VOLUME
22/2014
No. 2

ISSN 1804-1388 (Print)
ISSN 2336-1298 (Online)

COMMUNICATIONS IN MATHEMATICS

Available online at
http://cm.osu.cz

UNIVERSITAS OSTRAVIENSIS

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The journal is reviewed in Zentralblatt für Mathematik and Mathematical Reviews.

ISSN 1804-1388 (Print), ISSN 2336-1298 (Online)

Communications in Mathematics 22 (2014) 107–132

Bounds for Convex Functions of Čebyšev Functional Via Sonin's Identity with Applications

Silvestru Sever Dragomir

Abstract. Some new bounds for the Čebyšev functional in terms of the Lebesgue norms

$$\left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}$$

and the Δ-seminorms

$$\|f\|_p^{\Delta} := \left(\int_a^b \int_a^b |f(t) - f(s)|^p \,\mathrm{d}t\,\mathrm{d}s \right)^{\frac{1}{p}}$$

are established. Applications for mid-point and trapezoid inequalities are provided as well.

1 Introduction

For two Lebesgue integrable functions $f, g\colon [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)\,\mathrm{d}t - \frac{1}{(b-a)^2}\int_a^b f(t)\,\mathrm{d}t \int_a^b g(t)\,\mathrm{d}t\,.$$

In 1935, Grüss [7] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n)\,, \tag{1}$$

provided that there exists the real numbers m, M, n, N such that

$$m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t \in [a,b]\,. \tag{2}$$

2010 MSC: 26D15, 25D10
Key words: Absolutely continuous functions, Convex functions, Integral inequalities, Čebyšev functional, Jensen's inequality, Lebesgue norms, Mid-point inequalities, Trapezoid inequalities

The constant $\frac{1}{4}$ is best possible in (1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882 [5], states that

$$|C(f,g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{3}$$

provided that f', g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

Čebyšev inequality (3) also holds if $f, g\colon [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a,b]$ while $\|f'\|_\infty = \operatorname{ess\,sup}_{t\in[a,b]} |f'(t)|$.

A mixture between Grüss' result (1) and Čebyšev's one (3) is the following inequality obtained by Ostrowski in 1970 [12]:

$$|C(f,g)| \le \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{4}$$

provided that f is *Lebesgue integrable* and satisfies (2) while g is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (4).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [9] in which he proved that

$$|C(f,g)| \le \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a), \tag{5}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, Cerone and Dragomir [2] have proved the following results:

$$|C(f,g)| \le \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \cdot \frac{1}{b-a}\left(\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right|^p \mathrm{d}t\right)^{\frac{1}{p}}, \tag{6}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, and

$$|C(f,g)| \le \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_1 \cdot \frac{1}{b-a} \operatorname*{ess\,sup}_{t\in[a,b]} \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right|,$$

provided that $f \in L_p[a,b]$ and $g \in L_q[a,b]$ ($p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$; $p = 1$, $q = \infty$ or $p = \infty$, $q = 1$).

Notice that for $q = \infty$, $p = 1$ in (6) we obtain

$$\begin{aligned} |C(f,g)| &\le \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right| \mathrm{d}t \\ &\le \|g\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right| \mathrm{d}t \end{aligned}$$

and if g satisfies (2), then

$$\begin{aligned}|C(f,g)| &\le \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right| \mathrm{d}t \\ &\le \left\| g - \frac{n+N}{2}\right\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right| \mathrm{d}t \\ &\le \frac{1}{2}(N-n)\cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right| \mathrm{d}t. \qquad (7)\end{aligned}$$

The inequality between the first and the last term in (7) has been obtained by Cheng and Sun in [6]. However, the sharpness of the constant $\frac{1}{2}$, a generalization for the abstract Lebesgue integral and the discrete version of it have been obtained in [3].

For other recent results on the Grüss inequality, see [8], [10] and [13] and the references therein.

In this paper, some new bounds for the Čebyšev functional in terms of the Lebesgue norms $\left\|f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right\|_{[a,b],p}$ and the Δ-seminorms are established. Applications for mid-point and trapezoid inequalities are provided as well.

2 Some Results Via Sonin's Identity

The following result for convex functions of Čebyšev functional holds.

Theorem 1. *Let $f,g\colon [a,b] \to \mathbb{R}$ be Lebesgue integrable functions on $[a,b]$. If $\Phi\colon \mathbb{R}\to\mathbb{R}$ is convex on $\mathbb{R}$ then we have the inequality*

$$\begin{aligned}\Phi[C(f,g)] &\le \frac{1}{b-a}\inf_{\lambda\in\mathbb{R}}\int_a^b \Phi\left[\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\big(g(x)-\lambda\big)\right]\mathrm{d}x \\ &\le \frac{1}{(b-a)^2}\inf_{\lambda\in\mathbb{R}}\int_a^b\int_a^b \Phi\Big[\big(f(x)-f(t)\big)\big(g(x)-\lambda\big)\Big]\,\mathrm{d}t\,\mathrm{d}x. \qquad (8)\end{aligned}$$

Proof. Start with Sonin's identity [11, p. 246]

$$C(f,g) = \frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\big(g(x)-\lambda\big)\,\mathrm{d}x$$

that holds for any $\lambda\in\mathbb{R}$.

If we use Jensen's integral inequality we have for any $\lambda\in\mathbb{R}$

$$\begin{aligned}\Phi[C(f,g)] &= \Phi\left[\frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\big(g(x)-\lambda\big)\,\mathrm{d}x\right] \\ &\le \frac{1}{b-a}\int_a^b \Phi\left[\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\big(g(x)-\lambda\big)\right]\mathrm{d}x \\ &= \frac{1}{b-a}\int_a^b \Phi\left[\frac{1}{b-a}\int_a^b \Big[\big(f(x)-f(t)\big)\big(g(x)-\lambda\big)\Big]\,\mathrm{d}t\right]\mathrm{d}x \\ &\le \frac{1}{(b-a)^2}\int_a^b\int_a^b \Phi\Big[\big(f(x)-f(t)\big)\big(g(x)-\lambda\big)\Big]\,\mathrm{d}t\,\mathrm{d}x\,.\end{aligned}$$

Taking the infimum over $\lambda\in\mathbb{R}$ we deduce the desired inequalities (8). $\square$

Remark 1. If we write inequality (8) for the convex function $\Phi(x) = |x|^p$, $p \geq 1$, then we get the inequality

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p}} \inf_{\lambda\in\mathbb{R}} \left\{ \int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^p \left| g(x) - \lambda \right|^p \mathrm{d}x \right\}^{1/p} \tag{9}$$
$$\leq \frac{1}{(b-a)^{2/p}} \inf_{\lambda\in\mathbb{R}} \left\{ \int_a^b \int_a^b \left| f(x) - f(t) \right|^p \left| g(x) - \lambda \right|^p \mathrm{d}t\,\mathrm{d}x \right\}^{1/p}.$$

Utilising Hölder's integral inequality we have

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^p \left| g(x) - \lambda \right|^p \mathrm{d}x$$
$$\leq \operatorname*{ess\,sup}_{x\in[a,b]} \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^p \int_a^b \left| g(x) - \lambda \right|^p \mathrm{d}x$$
$$= \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty}^p \|g - \lambda\|_{[a,b],p}^p,$$

b) for $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^p \left| g(x) - \lambda \right|^p \mathrm{d}x$$
$$\leq \left(\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^{p\beta} \mathrm{d}x \right)^{1/\beta} \left(\int_a^b \left| g(x) - \lambda \right|^{p\alpha} \mathrm{d}x \right)^{1/\alpha}$$
$$= \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta}^p \|g - \lambda\|_{[a,b],p\alpha}^p,$$

c) for $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^p \left| g(x) - \lambda \right|^p \mathrm{d}x$$
$$\leq \operatorname*{ess\,sup}_{x\in[a,b]} \left| g(x) - \lambda \right|^p \int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^p \mathrm{d}x$$
$$= \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}^p \|g - \lambda\|_{[a,b],\infty}^p.$$

Utilising (9) we can state the following result.

Theorem 2. *Let $f, g\colon [a,b] \to \mathbb{R}$ be Lebesgue measurable functions on $[a,b]$. Then*

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p}} \inf_{\lambda\in\mathbb{R}} \|g - \lambda\|_{[a,b],p} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty},$$

b) *for* $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \inf_{\lambda \in \mathbb{R}} \|g - \lambda\|_{[a,b],p\alpha} \left\| f - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta},$$

c) *for* $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \inf_{\lambda \in \mathbb{R}} \|g - \lambda\|_{[a,b],\infty} \left\| f - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}.$$

We have the following particular cases of interest.

Corollary 1. *Let* $f, g \colon [a,b] \to \mathbb{R}$ *be Lebesgue measurable functions on* $[a,b]$*. Then*

a) *for* $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \left\| g - \frac{1}{b-a} \int_a^b g(t)\,\mathrm{d}t \right\|_{[a,b],p} \left\| f - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty},$$

b) *for* $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \left\| g - \frac{1}{b-a} \int_a^b g(t)\,\mathrm{d}t \right\|_{[a,b],p\alpha} \left\| f - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta},$$

c) *for* $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \left\| g - \frac{1}{b-a} \int_a^b g(t)\,\mathrm{d}t \right\|_{[a,b],\infty} \left\| f - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}.$$

If one function is bounded, then we can state the following result.

Corollary 2. *Assume that* $f, g \colon [a,b] \to \mathbb{R}$ *are Lebesgue measurable functions on* $[a,b]$*. If there exist constants* n, N *such that* $n \le g(t) \le N$ *for a.e.* $t \in [a,b]$*, then*

a) *for* $f \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{2}(N-n) \left\| f - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty},$$

b) *for* $f \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le \frac{1}{2}(N-n) \frac{1}{(b-a)^{1/p\beta}} \left\| f - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta},$$

c) for $f \in L_p[a,b]$

$$|C(f,g)| \leq \frac{1}{2}(N-n)\frac{1}{(b-a)^{1/p}}\left\|f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right\|_{[a,b],p}.$$

Proof. We observe that

$$\begin{aligned}
\left\|g - \frac{n+N}{2}\right\|_{[a,b],p} &= \left(\int_a^b \left|g(t) - \frac{n+N}{2}\right|^p \mathrm{d}t\right)^{1/p} \\
&\leq \left(\int_a^b \left(\frac{N-n}{2}\right)^p \mathrm{d}t\right)^{1/p} = \frac{N-n}{2}(b-a)^{1/p}, \\
\left\|g - \frac{n+N}{2}\right\|_{[a,b],p\alpha} &= \left(\int_a^b \left|g(t) - \frac{n+N}{2}\right|^{p\alpha} \mathrm{d}t\right)^{1/p\alpha} \\
&\leq \frac{N-n}{2}(b-a)^{1/p\alpha}
\end{aligned}$$

and

$$\left\|g - \frac{n+N}{2}\right\|_{[a,b],\infty} \leq \frac{N-n}{2}.$$

Utilising Theorem 2 we deduce the desired result of Corollary 2. □

When one function is of bounded variation, then we can state the following result.

Corollary 3. *If $f\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable and $g\colon [a,b] \to \mathbb{R}$ is of bounded variation, then*

a) *for $f \in L_\infty[a,b]$*

$$|C(f,g)| \leq \frac{1}{2}\bigvee_a^b(g)\left\|f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right\|_{[a,b],\infty},$$

b) *for $f \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$*

$$|C(f,g)| \leq \frac{1}{2}\bigvee_a^b(g)\frac{1}{(b-a)^{1/p\beta}}\left\|f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right\|_{[a,b],p\beta},$$

c) *for $f \in L_p[a,b]$*

$$|C(f,g)| \leq \frac{1}{2}\bigvee_a^b(g)\frac{1}{(b-a)^{1/p}}\left\|f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right\|_{[a,b],p},$$

where $\bigvee_a^b(g)$ is the total variation of the function g on the interval $[a,b]$.

Proof. Since $g\colon [a,b] \to \mathbb{R}$ is of bounded variation, then for any $t \in [a,b]$ we have

$$\left| g(t) - \frac{g(a)+g(b)}{2} \right| = \left| \frac{g(t)-g(a)+g(t)-g(b)}{2} \right| \leq \frac{1}{2}\Big[\big|g(t)-g(a)\big| + \big|g(b)-g(t)\big| \Big] \leq \frac{1}{2} \bigvee_a^b (g)\,.$$

Then

$$\left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],p} = \left(\int_a^b \left| g(t) - \frac{g(a)+g(b)}{2} \right|^p \mathrm{d}t \right)^{1/p} \leq \left(\int_a^b \left(\frac{1}{2} \bigvee_a^b (g) \right)^p \mathrm{d}t \right)^{1/p} = \frac{1}{2} \bigvee_a^b (g)\, (b-a)^{1/p},$$

$$\left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],p\alpha} \leq \frac{1}{2} \bigvee_a^b (g)\, (b-a)^{1/p\alpha},$$

and

$$\left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],\infty} \leq \frac{1}{2} \bigvee_a^b (g)\,.$$

Utilising Theorem 2 we deduce the desired result of Corollary 3. □

For functions h that are *Lipschitzian in the middle point* with the constant $L_{\frac{a+b}{2}}$ and the exponent $q > 0$, i.e. satisfying the condition

$$\left| h(t) - h\!\left(\frac{a+b}{2} \right) \right| \leq L_{\frac{a+b}{2}} \left| t - \frac{a+b}{2} \right|^q$$

for any $t \in [a,b]$, we have the following result as well.

Corollary 4. *If $f\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable and $g\colon [a,b] \to \mathbb{R}$ is Lipschitzian in the middle point with the constant $L_{\frac{a+b}{2}}$ and the exponent $q > 0$, then*

a) for $f \in L_\infty[a,b]$

$$|C(f,g)| \leq L_{\frac{a+b}{2}} \frac{(b-a)^q}{2^q (qp+1)^{1/p}} \left\| f - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty}, \tag{10}$$

b) for $f \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \leq L_{\frac{a+b}{2}} \frac{(b-a)^{q-1/p\beta}}{2^q (qp\alpha+1)^{1/p\alpha}} \left\| f - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta}, \tag{11}$$

c) for $f \in L_p[a,b]$

$$|C(f,g)| \leq L_{\frac{a+b}{2}} \frac{(b-a)^{q-1/p}}{2^q} \left\| f - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}. \tag{12}$$

Proof. We have

$$\begin{aligned} \left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p} &= \left(\int_a^b \left| g(t) - g\left(\frac{a+b}{2}\right) \right|^p \mathrm{d}t \right)^{1/p} \\ &\leq \left(\int_a^b L^p_{\frac{a+b}{2}} \left| t - \frac{a+b}{2} \right|^{qp} \mathrm{d}t \right)^{1/p} \\ &= L_{\frac{a+b}{2}} \left(\int_a^b \left| t - \frac{a+b}{2} \right|^{qp} \mathrm{d}t \right)^{1/p}. \end{aligned} \tag{13}$$

Observe that

$$\begin{aligned} &\left(\int_a^b \left| t - \frac{a+b}{2} \right|^{qp} \mathrm{d}t \right)^{1/p} \\ &= \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right)^{qp} \mathrm{d}t + \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right)^{qp} \mathrm{d}t \right)^{1/p} \\ &= \left(2 \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right)^{qp} \mathrm{d}t \right)^{1/p} = \left(2 \frac{\left(t - \frac{a+b}{2}\right)^{qp+1}}{qp+1} \Bigg|_{\frac{a+b}{2}}^b \right)^{1/p} \\ &= \left(2 \frac{\left(\frac{b-a}{2}\right)^{qp+1}}{qp+1} \right)^{1/p} = \left(\frac{(b-a)^{qp+1}}{2^{qp}(qp+1)} \right)^{1/p} = \frac{(b-a)^{q+1/p}}{2^q (qp+1)^{1/p}}. \end{aligned}$$

Then by (13) we have

$$\left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p} \leq L_{\frac{a+b}{2}} \frac{(b-a)^{q+1/p}}{2^q (qp+1)^{1/p}}.$$

Also

$$\left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p\alpha} \leq L_{\frac{a+b}{2}} \frac{(b-a)^{q+1/p\alpha}}{2^q (qp\alpha+1)^{1/p\alpha}}$$

and

$$\left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],\infty} \leq L_{\frac{a+b}{2}} \frac{(b-a)^q}{2^q}.$$

Utilising Theorem 2 we obtain

a) for $f \in L_\infty[a,b]$

$$\begin{aligned}|C(f,g)| &\le \frac{1}{(b-a)^{1/p}} \left\| g - g\left(\frac{a+b}{2}\right)\right\|_{[a,b],p} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty} \\ &\le \frac{1}{(b-a)^{1/p}} L_{\frac{a+b}{2}} \frac{(b-a)^{q+1/p}}{2^q (qp+1)^{1/p}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty} \\ &= L_{\frac{a+b}{2}} \frac{(b-a)^q}{2^q(qp+1)^{1/p}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty},\end{aligned}$$

b) for $f \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\begin{aligned}|C(f,g)| &\le \frac{1}{(b-a)^{1/p}} \left\| g - g\left(\frac{a+b}{2}\right)\right\|_{[a,b],p\alpha} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta} \\ &\le \frac{1}{(b-a)^{1/p}} L_{\frac{a+b}{2}} \frac{(b-a)^{q+1/p\alpha}}{2^q (qp\alpha+1)^{1/p\alpha}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta} \\ &= L_{\frac{a+b}{2}} \frac{(b-a)^{q-1/p\beta}}{2^q(qp\alpha+1)^{1/p\alpha}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta},\end{aligned}$$

c) and for $f \in L_p[a,b]$

$$\begin{aligned}|C(f,g)| &\le \frac{1}{(b-a)^{1/p}} \left\| g - g\left(\frac{a+b}{2}\right)\right\|_{[a,b],\infty} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p} \\ &\le \frac{1}{(b-a)^{1/p}} L_{\frac{a+b}{2}} \frac{(b-a)^{q}}{2^q} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p} \\ &= L_{\frac{a+b}{2}} \frac{(b-a)^{q-1/p}}{2^q} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}.\end{aligned}$$

Thus the inequalities (10)–(12) are proved. □

Remark 2. If the function g is Lipschitzian with the constant $L > 0$, then

a) for $f \in L_\infty[a,b]$

$$|C(f,g)| \le L \frac{b-a}{2(p+1)^{1/p}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty}, \tag{14}$$

b) for $f \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le L \frac{(b-a)^{1-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta}, \tag{15}$$

c) for $f \in L_p[a,b]$

$$|C(f,g)| \le L\frac{(b-a)^{1-1/p}}{2}\left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}. \tag{16}$$

3 Δ-Seminorms and Related Inequalities

For $f \in L_p[a,b]$, $p \in [1,\infty)$, we can define the functional (see [1] and [4])

$$\|f\|_p^{\Delta} := \left(\int_a^b \int_a^b \left| f(t) - f(s) \right|^p \mathrm{d}t\,\mathrm{d}s \right)^{\frac{1}{p}}$$

and for $f \in L_\infty[a,b]$, we can define

$$\|f\|_\infty^{\Delta} := \operatorname*{ess\,sup}_{(t,s)\in[a,b]^2} \left| f(t) - f(s) \right|.$$

If we consider $f_\Delta \colon [a,b]^2 \to \mathbb{R}$,

$$f_\Delta(t,s) = f(t) - f(s),$$

then obviously

$$\|f\|_p^{\Delta} = \|f_\Delta\|_p, \qquad p \in [1,\infty],$$

where $\|\cdot\|_p$ are the usual Lebesgue p-norms on $[a,b]^2$.

