critical Hamiltonians and feedback invariants. *Geometry of Feedback and Optimal Control* (1998) 219–256. In: B. Jakubczyk, W. Respondek (eds.)

[13] B. Jakubczyk, W. Respondek: On linearization of control systems. *Bull. Acad. Polon. Sci. Ser. Sci. Math.* 28 (1980) 517–522.

[14] V. Jurdjevic: *Geometric Control Theory*. Cambridge University Press (1997).

[15] V. Jurdjevic, H.J. Sussmann: Control systems on Lie groups. *J. Diff. Equations* 12 (1972) 313–329.

[16] A.J. Krener: On the equivalence of control systems and the linearization of nonlinear systems. *SIAM J. Control* 11 (1973) 670–676.

[17] C.C. Remsing: Optimal control and Hamilton-Poisson formalism. *Int. J. Pure Appl. Math.* 59 (2010) 11–17.

[18] W. Respondek, I.A. Tall: Feedback equivalence of nonlinear control systems: a survey on formal approach. *Chaos in Automatic Control* (2006) 137–262. In: W. Perruquetti, J.-P. Barbot (eds.)

[19] Y.L. Sachkov: Control theory on Lie groups. *J. Math. Sci.* 156 (2009) 381–439.

[20] H.J. Sussmann: An extension of a theorem of Nagano on transitive Lie algebras. *Proc. Amer. Math. Soc.* 45 (1974) 349–356.

[21] H.J. Sussmann: Lie brackets, real analyticity and geometric control. *Differential Geometric Control Theory* (1983) 1–116. In: R.W. Brockett, R.S. Millman, H.J. Sussmann (eds.)

Authors' address:
Department of Mathematics (Pure and Applied), Rhodes University, PO Box 94, 6140 Grahamstown, South Africa

E-mail: `rorybiggs@gmail.com`, `c.c.remsing@ru.ac.za`

Received: 19 November, 2014
Accepted for publication: 23 November, 2015
Communicated by: Anthony Bloch

Communications in Mathematics 23 (2015) 131–142

Partial Fuzzy Metric Space and Some Fixed Point Results

Shaban Sedghi, Nabi Shobkolaei, Ishak Altun

Abstract. In this paper, we introduce the concept of partial fuzzy metric on a nonempty set X and give the topological structure and some properties of partial fuzzy metric space. Then some fixed point results are provided.

1 Introduction and preliminaries

We recall some basic definitions and results from the theory of fuzzy metric spaces, used in the sequel.

Definition 1. [5] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if it satisfies the following conditions:

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0,1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norms are $a*b = ab$ and $a*b = \min\{a,b\}$.

Definition 2. [1] A triple $(X, M, *)$ is called a *fuzzy metric space* (in the sense of George and Veeramani) if X is a nonempty set, $*$ is a continuous t-norm and $M : X^2 \times (0,\infty) \to [0,1]$ is a fuzzy set satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

1. $M(x,y,t) > 0$,
2. $M(x,y,t) = 1 \Leftrightarrow x = y$,

2010 MSC: 54H25, 47H10
Key words: Fixed point, Fuzzy metric space, Partial fuzzy metric space.

3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
5. $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is a continuous mapping

If the fourth condition is replaced by

4′. $M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s)$,

then the space $(X, M, *)$ is said to be a *non-Archimedean fuzzy metric space*. It should be noted that any non-Archimedean fuzzy metric space is a fuzzy metric space.

The following properties of M noted in the theorem below are easy consequences of the definition.

Theorem 1. *Let $(X, M, *)$ be a fuzzy metric space.*

1. *$M(x, y, t)$ is nondecreasing with respect to t for each $x, y \in X$,*
2. *If M is non-Archimedean, then $M(x, y, t) \geq M(x, z, t) * M(z, y, t)$ for all $x, y, z \in X$ and $t > 0$.*

Example 1. Let (X, d) be an ordinary metric space and $a*b = ab$ for all $a, b \in [0, 1]$. Then the fuzzy set M on $X^2 \times (0, \infty)$ defined by

$$M(x, y, t) = \exp\left(-\frac{d(x, y)}{t}\right),$$

is a fuzzy metric on X.

Example 2. Let $a * b = ab$ for all $a, b \in [0, 1]$ and M be the fuzzy set on $\mathbb{R}^+ \times \mathbb{R}^+ \times (0, \infty)$ (where $\mathbb{R}^+ = (0, \infty)$) defined by

$$M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}},$$

for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, M, *)$ is a fuzzy metric space.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with centre $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. This definition of Cauchy sequence is identical with that given by George and Veeramani.

The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent.

The fixed point theory in fuzzy metric spaces started with the paper of Grabiec [2]. Later on, the concept of fuzzy contractive mappings, initiated by Gregori and Sapena in [3], have become of interest for many authors, see, e.g., the papers [3], [7], [8], [9], [10], [11].

In our paper we present the concept of partial fuzzy metric space and some properties of it. Then we give some fundamental fixed point theorem on complete partial fuzzy metric space.

2 Partial fuzzy metric space

In this section we introduce the concept of partial fuzzy metric space and give its properties.

Definition 3. A *partial fuzzy metric* on a nonempty set X is a function

$$P_M : X \times X \times (0, \infty) \to [0, 1]$$

such that for all $x, y, z \in X$ and $t, s > 0$

(PM1) $x = y \Leftrightarrow P_M(x, x, t) = P_M(x, y, t) = P_M(y, y, t)$,

(PM2) $P_M(x, x, t) \geq P_M(x, y, t)$,

(PM3) $P_M(x, y, t) = P_M(y, x, t)$,

(PM4) $P_M(x, y, \max\{t, s\}) * P_M(z, z, \max\{t, s\}) \geq P_M(x, z, t) * P_M(z, y, s)$.

(PM5) $P_M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

A partial fuzzy metric space is a 3-tuple $(X, P_M, *)$ such that X is a nonempty set and P_M is a partial fuzzy metric on X. It is clear that, if $P_M(x, y, t) = 1$, then from (PM1) and (PM2) $x = y$. But if $x = y$, $P_M(x, y, t)$ may not be 1. A basic example of a partial fuzzy metric space is the 3-tuple $(\mathbb{R}^+, P_M, *)$, where

$$P_M(x, y, t) = \frac{t}{t + \max\{x, y\}}$$

for all $t > 0$, $x, y \in \mathbb{R}^+$ and $a * b = ab$.

From (PM4) for all $x, y, z \in X$ and $t > 0$, we have:

$$P_M(x, y, t) * P_M(z, z, t) \geq P_M(x, z, t) * P_M(z, y, t).$$

Let $(X, M, *)$ and $(X, P_M, *)$ be a fuzzy metric space and partial fuzzy metric space, respectively. Then mappings $P_{M_i} : X \times X \times (0, \infty) \to [0, 1]$ $(i \in \{1, 2\})$ defined by

$$P_{M_1}(x, y, t) = M(x, y, t) * P_M(x, y, t)$$

and

$$P_{M_2}(x, y, t) = M(x, y, t) * a$$

are partial fuzzy metrics on X, where $0 < a < 1$.

Theorem 2. *The partial fuzzy metric $P_M(x, y, t)$ is nondecreasing with respect to t for each $x, y \in X$ and $t > 0$, if the continuous t-norm $*$ satisfies the following condition for all $a, b, c \in [0, 1]$*

$$a * b \geq a * c \Rightarrow b \geq c.$$

Proof. From (PM4) for all $x, y, z \in X$ and $t, s > 0$, we have:

$$P_M(x, y, \max\{t, s\}) * P_M(z, z, \max\{t, s\}) \geq P_M(x, z, s) * P_M(z, y, t).$$

Let $t > s$, then taking $z = y$ in above inequality we have

$$P_M(x, y, t) * P_M(y, y, t) \geq P_M(x, y, s) * P_M(y, y, t),$$

hence by assume we get $P_M(x, y, t) \geq P_M(x, y, s)$. □

It is easy to see that every fuzzy metric is a partial fuzzy metric, but the converse may not be true. In the following examples, the partial fuzzy metrics fails to satisfy properties of fuzzy metric.

Example 3. Let (X, p) is a partial metric space in the sense of Matthews [6] and $P_M : X \times X \times (0, \infty) \to [0, 1]$ be a mapping defined as

$$P_M(x, y, t) = \frac{t}{t + p(x, y)},$$

or

$$P_M(x, y, t) = \exp\left(-\frac{p(x, y)}{t}\right).$$

If $a * b = ab$ for all $a, b \in [0, 1]$, then clearly P_M is a partial fuzzy metric, but it may not be a fuzzy metric.

Lemma 1. *Let $(X, P_M, *)$ be a partial fuzzy metric space with $a * b = ab$ for all $a, b \in [0, 1]$. If we define $p : X^2 \to [0, \infty)$ by*

$$p(x, y) = \sup_{\alpha \in (0,1)} \int_{\alpha}^{1} \log_a(P_M(x, y, t))\,\mathrm{d}t,$$

then p is a partial metric on X for fixed $0 < a < 1$.

Proof. It is clear from the definition that $p(x, y)$ is well defined for each $x, y \in X$ and $p(x, y) \geq 0$ for all $x, y \in X$.

1. For all $t > 0$

$$p(x, x) = p(x, y) = p(y, y) \Leftrightarrow P_M(x, x, t) = P_M(x, y, t) = P_M(y, y, t) \Leftrightarrow x = y.$$

2.
$$\begin{aligned} p(x, x) &= \sup_{\alpha \in (0,1)} \int_{\alpha}^{1} \log_a(P_M(x, x, t))\,\mathrm{d}t \\ &\leq \sup_{\alpha \in (0,1)} \int_{\alpha}^{1} \log_a(P_M(x, y, t))\,\mathrm{d}t \\ &= p(x, y). \end{aligned}$$

3.
$$p(x,y) = \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a(P_M(x,y,t))\,\mathrm{d}t = \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a(P_M(y,x,t))\,\mathrm{d}t = p(y,x).$$

4. Since
$$P_M(x,y,t)P_M(z,z,t) \geq P_M(x,z,t)P_M(z,y,t),$$
and $\log_a$ is decreasing, it follows that
$$\log_a(P_M(x,y,t)) + \log_a(P_M(z,z,t)) \leq \log_a(P_M(x,z,t)) + \log_a(P_M(z,y,t)),$$
hence
$$\begin{aligned} p(x,y)+p(z,z) &= \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a(P_M(x,y,t))\,\mathrm{d}t + \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a(P_M(z,z,t))\,\mathrm{d}t \\ &\leq \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a(P_M(x,z,t))\,\mathrm{d}t + \sup_{\alpha\in(0,1)} \int_\alpha^1 \log_a(P_M(z,y,t))\,\mathrm{d}t \\ &= p(x,z)+p(z,y). \end{aligned}$$

This proves that p is a partial metric on X. □

Definition 4. Let $(X, P_M, *)$ be a partial fuzzy metric space.

1. A sequence $\{x_n\}$ in a partial fuzzy metric space $(X, P_M, *)$ *converges* to x if and only if $P_M(x,x,t) = \lim_{n\to\infty} P_M(x_n,x,t)$ for every $t > 0$.
2. A sequence $\{x_n\}$ in a partial fuzzy metric space $(X, P_M, *)$ is called a *Cauchy sequence* if $\lim_{n,m\to\infty} P_M(x_n,x_m,t)$ exists.
3. A partial fuzzy metric space $(X, P_M, *)$ is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$.

Suppose that $\{x_n\}$ is a sequence in partial fuzzy metric space $(X, P_M, *)$, then we define $L(x_n) = \{x \in X : x_n \to x\}$. In the following example shows that every convergent sequence $\{x_n\}$ in a partial fuzzy metric space $(X, P_M, *)$ fails to satisfy Cauchy sequence. In particular, it shows that the limit of a convergent sequence is not unique.

Example 4. Let $X = [0,\infty)$ and $P_M(x,y,t) = \frac{t}{t+\max\{x,y\}}$, then it is clear that $(X, P_M, *)$ is a partial fuzzy metric space where $a * b = ab$ for all $a, b \in [0,1]$. Let $\{x_n\} = \{1,2,1,2,\dots\}$. Then clearly it is convergent sequence and for every $x \geq 2$ we have
$$\lim_{n\to\infty} P_M(x_n,x,t) = P_M(x,x,t),$$
therefore
$$L(x_n) = \{x \in X : x_n \to x\} = [2,\infty).$$
but $\lim_{n,m\to\infty} P_M(x_n,x_m,t)$ is not exist, that is, $\{x_n\}$ is not Cauchy sequence.

The following Lemma shows that under certain conditions the limit of a convergent sequence is unique.

Lemma 2. *Let $\{x_n\}$ be a convergent sequence in partial fuzzy metric space $(X, P_M, *)$ such that $a * b \geq a * c \Rightarrow b \geq c$ for all $a, b, c \in [0, 1]$, $x_n \to x$ and $x_n \to y$. If*

$$\lim_{n\to\infty} P_M(x_n, x_n, t) = P_M(x, x, t) = P_M(y, y, t),$$

then $x = y$.

Proof. As

$$P_M(x, y, t) * P_M(x_n, x_n, t) \geq P_M(x, x_n, t) * P_M(y, x_n, t),$$

taking limit as $n \to \infty$, we have

$$P_M(x, y, t) * P_M(x, x, t) \geq P_M(x, x, t) * P_M(y, y, t).$$

By given assumptions and from (PM2), we have

$$P_M(y, y, t) \geq P_M(x, y, t) \geq P_M(y, y, t),$$

which shows that $P_M(x, y, t) = P_M(y, y, t) = P_M(x, x, t)$, therefore $x = y$. □

Lemma 3. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in partial fuzzy metric space $(X, P_M, *)$ such that $a * b \geq a * c \Rightarrow b \geq c$ for all $a, b, c \in [0, 1]$,*

$$\lim_{n\to\infty} P_M(x_n, x, t) = \lim_{n\to\infty} P_M(x_n, x_n, t) = P_M(x, x, t),$$

and

$$\lim_{n\to\infty} P_M(y_n, y, t) = \lim_{n\to\infty} P_M(y_n, y_n, t) = P_M(y, y, t),$$

then $\lim_{n\to\infty} P_M(x_n, y_n, t) = P_M(x, y, t)$. In particular, for every $z \in X$

$$\lim_{n\to\infty} P_M(x_n, z, t) = \lim_{n\to\infty} P_M(x, z, t).$$

Proof. As

$$P_M(x_n, y_n, t) * P_M(x, x, t) \geq P_M(x_n, x, t) * P_M(x, y_n, t),$$

therefore

$$\begin{aligned} P_M(x_n, y_n, t) * P_M(x, x, t) * P_M(y, y, t) &\geq P_M(x_n, x, t) * P_M(x, y_n, t) * P_M(y, y, t) \\ &\geq P_M(x_n, x, t) * P_M(x, y, t) * P_M(y, y_n, t). \end{aligned}$$

Thus

$$\begin{aligned} &\limsup_{n\to\infty} P_M(x_n, y_n, t) * P_M(x, x, t) * P_M(y, y, t) \\ &\qquad \geq \limsup_{n\to\infty} P_M(x_n, x, t) * P_M(x, y, t) * \limsup_{n\to\infty} P_M(y, y_n, t) \\ &\qquad = P_M(x, x, t) * P_M(x, y, t) * P_M(y, y, t), \end{aligned}$$

hence

$$\limsup_{n\to\infty} P_M(x_n, y_n, t) \geq P_M(x, y, t).$$

Also, as

$$P_M(x, y, t) * P_M(x_n, x_n, t) \geq P_M(x, x_n, t) * P_M(x_n, y, t),$$

therefore

$$\begin{aligned} P_M(x, y, t) * P_M(x_n, x_n, t) * P_M(y_n, y_n, t) \\ &\geq P_M(x, x_n, t) * P_M(x_n, y, t) * P_M(y_n, y_n, t) \\ &\geq P_M(x, x_n, t) * P_M(x_n, y_n, t) * P_M(y_n, y, t) \end{aligned}$$

Thus

$$\begin{aligned} P_M(x, y, t) &* P_M(x, x, t) * P_M(y, y, t) \\ &= P_M(x, y, t) * \limsup_{n\to\infty} P_M(x_n, x_n, t) * \limsup_{n\to\infty} P_M(y_n, y_n, t) \\ &\geq \limsup_{n\to\infty} P_M(x, x_n, t) * \limsup_{n\to\infty} P_M(x_n, y_n, t) * \limsup_{n\to\infty} P_M(y_n, y, t) \\ &= P_M(x, x, t) * \limsup_{n\to\infty} P_M(x_n, y_n, t) * P_M(y, y, t). \end{aligned}$$

Therefore

$$P_M(x, y, t) \geq \limsup_{n\to\infty} P_M(x_n, y_n, t).$$

That is,

$$\limsup_{n\to\infty} P_M(x_n, y_n, t) = P_M(x, y, t).$$

Similarly, we have

$$\limsup_{n\to\infty} P_M(x_n, y_n, t) = P_M(x, y, t).$$

Hence the result follows. □

Definition 5. Let $(X, P_M, *)$ be a partial fuzzy metric space. P_M is said to be upper semicontinuous on X if for every $x \in X$,

$$P_M(p, x, t) \geq \limsup_{n\to\infty} P_M(x_n, x, t),$$

whenever $\{x_n\}$ is a sequence in X which converges to a point $p \in X$.

3 Fixed point results

Let $(X, P_M, *)$ be a partial fuzzy metric space and $\emptyset \neq S \subseteq X$. Define

$$\delta_{P_M}(S, t) = \inf\{P_M(x, y, t) : x, y \in S\}$$

for all $t > 0$. For an $A_n = \{x_n, x_{n+1}, \dots\}$ in partial fuzzy metric space $(X, P_M, *)$, let $r_n(t) = \delta_{P_M}(A_n, t)$. Then $r_n(t)$ is finite for all $n \in \mathbb{N}$, $\{r_n(t)\}$ is nonincreasing, $r_n(t) \to r(t)$ for some $0 \leq r(t) \leq 1$ and also $r_n(t) \leq P_M(x_l, x_k, t)$ for all $l, k \geq n$.

Let $\mathcal{F}$ be the set of all continuous functions $F : [0,1]^3 \times [0,1] \to [-1,1]$ such that F is nondecreasing on $[0,1]^3$ satisfying the following condition:

- $F((u,u,u),v) \leq 0$ implies that $v \geq \gamma(u)$ where $\gamma : [0,1] \to [0,1]$ is a nondecreasing continuous function with $\gamma(s) > s$ for $s \in [0,1)$.

Example 5. Let $\gamma(s) = s^h$ for $0 < h < 1$, then the functions F defined by

$$F\big((t_1,t_2,t_3),t_4\big) = \gamma\big(\min\{t_1,t_2,t_3\}\big) - t_4$$

and

$$F\big((t_1,t_2,t_3),t_4\big) = \gamma\left(\sum_{i=1}^{3} a_i t_i\right) - t_4,$$

where $a_i \geq 0$, $\sum\limits_{i=1}^{3} a_i = 1$, belong to $\mathcal{F}$.

Now we give our main theorem.

Theorem 3. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space, P_M is upper semicontinuous function on X and T be a self map of X satisfying*

$$F\big(P_M(x,y,t), P_M(Tx,x,t), P_M(Tx,y,t), P_M(Tx,Ty,t)\big) \leq 0 \tag{1}$$

for all $x, y \in X$, where $F \in \mathcal{F}$. Then T has a unique fixed point p in X and T is continuous at p.

Proof. Let $x_0 \in X$ and $Tx_n = x_{n+1}$. Let $r_n(t) = \delta_{P_M}(A_n,t)$, where $A_n = \{x_n, x_{n+1}, \dots\}$. Then we know $\lim\limits_{n\to\infty} r_n(t) = r(t)$ for some $0 \leq r(t) \leq 1$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then T has a fixed point. Assume that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ be fixed. Taking $x = x_{n-1}$, $y = x_{n+m-1}$ in (1) where $n \geq k$ and $m \in \mathbb{N}$, we have

$$\begin{aligned} F&\begin{pmatrix} P_M(x_{n-1}, x_{n+m-1}, t), P_M(Tx_{n-1}, x_{n-1}, t), \\ P_M(Tx_{n-1}, x_{n+m-1}, t), P_M(Tx_{n-1}, Tx_{n+m-1}, t) \end{pmatrix} \\ &= F\begin{pmatrix} P_M(x_{n-1}, x_{n+m-1}, t), P_M(x_n, x_{n-1}, t), \\ P_M(x_n, x_{n+m-1}, t), P_M(x_n, x_{n+m}, t) \end{pmatrix} \leq 0 \end{aligned}$$

Thus we have

$$F\big(r_{n-1}(t), r_{n-1}(t), r_n(t), P_M(x_n, x_{n+m}, t)\big) \leq 0,$$

since F is nondecreasing on $[0,1]^3$. Also, since $r_n(t)$ is nonincreasing, we have

$$F\big(r_{k-1}(t), r_{k-1}(t), r_{k-1}(t), P_M(x_n, x_{n+m}, t)\big) \leq 0,$$

which implies that

$$P_M(x_n, x_{n+m}, t) \geq \gamma(r_{k-1}(t)).$$

Thus for all $n \geq k$, we have

$$\inf_{n\geq k}\big\{P_M(x_n, x_{n+m}, t)\big\} = r_k(t) \geq \gamma(r_{k-1}(t)).$$

Letting $k \to \infty$, we get $r(t) \geq \gamma(r(t))$. If $r(t) \neq 1$, then $r(t) \geq \gamma(r(t)) > r(t)$, which is a contradiction. Thus $r(t) = 1$ and hence $\lim_{n\to\infty} \gamma_n(t) = 1$. Thus given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $r_n(t) > 1 - \varepsilon$. Then we have for $n \geq n_0$ and $m \in \mathbb{N}$, $P_M(x_n, x_{n+m}, t) > 1 - \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists a $p \in X$ such that

$$\lim_{n\to\infty} P_M(x_n, p, t) = P_M(p, p, t).$$

Taking $x = x_n, y = p$ in (1), we have

$$\begin{aligned}F\big(&P_M(x_n, p, t), P_M(Tx_n, p, t), P_M(Tx_n, x_n, t), P_M(Tx_n, Tp, t)\big)\\ &= F\big(P_M(x_n, p, t), P_M(x_{n+1}, p, t), P_M(x_{n+1}, x_n, t), P_M(x_{n+1}, Tp, t)\big) \leq 0.\end{aligned}$$

Hence, we have

$$\begin{aligned}\limsup_{n\to\infty} &F\big(P_M(x_n, p, t), P_M(x_{n+1}, p, t), P_M(x_{n+1}, x_n, t), P_M(x_{n+1}, Tp, t)\big)\\ &= F\big(P_M(p, p, t), P_M(p, p, t), 1, \limsup_{n\to\infty} P_M(x_{n+1}, Tp, t)\big) \leq 0.\end{aligned}$$

Since

$$\begin{aligned}F\big(&P_M(p, p, t), P_M(p, p, t), P_M(p, p, t), \limsup_{n\to\infty} P_M(x_{n+1}, Tp, t)\big)\\ &\leq F\big(P_M(p, p, t), P_M(p, p, t), 1, \limsup_{n\to\infty} P_M(x_{n+1}, Tp, t)\big) \leq 0,\end{aligned}$$

which implies

$$P_M(p, Tp, t) \geq \limsup_{n\to\infty} P_M(x_{n+1}, Tp, t) \geq \gamma(P_M(p, p, t)).$$

On the other hand, we have

$$P_M(p, p, t) \geq P_M(p, Tp, t) \geq \gamma(P_M(p, p, t)).$$

Hence $P_M(p, p, t) = 1$. Also, since

$$P_M(p, Tp, t) \geq \gamma(P_M(p, p, t)) = \gamma(1) = 1,$$

this implies that $P_M(p, Tp, t) = 1$, therefore, we get $Tp = p$.

For the uniqueness, let p and w be fixed points of T. Taking $x = p, y = w$ in (1), we have

$$\begin{aligned}F\big(&P_M(p, w, t), P_M(Tp, p, t), P_M(Tp, w, t), P_M(Tp, Tw, t)\big)\\ &= F\big(P_M(p, w, t), P_M(p, p, t), P_M(p, w, t), P_M(p, w, t)\big) \leq 0.\end{aligned}$$

Since F is nondecreasing on $[0,1]^3$, we have

$$F\big(P_M(p,w,t), P_M(p,w,t), P_M(p,w,t), P_M(p,w,t)\big) \le 0,$$

which implies

$$P_M(p,w,t) \ge \gamma(P_M(p,w,t)) > P_M(p,w,t)$$

which is a contradiction. Thus we have $P_M(p,w,t) = 1$, therefore, $p = w$. Now, we show that T is continuous at p. Let $\{y_n\}$ be a sequence in X and $\lim_{n\to\infty} y_n = p$. Taking $x = p, y = y_n$ in (1), we have

$$\begin{aligned} &F\big(P_M(p,y_n,t), P_M(Tp,p,t), P_M(Tp,y_n,t), P_M(Tp,Ty_n,t)\big) \\ &\qquad = F\big(P_M(p,y_n,t), P_M(p,p,t), P_M(p,y_n,t), P_M(p,Ty_n,t)\big) \le 0, \end{aligned}$$

hence

$$\begin{aligned} &F\big(P_M(p,p,t), P_M(p,p,t), P_M(p,p,t), \limsup_{n\to\infty} P_M(p,Ty_n,t)\big) \\ &\qquad = F\left(\begin{array}{c} \limsup_{n\to\infty} P_M(p,y_n,t), \limsup_{n\to\infty} P_M(p,p,t), \\ \limsup_{n\to\infty} P_M(p,y_n,t), \limsup_{n\to\infty} P_M(p,Ty_n,t) \end{array}\right) \le 0, \end{aligned}$$

which implies

$$\limsup_{n\to\infty} P_M(p,Ty_n,t)) \ge \gamma(P_M(p,p,t)) = \gamma(1) = 1.$$

Thus,

$$\limsup_{n\to\infty} P_M(p,Ty_n,t) = 1.$$

Similarly, taking limit inf, we have

$$\limsup_{n\to\infty} P_M(p,Ty_n,t) = 1.$$

Therefore, $\limsup_{n\to\infty} P_M(Ty_n,p,t) = 1$, this implies that

$$\limsup_{n\to\infty} P_M(Ty_n,Tp,t) = 1 = P_M(p,p,t) = P_M(Tp,Tp,t).$$

Thus $\lim_{n\to\infty} Ty_n = p = Tp$. Hence T is continuous at p. □

Corollary 1. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space, $m \in \mathbb{N}$ and T be a self map of X satisfying for all $x, y \in X$,*

$$F\big(P_M(x,y,t), P_M(T^m x,x,t), P_M(T^m x,y,t), P_M(T^m x,T^m y,t)\big) \le 0$$

where $F \in \mathcal{F}$. Then T has a unique fixed point p in X and T^m is continuous at p.