Using the properties of the Lebesgue p-norms, we may deduce the following seminorm properties for $\|\cdot\|_p^{\Delta}$:

(i) $\|f\|_p^{\Delta} \ge 0$ for $f \in L_p[a,b]$ and $\|f\|_p^{\Delta} = 0$ implies that $f = c$ (c is a constant) a.e. in $[a,b]$,

(ii) $\|f+g\|_p^{\Delta} \le \|f\|_p^{\Delta} + \|g\|_p^{\Delta}$ if $f, g \in L_p[a,b]$,

(iii) $\|\alpha f\|_p^{\Delta} = |\alpha|\,\|f\|_p^{\Delta}$.

We call $\|\cdot\|_p^{\Delta}$ as Δ-seminorms.

We note that if $p = 2$, then

$$\begin{aligned}\|f\|_2^{\Delta} &= \left(\int_a^b \int_a^b \big(f(t) - f(s)\big)^2 \mathrm{d}t\,\mathrm{d}s \right)^{\frac{1}{2}} \\ &= \sqrt{2}\left((b-a)\|f\|_2^2 - \left(\int_a^b f(t)\,\mathrm{d}t \right)^2 \right)^{\frac{1}{2}}.\end{aligned}$$

Using the inequalities (1), (3) and (5), we obtain the following estimate for $\|\cdot\|_2^{\Delta}$:

a) for $m \le f \le M$

$$\|f\|_2^{\Delta} \le \frac{\sqrt{2}}{2}(M-m)(b-a)\,,$$

b) for $f' \in L_\infty[a,b]$

$$\|f\|_2^\Delta \le \frac{\sqrt{2}}{2\sqrt{3}} \|f'\|_\infty (b-a)^2 \,,$$

c) for $f' \in L_2[a,b]$

$$\|f\|_2^\Delta \le \frac{\sqrt{2}}{\pi} \|f'\|_2 (b-a)^{\frac{3}{2}} \,,$$

since

$$\|f\|_2^\Delta = \sqrt{2}(b-a)[C(f,f)]^{\frac{1}{2}} .$$

If $f\colon [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then we can point out the following bounds for $\|f\|_p^\Delta$ in terms of $\|f'\|_p$.

Theorem 3. *Assume that $f\colon [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$.*

(i) If $p \in [1,\infty)$, then we have the inequality

a) for $f' \in L_\infty[a,b]$

$$\|f\|_p^\Delta \le \frac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{\big[(p+1)(p+2)\big]^{\frac{1}{p}}} \|f'\|_\infty \,,$$

b) for $f' \in L_\alpha[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\|f\|_p^\Delta \le \frac{(2\beta^2)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{\big[(p+\beta)(p+2\beta)\big]^{\frac{1}{p}}} \|f'\|_\alpha \,,$$

c) for $f' \in L_1[a,b]$

$$\|f\|_p^\Delta \le (b-a)^{\frac{2}{p}} \|f'\|_1 \,.$$

(ii) If $p = \infty$, then we have the inequality

a) for $f' \in L_\infty[a,b]$

$$\|f\|_\infty^\Delta \le (b-a)\|f'\|_\infty \,,$$

b) for $f' \in L_\alpha[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\|f\|_\infty^\Delta \le (b-a)^{\frac{1}{\beta}} \|f'\|_\alpha \,,$$

c) for $f' \in L_1[a,b]$

$$\|f\|_\infty^\Delta \le \|f'\|_1 \,.$$

The following result of Grüss type holds, see [4].

Theorem 4. *Let $f, g\colon [a,b] \to \mathbb{R}$ be measurable on $[a,b]$. Then we have the inequality*

$$|C(f,g)| \le \frac{1}{2(b-a)^2} \|f\|_p^{\Delta} \|g\|_q^{\Delta},$$

where $p = 1$, $q = \infty$, or $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, or $q = 1$, $p = \infty$, provided all integrals involved exist.

The inequality is sharp in the sense that if we take $f(x) = g(x) = \operatorname{sgn}(x-\alpha)$ with $\alpha = \frac{a+b}{2}$, then the equality results.

Making use of the double integral inequality

$$|C(f,g)| \le \frac{1}{(b-a)^{2/p}} \inf_{\lambda \in \mathbb{R}} \left\{ \int_a^b \int_a^b \bigl|f(x) - f(t)\bigr|^p \bigl|g(x) - \lambda\bigr|^p \,\mathrm{d}t\,\mathrm{d}x \right\}^{1/p},$$

obtained in (9) we can state the following result as well.

Theorem 5. *Let $f, g\colon [a,b] \to \mathbb{R}$ be Lebesgue measurable functions on $[a,b]$. Then*

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \inf_{\lambda \in \mathbb{R}} \|g - \lambda\|_{[a,b],p} \|f\|_\infty^{\Delta}, \tag{17}$$

b) for $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$ $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p + 1/p\beta}} \inf_{\lambda \in \mathbb{R}} \|g - \lambda\|_{[a,b],p\alpha} \|f\|_{p\beta}^{\Delta}, \tag{18}$$

c) for $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{2/p}} \inf_{\lambda \in \mathbb{R}} \|g - \lambda\|_{[a,b],\infty} \|f\|_p^{\Delta}. \tag{19}$$

Proof. Utilising Hölder's inequality for double integrals, we have

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$\begin{aligned}
\int_a^b \int_a^b \bigl|f(x) - f(t)\bigr|^p \bigl|g(x) - \lambda\bigr|^p \,\mathrm{d}t\,\mathrm{d}x &\le \operatorname*{ess\,sup}_{(x,t) \in [a,b]^2} \bigl|f(x) - f(t)\bigr|^p \\
&\quad \times \int_a^b \int_a^b \bigl|g(x) - \lambda\bigr|^p \,\mathrm{d}t\,\mathrm{d}x \\
&= \bigl(\|f\|_\infty^{\Delta}\bigr)^p (b-a) \|g - \lambda\|_{[a,b],p}^p.
\end{aligned}$$

Then

$$\begin{aligned}
|C(f,g)|^p &\le \frac{1}{(b-a)^2} \bigl(\|f\|_\infty^{\Delta}\bigr)^p (b-a) \|g - \lambda\|_{[a,b],p}^p \\
&= \frac{1}{b-a} \bigl(\|f\|_\infty^{\Delta}\bigr)^p \|g - \lambda\|_{[a,b],p}^p.
\end{aligned}$$

b) For $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have

$$\begin{aligned}\int_a^b \int_a^b \left|f(x) - f(t)\right|^p \left|g(x) - \lambda\right|^p \mathrm{d}t\,\mathrm{d}x &\leq \left(\int_a^b \int_a^b \left|f(x) - f(t)\right|^{p\beta} \mathrm{d}t\,\mathrm{d}x\right)^{1/\beta} \\ &\quad\times \left(\int_a^b \int_a^b \left|g(x) - \lambda\right|^{p\alpha} \mathrm{d}t\,\mathrm{d}x\right)^{1/\alpha} \\ &= \left(\|f\|_{p\beta}^{\Delta}\right)^p (b-a)^{1/\alpha} \|g - \lambda\|_{[a,b],p\alpha}^p .\end{aligned}$$

Then

$$\begin{aligned}|C(f,g)|^p &\leq \frac{1}{(b-a)^2} \left(\|f\|_{p\beta}^{\Delta}\right)^p (b-a)^{1/\alpha} \|g - \lambda\|_{[a,b],p\alpha}^p \\ &= \frac{1}{(b-a)^{1+1/\beta}} \left(\|f\|_{p\beta}^{\Delta}\right)^p \|g - \lambda\|_{[a,b],p\alpha}^p .\end{aligned}$$

c) For $f \in L_p[a,b]$, $g \in L_\infty[a,b]$ we have

$$\begin{aligned}\int_a^b \int_a^b \left|f(x) - f(t)\right|^p \left|g(x) - \lambda\right|^p \mathrm{d}t\,\mathrm{d}x &\leq \operatorname*{ess\,sup}_{x \in [a,b]} \left|g(x) - \lambda\right|^p \\ &\quad\times \int_a^b \int_a^b \left|f(x) - f(t)\right|^p \mathrm{d}t\,\mathrm{d}x \\ &= \|g - \lambda\|_{[a,b],\infty}^p \left(\|f\|_p^{\Delta}\right)^p .\end{aligned}$$

Then

$$|C(f,g)|^p \leq \frac{1}{(b-a)^2} \|g - \lambda\|_{[a,b],\infty}^p \left(\|f\|_p^{\Delta}\right)^p .$$

Taking the power $\frac{1}{p}$ and then the infimum over $\lambda \in \mathbb{R}$, we get the desired results. $\square$

Some particular cases of interest are as follows.

Corollary 5. *Let $f, g\colon [a,b] \to \mathbb{R}$ be Lebesgue measurable functions on $[a,b]$. Then*

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p}} \left\| g - \frac{1}{b-a} \int_a^b g(t)\,\mathrm{d}t \right\|_{[a,b],p} \|f\|_\infty^{\Delta} ,$$

b) for $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p+1/p\beta}} \left\| g - \frac{1}{b-a} \int_a^b g(t)\,\mathrm{d}t \right\|_{[a,b],p\alpha} \|f\|_{p\beta}^{\Delta}$$

c) *for* $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{2/p}} \left\| g - \frac{1}{b-a}\int_a^b g(t)\,\mathrm{d}t \right\|_{[a,b],\infty} \|f\|_p^\Delta .$$

The case when one function is bounded is as follows.

Corollary 6. *Assume that* $f, g\colon [a,b] \to \mathbb{R}$ *are Lebesgue integrable functions on* $[a,b]$. *If there exist constants* n, N *such that* $n \le g(t) \le N$ *for a.e.* $t \in [a,b]$, *then*

a) *for* $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{2}(N-n)\|f\|_\infty^\Delta , \tag{20}$$

b) *for* $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le \frac{1}{2}(N-n)\frac{1}{(b-a)^{2/p\beta}}\|f\|_{p\beta}^\Delta \tag{21}$$

c) *for* $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{2}(N-n)\frac{1}{(b-a)^{2/p}}\|f\|_p^\Delta . \tag{22}$$

Proof. From (17)–(19) we have

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \left\| g - \frac{N+n}{2} \right\|_{[a,b],p} \|f\|_\infty^\Delta . \tag{23}$$

Since

$$\left\| g - \frac{n+N}{2} \right\|_{[a,b],p} \le \frac{N-n}{2}(b-a)^{1/p}$$

then by (23) we get (20).

b) For $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p+1/p\beta}} \left\| g - \frac{N+n}{2} \right\|_{[a,b],p\alpha} \|f\|_{p\beta}^\Delta . \tag{24}$$

Since

$$\left\| g - \frac{n+N}{2} \right\|_{[a,b],p\alpha} \le \frac{N-n}{2}(b-a)^{1/p\alpha}$$

then by (24) we get (21).

c) For $f \in L_p[a,b]$, $g \in L_\infty[a,b]$ we have

$$|C(f,g)| \le \frac{1}{(b-a)^{2/p}} \left\| g - \frac{N+n}{2} \right\|_{[a,b],\infty} \|f\|_p^\Delta . \tag{25}$$

Since

$$\left\| g - \frac{n+N}{2} \right\|_{[a,b],\infty} \le \frac{N-n}{2},$$

then by (25) we get (22).

□

The case when one function is of bounded variation, is as follows.

Corollary 7. *If $f\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable and $g\colon [a,b] \to \mathbb{R}$ is of bounded variation, then*

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{2} \bigvee_a^b (g) \, \|f\|_\infty^\Delta ,$$

b) for $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le \frac{1}{2} \bigvee_a^b (g) \, \frac{1}{(b-a)^{2/p\beta}} \|f\|_{p\beta}^\Delta ,$$

c) for $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{2} \bigvee_a^b (g) \, \frac{1}{(b-a)^{2/p}} \|f\|_p^\Delta .$$

Proof. From (17)–(19) we have

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],p} \|f\|_\infty^\Delta . \tag{26}$$

Since

$$\left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],p} \le \frac{1}{2} \bigvee_a^b (g) \, (b-a)^{1/p} ,$$

then by (26) we get the desired result.

b) For $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p+1/p\beta}} \left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],p\alpha} \|f\|^{\Delta}_{p\beta} . \tag{27}$$

Since

$$\left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],p\alpha} \le \frac{1}{2} \bigvee_a^b (g)\, (b-a)^{1/p\alpha} ,$$

then by (27) we get the desired result.

c) For $f \in L_p[a,b]$, $g \in L_\infty[a,b]$ we have

$$|C(f,g)| \le \frac{1}{(b-a)^{2/p}} \left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],\infty} \|f\|^{\Delta}_{p} . \tag{28}$$

Since

$$\left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],\infty} \le \frac{1}{2} \bigvee_a^b (g) ,$$

then by (28) we get the desired result.

$\square$

Corollary 8. *If $f\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable and $g\colon [a,b] \to \mathbb{R}$ is Lipschitzian in the middle point with the constant $L_{\frac{a+b}{2}}$ and the exponent $q > 0$, then*

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{2^q} L_{\frac{a+b}{2}} \frac{(b-a)^q}{(qp+1)^{1/p}} \|f\|^{\Delta}_{\infty} ,$$

b) for $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le \frac{1}{2^q} L_{\frac{a+b}{2}} \frac{(b-a)^{q-2/p\beta}}{(qp\alpha+1)^{1/p\alpha}} \|f\|^{\Delta}_{p\beta} ,$$

c) for $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{2^q} L_{\frac{a+b}{2}} (b-a)^{q-2/p} \|f\|^{\Delta}_{p} .$$

Proof. From (17)–(19) we have

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p} \|f\|^{\Delta}_{\infty} . \tag{29}$$

Since

$$\left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p} \le L_{\frac{a+b}{2}} \frac{(b-a)^{q+1/p}}{2^q (qp+1)^{1/p}} ,$$

then from (29) we deduce the desired result.

b) For $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, we have

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p+1/p\beta}} \left\| g - g\left(\frac{a+b}{2}\right)\right\|_{[a,b],p\alpha} \|f\|_{p\beta}^{\Delta} . \tag{30}$$

Since

$$\left\| g - g\left(\frac{a+b}{2}\right)\right\|_{[a,b],\alpha} \leq L_{\frac{a+b}{2}} \frac{(b-a)^{q+1/p\alpha}}{2^q (qp\alpha+1)^{1/p\alpha}} ,$$

then from (30) we deduce the desired result.

c) For $f \in L_p[a,b]$, $g \in L_\infty[a,b]$ we have

$$|C(f,g)| \leq \frac{1}{(b-a)^{2/p}} \left\| g - g\left(\frac{a+b}{2}\right)\right\|_{[a,b],\infty} \|f\|_{p}^{\Delta} . \tag{31}$$

Since

$$\left\| g - g\left(\frac{a+b}{2}\right)\right\|_{[a,b],\infty} \leq L_{\frac{a+b}{2}} \frac{(b-a)^{q}}{2^q} ,$$

then from (31) we deduce the desired result.

□

Remark 3. If the function g is Lipshitzian with the constant $L > 0$, then

a) for $f \in L_\infty[a,b]$

$$|C(f,g)| \leq \frac{1}{2} L \frac{b-a}{(p+1)^{1/p}} \|f\|_{\infty}^{\Delta} ,$$

b) for $f \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$|C(f,g)| \leq \frac{1}{2} L \frac{(b-a)^{1-2/p\beta}}{(p\alpha+1)^{1/p\alpha}} \|f\|_{p\beta}^{\Delta} ,$$

c) for $f \in L_p[a,b]$

$$|C(f,g)| \leq \frac{1}{2} L (b-a)^{1-2/p} \|f\|_{p}^{\Delta} .$$

4 Applications for Mid-point Inequalities

Consider absolutely continuous function $h\colon [a,b] \to \mathbb{R}$. We have the following well known representation

$$h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t = \frac{1}{b-a}\int_a^b K(t)h'(t)\,\mathrm{d}t,$$

where the kernel $K\colon [a,b] \to \mathbb{R}$ is defined by

$$K(t) := \begin{cases} t-a & \text{if } t \in \left[a, \frac{a+b}{2}\right], \\ t-b & \text{if } t \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Since $\int_a^b K(t)\,\mathrm{d}t = 0$, then

$$\frac{1}{b-a}\int_a^b K(t)h'(t)\,\mathrm{d}t = C(K,h').$$

Utilising Corollary 1 we have

a) for $h' \in L_\infty[a,b]$

$$\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{(b-a)^{1/p}}\|K\|_{[a,b],p}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty}, \quad (32)$$

b) for $h' \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta} = 1$

$$\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{(b-a)^{1/p}}\|K\|_{[a,b],p\alpha}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p\beta}, \quad (33)$$

c) for $h' \in L_p[a,b]$

$$\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{(b-a)^{1/p}}\|K\|_{[a,b],\infty}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p}. \quad (34)$$

Observe that for $q > 0$ we have

$$\begin{aligned}
\|K\|_{[a,b],q} &= \left[\int_a^b |K(t)|^q\,\mathrm{d}t\right]^{1/q} \\
&= \left[\int_a^{\frac{a+b}{2}} (t-a)^q\,\mathrm{d}t + \int_{\frac{a+b}{2}}^b (b-t)^q\,\mathrm{d}t\right]^{1/q} \\
&= \left[\frac{(t-a)^{q+1}}{q+1}\Bigg|_a^{\frac{a+b}{2}} - \frac{(b-t)^{q+1}}{q+1}\Bigg|_{\frac{a+b}{2}}^b\right]^{1/q} \\
&= \left[\frac{\left(\frac{b-a}{2}\right)^{q+1}}{q+1} + \frac{\left(\frac{b-a}{2}\right)^{q+1}}{q+1}\right]^{1/q} = \frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}}.
\end{aligned}$$

Then

$$\|K\|_{[a,b],p} = \frac{(b-a)^{1+1/p}}{2(p+1)^{1/p}}, \quad \|K\|_{[a,b],p\alpha} = \frac{(b-a)^{1+1/p\alpha}}{2(p\alpha+1)^{1/p\alpha}}.$$

We also have

$$\|K\|_{[a,b],\infty} = \frac{1}{2}(b-a).$$

Making use of (32)–(34) we get

a) for $h' \in L_\infty[a,b]$

$$\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{b-a}{2(p+1)^{1/p}}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty},$$

b) for $h' \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{(b-a)^{1-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}} \left\| h' - \frac{h(b)-h(a)}{b-a} \right\|_{[a,b],p\beta},$$

c) for $h' \in L_p[a,b]$

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{2}(b-a)^{1-1/p} \left\| h' - \frac{h(b)-h(a)}{b-a} \right\|_{[a,b],p}.$$

For $p = 1$ we get the simpler inequalities

a) for $h' \in L_\infty[a,b]$

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{4}(b-a) \left\| h' - \frac{h(b)-h(a)}{b-a} \right\|_{[a,b],\infty},$$

b) for $h' \in L_1[a,b]$

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{2}(b-a) \left\| h' - \frac{h(b)-h(a)}{b-a} \right\|_{[a,b],1}.$$

Utilising Corollary 2 we have

a)

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{2}(\Gamma-\gamma)\|K\|_{[a,b],\infty},$$

b) for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{2}(\Gamma-\gamma)\frac{1}{(b-a)^{1/p\beta}}\|K\|_{[a,b],p\beta},$$

c)

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{2}(\Gamma-\gamma)\frac{1}{(b-a)^{1/p}}\|K\|_{[a,b],p},$$

provided that $\gamma \le h'(t) \le \Gamma$ for a.e. $t \in [a,b]$.

Utilising the above calculations we then have

a)

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le (\Gamma-\gamma)(b-a), \tag{35}$$

b) for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{2}(\Gamma-\gamma)\frac{(b-a)^{1+1/p\alpha-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}}, \tag{36}$$

c)
$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{2}(\Gamma-\gamma)\frac{b-a}{2(p+1)^{1/p}}\,,\tag{37}$$

provided that $\gamma\le h'(t)\le\Gamma$ for a.e. $t\in[a,b]$.

In particular, for $p=1$ in (37) we have
$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{8}(\Gamma-\gamma)(b-a)\,,$$

which is the best inequality one can get from (35)–(37).

If we use Corollary 3 and assume that h' is of bounded variation on $[a,b]$, then

a)
$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\bigvee_a^b(h')\,(b-a)\,,$$

b) for $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$
$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{2}\bigvee_a^b(h')\,\frac{(b-a)^{1+1/p\alpha-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}}\,,$$

c)
$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{2}\bigvee_a^b(h')\,\frac{b-a}{2(p+1)^{1/p}}\,.\tag{38}$$

From (38) for $p=1$ we get
$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{8}(b-a)\bigvee_a^b(h').$$

If we use inequalities (14)–(16) and assume that h' is Lipschitzian with the constant $U>0$, namely
$$\left|h'(t)-h'(s)\right|\le U|t-s|\text{ for }t,s\in(a,b),$$

then

a)
$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le U\frac{1}{4}\frac{(b-a)^2}{(p+1)^{1/p}}\,,$$

b)
$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le U\frac{1}{4}\frac{(b-a)^{2-1/p\beta+1/p\alpha}}{(p\alpha+1)^{2/p\alpha}}\,,$$

c)

$$\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le U\frac{1}{4}\frac{(b-a)^2}{(p+1)^{1/p}}\,.$$

In particular, we get for $p=1$

$$\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{8}(b-a)^2U.$$

5 Applications for Trapezoid Inequalities

Consider absolutely continuous function $h\colon [a,b]\to\mathbb{R}$. We have the following well known representation

$$\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t = \frac{1}{b-a}\int_a^b V(t)h'(t)\,\mathrm{d}t$$

where the kernel $V\colon [a,b]\to\mathbb{R}$ is defined by

$$V(t) := t - \frac{a+b}{2}.$$

Since $\int_a^b V(t)\,\mathrm{d}t = 0$, then

$$\frac{1}{b-a}\int_a^b V(t)h'(t)\,\mathrm{d}t = C(V,h').$$

Utilising Corollary 1 we have

a) for $h'\in L_\infty[a,b]$

$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{(b-a)^{1/p}}\|V\|_{[a,b],p}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty}, \tag{39}$$

b) for $h'\in L_{p\beta}[a,b]$, $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{(b-a)^{1/p}}\|V\|_{[a,b],p\alpha}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p\beta}, \tag{40}$$

c) for $h'\in L_p[a,b]$

$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{(b-a)^{1/p}}\|V\|_{[a,b],\infty}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p}. \tag{41}$$

Observe that, for $q>0$ we have

$$\begin{aligned}\|V\|_{[a,b],q} &= \left[\int_a^b |V(t)|^q\,\mathrm{d}t\right]^{1/q} = \left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^q\mathrm{d}t + \int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^q\mathrm{d}t\right]^{1/q}\\ &= \left[2\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^q\mathrm{d}t\right]^{1/q} = \left[\frac{2\left(\frac{b-a}{2}\right)^{q+1}}{q+1}\right]^{1/q} = \frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}}.\end{aligned}$$

Then
$$\|V\|_{[a,b],p} = \frac{(b-a)^{1+1/p}}{2(p+1)^{1/p}}, \qquad \|V\|_{[a,b],p\alpha} = \frac{(b-a)^{1+1/p\alpha}}{2(p\alpha+1)^{1/p\alpha}}.$$
We also have
$$\|V\|_{[a,b],\infty} = \frac{1}{2}(b-a).$$

Making use of (39)–(41) we get

a) for $h' \in L_\infty[a,b]$
$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{b-a}{2(p+1)^{1/p}}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty},$$

b) for $h' \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$
$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{(b-a)^{1-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p\beta},$$

c) for $h' \in L_p[a,b]$
$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{2}(b-a)^{1-1/p}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p}.$$

For $p=1$ we get the simpler inequalities

a) for $h' \in L_\infty[a,b]$
$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{4}(b-a)\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty},$$

b) for $h' \in L_1[a,b]$
$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{2}(b-a)\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p}.$$

Since the p-norms of the kernel V are the same as of K, then we can state the following results as well.

If $\gamma \le h'(t) \le \Gamma$ for a.e. $t \in [a,b]$, then we have

a)
$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le (\Gamma-\gamma)(b-a), \tag{42}$$

b) for $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$
$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{2}(\Gamma-\gamma)\frac{(b-a)^{1+1/p\alpha-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}}, \tag{43}$$

c)

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{2}(\Gamma-\gamma)\frac{b-a}{2(p+1)^{1/p}}\,. \tag{44}$$

In particular, for $p=1$ in (44) we have

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{8}(\Gamma-\gamma)(b-a),$$

which is the best inequality one can get from (42)–(44).

If h' is of bounded variation on $[a,b]$, then

a)

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \bigvee_a^b(h')\,(b-a)\,,$$

b) for $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{2}\bigvee_a^b(h')\,\frac{(b-a)^{1+1/p\alpha-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}}\,,$$

c)

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{2}\bigvee_a^b(h')\,\frac{b-a}{2(p+1)^{1/p}}\,.$$

From (38) for $p=1$ we get

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{8}(b-a)\bigvee_a^b(h').$$

Assume that h' is Lipschitzian with the constant $U>0$ then

a)

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le U\frac{1}{4}\frac{(b-a)^2}{(p+1)^{1/p}}\,,$$

b)

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le U\frac{1}{4}\frac{(b-a)^{2-1/p\beta+1/p\alpha}}{(p\alpha+1)^{2/p\alpha}}\,,$$

c)

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le U\frac{1}{4}\frac{(b-a)^2}{(p+1)^{1/p}}\,.$$

In particular, we get for $p=1$

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{8}(b-a)^2U.$$

Some similar inequalities may be stated in terms of the Δ-seminorms. However the details are omitted.

6 Some Exponential Inequalities

We can state the following result.

Theorem 6. *Let $f, g\colon [a,b] \to \mathbb{R}$ be Lebesgue integrable functions on $[a,b]$. If $\Phi\colon \mathbb{R} \to \mathbb{R}$ is convex and monotonic nondecreasing on $\mathbb{R}$ then we have the inequality*

$$\Phi[C(f,g)] \le \frac{1}{b-a} \inf_{\mu\in\mathbb{R}} \int_a^b \Phi\left[\left(\frac{f(x)+g(x)}{2} - \mu\right)^2\right] \mathrm{d}x. \tag{45}$$

Proof. From Theorem 1 we have

$$\begin{aligned}&\Phi[C(f,g)]\\ &\le \frac{1}{b-a}\int_a^b \Phi\left[\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\left(g(x) - 2\mu + \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\right]\mathrm{d}x\end{aligned}$$

for any $\mu \in \mathbb{R}$.

Utilising the elementary inequality

$$\alpha\beta \le \left(\frac{\alpha+\beta}{2}\right)^2$$

that holds for any $\alpha, \beta \in \mathbb{R}$, we have

$$\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\left(g(x) - 2\mu + \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right) \le \left(\frac{f(x)+g(x)}{2} - \mu\right)^2$$

for any $x \in [a,b]$.

Since $\Phi\colon \mathbb{R} \to \mathbb{R}$ is monotonic nondecreasing on $\mathbb{R}$ then

$$\begin{aligned}\Phi&\left[\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\left(g(x) - 2\mu + \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\right]\\ &\qquad\le \Phi\left[\left(\frac{f(x)+g(x)}{2} - \mu\right)^2\right]\end{aligned} \tag{46}$$

for any $x \in [a,b]$.

Integrating (46) over x in $[a,b]$ and taking the infimum over $\mu \in \mathbb{R}$, we deduce the desired result (45). □

Remark 4. Writing the inequality (45) for $\Phi\colon \mathbb{R} \to \mathbb{R}$, $\Phi(x) = \exp x$ we have

$$\exp[C(f,g)] \le \frac{1}{b-a} \inf_{\mu\in\mathbb{R}} \int_a^b \exp\left[\left(\frac{f(x)+g(x)}{2} - \mu\right)^2\right] \mathrm{d}x. \tag{47}$$

This inequality can provide some exponential inequalities as follows.

Assume that $f\colon [a,b] \to \mathbb{R}$ is Lipschitzian with constant $L > 0$ and $g\colon [a,b] \to \mathbb{R}$ is Lipschitzian with constant $K > 0$. Then by taking

$$\mu = \frac{f\left(\frac{a+b}{2}\right) + g\left(\frac{a+b}{2}\right)}{2}$$

we have

$$\left(\frac{f(x)+g(x)}{2}-\frac{f\left(\frac{a+b}{2}\right)+g\left(\frac{a+b}{2}\right)}{2}\right)^2 \le \left(\frac{L+K}{2}\right)^2\left(x-\frac{a+b}{2}\right)^2$$

and by (47) we have

$$\exp[C(f,g)] \le \frac{1}{b-a}\int_a^b \exp\left[\left(\frac{L+K}{2}\right)^2\left(x-\frac{a+b}{2}\right)^2\right] \mathrm{d}x\,.$$

Acknowledgement

The author would like to thank the anonymous referees for their valuable comments that have been implemented in the final version of the paper.

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Author's address:
Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia
School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa
E-mail: `sever.dragomir@vu.edu.au`

Received: 28 January, 2014
Accepted for publication: 12 April, 2014
Communicated by: Olga Rossi

Communications in Mathematics 22 (2014) 133–140

A determinant formula for the relative class number of an imaginary abelian number field

Mikihito Hirabayashi

Abstract. We give a new formula for the relative class number of an imaginary abelian number field K by means of determinant with elements being integers of a cyclotomic field generated by the values of an odd Dirichlet character associated to K. We prove it by a specialization of determinant formula of Hasse.

1 Introduction

There are lots of formulas for the relative class number of an imaginary abelian number field K by means of determinant (see [5] for bibliography). In this paper we give such a new formula. We prove it by a specialization of the determinant formula for generalized group matrix which appears in [2, § 13]. The key idea is a transformation of generalized Bernoulli numbers and a transformation of their product over the odd characters to one over the even characters. In our formula, elements of the determinant are integers of a cyclotomic field generated by the values of an odd Dirichlet character associated to K, whereas elements of the determinants are rational numbers for known formulas. We may regard our formula as an imaginary version of Hasse's formula [2, § 16, (3)], which expresses the class number of a real abelian number field by means of determinant with elements being logarithms of cyclotomic units of its cyclic subfields.

2 Results

Let K be an imaginary abelian number field of degree n and with conductor f, and let K_0 be the maximal real subfield of K. Let H_0 be the subgroup of the group $(\mathbf{Z}/f\mathbf{Z})^\times$ of reduced residue classes modulo f corresponding to K_0. Let X_0 be the set of Dirichlet characters associated to K_0.

2010 MSC: Primary 11R29; Secondary 11R20
Key words: imaginary abelian number field, relative class number, determinant, class number formula

We assume that the Dirichlet characters χ associated to K, which we call characters of K for short, are primitive and that, as usual, $\chi(x) = 0$ for an integer x not relatively prime to the conductor $f(\chi)$ of χ.

We classify the group X_0 by the following equivalence $\sim$: for characters χ, $\psi \in X_0$ let $\chi \sim \psi$ if and only if there exists an integer m such that m is relatively prime to n_χ and that $\psi = \chi^m$, where n_χ is the order of χ. We call the classes classified by this equivalence Frobenius classes. Let $\{\psi_0\}$ be a system of representatives of the Frobenius classes. For a representative ψ_0 let t_{ψ_0} be an integer such that the quotient group $(\mathbf{Z}/f\mathbf{Z})^\times/H_{\psi_0}$ is generated by a class represented by $t_{\psi_0} \bmod f$, where $H_{\psi_0} = \{x \bmod f \in (\mathbf{Z}/f\mathbf{Z})^\times \,;\, \psi_0(x) = 1\}$.

We fix an odd character χ_1^* of K. As we will see, the elements of the determinant of our formula are integers of the field generated by the values of the character χ_1^*.

For an even character χ_0 of K and for an element $a \bmod f$ of $(\mathbf{Z}/f\mathbf{Z})^\times$ let

$$u_{\chi_0}(a) = -\chi_1^*(a) \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{f} \chi_1^*(x) R_f(ax)\,,$$

where $R_f(a)$ is the least positive residue modulo f of a. Then we define a matrix U by

$$U = (u_{\psi_0}(st_{\psi_0}^{-k}))_{(s \bmod f)H_0\,;\,\psi_0,\, 0 \le k \le \varphi(n_{\psi_0})-1}\,,$$

where $(s \bmod f)H_0$ runs in the rows over the quotient group $(\mathbf{Z}/f\mathbf{Z})^\times/H_0$, which is isomorphic to the Galois group G_0 of K_0; ψ_0 and k run in the columns: $\{\psi_0\}$ is a system defined above and φ is the Euler totient function. Here, $t_{\psi_0}^{-k} \bmod f$ is the inverse of $t_{\psi_0}^{k} \bmod f$, i.e., $t_{\psi_0}^{-k}$ is an integer satisfying $t_{\psi_0}^{-k} t_{\psi_0}^{k} \equiv 1 \pmod f$.