Proof. From Theorem 3, T^m has a unique fixed point p in X and T^m is continuous at p. Since $Tp = TT^m p = T^m Tp$, Tp is also a fixed point of T^m, By the uniqueness it follows $Tp = p$. □

In Theorem 3, if we take $F((t_1, t_2, t_3), t_4) = \gamma(\min\{t_1, t_2, t_3\}) - t_4$ then we have the next result.

Corollary 2. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space and T be a self map of X satisfying for all $x, y \in X$,*

$$P_M(Tx, Ty, t) \geq \gamma\big(\min\big\{P_M(x, y, t), P_M(Tx, x, t), P_M(Tx, y, t)\big\}\big).$$

Then T has a unique fixed point p in X and T is continuous at p.

Example 6. Let $X = \mathbb{R}^+$. Define $P_M : X^2 \times [0, \infty) \to [0, 1]$ by

$$P_M(x, y, t) = \exp\Big(-\frac{\max\{x, y\}}{t}\Big)$$

for all $x, y \in X$ and $t > 0$. Then $(X, P_M, *)$ is a complete partial fuzzy metric space where $a * b = ab$. Define map $T : X \to X$ by $Tx = \frac{x}{2}$ for $x \in X$ and let $\gamma : [0, 1] \to [0, 1]$ defined by $\gamma(s) = s^{\frac{1}{2}}$. It is easy to see that

$$\begin{aligned} P_M(Tx, Ty, t) &= \exp\Big(-\frac{\max\{\frac{x}{2}, \frac{y}{2}\}}{t}\Big) \\ &= \sqrt{\exp\Big(-\frac{\max\{x, y\}}{t}\Big)} \\ &= \sqrt{P_M(x, y, t)} \\ &\geq \sqrt{\min\big\{P_M(x, y, t), P_M(Tx, x, t), P_M(Tx, y, t)\big\}}. \end{aligned}$$

Thus T satisfy all the hypotheses of Corollary 2 and hence T has a unique fixed point.

Corollary 3. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space, $m \in \mathbb{N}$ and T be a self map of X satisfying for all $x, y \in X$,*

$$P_M(T^m x, T^m y, t) \geq \gamma\big(\min\big\{P_M(x, y, t), P_M(T^m x, x, t), P_M(T^m x, y, t)\big\}\big).$$

Then T has a unique fixed point p in X and and T^m is continuous at p.

Corollary 4. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space and T be a self map of X satisfying for all $x, y \in X$,*

$$P_M(Tx, Ty, t) \geq \sqrt{a_1 P_M(x, y, t) + a_2 P_M(Tx, x, t) + a_3 P_M(Tx, y, t)},$$

such that for every $a_i \geq 0$, $\sum_{i=1}^{3} a_i = 1$. Then T has a unique fixed point p in X and T is continuous at p.

Corollary 5. *Let $(X, M, *)$ be a complete bounded fuzzy metric space and T be a self map of X satisfying for all $x, y \in X$ the*

$$F\big(M(x, y, t), M(Tx, x, t), M(Tx, y, t), M(Tx, Ty, t)\big) \leq 0$$

where $F \in \mathcal{F}$. Then T has a unique fixed point p in X and T is continuous at p.

References

[1] A. George, P. Veeramani: On some results in fuzzy metric spaces. *Fuzzy Sets and Systems* 64 (1994) 395–399.

[2] M. Grabiec: Fixed points in fuzzy metric spaces. *Fuzzy Sets and Systems* 27 (1988) 385–389.

[3] V. Gregori, A. Sapena: On fixed point theorems in fuzzy metric spaces. *Fuzzy Sets and Systems* 125 (2002) 245–252.

[4] O. Kaleva, S. Seikkala: On fuzzy metric spaces. *Fuzzy Sets and Systems* 12 (1984) 215–229.

[5] I. Kramosil, J. Michálek: Fuzzy metrics and statistical metric spaces. *Kybernetika (Prague)* 11 (1975) 336–344.

[6] S. G. Matthews: Partial metric topology. *Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci.* 728 (1994) 183–197.

[7] D. Miheţ: A Banach contraction theorem in fuzzy metric spaces. *Fuzzy Sets and Systems* 144 (2004) 431–439.

[8] D. Miheţ: On fuzzy contractive mappings in fuzzy metric spaces. *Fuzzy Sets and Systems* 158 (2007) 915–921.

[9] D. Miheţ: Fuzzy ψ-contractive mappings in non-Archimedean fuzzy metric spaces. *Fuzzy Sets and Systems* 159 (2008) 739–744.

[10] D. Miheţ: Fuzzy quasi-metric versions of a theorem of Gregori and Sapena. *Iranian Journal of Fuzzy Systems* 7 (2010) 59–64.

[11] C. Vetro: Fixed points in weak non-Archimedean fuzzy metric spaces. *Fuzzy Sets and Systems* 162 (2011) 84–90.

Authors' address
Shaban Sedghi, Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran

Nabi Shobkolaei, Department of Mathematics, Babol Branch, Islamic Azad University, Babol, Iran

Ishak Altun, Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey

E-mail: sedghi_gh@yahoo.com, nabi_shobe@yahoo.com, ishakaltun@yahoo.com

Received: 18 December, 2014
Accepted for publication: 1 November, 2015
Communicated by: Vilém Novák

Communications in Mathematics 23 (2015) 143–161

Zeros of Solutions and Their Derivatives of Higher Order Non-homogeneous Linear Differential Equations

Zinelâabidine Latreuch and Benharrat Belaïdi

Abstract. This paper is devoted to studying the growth and oscillation of solutions and their derivatives of higher order non-homogeneous linear differential equations with finite order meromorphic coefficients. Illustrative examples are also treated.

1 Introduction and main results

We assume that the reader is familiar with the usual notations and basic results of the Nevanlinna theory [9], [11], [17]. Let f be a meromorphic function, we define

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\varphi})|\,d\varphi\,,$$
$$N(r,f) = \int_0^r \frac{n(t,f)-n(0,f)}{t}\,dt + n(0,f)\log r\,,$$

and

$$T(r,f) = m(r,f) + N(r,f) \quad (r>0)$$

is the Nevanlinna characteristic function of f, where $\log^+ x = \max(0,\log x)$ for $x \geq 0$, and $n(t,f)$ is the number of poles of $f(z)$ lying in $|z| \leq t$, counted according to their multiplicity. Also, we define

$$N\left(r,\frac{1}{f}\right) = \int_0^r \frac{n\left(t,\frac{1}{f}\right)-n\left(0,\frac{1}{f}\right)}{t}\,dt + n\left(0,\frac{1}{f}\right)\log r,$$
$$\overline{N}\left(r,\frac{1}{f}\right) = \int_0^r \frac{\overline{n}\left(t,\frac{1}{f}\right)-\overline{n}\left(0,\frac{1}{f}\right)}{t}\,dt + \overline{n}\left(0,\frac{1}{f}\right)\log r,$$

2010 MSC: 34M10, 30D35

Key words: Linear differential equations, Meromorphic functions, Exponent of convergence of the sequence of zeros

where $n\left(t,\frac{1}{f}\right)$ $\left(\overline{n}\left(t,\frac{1}{f}\right)\right)$ is the number of zeros (distinct zeros) of $f(z)$ lying in $|z|\leq t$, counted according to their multiplicity. In addition, we will use

$$\lambda(f)=\limsup_{r\to+\infty}\frac{\log N\left(r,\frac{1}{f}\right)}{\log r}$$

and

$$\overline{\lambda}(f)=\limsup_{r\to+\infty}\frac{\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r}$$

to denote respectively the exponents of convergence of the zero-sequence and distinct zeros of $f(z)$. In the following, we give the necessary notations and basic definitions.

Definition 1. [9], [17] Let f be a meromorphic function. Then the order $\rho(f)$ and the lower order $\mu(f)$ of $f(z)$ are defined respectively by

$$\rho(f)=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r}$$

and

$$\mu(f)=\liminf_{r\to+\infty}\frac{\log T(r,f)}{\log r}.$$

Definition 2. [7], [17] Let f be a meromorphic function. Then the hyper-order of $f(z)$ is defined by

$$\rho_2(f)=\limsup_{r\to+\infty}\frac{\log\log T(r,f)}{\log r}.$$

Definition 3. [9], [13] The type of a meromorphic function f of order ρ $(0<\rho<\infty)$ is defined by

$$\tau(f)=\limsup_{r\to+\infty}\frac{T(r,f)}{r^{\rho}}.$$

Definition 4. [7] Let f be a meromorphic function. Then the hyper-exponent of convergence of zero-sequence of $f(z)$ is defined by

$$\lambda_2(f)=\limsup_{r\to+\infty}\frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r}.$$

Similarly, the hyper-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f)=\limsup_{r\to+\infty}\frac{\log\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r}.$$

The study of oscillation of solutions of linear differential equations has attracted many interests since the work of Bank and Laine [1], [2], for more details see [11]. One of the main subject of this research is the zeros distribution of solutions and their derivatives of linear differential equations. In this paper, we first discuss the growth of solutions of higher-order linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = F(z), \tag{1}$$

where $A_j(z)$ $(j = 1, \ldots, k-1)$, $A_0(z) \not\equiv 0$ and $F(z)(\not\equiv 0)$ are meromorphic functions of finite order. Some results on the growth of entire and meromorphic solutions of (1) have been obtained by several researchers (see [5], [6], [10], [11], [16]). In the case that the coefficients $A_j(z)$ $(j = 0, \ldots, k-1)$ are polynomials and $F(z) \equiv 0$, the growth of solutions of (1) has been extensively studied (see [8]). In 1992, Hellerstein et al. (see [10]) proved that every transcendental solution of (1) is of infinite order, if there exists some $d \in \{0, 1, \ldots, k-1\}$ such that

$$\max_{j \neq d}\{\rho(A_j), \rho(F)\} < \rho(A_d) \leq \frac{1}{2}.$$

Recently, Wang and Liu proved the following.

Theorem 1. [16, Theorem 1.6] *Suppose that $A_0, A_1, \ldots, A_{k-1}, F(z)$ are meromorphic functions of finite order. If there exists some A_s $(0 \leq s \leq k-1)$ such that*

$$b = \max_{j \neq s}\left\{\rho(F), \rho(A_j), \lambda\left(\frac{1}{A_s}\right)\right\} < \mu(A_s) < \frac{1}{2}.$$

Then

1. *Every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicities, of (1) satisfies $\mu(A_s) \leq \rho_2(f) \leq \rho(A_s)$. Furthermore, if $F \not\equiv 0$, then we have $\mu(A_s) \leq \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) \leq \rho(A_s)$.*

2. *If $s \geq 2$, then every non-transcendental meromorphic solution f of (1) is a polynomial with degree $\deg f \leq s-1$. If $s = 0$ or 1 then every nonconstant solution of (1) is transcendental.*

For more details there are many interesting papers, please see [11] and references contained in it. Recently, in [12], the authors studied equations of type

$$f'' + A(z)f' + B(z)f = F(z), \tag{2}$$

where $A(z), B(z)\,(\not\equiv 0)$ and $F(z)\,(\not\equiv 0)$ are meromorphic functions of finite order. They proved under different conditions that every nontrivial meromorphic solution f of (2) satisfies

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = \rho(f) = +\infty \quad (j \in \mathbb{N})$$

with at most one exception. It's interesting now to study the stability of the exponent of convergence of the sequence of zeros (resp. distinct zeros) of solutions for higher order differential equation (1) with their derivatives. The main purpose of this paper is to deal with this problem. Before we state our results we need to define the following notations.

$$A_i^j(z) = A_i^{j-1}(z) + \left(A_{i+1}^{j-1}(z)\right)' - A_{i+1}^{j-1}(z)\frac{\left(A_0^{j-1}(z)\right)'}{A_0^{j-1}(z)}, \text{ for } j = 1, 2, 3, \dots, \tag{3}$$

where $i = 0, 1, \dots, k-1$ and

$$F^j(z) = \left(F^{j-1}(z)\right)' - F^{j-1}(z)\frac{\left(A_0^{j-1}(z)\right)'}{A_0^{j-1}(z)}, \text{ for } j = 1, 2, 3, \dots, \tag{4}$$

where $A_i^0(z) = A_i(z)$ $(i = 0, 1, \dots, k-1)$, $F^0(z) = F(z)$ and $A_k^j(z) = 1$. We obtain the following results.