With the notation above we have the following

Theorem 1. *For an imaginary abelian number field K of degree n and with conductor f, we have*

$$\det U = \pm \frac{(2f)^{n/2} c\, g^*}{Q\, w} h^*$$

where h^* is the relative class number of K, Q is the Hasse unit index of K, w is the number of roots of unity in K, and g^* is defined by

$$g^* = \prod_{\chi_1} \prod_{p|f} \bigl(1 - \chi_1(p)\bigr)$$

where the products $\prod_{\chi_1}$ and $\prod_{p|f}$ are taken over the odd characters χ_1 of K and the prime numbers p dividing f, respectively, and c is a natural number expressed by

$$c = \prod_{p|n_0} p^{\frac{1}{2} \sum_{p^k|n_0} \left(q(\frac{n_0}{p^k}) - \frac{n_0}{p^k}\right)},$$

where the product $\prod_{p|n_0}$ and the sum $\sum_{p^\mu|n_0}$ are taken over prime numbers p dividing $n_0 = n/2$ and the powers of p dividing n_0, respectively, and $q(m)$ is the number of solutions of $x^m = 1$ in G_0.

We remark here that the elements $u_{\chi_0}(a)$ and the matrix U depend on the character χ_1^*, as we see in the examples below, and that, in addition, U depends on the choice of integers t_{ψ_0}. In fact, we have different U's for different t_{ψ_0}'s in the case of $K = \mathbf{Q}(\zeta_7)$, the 7th cyclotomic field. Moreover, we note that the matrix U never coincides with any matrix in known formulas, because U always contains a constant column corresponding to the principal character $\psi_0 = 1$.

As seen by definition, the number g^* may be zero and then remains a problem of how to construct such a formula in Theorem 1 in case of $g^* = 0$.

For the cyclotomic fields of prime power conductor we have the following corollaries.

Corollary 1. *For the cyclotomic field $K = \mathbf{Q}(\zeta_{p^\rho})$ of conductor p^ρ $(\rho \geq 1)$, p an odd prime, we have*

$$\begin{aligned}\det U &= \det\bigl(u_{\psi_0}(g^i t_{\psi_0}^{-k})\bigr)_{0\leq i\leq \frac{p^{\rho-1}(p-1)}{2}-1\,;\,\psi_0, 0\leq k\leq \varphi(n_{\psi_0})-1} \\ &= \pm(2p^\rho)^{\frac{p^{\rho-1}(p-1)}{2}-1} h^*,\end{aligned}$$

where g is a primitive root modulo p^ρ.

For the field $K = \mathbf{Q}(\zeta_{p^\rho})$ we can take $t_{\psi_0} = g$ for every $\psi_0 \neq 1$ and $t_{\psi_0} = 1$ for $\psi_0 = 1$.

Corollary 2. *For the cyclotomic field $K = \mathbf{Q}(\zeta_{2^\rho})$ of conductor 2^ρ $(\rho \geq 2)$ we have*

$$\begin{aligned}\det U &= \bigl(u_{\psi_0}(5^i t_{\psi_0}^{-k})\bigr)_{0\leq i\leq 2^{\rho-2}-1\,;\,\psi_0, 0\leq k\leq \varphi(n_{\psi_0})-1} \\ &= \pm\, 2^{(\rho+1)2^{\rho-2}-\rho}\, h^*.\end{aligned}$$

For the field $K = \mathbf{Q}(\zeta_{2^\rho})$ we can take $t_{\psi_0} = 5$ for every $\psi_0 \neq 1$ and $t_{\psi_0} = 1$ for $\psi_0 = 1$.

Here we give examples. We adopt the basic characters which Hasse used in [2]. For an odd prime p let χ_p be an odd character modulo p of order $p-1$ and ψ_{p^ρ} $(\rho \geq 2)$ an even character modulo p^ρ of order $p^{\rho-1}$; in addition $\psi_{p^\rho}^p = \psi_{p^{\rho-1}}$. For the prime 2 let χ_4 be the odd character modulo 4 and ψ_{2^ρ} $(\rho \geq 3)$ an even character modulo 2^ρ of order $2^{\rho-2}$; in addition $\psi_{2^\rho}^2 = \psi_{2^{\rho-1}}$. The subscript of a basic character denotes the conductor.

For the following calculation of the values of $u_{\chi_0}(a)$, we use the identity

$$\sum_{\substack{x=1\\(x,f)=1\\\chi_0(x)=1}}^{f} \chi_1^*(x) R_f(ax) = \sum_{\substack{x=1\\(x,f)=1\\\chi_0(x)=1}}^{[f/2]} \chi_1^*(x)\bigl(2R_f(ax) - f\bigr)\,.$$

Example 1. Let $K = \mathbf{Q}(\zeta_5)$, i.e., $p = 5$, $\rho = 1$. Take $g = 2$ and $\chi_1^* = \chi_5$. Then $\{\psi_0\} = \{1, \chi_5^2\}$ and

$$\begin{aligned}u_1(a) &= -\chi_5(a)\bigl(2R_5(a) - 5 + i\bigl(2R_5(2a) - 5\bigr)\bigr)\,, \\ u_{\chi_5^2}(a) &= -\chi_5(a)\bigl(2R_5(a) - 5\bigr)\,.\end{aligned}$$

Consequently

$$U=\begin{pmatrix} u_1(1) & u_{\chi_5^2}(1) \\ u_1(2) & u_{\chi_5^2}(2) \end{pmatrix}=\begin{pmatrix} 3+i & 3 \\ 3+i & i \end{pmatrix}$$

and hence $\det U = -2\cdot 5$. Otherwise, by Corollary 1 and [2, Tafel II], $\det U = \pm(2\cdot 5)^{\frac{5-1}{2}-1}\cdot 1 = \pm 2\cdot 5$.

Taking $g=2$ and $\chi_1^* = \chi_5^3$, we have

$$U=\begin{pmatrix} 3-i & 3 \\ 3-i & -i \end{pmatrix}$$

and hence $\det U = -2\cdot 5$.

Example 2. Let $K=\mathbf{Q}(\zeta_{2^3})$, i.e., $p=2, \rho=3$. Take $\chi_1^*=\chi_4$. Then $\{\psi_0\}=\{1,\psi_{2^3}\}$ and

$$\begin{aligned} u_1(a) &= -2\chi_4(a)\bigl(R_{2^3}(a)-R_{2^3}(3a)\bigr)\,, \\ u_{\psi_{2^3}}(a) &= -2\chi_4(a)\bigl(R_{2^3}(a)-4\bigr)\,. \end{aligned}$$

Consequently

$$U=\begin{pmatrix} u_1(1) & u_{\psi_{2^3}}(1) \\ u_1(5) & u_{\psi_{2^3}}(5) \end{pmatrix}=\begin{pmatrix} 4 & 6 \\ 4 & -2 \end{pmatrix}$$

and hence $\det U = -2^5$. Otherwise, by Corollary 2 and [2, Tafel II], $\det U = \pm 2^{(3+1)2^{3-2}-3}\cdot 1 = \pm 2^5$.

Taking $\chi_1^* = \chi_4\psi_8$, we have

$$U=\begin{pmatrix} 8 & 6 \\ 8 & 2 \end{pmatrix}$$

and hence $\det U = -2^5$.

Example 3. Let $K=\mathbf{Q}(\sqrt{-3},\sqrt{5})$. Take $\chi_1^*=\chi_3$. Then $\{\psi_0\}=\{1,\chi_5^2\}$ and

$$\begin{aligned} u_1(a) &= -2\chi_3(a)\bigl(R_{15}(a)-R_{15}(2a)+R_{15}(4a)+R_{15}(7a)-15\bigr)\,, \\ u_{\chi_5^2}(a) &= -2\chi_3(a)\bigl(R_{15}(a)+R_{15}(4a)-15\bigr)\,. \end{aligned}$$

Consequently

$$U=\begin{pmatrix} u_1(1) & u_{\chi_5^2}(1) \\ u_1(2) & u_{\chi_5^2}(2) \end{pmatrix}=\begin{pmatrix} 10 & 20 \\ 10 & -10 \end{pmatrix}$$

and hence $\det U = -2^2\cdot 3\cdot 5^2$. Otherwise, since $c=1$, $g^*=2$, $w=2\cdot 3$ and $Q=1$, which is obtained by [2, Tafel II], we have by Theorem 1

$$\det U = \pm\frac{(2f)^{n/2}c\,g^*}{Qw}h^* = \pm\frac{(2\cdot 15)^2\cdot 1\cdot 2}{1\cdot 2\cdot 3}\cdot 1 = \pm 2^2\cdot 3\cdot 5^2.$$

Taking $\chi_1^*=\chi_3\chi_5^2$, we have

$$U=\begin{pmatrix} 30 & 20 \\ 30 & 10 \end{pmatrix}$$

and hence $\det U = -2^2\cdot 3\cdot 5^2$.

3 The determinant of a generalized group matrix

In the second chapter of the book [2] Hasse gave two transformations of the class number formula for a real abelian number field; the first transformation is an application of summations $\sum_x \chi(x)A_f(x)$ to the group matrix, $A_f(x)$ an ordinary distribution (cf. [2, p. 18] or [4, Lemma 12.15]), and the second transformation is one for summations $\sum_s \chi(s)u_\chi(s)$ and for the matrix $U_{\mathfrak{G}}$ (see Lemma 1).

By the first transformation, replacing the distribution $A_f(x)$ in [2, p. 18] with

$$A_f(x) = -\left(\frac{R_f(x)}{f} - \frac{1}{2}\right),$$

we can obtain the formula of Girstmair [1] with Maillet determinant for the relative class number of an imaginary abelian number field with conductor f.

For the proof of our formula we need the following lemmas. Let $\mathfrak{G}$ be an abelian group of order n and $\mathfrak{X}$ the group of characters of $\mathfrak{G}$. For $\chi \in \mathfrak{X}$ let

$$\mathfrak{H}_\chi = \{x \in \mathfrak{G}; \chi(x) = 1\}.$$

For $s \in \mathfrak{G}$ and $\chi \in \mathfrak{X}$ let $u_\chi(s)$ be a complex-valued function satisfying the following conditions:

(i) $u_\chi(s) = u_{\chi^\nu}(s)$ for $s \in \mathfrak{G}$ and $\nu \in \mathbf{Z}$ relatively prime to the order n_χ of χ.

(ii) $u_\chi(s) = u_\chi(s')$ for $s,\ s' \in \mathfrak{G}$ with $\chi(s) = \chi(s')$.

We classify the group $\mathfrak{X}$ by the Frobenius equivalence defined as in § 2. Let $\{\psi\}$ be a system of representatives of the Frobenius classes of $\mathfrak{X}$. For a character ψ let t_ψ be a representative of a generator $t_\psi\mathfrak{H}_\psi$ of the cyclic group $\mathfrak{G}/\mathfrak{H}_\psi$. Then we define a matrix $U_{\mathfrak{G}}$ by

$$U_{\mathfrak{G}} = \left(u_\psi(st_\psi^{-k})\right)_{s\in\mathfrak{G};\psi,0\le k\le\varphi(n_\psi)-1},$$

where s runs in the rows, and ψ and k run in the columns.

Lemma 1. *[2, §14] For the matrix $U_{\mathfrak{G}}$ we have*

$$\det U_{\mathfrak{G}} = \pm c_{\mathfrak{G}} \prod_{\chi\in\mathfrak{X}} \sum_{s \bmod \mathfrak{H}_\chi} \chi(s)u_\chi(s),$$

where $c_{\mathfrak{G}}$ is a positive number defined by

$$c_{\mathfrak{G}} = \pm \frac{1}{\det(\chi(s))_{s\in\mathfrak{G},\chi\in\mathfrak{X}}} \prod_\psi \left(\left(\frac{n}{n_\psi}\right)^{\varphi(n_\psi)} \det\left(\psi(t_\psi)^{ik}\right)_{\substack{1\le i\le n_\psi \\ (i,n_\psi)=1 \\ 0\le k\le\varphi(n_\psi)-1}} \right)$$

and $s \bmod \mathfrak{H}_\chi$ in the sum $\sum_{s \bmod \mathfrak{H}_\chi}$ runs over the quotient group $\mathfrak{G}/\mathfrak{H}_\chi$.

Lemma 2. *[2, §14 and §15] For an abelian group $\mathfrak{G}$ of order n the number $c_{\mathfrak{G}}$ is a natural number and holds*

$$c_{\mathfrak{G}} = \prod_{p|n} p^{\frac{1}{2}\sum_{p^k|n}\left(q(\frac{n}{p^k}) - \frac{n}{p^k}\right)},$$

where the product and summation are taken over the prime numbers p dividing n and over the powers of p dividing n, and $q(m)$ is the number of solutions of $x^m = 1$ in $\mathfrak{G}$. Therefore $c_{\mathfrak{G}} = 1$ if and only if $\mathfrak{G}$ is cyclic.

4 Proof of Theorem 1

Proof of Theorem 1. We start with the arithmetic class number formula for h^*,

$$h^* = Qw \prod_{\chi_1} \left(-\frac{1}{2} B_{1,\chi_1} \right).$$

For any odd character χ_1 of K we have

$$B_{1,\chi_1} = \frac{1}{f(\chi_1)} \sum_{a=1}^{f(\chi_1)} \chi_1(a)a = \frac{1}{f} \sum_{a=1}^{f} \chi_1(a)a$$

and like as [4, Lemma 8.7] we have

$$\sum_{\substack{a=1 \\ (a,f)=1}}^{f} \chi_1(a)a = \prod_{p|f} \big(1 - \chi_1(p)\big) \cdot \sum_{a=1}^{f} \chi_1(a)a \,.$$

In fact, if $p \mid f$, we have $\chi(p) \sum_{a=1}^{f} \chi(a)a = \sum_{b=1}^{f/p} \chi(pb)(pb)$ and hence

$$\begin{aligned} \prod_{p|f}(1 - \chi_1(p)) \cdot \sum_{a=1}^{f} \chi_1(a)a &= \sum_{a=1}^{f} \chi(a)a + \sum_{\substack{d|f \\ d>1}} \Big(\sum_{\substack{d'|d \\ d'>1}} \mu(d') \Big) \chi(d)d \\ &= \sum_{a=1}^{f} \chi(a)a - \sum_{\substack{d|f \\ d>1}} \chi(d)d = \sum_{\substack{a=1 \\ (a,f)=1}}^{f} \chi_1(a)a \,, \end{aligned}$$

where $\mu(\cdot)$ is the Möbius function.

Therefore, putting

$$S(\chi_1) = \sum_{\substack{a=1 \\ (a,f)=1}}^{f} \chi_1(a)a \,,$$

we have by the arithmetic class number formula for h^*

$$\frac{(-2f)^{n/2} g^* h^*}{Q\, w} = \prod_{\chi_1} S(\chi_1)$$

and hence our task is to show that the product of the right-hand side is $\pm c^{-1} \det U$.

Recall that χ_1^* is a fixed odd character of K. For an even character χ_0 of K let

$$H_{\chi_0} = \{ x \bmod f \in (\mathbf{Z}/f\mathbf{Z})^\times : \chi_0(x) = 1 \} \,.$$

Choose a system of representatives $s \bmod f$ of $(\mathbf{Z}/f\mathbf{Z})^\times / H_{\chi_0}$. Then, for an odd character $\chi_1 = \chi_0 \chi_1^*$ of K we have

$$S(\chi_1) = S(\chi_0 \chi_1^*) = \sum_{s \bmod H_{\chi_0}} \chi_0(s) u_{\chi_0}(s) \,,$$

where

$$u_{\chi_0}(s) = \chi_1^*(s) \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{f} \chi_1^*(x) R_f(sx)\,.$$

Therefore we have

$$\prod_{\chi_1} S(\chi_1) = \prod_{\chi_0} \sum_{s \bmod H_{\chi_0}} \chi_0(s) u_{\chi_0}(s)\,,$$

where the product $\prod_{\chi_0}$ is taken over the even characters χ_0 of K.

Here we use Lemmas 1 and 2 by letting $\mathfrak{G}$ be the group $(\mathbf{Z}/f\mathbf{Z})^\times/H_0$ and by replacing n by $n/2$, χ by χ_0, $U_{\mathfrak{G}}$ by U, $c_{\mathfrak{G}}$ by c, and $u_\psi(s)$ by $u_{\psi_0}(s)$.

To use Lemma 1, we need to check the $u_{\chi_0}(s)$ for meeting the conditions (i) and (ii) in § 3. First let ν be an integer relatively prime to the order of χ_0. Then $\chi_0^\nu(x) = 1$ if and only if $\chi_0(x) = 1$. Hence

$$\begin{aligned} u_{\chi_0^\nu}(s) &= \chi_1^*(s) \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0^\nu(x)=1}}^{f} \chi_1^*(x) R_f(sx) \\ &= u_{\chi_0}(s). \end{aligned}$$

Secondly let s, s' be integers relatively prime to f satisfying $\chi_0(s) = \chi_0(s')$. Hence

$$\begin{aligned} u_{\chi_0}(s') &= \chi_1^*(s') \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{f} \chi_1^*(x) R_f(s'x) \\ &= \chi_1^*(s') \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{f} \chi_1^*\big(s(s')^{-1}x\big) R_f\big(s' \cdot s(s')^{-1}x\big) \\ &= \chi_1^*(s') \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{f} \chi_1^*(s)\chi_1^*(s')^{-1}\chi_1^*(x) R_f(sx) \\ &= \chi_1^*(s) \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{f} \chi_1^*(x) R_f(sx) \\ &= u_{\chi_0}(s)\,. \end{aligned}$$

Here $(s')^{-1} \bmod f$ is the inverse of $s' \bmod f$. Therefore we have checked the conditions.

Consequently, by Lemma 1 we obtain

$$\frac{(-2f)^{n/2}g^*h^*}{Q\,w} = \prod_{\chi_1} S(\chi_1) = \frac{1}{\pm c}\det U\,,$$

that is,

$$\det U = \pm\frac{(2f)^{n/2}c\,g^*}{Q\,w}h^*$$

and by Lemma 2 we immediately obtain the expression of c. This completes the proof.

Corollaries 1 and 2 are directly obtained by Theorem 1, because for the cyclotomic fields K of prime power conductors we have $g^* = 1$ by definition, $c = 1$ by Lemma 2 and $Q = 1$ by [2, Satz 27].

Acknowledgements

The author is grateful to the anonymous referees for their careful readings of the first version of this paper and for their many valuable suggestions and comments, which improved significantly the presentation of the paper.

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Author's address:
Kanazawa Institute of Technology, 7-1 Ohgigaoka, Nonoichi, Ishikawa 921-8501 Japan

E-mail: `hira@neptune.kanazawa-it.ac.jp`

Received: 7 March, 2014
Accepted for publication: 16 September, 2014
Communicated by: Attila Bérczes

Communications in Mathematics 22 (2014) 141–149

Discontinuity of the Fuglede-Kadison determinant on a group von Neumann algebra

Benjamin Küster

Abstract. We show that in contrast to the case of the operator norm topology on the set of regular operators, the Fuglede-Kadison determinant is not continuous on isomorphisms in the group von Neumann algebra $\mathcal{N}(\mathbb{Z})$ with respect to the strong operator topology. Moreover, in the weak operator topology the determinant is not even continuous on isomorphisms given by multiplication with elements of $\mathbb{Z}[\mathbb{Z}]$. Finally, we define $T \in \mathcal{N}(\mathbb{Z})$ such that for each $\lambda \in \mathbb{R}$ the operator $T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}$ is a self-adjoint weak isomorphism of determinant class but $\lim_{\lambda\to 0} \det(T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}) \neq \det(T)$.

1 Introduction

Fuglede and Kadison [1] introduce their determinant for operators in a finite factor. They prove that, for regular (i.e. invertible) operators, the new determinant shares many algebraic and analytic properties with the usual matrix determinant (which it generalises). That includes continuity with respect to the operator norm. We consider the continuity properties of the generalised Fuglede-Kadison determinant which is used for example by Lück [4, p. 127] to define the topological invariant "L^2-torsion". Let f be an element of a finite von Neumann algebra (N, τ). The (generalised) Fuglede-Kadison determinant of f is

$$\det(f) := \begin{cases} \exp\left(\int_{0+}^{\infty} \ln(\lambda)\,\mathrm{d}F(f)\right), & \text{if } \int_{0+}^{\infty} \ln(\lambda)\,\mathrm{d}F(f) > -\infty, \\ 0\,, & \text{otherwise.} \end{cases} \tag{1}$$

In this definition, $F(f)\colon [0,\infty) \to [0,\infty)$ is the spectral density function of f which is defined by $F(f)(\lambda) = \tau(E_{\lambda^2}^{f^*f})$, where $E_{\lambda^2}^{f^*f}$ is a spectral projection of the self-adjoint operator f^*f. The associated measure on the Borel σ-algebra of $\mathbb{R}$ is given by $\mathrm{d}F(f)\,((a,b]) = F(f)(b) - F(f)(a)$ for $a, b \in \mathbb{R}, a < b$. The notation "0+" in 1 means that we omit the possible atom 0 in the domain of integration. The

2010 MSC: 47C15
Key words: Fuglede-Kadison determinant, group von Neumann algebra

omission of that atom is the reason why definition (1) is slightly more general than the original analytic extension of the Fuglede-Kadison determinant to non-regular operators [1, p. 528]. In that extension, all operators with non-zero kernel have determinant zero. In contrast, the generalised Fuglede-Kadison determinant (1) completely ignores the kernel, which leads for example to the odd equation $\det 0 = 1$. However, for injective operators in a finite factor, the original Fuglede-Kadison determinant and its generalisation (1) agree.

Applications

An example of a finite von Neumann algebra is the group von Neumann algebra $\mathcal{N}(G)$ of a discrete group G. It is defined as the set of all operators in $B(l^2(G))$ that commute with the G-action on $l^2(G)$ given by left multiplication. The trace is

$$\begin{aligned} \tau = \mathrm{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) &\to \mathbb{C}\,, \\ T &\mapsto \langle Te, e \rangle_{l^2(G)}\ , \end{aligned} \tag{2}$$

where e is the neutral element of G. In [4, chapter 1], Lück extends that example to the more general theory of morphisms of finite-dimensional Hilbert $\mathcal{N}(G)$-modules. In that context, the Fuglede-Kadison determinant is the main technical ingredient in the definition of L^2-torsion (see [4, chapter 3]).

An important class of operators in $\mathcal{N}(G)$ are those which are given by left multiplication with an element of the integer group ring $\mathbb{Z}G$. Let $(a\colon G \to \mathbb{Z}) \in \mathbb{Z}G$, i.e. $a(g) \neq 0$ for only finitely many $g \in G$. The operator in $\mathcal{N}(G)$ defined by a is

$$\begin{aligned} &A\colon l^2(G) \to l^2(G) \\ &(c_g)_{g\in G} \mapsto \big((Ac)_g\big)_{g\in G}\,, \quad (Ac)_g = \sum_{h\in G} a_h c_{h^{-1}g}\,. \end{aligned}$$

Matrices of such operators are exactly the morphisms of Hilbert $\mathcal{N}(G)$-modules that occur in the study of L^2-invariants of finite free G-CW-complexes. Therefore, determinants of those operators are an important special case of research.

A different example of application of the determinant is the case of the von Neumann algebra associated to an equivalence relation in a probability space, see [2].

Motivation

The motivation to study the continuity properties of the determinant springs from the desire to understand the behaviour under limits of all constructions that use the determinant, e.g. the L^2-torsion invariant.

The few positive results about the continuity properties of the determinant of morphisms of finite-dimensional Hilbert $\mathcal{N}(G)$-modules [4, p. 129] all consider operator norm convergence and follow essentially from the classical dominated or monotone convergence theorems of Lebesgue and Levi. For example, there is the result that for an injective positive morphism $f\colon U \to U$ in a finite-dimensional Hilbert $\mathcal{N}(G)$-module, we have

$$\lim_{\lambda\to 0^+} \det(f + \lambda\cdot \mathrm{id}_U) = \det(f)\,. \tag{3}$$

Naturally, for such continuity results one expects non-trivial counterexamples when the conditions of no classical convergence theorem for integrals are fulfilled. By non--trivial counterexamples, we mean operators which are as regular as possible. By the latter we shall mean that the kernel and the cokernel are as small as possible, the operator has useful properties such as being self-adjoint, and the operator is of determinant class, i.e. has strictly positive determinant.

For the strong and weak operator topologies, positive results are much harder to obtain since the convergence of operators in those topologies does not imply convergence of the spectral density functions of the operators in any usable sense. We are not aware of any published research in the study of the continuity of the determinant with respect to other topologies than the one induced by the operator norm.

Main results

Our three main results are: The determinant is not continuous on all isomorphisms in $\mathcal{N}(\mathbb{Z})$ with respect to the strong operator topology. In the case of the weak operator topology, the example of discontinuity can be constructed within the class of operators in $\mathcal{N}(\mathbb{Z})$ given by left multiplication with elements of $\mathbb{Z}[\mathbb{Z}]$. Considering the operator norm topology, the Fuglede-Kadison determinant can be discontinuous at $\lambda = 0$ on a line $\{T+\lambda\cdot\mathrm{id}_{l^2(\mathbb{Z})} \mid \lambda \in \mathbb{R}\}$ that consists entirely of weak isomorphisms of determinant class. That is a non-trivial counterexample to (3) in absence of positivity. In all cases the operators are constructed explicitly and the short proofs of their properties suggest how one might construct similar "pathologic" examples in other situations.

Method

The basis for the construction of our examples is the following model for the group von Neumann algebra of the integers. Lück remarks in [4, p. 15] that there is an isometric $\star$-algebra-isomorphism $\mathcal{N}(\mathbb{Z}) \cong L^\infty(S^1)$, where $L^\infty(S^1)$ is identified with the set of pointwise multiplication operators $\{M_g \mid g \in L^\infty(S^1)\} \subset B(L^2(S^1))$ and the involution on $L^\infty(S^1)$ is pointwise complex conjugation. That isomorphism of algebras is induced by an isometry of Hilbert spaces

$$\begin{aligned} l^2(\mathbb{Z}) &\xrightarrow{\cong} L^2(S^1)\,, \\ (a_k)_{k\in\mathbb{Z}} &\longmapsto \Big(z \mapsto \sum_{k\in\mathbb{Z}} a_k z^k\Big), \quad z = \mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{C}\,. \end{aligned} \tag{4}$$

Note that (4) implies that an operator in $\mathcal{N}(\mathbb{Z})$ given by left multiplication with an element $(a_k)_{k\in\mathbb{Z}} \in \mathbb{C}[\mathbb{Z}]$ is identified with the polynomial $\sum_{k\in\mathbb{Z}} a_k z^k$ in $L^\infty(S^1)$. The identification $\mathcal{N}(\mathbb{Z}) \cong L^\infty(S^1)$ allows for simple constructions of concrete morphisms in $\mathcal{N}(\mathbb{Z})$ with prescribed spectral density functions. Moreover, under the identification there is the following simple formula for the determinant [4, p. 128]:

$$\ln\det\, g = \int_{S^1} \ln|g(z)| \cdot \chi_{\{u\in S^1 \mid g(u)\neq 0\}}\,\mathrm{d}\,\mathrm{vol}_z\,, \quad g \in L^\infty(S^1) \tag{5}$$

where $\mathrm{d}\,\mathrm{vol}_z$ is the usual "round" measure on S^1, scaled such that $\mathrm{vol}(S^1) = 1$.

At this point we would like to remark that although $\mathcal{N}(\mathbb{Z}) \cong L^\infty(S^1)$ is not a type II factor (because $\mathbb{Z}$ has no infinite conjugacy classes, see [3, Theorem 6.7.5]), $L^\infty(S^1)$ can be embedded into a type II factor as a maximal commutative subalgebra, by the classical group measure space construction. Therefore, operators in $\mathcal{N}(\mathbb{Z})$ can be regarded as elements in a type II factor. Moreover, since all operators involved in our counterexamples are injective, their determinant agrees in fact with the original Fuglede-Kadison determinant from [1], which means that our results apply in particular to the original Fuglede-Kadison determinant.

2 Discontinuity in the Weak Operator Topology

Proposition 1. *There is a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{N}(\mathbb{Z})$ of isomorphisms, given by left multiplication with elements in $\mathbb{Z}[\mathbb{Z}]$, which converges to $\mathrm{id}_{l^2(\mathbb{Z})}$ with respect to the weak operator topology but $\lim_{n\to\infty} \det(A_n) \neq 1 = \det(\mathrm{id}_{l^2(\mathbb{Z})})$.*

Proof. Define A_n to be left multiplication with $({}^n a_k)_{k\in\mathbb{Z}}$, where ${}^n a_0 = 1$, ${}^n a_n = 2$ and ${}^n a_k = 0$ for all $k \in \mathbb{Z}$ other than 0 and n. Under the isometric isomorphism $\mathcal{N}(\mathbb{Z}) \cong L^\infty(S^1)$, A_n corresponds to the polynomial

$$p_n(z) := 1 + 2z^n, \quad z \in S^1 \subset \mathbb{C}\,.$$

The polynomial $1 + 2z^n$ on S^1 is bounded away from zero for each $n \in \mathbb{N}$. Hence A_n is invertible with inverse the operator corresponding to $1/p_n \in L^\infty(S^1)$. In [4, p. 136], Lück proves an example that implies $\det(A_n) = 2$ for all $n \in \mathbb{N}$. From (2) follows immediately that $\det(\mathrm{id}_{l^2(\mathbb{Z})}) = 1$. What is left to show is that $(A_n)_{n\in\mathbb{N}}$ converges to $\mathrm{id}_{l^2(\mathbb{Z})}$ in the weak operator topology as $n \to \infty$. Let $f, g \in C^\infty(S^1)$. Then we have

$$\begin{aligned}
\left|\frac{1}{2}\big\langle (p_n - 1)f, g\big\rangle_{L^2(S^1)}\right| &= \left|\int_{S^1} z^n f(z)\,\overline{g(z)}\,\mathrm{d}\,\mathrm{vol}_z\right| \\
&= \left|\int_0^{2\pi} \mathrm{e}^{\mathrm{i}nt} f\big(\exp(\mathrm{i}t)\big)\,\overline{g\big(\exp(\mathrm{i}t)\big)}\,\mathrm{d}t\right| \\
&= \left|\int_0^{2\pi} \frac{1}{\mathrm{i}n}\mathrm{e}^{\mathrm{i}nt}\frac{\mathrm{d}}{\mathrm{d}t}(f\overline{g})\big(\exp(\mathrm{i}t)\big)\,\mathrm{d}t\right| \\
&\leq \frac{1}{n}\big\|(fg)'\big\|_\infty \overset{n\to\infty}{\longrightarrow} 0\,.
\end{aligned} \tag{6}$$

Now, since $C^\infty(S^1)$ is dense in $L^2(S^1)$ with respect to its Hilbert space norm, we can conclude from (6) that the sequence of operators in $B(L^2(S^1))$ given by pointwise multiplication with p_n converges to $\mathrm{id}_{L^2(S^1)}$ in the weak operator topology as $n \to \infty$. The corresponding claim about the sequence $(A_n)_{n\in\mathbb{N}}$ follows immediately. □

Note that the sequence $(A_n)_{n\in\mathbb{N}}$ from the previous proof does not converge to $\mathrm{id}_{l^2(\mathbb{Z})}$ in the strong operator topology: For each polynomial p_n corresponding to A_n, we have $\|(p_n - 1)f\|_{L^2(S^1)} = 2\,\|f\|_{L^2(S^1)}$ for all $f \in L^2(S^1)$.

3 Discontinuity in the Strong Operator Topology

Proposition 2. *Let $r = (r_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers (e.g. $r_n = \sin(n) + 1$). There is a sequence of isomorphisms $(f_n^r)_{n\in\mathbb{N}} \subset \mathcal{N}(\mathbb{Z})$ such that $f_n^r \to \mathrm{id}_{l^2(\mathbb{Z})}$ in the strong operator topology as $n \to \infty$ and $\det(f_n^r) = \exp(-r_n)$.*

Proof. We use the identification $L^\infty(S^1) \cong \mathcal{N}(\mathbb{Z})$. Let for $n \in \mathbb{N}$ the operator f_n^r correspond to the function $g_n^r \in L^\infty(S^1)$ given by

$$g_n^r\left(\exp(2\pi it)\right) := \begin{cases} \exp\left(-n\cdot r_n\right), & 0 < t \le \frac{1}{n}, \\ 1, & \frac{1}{n} < t \le 1. \end{cases}$$

Then f_n^r is self-adjoint, as g_n^r is real, and invertible with inverse the operator corresponding to the well-defined function $1/g_n^r \in L^\infty(S^1)$.

We prove first that f_n^r converges to $\mathrm{id}_{l^2(\mathbb{Z})}$ in the strong operator topology. This is equivalent to proving that the pointwise multiplication operator $M_{g_n^r} \in B(L^2(S^1))$ converges to $\mathrm{id}_{L^2(S^1)}$.