Theorem 2. (Main Theorem) *Let $A_0(z)\,(\not\equiv 0), A_1(z), \dots, A_{k-1}(z)$ and $F(z)\,(\not\equiv 0)$ be meromorphic functions of finite order such that $A_0^j(z) \not\equiv 0$ and $F^j(z) \not\equiv 0$, where $j \in \mathbb{N}$. If f is a meromorphic solution of* (1) *with $\rho(f) = \infty$ and $\rho_2(f) = \rho$, then f satisfies*

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = +\infty \quad (j = 0, 1, 2, \dots)$$

and

$$\overline{\lambda}_2\left(f^{(j)}\right) = \lambda_2\left(f^{(j)}\right) = \rho \quad (j = 0, 1, 2, \dots).$$

Furthermore, if f is of finite order with

$$\rho(f) > \max_{i=0,\dots,k-1}\left\{\rho(A_i), \rho(F)\right\},$$

then

$$\overline{\lambda}\left(f^{(j)}\right) = \lambda\left(f^{(j)}\right) = \rho(f) \quad (j = 0, 1, 2, \dots).$$

Remark 1. The condition "$A_0^j(z) \not\equiv 0$ and $F^j(z) \not\equiv 0$ where $j \in \mathbb{N}$" in Theorem 2 is necessary. For example, the entire function $f(z) = \mathrm{e}^{\mathrm{e}^z} - 1$ satisfies

$$f^{(3)} - \mathrm{e}^z f'' - f' - \mathrm{e}^{2z} f = \mathrm{e}^{2z},$$

where $A_2(z) = -\mathrm{e}^z$, $A_1(z) = -1$, $A_0(z) = -\mathrm{e}^{2z}$ and $F(z) = \mathrm{e}^{2z}$. So

$$A_0^1(z) = -\mathrm{e}^{2z} + 2 \quad F^1(z) \equiv 0.$$

On the other hand, we have $\lambda(f') = 0 < \lambda(f) = \infty$.

Here, we will give some sufficient conditions on the coefficients which guarantee $A_0^j(z) \not\equiv 0$ and $F^j(z) \not\equiv 0, (j = 1, 2, 3, \dots)$.

Theorem 3. *Let $A_0(z)\,(\not\equiv 0), A_1(z), \dots, A_{k-1}(z)$ and $F(z)\,(\not\equiv 0)$ be entire functions of finite order such that $\rho(A_0) > \max\limits_{i=1,\dots,k-1}\{\rho(A_i), \rho(F)\}$. Then all nontrivial solutions of (1) satisfy*

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, 2, \dots)$$

with at most one possible exceptional solution f_0 such that

$$\rho(f_0) = \max\{\overline{\lambda}(f_0), \rho(A_0)\}.$$

Furthermore, if $\rho(A_0) \le \frac{1}{2}$, then every transcendental solution of (1) satisfies

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, 2, \dots).$$

Remark 2. The condition $\rho(A_0) > \max\limits_{i=1,\dots,k-1}\{\rho(A_i), \rho(F)\}$ does not ensure that all solutions of (1) are of infinite order. For example, we can see that $f_0(z) = \mathrm{e}^{-z^2}$ satisfies the differential equation

$$f^{(3)} + 2zf'' + 3f' + \big(\mathrm{e}^{z^2} - 2z\big)f = 1,$$

where

$$\overline{\lambda}(f_0) = 0 < \rho(f_0) = \rho(A_0) = 2.$$

Combining Theorem 1 and Theorem 3, we obtain the following result.

Corollary 1. *Let $A_0(z)\,(\not\equiv 0), A_1(z), \dots, A_{k-1}(z)$ and $F(z)\,(\not\equiv 0)$ be entire functions of finite order such that*

$$\max_{i=1,\dots,k-1}\{\rho(F), \rho(A_i)\} < \mu(A_0) < \frac{1}{2}.$$

Then, every transcendental solution f of (1) satisfies

$$\mu(A_0) \le \overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho_2(f) \le \rho(A_0) \quad (j \in \mathbb{N}).$$

Furthermore, if

$$\max_{i=1,\dots,k-1}\{\rho(F), \rho(A_i)\} < \mu(A_0) = \rho(A_0) < \frac{1}{2},$$

then every transcendental solution f of (1) satisfies

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho_2(f) = \rho(A_0)\,.$$

Theorem 4. *Let $A_0(z)\,(\not\equiv 0), A_1(z), \dots, A_{k-1}(z)$ and $F(z)\,(\not\equiv 0)$ be entire functions of finite order such that $A_1, \dots, A_{k-1}$ and F are polynomials and A_0 is transcendental. Then all nontrivial solutions of (1) satisfy*

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, 2, \dots)$$

with at most one possible solution f_0 of finite order.

Corollary 2. *Let P be a nonconstant entire function, let Q be a nonzero polynomial, and let f be any entire solution of the differential equation*

$$f^{(k)} + e^{P(z)} f = Q(z) \quad (k \in \mathbb{N}).$$

1. *If P is polynomial, then*

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = \rho(f) = \infty \quad (j = 0, 1, 2, \dots)$$

and

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho \quad (j = 0, 1, 2, \dots),$$

where ρ is positive integer not exceeding the degree of P.

2. *If P is transcendental with $\rho(P) < \frac{1}{2}$, then*

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \infty \quad (j = 0, 1, 2, \dots).$$

Theorem 5. *Let $j \geq 1$ be an integer, let $A_0(z) \, (\not\equiv 0), A_1(z), \dots, A_{k-1}(z)$ and $F(z) \, (\not\equiv 0)$ be entire functions of finite order such that $\rho(F) < \rho(A_i) \leq \rho(A_0)$ $(i = 1, \dots, k-1)$ and*

$$\tau(A_0) > \begin{cases} \sum\limits_{l \in I_j} \beta_l \tau(A_{l+1}) & \text{if } j < k, \\ \sum\limits_{l \in I_k} \beta_l \tau(A_{l+1}) & \text{if } j \geq k, \end{cases}$$

where $\beta_l = \sum_{p=l}^{j-1} C_p^l$ with $C_p^l = \frac{p!}{(p-l)!\, l!}$,

$$I_k = \big\{0 \leq l \leq k-2 : \rho(A_{l+1}) = \rho(A_0)\big\}$$

and

$$I_j = \big\{0 \leq l \leq j-1 : \rho(A_{l+1}) = \rho(A_0)\big\}.$$

If f is a nontrivial solution of (1) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$, then f satisfies

$$\overline{\lambda}\big(f^{(m)}\big) = \lambda\big(f^{(m)}\big) = +\infty \quad (m = 0, 1, 2, \dots, j)$$

and

$$\overline{\lambda}_2\big(f^{(m)}\big) = \lambda_2\big(f^{(m)}\big) = \rho \quad (m = 0, 1, 2, \dots, j).$$

From Theorem 5, we obtain the following result of paper [12].

Corollary 3. [12] *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be entire functions with finite order such that $\rho(B) = \rho(A) > \rho(F)$ and $\tau(B) > k\tau(A)$, $k \geq 1$ is an integer. If f is a nontrivial solution of (2) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$, then f satisfies*

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, \dots, k)$$

and

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho \quad (j = 0, 1, \dots, k).$$

In the next theorem, we denote by $\sigma(f)$ the following quantity

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log m(r,f)}{\log r}.$$

Theorem 6. *Let $A_0(z)\,(\not\equiv 0), A_1(z), \dots, A_{k-1}(z)$ and $F(z)\,(\not\equiv 0)$ be meromorphic functions of finite order such that $\sigma(A_0) > \max_{i=1,\dots,k-1}\{\sigma(A_i), \sigma(F)\}$. If f is a meromorphic solution of (1) with $\rho(f) = \infty$ and $\rho_2(f) = \rho$, then f satisfies*

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, 2, \dots)$$

and

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho \quad (j = 0, 1, 2, \dots).$$

In the following, we mean by two meromorphic functions f and g share a finite value a CM (counting multiplicities) when $f-a$ and $g-a$ have the same zeros with the same multiplicities. It is well-known that if f and g share four distinct values CM, then f is a Möbius transformation of g. Rubel and Yang [14], [17] proved that if f is an entire function and shares two finite values CM with its derivative, then $f = f'$. We give here a different result.

Theorem 7. *Let k be a positive integer and let f be entire function. If f and $f^{(k)}$ share the value $a \neq 0$ CM, then*

(a) *$\rho(f) = 1$ or*

(b) *with at most one exception*

$$\overline{\lambda}(f-a) = \overline{\lambda}\big(f^{(j)}\big) = \infty \quad (j = 1, 2, \dots).$$

2 Preliminary lemmas

Lemma 1. [9] *Let f be a meromorphic function and let $k \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O\big(\log T(r,f) + \log r\big)$, possibly outside of an exceptional set $E \subset (0,+\infty)$ with finite linear measure. If f is of finite order of growth, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 2. [3], [5] *Let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions.*

1. *If f is a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = F \tag{5}$$

with $\rho(f) = +\infty$, then f satisfies

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty.$$

2. *If f is a meromorphic solution of* (5) *with $\rho(f) = +\infty$ and $\rho_2(f) = \rho$, then f satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty \quad \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

Lemma 3. [15] *Let $A_0, A_1, \dots, A_{k-1}$, $F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of equation* (5) *with*

$$\max_{j=0,1,\dots,k-1} \big\{\rho(A_j), \rho(F)\big\} < \rho(f) < +\infty,$$

then

$$\overline{\lambda}(f) = \lambda(f) = \rho(f).$$

Lemma 4. [6] *Let $A, B_1, \dots, B_{k-1}$, $F \not\equiv 0$ be entire functions of finite order, where $k \geq 2$. Suppose that either* (a) *or* (b) *below holds:*

(a) $\rho(B_j) < \rho(A)$ $(j = 1, \dots, k-1)$;

(b) $B_1, \dots, B_{k-1}$ *are polynomials and A is transcendental.*

Then we have

1. *All solutions of the differential equation*

$$f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_1 f' + A f = F$$

satisfy

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$$

with at most one possible solution f_0 of finite order.

2. *If there exists an exceptional solution f_0 in case 1, then f_0 satisfies*

$$\rho(f_0) \leq \max\big\{\rho(A), \rho(F), \overline{\lambda}(f_0)\big\} < \infty. \tag{6}$$

Furthermore, if $\rho(A) \neq \rho(F)$ and $\overline{\lambda}(f_0) < \rho(f_0)$, then

$$\rho(f_0) = \max\big\{\rho(A), \rho(F)\big\}.$$

Lemma 5. *Let $A_0, A_1, \dots, A_{k-1}$ be the coefficients of* (1). *For any integer j, the following inequalities hold*

$$m\big(r, A_1^j\big) \leq \begin{cases} \sum\limits_{i=0}^{j} C_j^i m(r, A_{i+1}) + O(\log r) & \text{if } j < k, \\ \sum\limits_{i=0}^{k-2} C_j^i m(r, A_{i+1}) + O(\log r) & \text{if } j \geq k, \end{cases} \tag{7}$$

where A_1^j is defined in (3).