Let $h \in L^2(S^1)$.

$$\begin{aligned}\left\|M_{g_n^r}(h) - h\right\|_{L^2(S^1)} &= \int_{S^1} \left|g_n^r(z)h(z) - h(z)\right|^2 \mathrm{d}\,\mathrm{vol}_z \\ &= \int_0^1 \left|g_n^r\big(\exp(2\pi it)\big)h\big(\exp(2\pi it)\big) - h\big(\exp(2\pi it)\big)\right|^2 \mathrm{d}t \\ &= \int_0^{1/n} \left|h\big(\exp(2\pi it)\big)\big(\exp(-n\cdot r_n) - 1\big)\right|^2 \mathrm{d}t \\ &\le \int_0^{1/n} \left|h\big(\exp(2\pi it)\big)\right|^2 \mathrm{d}t\,.\end{aligned}$$

The final integral converges to zero as $n \to \infty$ due to σ-additivity of Lebesgue measure. The calculation of the determinant of f_n^r is a very easy task using (5):

$$\begin{aligned}\ln\det(f_n^r) &= \int_{S^1} \ln\big(|g_n^r(z)|\big)\cdot\chi_{\{u\in S^1 \mid g_n^r(u)\neq 0\}}\,\mathrm{d}\,\mathrm{vol}_z \\ &= \int_0^1 \ln\big(g_n^r(\exp(2\pi it))\big)\mathrm{d}t \\ &= \int_0^{1/n} \ln\big(\exp\left(-n\cdot r_n\right)\big)\mathrm{d}t \\ &= -r_n\,.\end{aligned}$$

□

4 Discontinuity in the Operator Norm Topology

Define $T \in \mathcal{N}(\mathbb{Z})$ as the operator corresponding to $g \in L^\infty(S^1)$, where

$$g\left(\exp(2\pi \mathrm{i} x)\right) := \left(\frac{1}{n} - x\right)^{n(n+1)} - \mathrm{e}^{-\sqrt{n-1}}, \quad x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], \ n \in \mathbb{N}. \tag{7}$$

The shape of the graph of g is illustrated in Figure 1 below. One property of g is that for all $x \in (\frac{1}{n+1}, \frac{1}{n}]$, $n \in \mathbb{N}$, there is a $\delta_n > 0$ such that

$$-\mathrm{e}^{-\sqrt{n}} - \delta_n \geq g\left(\exp(2\pi \mathrm{i} x)\right) \geq -\mathrm{e}^{-\sqrt{n-1}}.$$

The statement in line (4) can be verified using

$$\begin{aligned}\left|\left(\frac{1}{n} - x\right)^{n(n+1)}\right| &\leq \left(\frac{1}{n} - \frac{1}{n+1}\right)^{n(n+1)} = \left(\frac{1}{n(n+1)}\right)^{n(n+1)} \\ &\leq \frac{1}{2}\left(\mathrm{e}^{-\sqrt{n-1}} - \mathrm{e}^{-\sqrt{n}}\right),\end{aligned}$$

where the second inequality is a straightforward check. For example, we can set $\delta_n := \frac{1}{3}(\mathrm{e}^{-\sqrt{n-1}} - \mathrm{e}^{-\sqrt{n}})$. Note that (4) implies that $x \mapsto g\left(\exp(2\pi \mathrm{i} x)\right)$ is a strictly decreasing function since it is strictly decreasing on each interval $(\frac{1}{n+1}, \frac{1}{n}]$.

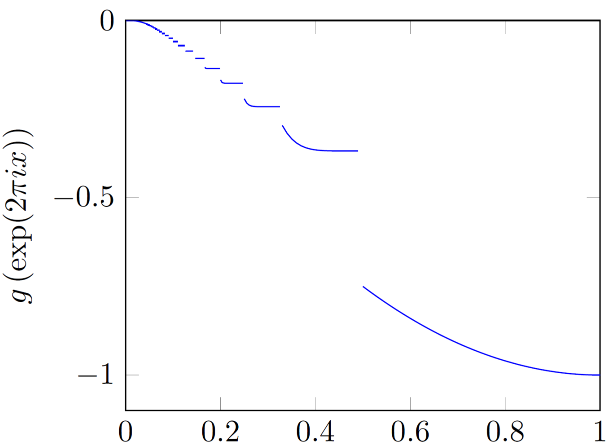

Figure 1: Qualitative picture of the graph of g. The slope of the $\left(\frac{1}{n} - x\right)^{n(n+1)}$-segments is strongly exaggerated.

4.1 Verification of the properties of T

Note that for $\lambda \in \mathbb{R}$, the operator $T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}$ corresponds to $g + \lambda \cdot 1$, where 1 is the “constant 1” function on S^1.

Proposition 3. *For each $\lambda \in \mathbb{R}$ the operator $T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}$ is a weak isomorphism.*

Proof. As T is self-adjoint, the claim is equivalent to the claim that $T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}$ is injective, i.e. for $\lambda \in \mathbb{R}$ the zero locus of $g + \lambda \cdot 1$ is a null set in S^1. Since $x \mapsto g(\exp(2\pi \mathrm{i} x))$ is strictly decreasing, $g + \lambda \cdot 1$ can have at most one zero. $\square$

Proposition 4. *For each $\lambda \in \mathbb{R}$ the operator $T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}$ is of determinant class.*

Proof. To simplify notation, set $\gamma_\lambda(x) := g(\exp(2\pi \mathrm{i} x)) + \lambda$ for $x \in (0,1]$.

Case $\lambda = 0$: We use equation (5).

$$\begin{aligned}
\ln \det(T) &= \int_{S^1} \ln(|g(z)|) \cdot \chi_{\{u \in S^1 \mid g(u) \neq 0\}} \,\mathrm{d}\,\mathrm{vol}_z = \sum_{n \in \mathbb{N}} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \ln|\gamma_\lambda(x)| \,\mathrm{d}x \\
&\overset{(4)}{\geq} \sum_{n \in \mathbb{N}} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \ln\left(\mathrm{e}^{-\sqrt{n}}\right) \mathrm{d}x \\
&= \sum_{n \in \mathbb{N}} \left(\frac{1}{n} - \frac{1}{n+1}\right) \cdot (-\sqrt{n}) = \sum_{n \in \mathbb{N}} \frac{-1}{\sqrt{n}(n+1)} \\
&> -\infty\,.
\end{aligned}$$

Case $\lambda = \mathrm{e}^{-\sqrt{m-1}}$, $m \in \mathbb{N}$: Again, we use equation (5).

$$\begin{aligned}
\ln \det\left(T + \lambda_m \cdot \mathrm{id}_{l^2(\mathbb{Z})}\right) &= \sum_{n \in \mathbb{N}} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \ln|\gamma_\lambda(x)| \,\mathrm{d}x \\
&\geq \int_{\frac{1}{m+1}}^{\frac{1}{m}} \ln|\gamma_\lambda(x)| \,\mathrm{d}x + \int_0^1 \ln\min\{\delta_m, \delta_{m-1}\} \,\mathrm{d}x \\
&= \int_{\frac{1}{m+1}}^{\frac{1}{m}} \ln\left(\left(\frac{1}{m} - x\right)^{m(m+1)}\right) \mathrm{d}x + \ln\min\{\delta_m, \delta_{m-1}\} \\
&= \ln\left(\frac{1}{m(m+1)}\right) - 1 + \ln\left(\min\{\delta_m, \delta_{m-1}\}\right) \\
&> -\infty\,.
\end{aligned} \tag{8}$$

Case $\lambda = \mathrm{e}^{-\sqrt{m-1}} + (\frac{1}{m} - \frac{1}{m+1})^{m(m+1)}$, $m \in \mathbb{N}$: The point $(\frac{1}{m+1}, 0)$ is a limit point of the graph of γ_λ. We have $|\gamma_\lambda(x)| \geq \delta_m$ for all $x \in (\frac{1}{m+2}, \frac{1}{m+1}]$. Note that $\gamma_\lambda|_{(\frac{1}{m+1}, \frac{1}{m}]}$ is a polynomial whose derivative has a right limit for $x \to \frac{1}{m+1}^+$ which is strictly greater than zero. If d_m is that limit, we can find $\varepsilon > 0$ such that $|\gamma_\lambda(x)| \geq \frac{d_m}{2}|x - \frac{1}{m+1}|$ for all $x \in [\frac{1}{m+1} - \varepsilon, \frac{1}{m+1} + \varepsilon]$. On $(0,1] \backslash [\frac{1}{m+1} - \varepsilon, \frac{1}{m+1} + \varepsilon]$,

γ_λ is bounded away from zero by some bound $\delta > 0$. We can estimate using (5):

$$\begin{aligned}
\ln\det\left(T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}\right) &= \sum_{n\in\mathbb{N}} \int\limits_{\frac{1}{n+1}}^{\frac{1}{n}} \ln|\gamma_\lambda(x)|\,\mathrm{d}x \\
&\geq \int\limits_{\frac{1}{m+1}-\varepsilon}^{\frac{1}{m+1}+\varepsilon} \ln\left|\frac{1}{2}d_m\left(x - \frac{1}{m+1}\right)\right|\,\mathrm{d}x + \int\limits_0^1 \ln\delta\,\mathrm{d}x \\
&= 2\varepsilon\left(\ln\left(\frac{1}{2}d_m\right) + \ln(\varepsilon) - 1\right) + \ln\delta \\
&> -\infty\,.
\end{aligned}$$

Case $\mathrm{e}^{-\sqrt{m-1}} < \lambda < \mathrm{e}^{-\sqrt{m-1}} + \left(\frac{1}{m} - \frac{1}{m+1}\right)^{m(m+1)}$, $m \in \mathbb{N}$: The graph of γ_λ cuts the x-axis at some $x_0 \in (\frac{1}{m+1}, \frac{1}{m})$. We can proceed as in the previous case.

For other $\lambda \in \mathbb{R}$: The function γ_λ is bounded away from 0 so the case is trivial. □

Proposition 5. *There is a sequence* $(\lambda_m)_{m\in\mathbb{N}} \subset (0,1]$ *converging to zero such that* $\det\left(T + \lambda_m \cdot \mathrm{id}_{l^2(\mathbb{Z})}\right) < \frac{1}{m(m+1)}$. *So* $\lim_{m\to\infty} \det\left(T + \lambda_m \cdot \mathrm{id}_{l^2(\mathbb{Z})}\right) = 0 \neq \det(T)$.

Proof. Set $\lambda_m := \mathrm{e}^{-\sqrt{m-1}}$. Similarly as in the previous proof, use (5):

$$\begin{aligned}
\ln\det\left(T + \lambda_m \cdot \mathrm{id}_{l^2(\mathbb{Z})}\right) &= \sum_{n\in\mathbb{N}} \int\limits_{\frac{1}{n+1}}^{\frac{1}{n}} \ln\Big|g\big(\exp(2\pi\mathrm{i}x)\big) + \lambda_m\Big|\,\mathrm{d}x \\
&\leq \int\limits_{\frac{1}{m+1}}^{\frac{1}{m}} \ln\Big|g\big(\exp(2\pi\mathrm{i}x)\big) + \lambda_m\Big|\,\mathrm{d}x \qquad (9) \\
&= \ln\left(\frac{1}{m(m+1)}\right) - 1\,. \qquad (10)
\end{aligned}$$

In line (9) we used that the summands in the previous line are non-positive. In line (10) we used the estimate (8). □

Acknowledgements

The author would like to thank Prof. Wolfgang Lück at Bonn University for his suggestion to study the continuity properties of the Fuglede-Kadison determinant and for helpful conversations as well as critical feedback in the *Master-Begleitseminar*. The research presented in this article was carried out while the author was financially supported by a Qualification Fellowship of the *Graduiertenkolleg 1150* at Bonn University, Germany.

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Author's address:
Department of Mathematics and Computer Science, University of Marburg, Hans-Meerwein-Str. 6, 35043 Marburg, Germany

E-mail: `bkuester@mathematik.uni-marburg.de`

Received: 26 March, 2014
Accepted for publication: 18 July, 2014
Communicated by: Stephen Glasby

Communications in Mathematics 22 (2014) 151–157

On a binary recurrent sequence of polynomials

Reinhardt Euler, Luis H. Gallardo, Florian Luca

Abstract. In this paper, we study the properties of the sequence of polynomials given by $g_0 = 0,\ g_1 = 1,\ g_{n+1} = g_n + \Delta g_{n-1}$ for $n \geq 1$, where $\Delta \in \mathbb{F}_q[t]$ is non-constant and the characteristic of $\mathbb{F}_q$ is 2. This complements some results from [2].

1 Introduction

Let $\mathbb{F}_q$ be the finite field with $q = 2^k$ elements for some $k \geq 1$. Given $\Delta \in \mathbb{F}_q[t]$ non constant define $\{g_n\}_{n\geq 0}$ by $g_0 = 0,\ g_1 = 1$ and

$$g_{n+2} = g_{n+1} + \Delta g_n \quad \text{for} \quad n \geq 0\,. \tag{1}$$

This sequence was studied in [2]. In this paper, we correct an oversight from [2], answer an open question about this sequence asked there and prove a few more properties of this sequence.

In [2], it was shown that $g_n = 0$ holds infinitely often. Here, we correct this statement and show that in fact $g_n = 1$ holds infinitely often and $g_n = 0$ for $n = 0$ only. At the end of [2] it was asked whether the sequence $\{g_n\}_{n\geq 0}$ is periodic. Here, we show that this is not the case by proving in fact that $\limsup_{n\to\infty} \deg(g_n) = \infty$. We also find explicit formulas for g_n when $n = 2^m,\ 2^m - 1,\ 2^m + 1$ for some $m \geq 0$. We also find more properties of the polynomials $\{g_n\}_{n\geq 0}$. For example, it is easy to show by induction that the degree of g_n is at most $n-1$ and that g_n is a polynomial in Δ with coefficients in $\{0, 1\}$. We let $\ell(g_n)$ be the *length* of g_n as a polynomial in $\mathbb{F}_q[\Delta]$, namely the sum of its coefficients and compute this number. We find that $\ell(g_n) = a_n$, where $\{a_n\}_{n\geq 0}$ is the Stern-Brocot sequence given by $a_0 = 0,\ a_1 = 1$ and

$$a_{2n} = a_n \qquad \text{and} \qquad a_{2n+1} = a_{n+1} + a_n \qquad \text{for all} \qquad n \geq 0\,.$$

We also compute how many of the a_n monomials in g_n have odd degree in Δ. Let b_n be this number. We find that $b_{2n} = 0$ and $b_{2n+1} = a_n$ for all $n \geq 0$.

2010 MSC: 11T55, 11T06, 11B39

Key words: sequences of binary polynomials, Stern-Brocot sequence, perfect fields of characteristic 2

All these results are summarized in the theorem below.

Theorem 1. *The following holds:*

(i) $g_{2^m} = 1$ *for all* $m \geq 0$,

(ii) $g_{2^m+1} = 1 + \Delta + \Delta^2 + \cdots + \Delta^{2^{m-1}}$ *for all* $m \geq 1$,

(iii) $g_{2^m-1} = 1 + \Delta + \Delta^3 + \cdots + \Delta^{2^{m-1}-1}$ *for all* $m \geq 1$,

(iv) $\ell(g_n) = a_n$,

(v) $b_{2n} = 0$,

(vi) $b_{2n+1} = a_n$ *for all* $n \geq 0$.

2 The proof of Theorem 1

We first prove a lemma.

Lemma 1. *For all* $n \geq 0$*:*

(i) $g_{2n+4} = g_{2n+2} + \Delta^2 g_{2n}$,

(ii) $g_{2n} = g_n^2$.

Proof. For (i), we write using (1) (with n replaced by $2n$ and by $2n+2$) and the fact that the characteristic of $\mathbb{F}_q$ is 2:

$$g_{2n+1} = g_{2n+2} + \Delta g_{2n} \qquad \text{and} \qquad g_{2n+3} = g_{2n+4} + \Delta g_{2n+2} \,. \tag{2}$$

Inserting the above relations into (1) with n replaced by $2n+1$, we get

$$g_{2n+4} + \Delta g_{2n+2} = g_{2n+3} = g_{2n+2} + \Delta g_{2n+1} = g_{2n+2} + \Delta(g_{2n+2} + \Delta g_{2n}) \,,$$

or

$$g_{2n+4} = g_{2n+2} + \Delta^2 g_{2n}$$

as desired. For (ii), we use induction on n. The cases $n = 0, 1$ are clear. Assuming that $n \geq 2$ and that (ii) holds for all $m \leq n$, we have, by (i),

$$g_{2n+2} = g_{2n} + \Delta^2 g_{2n-2} = g_n^2 + \Delta^2 g_{n-1}^2 = (g_n + \Delta g_{n-1})^2 = g_{n+1}^2 \,,$$

which completes the induction and the proof of (ii). $\square$

We are now ready to prove Theorem 1. We first prove (i)–(iii) by induction on $m \geq 0$. The cases $m = 0, 1$ can be verified by hand. Assume that $m \geq 2$ and (i)–(iii) hold for all $n < m$. Then, by Lemma 1 (ii) and the induction hypothesis, we have

$$g_{2^m} = (g_{2^{m-1}})^2 = 1^2 = 1 \,.$$

Further,

$$1 = g_{2^m} = g_{2^m-1} + \Delta g_{2^m-2} = g_{2^m-1} + \Delta(g_{2^{m-1}-1})^2,$$

so

$$\begin{aligned} g_{2^m-1} &= 1 + \Delta g_{2^{m-1}-1}^2 \\ &= 1 + \Delta(1 + \Delta + \Delta^3 + \cdots + \Delta^{2^{m-2}-1})^2 \\ &= 1 + \Delta + \Delta^3 + \cdots + \Delta^{2^{m-1}-1}. \end{aligned}$$

Finally,

$$\begin{aligned} g_{2^m+1} &= g_{2^m} + \Delta g_{2^m-1} \\ &= 1 + \Delta(1 + \Delta + \Delta^3 + \cdots + \Delta^{2^{m-1}-1}) \\ &= 1 + \Delta + \Delta^2 + \cdots + \Delta^{2^{m-1}}. \end{aligned}$$

For (iv), we check that the statement is true for $n = 0, 1$. Since

$$g_{2n} = g_n^2$$

we have $a_{2n} = \ell(g_{2n}) = \ell(g_n^2) = \ell(g_n) = a_n$. Since

$$g_{2n+1} = g_{2n+2} + \Delta g_{2n} = g_{n+1}^2 + \Delta g_n^2 \tag{3}$$

and every monomial appearing in either g_{n+1}^2 or g_n^2 appears with even degree, we have that

$$\ell(g_{2n+1}) = \ell(g_{n+1}^2) + \ell(g_n^2) = \ell(g_{n+1}) + \ell(g_n) = a_{n+1} + a_n\,,$$

which is what we wanted.

We now prove (v) and (vi). By (ii) of Lemma 1, we have that

$$g_{2n} = g_n^2$$

is a polynomial in Δ whose monomials have even degree. Hence, $b_{2n} = 0$. For the odd n, note that $b_n = \ell(g_n')$, where g_n' denotes the derivative of g_n as a polynomial in Δ. Taking the derivative in relation (1) and using the fact that the characteristic of $\mathbb{F}_q$ is 2, we get

$$g_n = g_{n+2}' + g_{n+1}' + \Delta g_n'\,.$$

Inserting the above relation with n replaced by $n+1$ and $n+2$ in (1), we get

$$\begin{aligned} g_{n+4}' + g_{n+3}' + \Delta g_{n+2}' &= g_{n+2} = g_{n+1} + \Delta g_n \\ &= g_{n+3}' + g_{n+2}' + \Delta g_{n+1}' + \Delta(g_{n+2}' + g_{n+1}' + \Delta g_n')\,, \end{aligned}$$

which leads to

$$g_{n+4}' = g_{n+2}' + \Delta^2 g_n'\,.$$

Since $g_0 = 0,\ g_1 = 1,\ g_2 = 1,\ g_3 = 1 + \Delta$, we have that $g_1' = 0$ and $g_3' = 1$. Thus, we get that $g_{2n+1}' = g_n(\Delta^2)$, where $g_n(\Delta^2)$ is the same sequence of polynomials $\{g_n\}_{n\geq 0}$ but with Δ replaced by Δ^2. Now (vi) follows from (iv).

A simpler argument for (vi) suggested by the referee goes as follows: since

$$g_{n+1}^2 = g_{2n+2} = g_{2n+1} + \Delta g_{2n} = g_{2n+1} + \Delta g_n^2 \,,$$

taking derivatives yields

$$0 = (g_{n+1}^2)' = g'_{2n+1} + g_n^2 + \Delta(g_n^2)' = g'_{2n+1} + g_n^2 \,,$$

and therefore $g'_{2n+1} = g_{2n}$. Hence,

$$b_{2n+1} = \ell(g'_{2n+1}) = \ell(g_{2n}) = a_{2n} = a_n \,.$$

Of course, the even case can be treated similarly:

$$b_{2n} = \ell(g'_{2n}) = \ell((g_n^2)') = \ell(0) = 0\,.$$

Remark 1. Another approach to (iv)–(vi) of Theorem 1 due to the referee is as follows. First let us define the sequence $\{g_n\}_{n\geq 0}$ of polynomials in $\mathbb{Z}[\Delta]$ given by the same recurrence

$$g_{n+2} = g_{n+1} + \Delta g_n$$

with $g_0 = 0,\ g_1 = 1$. Then we have the following representation of the general term g_n.

Lemma 2. *We have for $n \geq 0$,*

$$g_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \Delta^k. \tag{4}$$

Proof. For $n = 0, 1$, we have $g_1 = 1,\ g_2 = 1 + \Delta$ which are consistent with what is shown at (4) when $n = 0, 1$. Assuming now that $n \geq 1$ and that (4) holds both for n and for n replaced by $n - 1$, then

$$\begin{aligned} g_{n+2} &= g_{n+1} + \Delta g_n \qquad (5)\\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \Delta^k + \Delta \left(\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-k}{k} \Delta^k \right) \\ &= \binom{n}{0} + \sum_{k=1}^{\lfloor n/2 \rfloor} \left(\binom{n-k}{k} + \binom{(n-1)-(k-1)}{k-1} \right) \Delta^k \\ &\quad + \sum_{k=\lfloor n/2 \rfloor + 1}^{\lfloor (n-1)/2 \rfloor + 1} \binom{n-1-(k-1)}{k-1} \Delta^k \\ &= 1 + \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n+1-k}{k} \Delta^k + \sum_{k=\lfloor n/2 \rfloor + 1}^{\lfloor (n-1)/2 \rfloor + 1} \binom{n-k}{k-1} \Delta^k. \qquad (6) \end{aligned}$$

In the above formula we used the fact that

$$\binom{n-k}{k} + \binom{(n-1)-(k-1)}{k-1} = \binom{n-k}{k} + \binom{n-k}{k-1} = \binom{n+1-k)}{k}.$$

The left-most term 1 in (5) equals $\binom{n+1-0}{0}$, the last term is 0 when n is even because then $\lfloor n/2\rfloor = \lfloor (n-1)/2\rfloor + 1 = \lfloor (n+1)/2\rfloor$, while in case when $n = 2m+1$ is odd, then the last term is the monomial in $k = m+1 = \lfloor (n+1)/2\rfloor$ with coefficient $\binom{2m-m}{m} = 1 = \binom{n+1-k}{k}$. This completes the induction. □

By Lemma 2, we have, in characteristic 2,

$$g_{n+1} = \sum_{k=0}^{\lfloor n/2\rfloor} \left[\binom{n-k}{k} \bmod 2\right] \Delta^k. \tag{7}$$

Hence,

$$\ell(g_{n+1}) = \sum_{k=0}^{\lfloor n/2\rfloor} \left[\binom{n-k}{k} \bmod 2\right] = a_{n+1}\,,$$

which is (iv) for all $n \geq 1$ (the fact that $\ell(g_0) = a_0 = 0$ is clear). The last equality is Theorem 4.1 in [4] (see also sequence A002487 in [5]). Letting

$$b_{n+1} := \sum_{\substack{k=0\\ k \text{ odd}}}^{\lfloor n/2\rfloor} \left[\binom{n-k}{k} \bmod 2\right],$$

we have, since $\binom{\text{even}}{\text{odd}} = \text{even}$ (which can be easily checked by invoking Lucas' theorem on binomial coefficients modulo p for the prime $p = 2$), we get

$$b_{2n} := \sum_{\substack{k=0\\ k \text{ odd}}}^{\lfloor n/2\rfloor} \left[\binom{2n-k-1}{k} \pmod 2\right] = 0\,,$$

which is (v). Further, because $\binom{2n}{2k} \equiv \binom{n}{k} \bmod 2$ (again by Lucas's theorem), we have

$$\begin{aligned} a_{2n+1} - b_{2n+1} &= \sum_{k=0}^{n} \left[\binom{2n-2k}{2k} \bmod 2\right] = \sum_{k=0}^{n} \left[\binom{n-k}{k} \bmod 2\right] \\ &= a_{n+1}\,, \end{aligned}$$

from where we get that $b_{2n+1} = a_{2n+1} - a_{n+1} = a_n$, which is (vi).

3 Comments and Open questions

First of all, observe that our results hold more generally for the finite field $\mathbb{F}_q$, with q even, replaced by any infinite field of characteristic 2, since we have not used the property $h^q = h$ for the elements h of our field. There are many questions one can ask about the sequence $\{g_n\}_{n\geq 0}$. For example, what can we say about the number of irreducible factors of g_n as a polynomial in Δ? Is it true that all roots of g_{2n+1} are simple? We leave such questions to the reader. As for the degree of g_n, writing $n = 2^a b$, where b is odd, gives $\deg(g_n) = 2^a(b-1)/2$. One may recognize this last quantity as $n * (n-1)/2$, where for nonnegative integers m and n, the quantity $m * n$ denotes the nonnegative integer whose binary representation is the bitwise AND operation of the binary representations of m and n. Indeed, since $g_{2n} = g_n^2$, we get that $g_n = g_{2^a b} = g_b^{2^a}$, so it suffices to show that if m is odd, then g_m has degree $(m-1)/2$. But this follows by replacing n by $m-1$ in (7):

$$g_m = \sum_{k=0}^{(m-1)/2} \left[\binom{m-1-k}{k} \bmod 2 \right] \Delta^k,$$

and noting that the last term of the above sum corresponding to $k = (m-1)/2$ has coefficient $\binom{(m-1)/2}{(m-1)/2} = 1$.

The above questions may be asked in the more general context of the field $\mathbb{F}[\Delta]$. A restriction to perfect fields of characteristic 2 may be useful since then we have for all polynomials $C \in \mathbb{F}[t]$ the simple relation

$$C = A^2 + tB^2$$

for some polynomials $A, B \in \mathbb{F}[t]$. By construction, the elements of our sequence with odd subscripts satisfy a relation of this type (see (3) in the proof of (iv)).

Observe also that this sequence can be easily dealt with over fields of characteristic $p > 2$ by the Binet formulae. However, in our case $p = 2$ and $\mathbb{F}$ finite, we were not able to use these formulae to describe our sequence since we do not know explicitly the solutions of the quadratic equation

$$x^2 + x + \Delta = 0$$

in the ring $\mathbb{F}_q[t]$. This motivates our new approach to study the sequence in the present paper.

Moreover, the reader may try to check which of the properties in [3], that hold for the classical case in which the coefficients are integers, are still true in our characteristic 2 case by using the tools of [1].

4 Acknowledgements

We thank the referee for comments which improved the quality of our paper and for providing proofs to some of our results which are alternative to our original ones. This paper was written during a visit of F. L. at the Mathematics Department of the Université de Bretagne Occidentale in Brest in February, 2014. He thanks the people of this institution for their hospitality. During the preparation of this paper, F. L. was also supported in part by Project PAPIIT IN104512 (UNAM).

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Authors' addresses:
Reinhardt Euler: Lab-STICC UMR CNRS 6285, University of Brest, 6, Avenue Le Gorgeu, C.S. 93837, 29238 Brest, Cedex 3, France

E-mail: `Reinhardt.Euler@univ-brest.fr`

Luis H. Gallardo: Department of Mathematics, University of Brest, 6, Avenue Le Gorgeu, C.S. 93837, 29238 Brest, Cedex 3, France

E-mail: `Luis.Gallardo@univ-brest.fr`, `gallardo@math.cnrs.fr`

Florian Luca: Mathematical Institute, UNAM Juriquilla, 76230 Santiago de Querétaro, México, and School of Mathematics, University of the Witwatersrand, P. O. Box Wits 2050, South Africa

E-mail: `fluca@matmor.unam.mx`

Received: 9 September, 2014
Accepted for publication: 27 November, 2014
Communicated by: Karl Dilcher

Communications in Mathematics 22 (2014) 159–184

Symmetries and currents in nonholonomic mechanics

Michal Čech, Jana Musilová

Abstract. In this paper we derive general equations for constraint Noether--type symmetries of a first order non-holonomic mechanical system and the corresponding currents, i.e. functions constant along trajectories of the nonholonomic system. The approach is based on a consistent and effective geometrical theory of nonholonomic constrained systems on fibred manifolds and their jet prolongations, first presented and developed by Olga Rossi. As a representative example of application of the geometrical theory and the equations of symmetries and conservation laws derived within this framework we present the Chaplygin sleigh. It is a mechanical system subject to one linear nonholonomic constraint enforcing the plane motion. We describe the trajectories of the Chaplygin sleigh and show that the usual kinetic energy conservation law holds along them, the time translation generator being the corresponding constraint symmetry and simultaneously the symmetry of nonholonomic equations of motion. Moreover, the expressions for two other currents are obtained. Remarkably, the corresponding constraint symmetries are not symmetries of nonholonomic equations of motion. The physical interpretation of results is emphasized.

1 Introduction

While a wide variety of problems within the mechanics of first order systems without constraints or with holonomic constraints is solved, mechanics of nonholonomic systems is still studied relatively intensively by various authors using various approaches. Bibliography concerning nonholonomic constraints is very rich, see e.g. famous books by Neimark and Fufaev [26], Bloch and coworkers [2], Cortés Monforte [7], and Bullo [3], and others, or many papers as e.g. [9], [23], [24], [29], [34], [35], [39], [40], or recently e.g. [28] (for nonlinear constraints), to mention just a few. Most of the above cited works are concerned with linear or affine nonholonomic constraints, relevant a.e. for technical applications. A geometrical theory of

2010 MSC: 49S05, 58E30
Key words: nonholonomic mechanical systems, nonholonomic constraint submanifold, canonical distribution, reduced equations of motion, symmetries of nonholonomic systems, conservation laws, Chaplygin sleigh

nonholonomic systems on fibred manifolds and their jet prolongations was proposed by Olga Rossi (Krupková) in [14] and elaborated in her later works among which we can emphasize e.g. [15], [16], [19]. This theory differs from other approaches by the idea that a *nonholonomic constraint is a fibred submanifold* of the first jet prolongation of the underlying fibred manifold. The nonholonomic mechanical system is considered as a *dynamical system on this constraint submanifold* which is its true phase space. The equations of motion called the *reduced equations* are equivalent with the well known Chetaev equations [6] based on the standardly used d'Alembert's principle. In this sense the geometrical model is a generalization of the d'Alembert's principle to nonlinear as well as higher order constraints. A detailed explanation of the theory based on the nonholonomic variational principle can be found in [19].

The geometrical theory is an effective tool for solving a wide variety of problems connected with nonholonomic systems. One of them is the nonholonomic inverse problem, see e.g. [22] and [30]. The relevance and applicability of the theory was verified on examples (see [37]) and practical situations (see [8], [10], [11], [12], [13]), including the experimental verification in [12] and [13]. An interesting realistic case of a nonlinear constraint is represented by the mechanical system consisting of a mass particle in the special relativity theory. This problem is solved in [21] and [31]. Explicit results of this kind should be compared with usually applied analytic and geometric techniques which provide mostly only conclusions concerning equilibria.

Some questions concerning nonholonomic systems are still not satisfactorily understood. One of them is the problem of nonholonomic symmetries and conservation laws. On the other hand, a proper understanding of symmetries and conservation laws is a key question in mechanics including nonholonomic systems in particular. Here we emphasize a new concept of nonholonomic symmetry of a Lagrangian system and generalization of Noether theorem formulated by Olga Rossi [18] within the framework of her geometrical theory. An interesting example of the projectile motion controlled by the constant speed constraint was discussed and completely solved in [38].

In the present paper we derive general equations of constraint Noether-type symmetries for a Lagrangian first order mechanical system subjected to a quite general nonholonomic constraint and the expressions for corresponding currents, i.e. quantities conserved along trajectories. It should be emphasized that the constraint symmetries of a Lagrangian in the generalized Noether theorem need not be symmetries of the constraint equations of motion. So they play similar role as "pseudosymmetries" in nonconservative mechanics (see [4], [33], [36]). More generally, in [36] the solution of the problem of symmetries is based on the idea of generating first integrals through so called *adjoint symmetries* (a dual concept of pseudosymmetries). We focus to Noether-type symmetries defined as vector fields leaving invariant (up to a constraint form) the constraint Lepage equivalent of a Lagrangian. We illustrate the results on an example interesting from the physical point of view: the Chaplygin sleigh. It appears that the solution of the problem is technically not so simple. We present the solutions of reduced equations of the sleigh including graphical outputs, as well as conservation laws and corresponding symmetries. Moreover, we find the (non-variational) Chetaev constraint forces explicitly and emphasize the physical interpretation of the results. A brief overview (following the page restriction requirements) has been submitted for publication in the proceedings of the VIII-th International Conference Differential Geometry and Dynamical Systems (DGDS) 2014 where the results were reported, see [5].

2 Elements of the geometrical theory of nonholonomic mechanics

In this section we summarize elements of the geometrical theory of first order nonholonomic mechanical systems arising from initially Lagrangian unconstrained ones.

2.1 Underlying structures and notations

The geometrical theory of nonholonomic mechanical systems is developed on an $(m+1)$-dimensional underlying fibred manifold $(Y,\, \pi,\, X)$ with the total space Y, the one-dimensional base X and the projection (surjective submersion) π. The dimension of fibres m represents the number of degrees of freedom of an unconstrained system. We use the standard notation for jet prolongations of this manifold, $(J^rY,\, \pi_r,\, X)$, $r = 0,\, 1,\, 2$, $Y = J^0Y$, $\pi = \pi_0$ and for fibred manifolds $(J^rY,\, \pi_{r,s},\, J^sY)$, $s = 0,\, 1$. We denote as $(V,\, \psi)$ a fibred chart on Y, where $V \subset Y$ is an open set, $\psi = (t,\, q^\sigma)$, $1 \le \sigma \le m$. Then $(U,\, \varphi)$, $U = \pi(V)$, $\varphi = (t)$, is the associated chart on X, and $(V_r,\, \psi_r)$, $V_r = \pi_{r,0}^{-1}(V)$, $\psi_1 = (t,\, q^\sigma,\, \dot{q}^\sigma)$, $\psi_2 = (t,\, q^\sigma,\, \dot{q}^\sigma,\, \ddot{q}^\sigma)$, are the associated fibred charts on J^1Y and J^2Y, respectively. Let $U \subset X$ be an open set. A section $\delta\colon U \ni t \to \delta(t) \in J^rY$, $r = 1,\, 2$, is called *holonomic* if there exists a section $\gamma\colon U \ni t \to \gamma(t) \in Y$ such that $\delta = J^r\gamma$.