Proof. First, we prove the case $j < k$. We have from (3)

$$A_i^j = A_i^{j-1} + A_{i+1}^{j-1}\left(\frac{\left(A_{i+1}^{j-1}\right)'}{A_{i+1}^{j-1}} - \frac{\left(A_0^{j-1}\right)'}{A_0^{j-1}}\right) \quad (i \in \mathbb{N}).$$

By using Lemma 1, we have for all $j \in \mathbb{N}$

$$m\big(r, A_i^j\big) \le m\big(r, A_i^{j-1}\big) + m\big(r, A_{i+1}^{j-1}\big) + O(\log r). \tag{8}$$

In order to prove Lemma 5, we apply mathematical induction. For $j = 2$, we have from (8)

$$\begin{aligned} m\big(r, A_1^2\big) &\le m\big(r, A_1^1\big) + m\big(r, A_2^1\big) + O(\log r) \\ &\le m(r, A_1) + 2m(r, A_2) + m(r, A_3) + O(\log r) \\ &= C_2^0 m(r, A_1) + C_2^1 m(r, A_2) + C_2^2 m(r, A_3) + O(\log r) \\ &= \sum_{i=0}^{2} C_2^i m(r, A_{i+1}) + O(\log r). \end{aligned}$$

Suppose that (7) is true and we show that for $j + 1 < k$

$$m\big(r, A_1^{j+1}\big) \le \sum_{i=0}^{j+1} C_{j+1}^i m(r, A_{i+1}) + O(\log r).$$

By using (3) and (7) we have

$$\begin{aligned} m\big(r, A_1^{j+1}\big) &\le m\big(r, A_1^j\big) + m\big(r, A_2^j\big) + O(\log r) \\ &\le \sum_{i=0}^{j} C_j^i m(r, A_{i+1}) + \sum_{i=0}^{j} C_j^i m(r, A_{i+2}) + O(\log r) \\ &= C_j^0 m(r, A_1) + \sum_{i=1}^{j} C_j^i m(r, A_{i+1}) \\ &\quad + \sum_{i=0}^{j-1} C_j^i m(r, A_{i+2}) + C_j^j m(r, A_{j+2}) + O(\log r) \\ &= C_j^0 m(r, A_1) + \sum_{i=1}^{j} C_j^i m(r, A_{i+1}) \\ &\quad + \sum_{i=1}^{j} C_j^{i-1} m(r, A_{i+1}) + C_j^j m(r, A_{j+2}) + O(\log r). \end{aligned}$$

Of course if $j + 1 \ge k$, then $m(r, A_{j+1}) = m(r, A_{j+2}) = 0$. Since $C_j^0 = C_{j+1}^0$ and $C_j^j = C_{j+1}^{j+1}$, then we have

$$\begin{aligned} m\big(r, A_1^{j+1}\big) \le C_{j+1}^0 m(r, A_1) &+ \sum_{i=1}^{j} \big(C_j^i + C_j^{i-1}\big) m(r, A_{i+1}) \\ &+ C_{j+1}^{j+1} m(r, A_{j+2}) + O(\log r). \end{aligned}$$

Using the identity $C_j^i + C_j^{i-1} = C_{j+1}^i$, we get

$$\begin{aligned} m\big(r, A_1^{j+1}\big) &\le C_{j+1}^0 m(r, A_1) + \sum_{i=1}^{j} C_{j+1}^i m(r, A_{i+1}) + C_{j+1}^{j+1} m(r, A_{j+2}) + O(\log r) \\ &= \sum_{i=0}^{j+1} C_{j+1}^i m(r, A_{i+1}) + O(\log r). \end{aligned}$$

For the case $j \ge k$, we need just to remark that $m(r, A_{i+1}) = 0$ when $i \ge k-1$ and by using the same procedure as before we obtain

$$m\big(r, A_1^j\big) \le \sum_{i=0}^{k-2} C_j^i m(r, A_{i+1}) + O(\log r). \qquad \square$$

Lemma 6. *Let $A_0, A_1, \dots, A_{k-1}$ be the coefficients of (1). For any integer j, the following inequalities hold*

$$\sum_{p=0}^{j-1} m\big(r, A_1^p\big) \le \begin{cases} \sum\limits_{i=0}^{j-1}\Big(\sum\limits_{p=i}^{j-1} C_p^i\Big) m(r, A_{i+1}) + O(\log r) & \text{if } j < k, \\ \sum\limits_{i=0}^{k-2}\Big(\sum\limits_{p=i}^{j-1} C_p^i\Big) m(r, A_{i+1}) + O(\log r) & \text{if } j \ge k. \end{cases} \tag{9}$$

Proof. We prove only the case $j < k$. By Lemma 5 we have

$$\sum_{p=0}^{j-1} m\big(r, A_1^p\big) \le \sum_{p=0}^{j-1}\Big(\sum_{i=0}^{p} C_p^i m(r, A_{i+1})\Big) + O(\log r). \tag{10}$$

The first term of the right hand of (10) can be expressed as

$$\begin{aligned} \sum_{p=0}^{j-1}\Big(\sum_{i=0}^{p} C_p^i m(r, A_{i+1})\Big) &= C_0^0 m(r, A_1) + \big(C_1^0 m(r, A_1) + C_1^1 m(r, A_2)\big) \\ &\quad + \big(C_2^0 m(r, A_1) + C_2^1 m(r, A_2) + C_2^2 m(r, A_3)\big) + \cdots \\ &\quad + \big(C_{j-1}^0 m(r, A_1) + C_{j-1}^1 m(r, A_2) + \cdots + C_{j-1}^{j-1} m(r, A_j)\big) \end{aligned}$$

which we can write as

$$\begin{aligned} \sum_{p=0}^{j-1}\Big(\sum_{i=0}^{p} C_p^i m(r, A_{i+1})\Big) &= \big(C_0^0 + C_1^0 + \cdots + C_{j-1}^0\big) m(r, A_1) \\ &\quad + \big(C_1^1 + C_2^1 + \cdots + C_{j-1}^1\big) m(r, A_2) + \cdots \\ &\quad + \big(C_{j-2}^{j-2} + C_{j-1}^{j-2}\big) m(r, A_{j-1}) + C_{j-1}^{j-1} m(r, A_j). \end{aligned}$$

Then

$$\sum_{p=0}^{j-1} m\big(r, A_1^p\big) \le \sum_{p=0}^{j-1}\bigg(\sum_{i=0}^{p} C_p^i m(r, A_{i+1})\bigg) + O(\log r)$$
$$= \sum_{i=0}^{j-1}\bigg(\sum_{p=i}^{j-1} C_p^i\bigg) m(r, A_{i+1}) + O(\log r).$$

By using the same procedure as above we can prove the case $j \ge k$. □

Lemma 7. [18] *Let $\phi(z)$ be a nonconstant entire function and k be a positive integer. Then, with at most one exception, every solution F of the differential equation*

$$F^{(k)} - \mathrm{e}^{\phi(z)} F = 1$$

satisfies $\rho_2(F) = \rho(\mathrm{e}^{\phi})$.

Lemma 8. [4] *Let P be a nonconstant entire function, let Q be a nonzero polynomial, and let f be any entire solution of the differential equation*

$$f^{(k)} + \mathrm{e}^{P(z)} f = Q(z) \quad (k \in \mathbb{N}).$$

If P is polynomial, then f has an infinite order and its hyper-order $\rho_2(f)$ is a positive integer not exceeding the degree of P. If P is transcendental with order less than $\frac{1}{2}$, then the hyper-order of f is infinite.

Lemma 9. *Let f be a meromorphic function with $\rho(f) = \rho \ge 0$. Then, there exists a set $E_1 \subset [1, +\infty)$ with infinite logarithmic measure*

$$\operatorname{lm}(E_1) = \int_1^{+\infty} \frac{\chi_{E_1}(t)}{t}\,\mathrm{d}t = \infty\,,$$

where $\chi_{E_1}(t)$ is the characteristic function of the set E_1, such that

$$\lim_{\substack{r\to+\infty\\ r\in E_1}} \frac{\log T(r,f)}{\log r} = \rho.$$

Proof. Since $\rho(f) = \rho$, then there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to $+\infty$ satisfying $\big(1 + \frac{1}{n}\big) r_n < r_{n+1}$ and

$$\lim_{r_n\to+\infty} \frac{\log T(r_n,f)}{\log r_n} = \rho(f).$$

So, there exists an integer n_1 such that for all $n \ge n_1$, for any $r \in \big[r_n, \big(1 + \frac{1}{n}\big) r_n\big]$, we have

$$\frac{\log T(r_n,f)}{\log\big(1+\frac{1}{n}\big) r_n} \le \frac{\log T(r,f)}{\log r} \le \frac{\log T\big(\big(1+\frac{1}{n}\big) r_n, f\big)}{\log r_n}\,.$$

Set $E_1 = \bigcup_{n=n_1}^{\infty} \left[r_n, \left(1+\frac{1}{n}\right)r_n\right]$, we obtain

$$\lim_{\substack{r\to+\infty \\ r\in E_1}} \frac{\log T(r,f)}{\log r} = \lim_{r_n\to+\infty} \frac{\log T(r_n,f)}{\log r_n},$$

and

$$\operatorname{lm}(E_1) = \sum_{n=n_1}^{\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum_{n=n_1}^{\infty} \log\left(1+\frac{1}{n}\right) = \infty.$$

Thus, the proof of the lemma is completed. □

Lemma 10. *Let f_1, f_2 be meromorphic functions satisfying $\rho(f_1) > \rho(f_2)$. Then there exists a set $E_2 \subset (1,+\infty)$ having infinite logarithmic measure such that for all $r \in E_2$, we have*

$$\lim_{r\to+\infty} \frac{T(r,f_2)}{T(r,f_1)} = 0.$$

Proof. Set $\rho_1 = \rho(f_1)$, $\rho_2 = \rho(f_2)$, $(\rho_1 > \rho_2)$. By Lemma 9, there exists a set $E_2 \subset (1,+\infty)$ having infinite logarithmic measure such that for any given $0 < \varepsilon < \frac{\rho_1-\rho_2}{2}$ and all sufficiently large $r \in E_2$

$$T(r,f_1) > r^{\rho_1-\varepsilon}$$

and for all sufficiently large r, we have

$$T(r,f_2) < r^{\rho_2+\varepsilon}.$$

From this we can get

$$\frac{T(r,f_2)}{T(r,f_1)} < \frac{r^{\rho_2+\varepsilon}}{r^{\rho_1-\varepsilon}} = \frac{1}{r^{\rho_1-\rho_2-2\varepsilon}} \quad (r \in E_2).$$

Since $0 < \varepsilon < \frac{\rho_1-\rho_2}{2}$, then we obtain

$$\lim_{\substack{r\to+\infty \\ r\in E_2}} \frac{T(r,f_2)}{T(r,f_1)} = 0.$$

□

3 Proofs of the Theorems and the Corollary

Proof of Theorem 2. For the proof, we use the principle of mathematical induction. Since $A_0(z) \not\equiv 0$ and $F(z) \not\equiv 0$, then by using Lemma 2 we have

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

Dividing both sides of (1) by A_0, we obtain

$$\frac{A_k}{A_0} f^{(k)} + \frac{A_{k-1}}{A_0} f^{(k-1)} + \cdots + \frac{A_1}{A_0} f' + f = \frac{F}{A_0}. \quad (11)$$

Differentiating both sides of equation (11), we have

$$\frac{A_k}{A_0} f^{(k+1)} + \left(\left(\frac{A_k}{A_0}\right)' + \frac{A_{k-1}}{A_0}\right) f^{(k)} + \cdots + \left(\left(\frac{A_1}{A_0}\right)' + 1\right) f' = \left(\frac{F}{A_0}\right)'. \quad (12)$$

Multiplying now (12) by A_0, we get

$$f^{(k+1)} + A^1_{k-1}(z) f^{(k)} + \cdots + A^1_0(z) f' = F^1(z), \quad (13)$$

where

$$\begin{aligned} A^1_i(z) &= A_0\left(\left(\frac{A_{i+1}(z)}{A_0(z)}\right)' + \frac{A_i(z)}{A_0(z)}\right) \\ &= A_i(z) + A'_{i+1}(z) - A_{i+1}(z)\frac{A'_0(z)}{A_0(z)} \quad (i = 0, \ldots, k-1) \end{aligned}$$

and

$$F^1(z) = A_0(z)\left(\frac{F(z)}{A_0(z)}\right)' = F'(z) - F(z)\frac{A'_0(z)}{A_0(z)}.$$

Since $A^1_0(z) \not\equiv 0$ and $F^1(z) \not\equiv 0$ are meromorphic functions with finite order, then by using Lemma 2 we obtain

$$\overline{\lambda}(f') = \lambda(f') = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2(f') = \lambda_2(f') = \rho_2(f) = \rho.$$

Dividing now both sides of (13) by A^1_0, we obtain

$$\frac{A^1_k}{A^1_0} f^{(k+1)} + \frac{A^1_{k-1}}{A^1_0} f^{(k)} + \cdots + \frac{A^1_1}{A^1_0} f'' + f' = \frac{F^1}{A^1_0}. \quad (14)$$

Differentiating both sides of equation (14) and multiplying by A^1_0, we get

$$f^{(k+2)} + A^2_{k-1}(z) f^{(k+1)} + \cdots + A^2_0(z) f'' = F^2(z), \quad (15)$$

where $A^2_0(z) \not\equiv 0$ and $F^2(z) \not\equiv 0$ are meromorphic functions defined in (3) and (4). By using Lemma 2, we obtain

$$\overline{\lambda}(f'') = \lambda(f'') = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2(f'') = \lambda_2(f'') = \rho_2(f) = \rho.$$

Suppose now that

$$\overline{\lambda}_i\big(f^{(k)}\big) = \lambda_i\big(f^{(k)}\big) = \rho_i(f) \quad (i = 1, 2) \quad (16)$$

for all $k = 0, 1, 2, \ldots, j-1$, and we prove that (16) is true for $k = j$. With the same procedure as before, we can obtain

$$f^{(k+j)} + A^j_{k-1}(z) f^{(k-1+j)} + \cdots + A^j_0(z) f^{(j)} = F^j(z),$$

where $A_0^j(z) \not\equiv 0$ and $F^j(z) \not\equiv 0$ are meromorphic functions defined in (3) and (4). By using Lemma 2, we obtain

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = \rho(f) = +\infty$$

and

$$\overline{\lambda}_2\big(f^{(j)}\big) = \lambda_2\big(f^{(j)}\big) = \rho_2(f) = \rho.$$

For the case $\rho(f) > \max\limits_{i=0,\dots,k-1}\big\{\rho(A_i), \rho(F)\big\}$ we use simply similar reasoning as above and by using Lemma 3, we obtain

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = \rho(f) \quad (j = 0, 1, 2, \dots).$$

This completes the proof of Theorem 2. □

Proof of Theorem 3. By Lemma 4, all nontrivial solutions of (1) are of infinite order with at most one exceptional solution f_0 of finite order. By (3) we have

$$\begin{aligned} A_0^j &= A_0^{j-1} + A_1^{j-1}\left(\frac{\big(A_1^{j-1}\big)'}{A_1^{j-1}} - \frac{\big(A_0^{j-1}\big)'}{A_0^{j-1}}\right) \\ &= A_0^{j-2} + A_1^{j-2}\left(\frac{\big(A_1^{j-2}\big)'}{A_1^{j-2}} - \frac{\big(A_0^{j-2}\big)'}{A_0^{j-2}}\right) + A_1^{j-1}\left(\frac{\big(A_1^{j-1}\big)'}{A_1^{j-1}} - \frac{\big(A_0^{j-1}\big)'}{A_0^{j-1}}\right) \\ &= A_0 + \sum_{p=0}^{j-1} A_1^p\left(\frac{\big(A_1^p\big)'}{A_1^p} - \frac{\big(A_0^p\big)'}{A_0^p}\right). \end{aligned} \tag{17}$$

Now, suppose that there exists $j \in \mathbb{N}$ such that $A_0^j(z) \equiv 0$. By (17) we obtain

$$-A_0 = \sum_{p=0}^{j-1} A_1^p\left(\frac{\big(A_1^p\big)'}{A_1^p} - \frac{\big(A_0^p\big)'}{A_0^p}\right). \tag{18}$$

Hence

$$m(r, A_0) \le \sum_{p=0}^{j-1} m\big(r, A_1^p\big) + O(\log r). \tag{19}$$

Using Lemma 6 and (19) we have

$$\begin{aligned} T(r, A_0) = m(r, A_0) &\le \begin{cases} \sum\limits_{i=0}^{j-1}\left(\sum\limits_{p=i}^{j-1} C_p^i\right) m(r, A_{i+1}) + O(\log r) & \text{if } j < k \\ \sum\limits_{i=0}^{k-2}\left(\sum\limits_{p=i}^{j-1} C_p^i\right) m(r, A_{i+1}) + O(\log r) & \text{if } j \ge k \end{cases} \\ &= \begin{cases} \sum\limits_{i=0}^{j-1}\left(\sum\limits_{p=i}^{j-1} C_p^i\right) T(r, A_{i+1}) + O(\log r) & \text{if } j < k \\ \sum\limits_{i=0}^{k-2}\left(\sum\limits_{p=i}^{j-1} C_p^i\right) T(r, A_{i+1}) + O(\log r) & \text{if } j \ge k \end{cases} \end{aligned} \tag{20}$$

which implies the contradiction

$$\rho(A_0) \le \max_{i=1,\dots,k-1} \rho(A_i),$$

and we can deduce that $A_0^j(z) \not\equiv 0$ for all $j \in \mathbb{N}$. Suppose now there exists $j \in \mathbb{N}$ which is the first index such that $F^j(z) \equiv 0$. From (4) we obtain

$$\big(F^{j-1}(z)\big)' - F^{j-1}(z)\frac{\big(A_0^{j-1}(z)\big)'}{A_0^{j-1}(z)} = 0$$

which implies

$$F^{j-1}(z) = cA_0^{j-1}(z), \tag{21}$$

where $c \in \mathbb{C} \setminus \{0\}$. By (17) and (21) we have

$$\frac{1}{c}F^{j-1} = A_0(z) + \sum_{p=0}^{j-2} A_1^p(z)\left(\frac{\big(A_1^p(z)\big)'}{A_1^p(z)} - \frac{\big(A_0^p(z)\big)'}{A_0^p(z)}\right). \tag{22}$$

On the other hand, we obtain from (4)

$$m(r, F^j) \le m(r, F) + O(\log r) \quad (j \in \mathbb{N}). \tag{23}$$

By (20), (22) and (23), we have

$$\begin{aligned} T(r, A_0) = m(r, A_0) &\le \sum_{p=0}^{j-2} m\big(r, A_1^p\big) + m(r, F^{j-1}) + O(\log r) \\ &\le \begin{cases} \sum\limits_{i=0}^{j-2}\Big(\sum\limits_{p=i}^{j-2} C_p^i\Big) m(r, A_{i+1}) + m(r, F) + O(\log r) & \text{if } j-1 < k \\ \sum\limits_{i=0}^{k-2}\Big(\sum\limits_{p=i}^{j-2} C_p^i\Big) m(r, A_{i+1}) + m(r, F) + O(\log r) & \text{if } j-1 \ge k \end{cases} \\ &= \begin{cases} \sum\limits_{i=0}^{j-2}\Big(\sum\limits_{p=i}^{j-2} C_p^i\Big) T(r, A_{i+1}) + T(r, F) + O(\log r), & \text{if } j-1 < k \\ \sum\limits_{i=0}^{k-2}\Big(\sum\limits_{p=i}^{j-2} C_p^i\Big) T(r, A_{i+1}) + T(r, F) + O(\log r), & \text{if } j-1 \ge k \end{cases} \end{aligned}$$

which implies the contradiction $\rho(A_0) \le \max_{i=1,\dots,k-1}\big\{\rho(A_i), \rho(F)\big\}$. Since $A_0^j \not\equiv 0$ and $F^j \not\equiv 0$ $(j \in \mathbb{N})$, then by applying Theorem 2 and Lemma 4 we have

$$\overline{\lambda}\big(f^{(j)}\big) = \lambda\big(f^{(j)}\big) = +\infty \quad (j = 0, 1, 2, \dots)$$

with at most one exceptional solution f_0 of finite order. Since

$$\rho(A_0) > \max_{i=1,\dots,k-1}\big\{\rho(A_i), \rho(F)\big\},$$

then by (6) we obtain

$$\rho(f_0) \le \max\{\rho(A_0), \overline{\lambda}(f_0)\}. \tag{24}$$

On the other hand by (1), we can write

$$A_0 = \frac{F}{f_0} - \left(\frac{f_0^{(k)}}{f_0} + A_{k-1}\frac{f_0^{(k-1)}}{f_0} + \cdots + A_1\frac{f_0'}{f_0}\right).$$

It follows that by Lemma 1

$$\begin{aligned} T(r, A_0) = m(r, A_0) &\le m\left(r, \frac{F}{f_0}\right) + \sum_{i=1}^{k-1} m(r, A_i) + O(\log r) \\ &\le T(r, f_0) + T(r, F) + \sum_{i=1}^{k-1} T(r, A_i) + O(\log r), \end{aligned}$$

which implies

$$\rho(A_0) \le \max_{i=1,\dots,k-1}\{\rho(f_0), \rho(A_i), \rho(F)\} = \rho(f_0). \tag{25}$$

Since $\overline{\lambda}(f_0) \le \rho(f_0)$, then by using (24) and (25) we obtain

$$\rho(f_0) = \max\{\rho(A_0), \overline{\lambda}(f_0)\}.$$

If $\rho(A_0) \le \frac{1}{2}$, then by the theorem of Hellerstein et al. (see [10]) every transcendental solution f of (1) is of infinite order without exceptions. So, by the same proof as before we obtain

$$\overline{\lambda}(f^{(j)}) = \lambda(f^{(j)}) = +\infty \quad (j = 0, 1, 2, \dots).$$

This completes the proof of Theorem 3. □

Proof of Theorem 4. Using the same proof as Theorem 3, we obtain Theorem 4. □

Proof of Corollary 2.

1. If P is polynomial, since $A_0(z) = e^{P(z)}$, $A_i(z) \equiv 0$ $(i = 1, \dots, k-1)$ and $F(z) = Q(z)$, then

$$\rho(A_0) > \max_{i=1,\dots,k-1}\{\rho(A_i), \rho(F)\},$$

hence $A_0^j(z) \not\equiv 0$ and $F^j(z) \not\equiv 0$, $j \in \mathbb{N}$. On the other hand, by Lemma 8 every solution f has an infinite order and its hyper-order $\rho_2(f)$ is a positive integer not exceeding the degree of P. So, by applying Theorem 2 we obtain

$$\overline{\lambda}(f^{(j)}) = \lambda(f^{(j)}) = \rho(f) = \infty \quad (j = 0, 1, 2, \dots)$$

and

$$\overline{\lambda}_2(f^{(j)}) = \lambda_2(f^{(j)}) = \rho \quad (j = 0, 1, 2, \dots),$$

where ρ is positive integer not exceeding the degree of P.

2. Using the same reasoning as in 1. for the case P is transcendental with $\rho(P) < \frac{1}{2}$. □

Proof of Theorem 5. First, we prove that $A_0^n(z) \not\equiv 0$ for all $n = 1, 2, \dots, j$. Suppose there exists $1 \leq s \leq j$ such that $A_0^s \equiv 0$. By (20), we have

$$T(r, A_0) = m(r, A_0) \leq \begin{cases} \sum\limits_{l=0}^{s-1} \left(\sum\limits_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + O(\log r) & \text{if } s < k \\ \sum\limits_{l=0}^{k-2} \left(\sum\limits_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + O(\log r) & \text{if } s \geq k \end{cases}$$

$$= \begin{cases} \sum\limits_{l \in I_s} \left(\sum\limits_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + \sum\limits_{l \in \{0,1,\dots,s-1\} - I_s} \left(\sum\limits_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + O(\log r) & \text{if } s < k \\ \sum\limits_{l \in I_k} \left(\sum\limits_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + \sum\limits_{l \in \{0,1,\dots,k-2\} - I_k} \left(\sum\limits_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + O(\log r) & \text{if } s \geq k. \end{cases}$$

Then, by using Lemma 10, there exists a set $E_2 \subset (1, +\infty)$ having infinite logarithmic measure such that for all $r \in E_2$, we have

$$T(r, A_0) = m(r, A_0) \leq \begin{cases} \sum\limits_{l \in I_s} \left(\sum\limits_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + o\big(T(r, A_0)\big) & \text{if } s < k, \\ \sum\limits_{l \in I_k} \left(\sum\limits_{p=l}^{s-1} C_p^l \right) T(r, A_{l+1}) + o\big(T(r, A_0)\big) & \text{if } s \geq k, \end{cases}$$

which implies the contradiction

$$\tau(A_0) \leq \begin{cases} \sum\limits_{l \in I_s} \beta_l \tau(A_{l+1}) & \text{if } s < k, \\ \sum\limits_{l \in I_k} \beta_l \tau(A_{l+1}) & \text{if } s \geq k, \end{cases}$$

where $\beta_l = \sum_{p=l}^{s-1} C_p^l$. Hence $A_0^n(z) \not\equiv 0$ for all $n = 1, 2, \dots, j$. By the same procedure we deduce that $F^n(z) \not\equiv 0$ for all $n = 1, 2, \dots, j$. Then, by Theorem 2 we have

$$\overline{\lambda}\big(f^{(m)}\big) = \lambda\big(f^{(m)}\big) = +\infty \quad (m = 0, 1, \dots, j)$$

and

$$\overline{\lambda_2}\big(f^{(m)}\big) = \lambda_2\big(f^{(m)}\big) = \rho \quad (m = 0, 1, \dots, j). \qquad \square$$

Proof of Theorem 6. By using the same reasoning as in the proof of Theorem 3, we can prove Theorem 6. $\square$

Proof of Theorem 7. Since f and $f^{(k)}$ share the value a CM, then

$$\frac{f^{(k)}-a}{f-a}=\mathrm{e}^{Q(z)},$$

where Q is entire function. Set $G=\frac{f}{a}-1$. Then G satisfies the following differential equation

$$G^{(k)}-\mathrm{e}^{Q(z)}G=1. \tag{26}$$

(a) If Q is constant, by solving (26) we obtain $\rho(G)=\rho(f)=1$.

(b) If Q is nonconstant, we know from Lemma 7 that $\rho_2(G)=\rho(\mathrm{e}^Q)$ with at most one exception, which means that G is of infinite order with one exception at most. On the other hand, $A_0(z)=-\mathrm{e}^{Q(z)}$, $A_i(z)\equiv 0$ $(i=1,\dots,k-1)$ and $F(z)=1$, then

$$\rho(A_0)>\max_{i=1,\dots,k-1}\bigl\{\rho(A_i),\rho(F)\bigr\}.$$

Hence $A_0^j(z)\not\equiv 0$ and $F^j(z)\not\equiv 0$, $j\in\mathbb{N}$. So, by applying Theorem 2, we obtain

$$\overline{\lambda}\bigl(G^{(j)}\bigr)=\lambda\bigl(G^{(j)}\bigr)=\rho(G)=\infty \quad (j=0,1,2,\dots)$$

with one exception at most. Since $G=\frac{f}{a}-1$, we deduce

$$\overline{\lambda}(f-a)=\overline{\lambda}\bigl(f^{(j)}\bigr)=\infty \quad (j=1,2,\dots)$$

with one exception at most. □

Acknowledgements

The authors want to thank the anonymous referee for his/her valuable suggestions.

References

[1] S. Bank, I. Laine: On the oscillation theory of $f''+Af=0$ where A is entire. *Trans. Amer. Math. Soc.* 273 (1) (1982) 351–363.

[2] S. Bank, I. Laine: On the zeros of meromorphic solutions of second order linear differential equations. *Comment. Math. Helv.* 58 (4) (1983) 656–677.

[3] B. Belaïdi: Growth and oscillation theory of solutions of some linear differential equations. *Mat. Vesnik* 60 (4) (2008) 233–246.

[4] T. B. Cao: Growth of solutions of a class of complex differential equations. *Ann. Polon. Math.* 95 (2) (2009) 141–152.

[5] Z. X. Chen: Zeros of meromorphic solutions of higher order linear differential equations. *Analysis* 14 (4) (1994) 425–438.

[6] Z. X. Chen, S. A. Gao: The complex oscillation theory of certain nonhomogeneous linear differential equations with transcendental entire coefficients. *J. Math. Anal. Appl.* 179 (2) (1993) 403–416.

[7] Z. X. Chen, C. C. Yang: Some further results on the zeros and growths of entire solutions of second order linear differential equations. *Kodai Math. J.* 22 (2) (1999) 273–285.

[8] G. G. Gundersen, E. M. Steinbart, S. Wang: The possible orders of solutions of linear differential equations with polynomial coefficients. *Trans. Amer. Math. Soc.* 350 (3) (1998) 1225–1247.

[9] W. K. Hayman: *Meromorphic functions*. Clarendon Press, Oxford (1964).

[10] S. Hellerstein, J. Miles, J. Rossi: On the growth of solutions of certain linear differential equations. *Ann. Acad. Sci. Fenn. Ser. A I Math.* 17 (2) (1992) 343–365.

[11] I. Laine: *Nevanlinna theory and complex differential equations*. de Gruyter Studies in Mathematics, 15. Walter de Gruyter & Co., Berlin-New York (1993).

[12] Z. Latreuch, B. Belaïdi: On the zeros of solutions and their derivatives of second order non-homogeneous linear differential equations. *Miskolc Math. Notes* 16 (1) (2015) 237–248.

[13] B. Ya. Levin: *Lectures on entire functions*. American Mathematical Society (1996).

[14] L. A. Rubel, C. C. Yang: Values shared by an entire function and its derivative. Complex analysis (Proc. Conf., Univ. Kentucky, Lexington, Ky., 1976). Lecture Notes in Math.

[15] J. Tu, H. Y. Xu, C. Y. Zhang: On the zeros of solutions of any order of derivative of second order linear differential equations taking small functions. *Electron. J. Qual. Theory Differ. Equ.* 23 (2011) 1–17.

[16] L. Wang, H. Liu: Growth of meromorphic solutions of higher order linear differential equations. *Electron. J. Differential Equations* 125 (2014) 1–11.

[17] C. C. Yang, H. X. Yi: Uniqueness theory of meromorphic functions. *Kluwer Academic Publishers Group, Dordrecht* 557 (2003).

[18] L. Z. Yang: The growth of linear differential equations and their applications. *Israel J. Math.* 147 (2005) 359–370.

Authors' address:
Department of Mathematics, Laboratory of Pure and Applied Mathematics, University of Mostaganem (UMAB), B. P. 227 Mostaganem, Algeria
E-mail: z.latreuch@gmail.com, benharrat.belaidi@univ-mosta.dz

Received: 25 March, 2015
Accepted for publication: 11 July, 2015
Communicated by: Geoff Prince

Communications in Mathematics 23 (2015) 163–169

Almost pseudo symmetric Sasakian manifold admitting a type of quarter symmetric metric connection

Vishnuvardhana S.V. and Venkatesha

Abstract. In the present paper we have obtained the necessary condition for the existence of almost pseudo symmetric and almost pseudo Ricci symmetric Sasakian manifold admitting a type of quarter symmetric metric connection.

1 Introduction

Cartan [2] initiated the study of Riemannian symmetric spaces and obtained a classification of these spaces. The class of Riemannian symmetric manifolds is a very natural generalization of the class of manifolds of constant curvature. Many authors have been studied the notion of locally symmetric manifolds by extending into several manifolds such as recurrent manifolds [20], pseudo-Riemannian manifold with recurrent concircular curvature tensor [14], semi-symmetric manifolds [17], pseudo symmetric manifolds [3], weakly symmetric manifolds [18], almost pseudo symmetric manifolds [8], etc.

A non-flat Riemannian manifold (M^n, g) $(n \geq 2)$ is said to be almost pseudo symmetric $(APS)_n$ [8], if the curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = [A(X)+B(X)]R(Y,Z)W + A(Y)R(X,Z)W + A(Z)R(Y,X)W + A(W)R(Y,Z)X + g(R(Y,Z)W,X)P\,, \tag{1}$$

where A, B are two nonzero 1-forms defined by

$$A(X) = g(X,P), \qquad B(X) = g(X,Q). \tag{2}$$

If in particular $A = B$ in (1) then the manifold reduces to a pseudo symmetric manifold introduced by M. C. Chaki [3].

2010 MSC: 53C05, 53C15, 53D10

Key words: Almost pseudo symmetric manifold, pseudo Ricci symmetric manifold, almost pseudo Ricci symmetric manifold, quarter-symmetric metric connection

Recently Gazi, Pal and Mallick with U.C. De studied almost pseudo conformally symmetric manifolds [9], almost pseudo-Z-symmetric manifolds [11] and almost pseudo concircularly symmetric manifolds [10]. Also Yilmaz in [21] studied decomposable almost pseudo conharmonically symmetric manifolds.

In 2007, Chaki and Kawaguchi [5] introduced the notion of almost pseudo Ricci symmetric manifolds as an extended class of pseudo symmetric manifold. A Riemannian manifold (M^n, g) is called an almost pseudo Ricci symmetric manifold $(APRS)_n$, if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = [A(X)+B(X)]S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y), \quad (3)$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and A, B are two nonzero 1-forms defined as in (2). If, in particular, $B = A$ then almost pseudo Ricci symmetric manifold reduces to pseudo Ricci symmetric manifold [4]. It may be mentioned that almost pseudo Ricci symmetric manifold is not a particular case of weakly Ricci symmetric manifold, introduced by Tamassy and Binh [19]. Since then, several papers [7], [13], [16] have appeared concerning different aspects of almost pseudo Ricci symmetric manifold.

Motivated by the above study, in the present paper we have studied the existence of almost pseudo symmetric and almost pseudo Ricci-symmetric Sasakian manifolds admitting a quarter-symmetric metric connection. The paper is organized as follows: In Section 2, we have given a brief introduction about Sasakian manifolds and some formulae for quarter-symmetric metric connection. In the next section, it is shown that almost pseudo symmetric Sasakian manifold satisfies cyclic Ricci tensor only when $3A(X) + B(X) = 0$. Section 4 is devoted to study of almost pseudo symmetric Sasakian manifold with respect to quarter symmetric metric connection, here we proved that there is no almost pseudo symmetric Sasakian manifold admitting a quarter symmetric metric connection, unless $3A+B$ vanishes everywhere. In the last section we studied almost pseudo Ricci symmetric Sasakian manifold with respect to quarter symmetric metric connection.

2 Preliminaries

It is known that in a Sasakian manifold M^n, the following relations hold [1], [15]:

$$\phi^2 = -I + \eta\, o\, \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad g(X,\xi) = \eta(X), \quad (4)$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad (5)$$

$$(\nabla_X \phi)Y = R(\xi, X)Y, \quad \nabla_X \xi = -\phi X, \quad (6)$$

$$d\eta(\phi X, \xi) = 0, \quad d\eta(X,\xi) = 0, \quad (7)$$

$$(a) \quad g(R(\xi,X)Y,\xi) = g(X,Y) - \eta(X)\eta(Y), \quad (b) \quad R(\xi,X)\xi = -X + \eta(X)\xi, \quad (8)$$

$$S(X,\xi) = (n-1)\eta(X), \quad (9)$$

$$g(R(X,Y)\xi, Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \quad (10)$$

for any vector fields X, Y, Z on M^n.

Here we consider a quarter symmetric metric connection $\tilde{\nabla}$ on a Sasakian manifold given by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{11}$$

The relation between curvature tensor $\tilde{R}(X,Y)Z$ of M^n with respect to quarter symmetric metric connection $\tilde{\nabla}$ and the Riemannian curvature tensor $R(X,Y)Z$ with respect to the connection ∇ is given by [12]

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z - 2d\eta(X,Y)\phi Z + \eta(X)g(Y,Z)\xi \\ - \eta(Y)g(X,Z)\xi + \{\eta(Y)X - \eta(X)Y\}\eta(Z)\,,\end{aligned} \tag{12}$$

where $R(X,Y)Z$ is the Riemannian curvature of the manifold. Also from (12) we obtain

$$\tilde{S}(X,Y) = S(Y,Z) - 2d\eta(\phi Z,Y) + g(Y,Z) + (n-2)\eta(Y)\eta(Z), \tag{13}$$

where $\tilde{S}$ and S are the Ricci tensors of the connections $\tilde{\nabla}$ and ∇ respectively. From (13) it is clear that in a Sasakian manifold the Ricci tensor with respect to the quarter-symmetric metric connection is symmetric.

Now contracting (13) we have

$$\tilde{r} = r + 2(n-1), \tag{14}$$

where $\tilde{r}$ and r are the scalar curvatures of the connections $\tilde{\nabla}$ and ∇ respectively.

3 Almost Pseudo Symmetric Sasakian manifold Satisfying Cyclic Ricci tensor

On taking the cyclic sum of (3), we get

$$\begin{aligned}(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = [3A(X) + B(X)]S(Y,Z) \\ + [3A(Y) + B(Y)]S(X,Z) + [3A(Z) + B(Z)]S(X,Y).\end{aligned} \tag{15}$$

Let M^n admits a cyclic Ricci tensor. Then (15) reduces to

$$[3A(X) + B(X)]S(Y,Z) + [3A(Y) + B(Y)]S(X,Z) + [3A(Z) + B(Z)]S(X,Y) = 0. \tag{16}$$

Taking $Z = \xi$ in (16) and using (9), we have

$$\begin{aligned}[3A(X) + B(X)](n-1)\eta(Y) + [3A(Y) + B(Y)](n-1)\eta(X) \\ + [3A(\xi) + B(\xi)]S(X,Y) = 0.\end{aligned} \tag{17}$$

Now putting $Y = \xi$ in the above equation and by making use of (2), (4) and (9), we obtain

$$(n-1)[3A(X) + B(X)] + [3\eta(P) + \eta(Q)](n-1)\eta(X) + [3\eta(P) + \eta(Q)]S(X,\xi) = 0. \tag{18}$$

Again taking $X = \xi$ in (18) and using (2), (4) and (9), we get

$$3\eta(P) + \eta(Q) = 0. \tag{19}$$

From equations (18) and (19), it follows that

$$3A(X) + B(X) = 0. \tag{20}$$

Thus we can state:

Theorem 1. *An almost pseudo symmetric Sasakian manifold satisfies cyclic Ricci tensor if and only if $3A(X) + B(X) = 0$ for any vector fields X, Y, Z on M^n.*

4 Almost pseudo symmetric Sasakian manifold with respect to quarter symmetric metric connection

Definition 1. A Sasakian manifold (M^n, g) $(n \geq 2)$ is said to be almost pseudo symmetric $(APS)_n$ with respect to quarter symmetric metric connection, if there exist 1-forms A and B and a vector field P such that

$$\begin{aligned}(\bar{\nabla}_X \bar{R})(Y, Z)W = {} & [A(X) + B(X)]\bar{R}(Y, Z)W \\ & + A(Y)\bar{R}(X, Z)W + A(Z)\bar{R}(Y, X)W \\ & + A(W)\bar{R}(Y, Z)X + g(\bar{R}(Y, Z)W, X)P,\end{aligned} \tag{21}$$

Theorem 2. *There is no almost pseudo symmetric Sasakian manifold admitting a quarter symmetric metric connection, unless $3A + B$ vanishes everywhere.*

Proof. Contracting (21), we get

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Z, W) = {} & [A(X) + B(X)]\tilde{S}(Z, W) + A(\tilde{R}(X, Z)W) \\ & + A(Z)\tilde{S}(X, W) + A(W)\tilde{S}(Z, X) + A(\tilde{R}(X, W)Z).\end{aligned} \tag{22}$$

Substituting $W = \xi$ in (22) and then using the relations (12) and (13), we have

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Z, \xi) = {} & 2n\eta(Z)A(X) + 2(n-1)\eta(Z)B(X) + 2(n-2)\eta(X)A(Z) \\ & + \eta(P)\big\{S(Z, X) - 2d\eta(\phi X, Z) + g(X, Z) + (n-2)\eta(X)\eta(Z)\big\} \\ & + 2\eta(Z)A(X) - 2g(X, Z)A(\xi).\end{aligned} \tag{23}$$

Now, we know that

$$(\tilde{\nabla}_X \tilde{S})(Z, Y) = \tilde{\nabla}_X(\tilde{S}(Z, Y)) - \tilde{S}(\tilde{\nabla}_X Z, Y) - \tilde{S}(Z, \tilde{\nabla}_X Y). \tag{24}$$

Replacing Y with ξ in the above equation and using (4), (11) and (13), we get

$$(\tilde{\nabla}_X \tilde{S})(Z, \xi) = S(Z, \phi X) + (1 - 2n)g(Z, \phi X). \tag{25}$$

By virtue of (23) and (25), we obtain

$$\begin{aligned}S(Z, \phi X) + (1 - 2n)g(Z, \phi X) = {} & 2n\eta(Z)A(X) \\ & + 2(n-1)\eta(Z)B(X) + 2(n-2)\eta(X)A(Z) \\ & + \eta(P)\big\{S(Z, X) - 2d\eta(\phi X, Z) + g(X, Z) + (n-2)\eta(X)\eta(Z)\big\} \\ & + 2\eta(Z)A(X) - 2g(X, Z)A(\xi).\end{aligned} \tag{26}$$

Taking $X = Z = \xi$ in (26) and using (4) and (9), we obtain

$$3A(\xi) + B(\xi) = 0. \tag{27}$$

Putting $Z = \xi$ in (22) and by making use of equations (4), (7), (12) and (13), we get

$$\begin{aligned} S(\phi X, W) + (1-2n)g(\phi X, W) &= 2(n+1)A(X)\eta(W) \\ &+ 2(n-1)B(X)\eta(W) - 2g(X,W)A(\xi) \\ &+ A(\xi)\{S(X,W) - 2d\eta(\phi W, X) + g(X,W) + (n-2)\eta(X)\eta(W)\} \\ &+ 2(n-2)\eta(X)A(W). \end{aligned} \tag{28}$$

By taking $X = \xi$ in (28) and then using (4) and (9), it follows that

$$0 = 2(2n-1)A(\xi)\eta(W) + 2(n-1)B(\xi)\eta(W) + 2(n-2)A(W). \tag{29}$$

Again putting $W = \xi$ in (28) and using (4) and (9), we have

$$0 = 2(n+1)A(X) + 2(n-1)B(X) + 4(n-2)A(\xi)\eta(X). \tag{30}$$

Adding (29) with (30) by replacing W by X and in view of (27), we get

$$4nA(X) - 2A(X) + 2(n-1)B(X) + 2(n-2)A(\xi)\eta(X) = 0. \tag{31}$$

Further replacing W by X in (29) and then adding with (31), in view of (27) we arrive at

$$3A(X) + B(X) = 0$$ □

5 Almost pseudo Ricci symmetric Sasakian manifold with respect to quarter symmetric metric connection

A non-flat n-dimensional Riemannian manifold M^n $(n \geq 2)$ is said to be almost pseudo Ricci symmetric Sasakian manifold with respect to quarter symmetric metric connection if there exist 1-forms A and B such that

$$(\tilde{\nabla}_X \tilde{S})(Y,Z) = [A(X) + B(X)]\tilde{S}(Y,Z) + A(Y)\tilde{S}(X,Z) + A(Z)\tilde{S}(X,Y). \tag{32}$$

Theorem 3. *There is no almost pseudo Ricci symmetric Sasakian manifold admitting a quarter symmetric metric connection, unless $3A + B = 0$ everywhere.*

Proof. Assume that M^n is an almost pseudo Ricci symmetric Sasakian manifold with respect to quarter symmetric metric connection. Replacing Z with ξ in (32) and then using (25), we get

$$\begin{aligned} &S(\phi X, Y) + (1-2n)g(\phi X, Y) \\ &\qquad = [A(X) + B(X)]\tilde{S}(Y,\xi) + A(Y)\tilde{S}(X,\xi) + A(\xi)\tilde{S}(X,Y). \end{aligned} \tag{33}$$

By substituting $X = Y = \xi$ in (33) and then using (13), one can get

$$3A(\xi) + B(\xi) = 0. \tag{34}$$

Taking $X = \xi$ in (33) and in view of (13), we have

$$0 = 2(n-1)\{2A(\xi)\eta(Y) + B(\xi)\eta(Y) + A(Y)\}. \tag{35}$$

Putting $Y = \xi$ in (33) and using (13), we get

$$0 = 2(n-1)\{A(X) + B(X) + 2A(\xi)\eta(X)\}. \tag{36}$$

Adding (35) with (36) by replacing Y by X and in view of (34), we obtain

$$2(n-1)\{2A(X) + B(X) + A(\xi)\eta(X)\} = 0. \tag{37}$$

Replacing Y by X in (35) and then adding with (37), in view of (34) we get

$$3A(X) + B(X) = 0.$$

□

Acknowledgement

Vishnuvardhana S.V. was supported by the Department of Science and Technology, INDIA though the JRF [IF140186] DST/INSPIRE FELLOWSHIP/2014/181.

References

[1] D.E. Blair: *Contact manifolds in Riemannian geometry*. Springer-Verlag, Berlin (1976). Lecture Notes in Mathematics, 509

[2] E. Cartan: Sur une classe remarquable d'espaces de Riemann. *Soc. Math., France* 54 (1926) 214–264.

[3] M.C. Chaki: On pseudo symmetric manifolds. *Analele Stiint, Univ.Al-I. Cuza* 33 (1987) 53–58.

[4] M.C. Chaki: On pseudo Ricci symmetric manifolds. *Bulg. J. Phys.* 15 (1988) 526–531.

[5] M.C. Chaki, T. Kawaguchi: On almost pseudo Ricci symmetric manifolds. *Tensor N. S.* 68 (2007) 10–14.

[6] A. De: On almost pseudo symmetric manifolds admitting a type of semi-symmetric non-metric connection. *Acta Math. Hungar.* 125 (2009) 183–190.

[7] U.C. De, A. De: On almost pseudo-conformally symmetric Ricci-recurrent manifolds with applications to relativity. *Czechoslovak Mathematical Journal* 62 (2012) 1055–1072.

[8] U.C. De, A.K. Gazi: On almost pseudo symmetric manifolds. *Ann. Univ. Sci. Budap.* 51 (2008) 53–68.

[9] U.C. De, A.K. Gazi: On almost pseudo conformally symmetric manifolds. *Demonstratio Mathematica* 4 (2009) 861–878.

[10] U.C. De, S. Mallick: On almost pseudo concircularly symmetric manifolds. *The Journal of Mathematics and Computer Science* 4 (3) (2012) 317–330.

[11] U.C. De, P. Pal: On Almost Pseudo-Z-symmetric Manifolds. *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica* 53 (1) (2014) 25–43.

[12] U.C. De, J. Sengupta: Quarter-symmetric metric connection on a Sasakian manifold. *Commun. Fac. Sci. Univ. Ank. Series A1.* 49 (2000) 7–13.

[13] S.K. Hui, F.O. Zengin: On almost pseudo concircular Ricci symmetric manifolds. *Boletin de la Asociacion Matematica Venezolana* 1 (2012) 57–69.

[14] K. Olszak, Z. Olszak: On pseudo-riemannian manifolds with recurrent concircular curvature tensor. *Acta Mathematica Hungar.* 137 (1-2) (2012) 64–71.

[15] S. Sasaki: *Lecture notes on almost contact manifolds, part 1.* Tohoku University (1975).

[16] A.A. Shaikh, S.K. Hui, C.S. Bagewadi: On Quasi-Conformally Flat Almost Pseudo Ricci Symmetric Manifolds. *Tamsui Oxford Journal of Mathematical Sciences* 26 (2) (2010) 203–219.

[17] Z.I. Szabo: Structure theorems on riemannian spaces satisfying $R(X,Y)R = 0$. *Journal of Differential Geometry* 17 (1982) 531–582.

[18] L. Tamassy, T.Q. Binh: On weakly symmetric and weakly projectively symmetric riemannian manifolds. *Colloquia Mathematica Societatis Janos Bolyai* 56 (1989) 663–670.

[19] L. Tamassy, T.Q. Binh: On weak symmetries of Einstein and Sasakian manifolds. *Tensor* 53 (1993) 140–148. International Conference on Differential Geometry and its Applications (Bucharest, 1992)

[20] A.G. Walker: On ruse's space of recurrent curvature. *Proceedings of the London Mathematical Society* 52 (1951) 36–64.

[21] H.B. Yilmaz: On Decomposable Almost Pseudo Conharmonically Symmetric Manifolds. *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica* 51 (1) (2012) 111–124.

Authors' address:
Department of Mathematics, Kuvempu University, Shankaraghatta, 577 451, Shimoga, Karnataka, India

E-mail: vensmath@gmail.com, svvishnuvardhana@gmail.com

Received: 24 August, 2015
Accepted for publication: 30 October, 2015
Communicated by: Olga Rossi

Forthcoming conferences

ALaNT 4 – Joint Conferences on Algebra, Logic and Number Theory

The joint conferences on Algebra, Logic and Number Theory (ALANT) unite two simultaneous events: the Czech, Polish and Slovak Conference on Number Theory and the Colloquiumfest.

Place: Telč (Czech Republic)

Date: June 13 – June 17, 2016

Web: `http://ntc.osu.cz/2016`

Differential Geometry and its Applications

The tradition of the international conference Differential Geometry and its Applications goes back more than 35 years and the forthcoming meeting will be the 13th one in the series.

Place: Brno (Czech Republic)

Date: July 11 – July 15, 2016

Web: `http://web.math.muni.cz/dga2016`

Programme Committee

J. Berndt (London)
M.G. Eastwood (Adelaide)
E. Garcia Rio (Santiago de Compostela)
P. Gilkey (Oregon)
O. Kowalski (Prague)
F. Pedit (Massachusetts)
Z. Shen (Indianapolis)
J.D. Stasheff (North Carolina)
A. Čap (Vienna)
V. Cortes (Hamburg)
S. Gindikin (Rutgers)
I. Kolář (Brno)
J.M. Landsberg (College Station, TX)
O. Rossi (Ostrava)
J. Slovák, Chair (Brno)
A.A. Vinogradov (Emilia-Romagna)

Contents of Previous Volumes*

*Full texts in pdf are available free on-line at http://cm.osu.cz.

No. 1, Vol. 23 (2015)

Research papers

Aims and Scope

Communications in Mathematics publishes original research papers and survey papers in pure and applied mathematics, preferably in the following areas: *algebraic structures*, *calculus of variations*, *combinatorics*, *control and optimization*, *cryptography*, *differential equations*, *differential geometry*, *fuzzy logic and fuzzy set theory*, *global analysis*, *history of mathematics*, *mathematical physics*, *number theory*.

To be acceptable for publication papers must make original and significant contributions to one or more of the topics within these areas. High quality review papers of interest to a wide range of scientists in mathematics and its applications are welcome.

Instructions for Authors

Articles submitted for publication must be written in English, and produced in TeX, preferably using Publisher's document style based upon LaTeX.

Articles should contain the title, authors' names and addresses, full contact details of the corresponding author, the abstract, keywords, and 2010 MS classification numbers.

Footnotes should be avoided if possible, but if they are included they should be brief, and numbered consecutively.

References must be as complete as possible and put in alphabetical order.

Submission to the journal proceeds totally on-line through the web page `http://cm.osu.cz`, and the author is guided stepwise through the procedure. To complete the submission successfully, the author is required to choose one of the Editors to handle the paper. He/she should choose an Editor whose areas fit as closely as possible with the topic of the paper. All correspondence, including notification of the Editor's decision, requests for revision, and proofs takes place by e-mail.

Communications in Mathematics, Volume 23, 2015, is the continuation of the journal *Acta Mathematica et Informatica Universitatis Ostraviensis*, ISSN 1211-4774 (1993–2003), and of the journal *Acta Mathematica Universitatis Ostraviensis*, ISSN 1214-8148 (2004–2009).

Published by The University of Ostrava
December 2015

Typeset by TeX

ISSN 1804-1388 (Print), ISSN 2336-1298 (Online)

Contents

Research papers

Available online at
http://cm.osu.cz