We also use the standard concept of a vector field on Y and its prolongations connected with the fibred structure. The standard concept of differential forms is used as well. A vector field ξ on J^rY is called π_r*-projectable* if there exists a vector field ξ_0 on X such that $T\pi_r\, \xi = \xi_0 \circ \pi_r$. A vector field ξ is called π_r*-vertical* if $T\pi_r\, \xi = 0$. A vector field ξ on J^rY is called $\pi_{r,s}$*-projectable* if there exists a vector field ζ on J^sY such that $T\pi_{r,s}\, \xi = \zeta \circ \pi_{r,s}$. A vector field on J^rY is called $\pi_{r,s}$*-vertical* if $T\pi_{r,s}\, \xi = 0$. The chart expressions of the above mentioned vector fields are (for $r = 0,\, 1,\, 2$, $s = 0,\, 1$, $s < r$)

$$\xi = \xi^0(t)\, \frac{\partial}{\partial t} + \sum_{j=0}^{r} \xi^\sigma_{(j)}(t, q^\nu, \dots, q^\nu_r)\, \frac{\partial}{\partial q^\sigma_j}\,,$$

with $\xi^0 = 0$ for a π_r-vertical vector field, and

$$\xi = \xi^0(t, q^\nu, \dots, q^\nu_s)\frac{\partial}{\partial t} + \sum_{j=0}^{s} \xi^\sigma_j(t, q^\nu, \dots, q^\nu_s)\, \frac{\partial}{\partial q^\sigma_j} + \sum_{j=s+1}^{r} \xi^\sigma_j(t, q^\nu, \dots, q^\nu_r)\, \frac{\partial}{\partial q^\sigma_j}\,,$$

with $\xi^0 = 0$ and $\xi^\sigma_j = 0$, $j = 0,\, \dots,\, s$, for a $\pi_{r,s}$-vertical vector field. In the preceding expressions we denoted $q^\sigma = q^\sigma_0$, $\dot{q}^\sigma = q^\sigma_1$, $\ddot{q}^\sigma = q^\sigma_2$.

A differential q-form η on J^rY is called π_r*-horizontal* if $i_\xi\eta = 0$ for every π_r-vertical vector field ξ on J^rY. A q-form η on J^rY is called $\pi_{r,s}$*-horizontal* if

$i_\xi \eta = 0$ for every $\pi_{r,s}$-vertical vector field ξ on $J^r Y$. π_r-horizontal 1-forms have a chart expression

$$\eta = \eta_0(t, q^\sigma, \dots, q^\sigma_r)\,\mathrm{d}t\,.$$

Every π-projectable vector field $\xi = \xi^0(t)\frac{\partial}{\partial t} + \xi^\sigma(t,\, q^\nu)\frac{\partial}{\partial q^\sigma}$ on Y can be prolonged on $J^r Y$, $r = 1,\, 2$,

$$J^1\xi = \xi^0\frac{\partial}{\partial t} + \xi^\sigma\frac{\partial}{\partial q^\sigma} + \tilde{\xi}^\sigma\frac{\partial}{\partial \dot{q}^\sigma}\,, \quad \text{or} \quad J^2\xi = \xi^0\frac{\partial}{\partial t} + \xi^\sigma\frac{\partial}{\partial q^\sigma} + \tilde{\xi}^\sigma\frac{\partial}{\partial \dot{q}^\sigma} + \hat{\xi}^\sigma\frac{\partial}{\partial \ddot{q}^\sigma}\,,$$

where $\tilde{\xi}^\sigma = \frac{\mathrm{d}\xi^\sigma}{\mathrm{d}t} - \dot{q}^\sigma\frac{\mathrm{d}\xi^0}{\mathrm{d}t}$, and $\hat{\xi}^\sigma = \frac{\mathrm{d}\tilde{\xi}^\sigma}{\mathrm{d}t} - \ddot{q}^\sigma\frac{\mathrm{d}\xi^0}{\mathrm{d}t}$. A q-form η on $J^r Y$ is called *contact* if $J^r\gamma^*\eta = 0$ for every section γ of π. Contact forms on $J^r Y$ form a differential ideal $\mathcal{I}_C$ called the *contact ideal*. For expressing differential forms in coordinates we use the basis of 1-forms adapted to the contact structure, $(t,\, \omega^\sigma,\, \mathrm{d}\dot{q}^\sigma)$ and $(t,\, \omega^\sigma,\, \dot{\omega}^\sigma,\, \mathrm{d}\ddot{q}^\sigma)$ on $J^1 Y$ and $J^2 Y$, respectively, where $\omega^\sigma = \mathrm{d}q^\sigma - \dot{q}^\sigma\,\mathrm{d}t$, $\dot{\omega}^\sigma = \mathrm{d}\dot{q}^\sigma - \ddot{q}^\sigma\,\mathrm{d}t$. There exists a unique decomposition of a q-form η on $J^r Y$ into its $(q-1)$-*contact* and q-*contact component* $\pi^*_{r+1,r}\eta = p_{q-1}\eta + p_q\eta$. The chart expression of $p_{q-1}\eta$ in the basis adapted to the contact structure is a linear combination of terms with just $(q-1)$ factors of the type ω^σ or $\dot{\omega}^\sigma$ and the chart expression of $p_q\eta$ is a linear combination of terms with just q such factors. (The only contact form on Y is the trivial (zero) one.) Notice that jet prolongations of π-projectable vector fields are closely related to the contact ideal being its symmetries: $\partial_{J^r\xi}\omega \in \mathcal{I}_C$ for every $\omega \in \mathcal{I}_C$. Here $\partial_{J^r\xi}$ denotes the Lie derivative along a vector field $J^r\xi$.

A *distribution* on $J^r Y$ is a mapping $\mathcal{D}\colon J^r Y \ni x \to \mathcal{D}(x) \subset T_x J^r Y$, where $\mathcal{D}(x)$ is a vector subspace of $T_x J^r Y$. A distribution is generated by local vector fields ξ_ι on $J^r Y$, $\iota \in \mathcal{I}$, where $\mathcal{I}$ is a set of indices. Equivalently, the distribution $\mathcal{D}$ can be annihilated by 1-forms η on $J^r Y$ such that $i_\xi\eta = 0$ for every vector field ξ belonging to the distribution $\mathcal{D}$.

2.2 Unconstrained systems

The geometrical theory of nonholonomic systems, as introduced in [14], is universal in the following sense: It concerns all types of nonholonomic mechanical systems given by equations of motion of the initial unconstrained system and the nonholonomic constraint, independently whether the equations of motion of the initial system are variational (Lagrangian) or not. In this paper we concentrate on the first of both situations because the concept of nonholonomic symmetries is formulated for constrained Lagrangians, not for equations.

Let λ be a first order Lagrangian, i.e. a horizontal form on $J^1 Y$, $\lambda = L(t, q^\sigma, \dot{q}^\sigma)\,\mathrm{d}t$. The pair $(\pi,\, \lambda)$ represents a *Lagrange structure*. The first order Lagrangean mechanics studies a.e. *extremals* of the Lagrange structure, i.e. sections γ of π representing critical sections γ of the variational integral (action function)

$$S_\Omega\colon\ \Gamma(\pi) \ni \gamma \to S_\Omega[\gamma] = \int_\Omega J^1\gamma^*\,\lambda$$

where $\Gamma(\pi)$ is a set of all sections of the projection π defined on open subsets of the base X, and Ω is a compact set included in the domain of γ. Critical sections of S are zero points of the variational derivative of S, i.e. integral

$$\left.\frac{\mathrm{d}S[\gamma_u]}{\mathrm{d}u}\right|_{u=0} = \int_\Omega J^1\gamma^*\,\partial_{J^1\xi}\lambda\,,$$

where ξ is a π-projectable vector field called the *variation* and $\{\gamma_u\}$, $u \in (-\varepsilon, \varepsilon)$, is a one-parameter system of sections generated by ξ such that $\gamma_0 = \gamma$, i.e. $\gamma_u = \phi_u \circ \gamma \circ \phi_{0u}^{-1}$, where (ϕ_u, ϕ_{0u}) is the one-parameter group of the vector field ξ. The variational derivative of the variational integral leads to the *first variation formula*

$$\int_\Omega J^1\gamma^*\partial_{J^1\xi}\lambda = \int_\Omega J^1\gamma^* i_{J^1\xi}\,\mathrm{d}\theta_\lambda + \int_{\partial\Omega} J^1\gamma^* i_{J^1\xi}\theta_\lambda\,, \tag{1}$$

where $\theta_\lambda = L\,\mathrm{d}t + \frac{\partial L}{\partial \dot q^\sigma}\,\omega^\sigma$ is the Lepage equivalent of the Lagrangian (the Poincaré--Cartan form). The condition for an extremal leads to *Euler-Lagrange equations*—equations of motion of the system. The coordinate free expression of these equations reads $J^1\gamma^*\,i_{J^1\xi}\,\mathrm{d}\theta_\lambda = 0$ or $J^2\gamma^* E_\lambda = 0$, where in coordinates

$$E_\lambda = E_\sigma\,\omega^\sigma \wedge \mathrm{d}t\,, \qquad E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q^\sigma}\,, \tag{2}$$

or equivalently

$$\begin{gathered} E_\sigma \circ J^2\gamma = (A_\sigma + B_{\sigma\nu}\ddot q^\nu) \circ J^2\gamma = 0\,, \\ A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}'}{\mathrm{d}t}\frac{\partial L}{\partial \dot q}\,, \qquad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot q^\sigma \partial \dot q^\nu}\,. \end{gathered} \tag{3}$$

Here

$$\frac{\mathrm{d}'}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} - \ddot q^\sigma \frac{\partial}{\partial \dot q^\sigma} = \frac{\partial}{\partial t} + \dot q \frac{\partial}{\partial q^\sigma}\,.$$

A π-projectable vector field ξ on Y is called a *symmetry* of the Lagrange structure (π, λ) if it holds $\partial_{J^1\xi}\lambda = 0$. This condition is the *Noether equation*. For a given Lagrangian it is interpreted as a set of equations for symmetries, for a given vector field ξ it represents a functional equation for Lagrangians having the symmetry ξ. (For our purposes the first of both interpretations will be relevant.) The chart expression of the Noether equation is

$$\frac{\partial L}{\partial t}\xi^0 + \frac{\partial L}{\partial q^\sigma}\xi^\sigma + \frac{\partial L}{\partial \dot q^\sigma}\left(\frac{\mathrm{d}\xi^\sigma}{\mathrm{d}t} - \dot q^\sigma \frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right) + L\frac{\mathrm{d}\xi^0}{\mathrm{d}t} = 0\,. \tag{4}$$

Taking into account the first variation formula we can see that if ξ is a symmetry of the Lagrange structure then the quantity

$$i_{J^1\xi}\theta_\lambda = \left(L - \dot q^\sigma \frac{\partial L}{\partial \dot q^\sigma}\right)\xi^0 + \frac{\partial L}{\partial \dot q^\sigma}\xi^\sigma \tag{5}$$

(called the *current*) is constant along extremals. This result representing conservation laws is well known as the *Emmy Noether theorem*.

2.3 Nonholonomic dynamics

Suppose that an unconstrained first order Lagrangian mechanical system is subjected to a nonholonomic constraint given by k equations, $1 \leq k \leq m-1$,

$$f^a(t,\, q^\sigma,\, \dot{q}^\sigma) = 0\,, \quad 1 \leq a \leq k\,, \quad \text{where} \quad \operatorname{rank}\left(\frac{\partial f^a}{\partial \dot{q}^\sigma}\right) = k\,,$$

or in a normal form

$$\dot{q}^{m-k+a} = g^a(t,\, q^\sigma,\, \dot{q}^l)\,, \quad 1 \leq l \leq m-k\,.$$

These equations define a *constraint submanifold* $Q \subset J^1Y$ of codimension k fibred over Y (and, of course, over X as well). The corresponding projections $\bar{\pi}_{1,0}$ and $\bar{\pi}_1$ are the mappings $\pi_{1,0}$ and π_1 restricted to Q, respectively. Denote

$$\iota\colon\ Q \ni (t,\, q^\sigma,\, \dot{q}^l) \longrightarrow (t,\, q^\sigma,\, \dot{q}^l,\, g^a(t,\, q^\nu,\, \dot{q}^s)) \in J^1Y$$

the canonical embedding of Q into J^1Y. On the submanifold Q there arise the *induced contact ideal* $\tilde{\mathcal{I}}_C$ generated by forms $\bar{\omega}^\sigma = \iota^*\omega^\sigma$ and the *canonical distribution*

$$\mathcal{C} = \{\operatorname{span}\varphi^a \mid 1 \leq a \leq k\}\,, \qquad \varphi^a = \iota^*\omega^{m-k+a} - \frac{\partial g^a}{\partial \dot{q}^l}\iota^*\,\omega^l. \tag{6}$$

The $\bar{\pi}_1$-projectable vector fields belonging to the canonical distribution are called *Chetaev vector fields*. They represent *admissible variations* in the nonholonomic variational principle (first introduced in [19]). Let us briefly recall this principle and its consequences. Let $(\pi,\, \lambda)$ be an unconstrained Lagrangian structure and θ_λ the corresponding Poincaré-Cartan form. By the *constraint system* on Q defined by λ we mean the differential form $\iota^*\theta_\lambda$. Denote $\bar{\lambda} = \iota^*\lambda = (L \circ \iota)\,\mathrm{d}t$ and $\theta_{\bar{\lambda}} = \theta_{\iota^*\lambda}$. Calculating $\iota^*\theta_\lambda$ we obtain

$$\iota^*\theta_\lambda = \bar{L}\,\mathrm{d}t + \frac{\partial \bar{L}}{\partial \dot{q}^l}\bar{\omega}^l + \bar{L}_a\varphi^a = \theta_{\iota^*\lambda} + \bar{L}_a\varphi^a\,,$$
$$\bar{L} = L \circ \iota\,, \qquad \bar{L}_a = \frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota\,.$$

Let δ be a section of the projection $\bar{\pi}_1\colon\ Q \to X$ defined on an open subset $U \subset X$ containing a compact set $\Omega \subset X$. Let $Z \in \mathcal{C}$ be a $\bar{\pi}_1$-projectable vector field and let $(\phi_u,\, \phi_{0u})$ its one-parameter group and $\{\delta_u\} = \{\phi_u \circ \delta \circ \phi_{0u}^{-1}\}$, $\delta_0 = \delta$, the one-parameter family of sections generated by Z. The constraint variational integral and its variational derivative are

$$S_\Omega[\delta] = \int\limits_\Omega \delta^*\iota^*\theta_\lambda, \quad \left.\frac{\mathrm{d}S[\delta_u]}{\mathrm{d}u}\right|_{u=0} = \int\limits_\Omega \delta^*\partial_Z\iota^*\theta_\lambda\,.$$

If we restrict to holonomic sections we obtain the variational derivative of the variational integral in the form

$$\left.\frac{\mathrm{d}S[\gamma_u]}{\mathrm{d}u}\right|_{u=0} = \int\limits_\Omega J^1\gamma^*\partial_Z\iota^*\theta_\lambda\,.$$

Nonholonomic first variation formula reads (taking into account that $i_Z(\bar{L}^a\varphi^a)=0$, because $Z\in\mathcal{C}$)

$$\int\limits_{\Omega} J^1\gamma^*\,\partial_Z\iota^*\theta_\lambda=\int\limits_{\Omega} J^1\gamma^* i_Z\,\mathrm{d}\iota^*\theta_\lambda+\int\limits_{\partial\Omega} J^1\gamma^* i_Z\theta_{\iota^*\lambda}\,. \tag{7}$$

By a direct calculation we can justify that the integrand in the first integral on the right-hand side of (7) depends only on components of Z on Y. The requirement of vanishing of this integral (for arbitrary Ω) leads to equations of motion

$$J^1\gamma^* i_Z\,\mathrm{d}\iota^*\theta_\lambda=0 \implies \big(\varepsilon_s(\bar{L})-\bar{L}_a\varepsilon_s(g^a)\big)\circ J^2\gamma=0\,, \tag{8}$$

for $1\le s\le m-k$. In the expressions of the type

$$\varepsilon_s(f)=\frac{\partial_c f}{\partial q^s}-\frac{\mathrm{d}_c}{\mathrm{d}t}\frac{\partial f}{\partial\dot{q}^s}\,,\quad \text{where}\quad f=f(t,\,q^\sigma,\,\dot{q}^l)\,,$$

the constraint derivative operators are used

$$\frac{\partial_c}{\partial q^s}=\frac{\partial}{\partial q^s}+\frac{\partial g^a}{\partial\dot{q}^s}\frac{\partial}{\partial q^{m-k+a}}\,,$$

$$\frac{\mathrm{d}_c}{\mathrm{d}t}=\frac{\partial}{\partial t}+\dot{q}^l\frac{\partial}{\partial q^l}+g^a\frac{\partial}{\partial q^{m-k+a}}+\ddot{q}^l\frac{\partial}{\partial\dot{q}^l}=\frac{\mathrm{d}'_c}{\mathrm{d}t}+\ddot{q}^l\frac{\partial}{\partial\dot{q}^l}\,.$$

Note that these operators have an important geometrical meaning: Vector fields

$$\frac{\partial_c}{\partial t}=\frac{\mathrm{d}'_c}{\mathrm{d}t}-\dot{q}^l\frac{\partial_c}{\partial q^l}\,,\quad \frac{\partial_c}{\partial q^l},\quad \frac{\partial}{\partial\dot{q}^l},\quad 1\le l\le m-k\,,$$

generate the canonical distribution $\mathcal{C}$. The equations (8) can be written as follows

$$\bar{A}_s+\bar{B}_{sr}\ddot{q}^r=0\,,\quad 1\le s\le m-k\,, \tag{9}$$

$$\bar{A}_s=\frac{\partial_c\bar{L}}{\partial q^s}-\frac{\mathrm{d}'_c}{\mathrm{d}t}\frac{\partial\bar{L}}{\partial\dot{q}^s}-\bar{L}_a\left(\frac{\partial_c g^a}{\partial q^s}-\frac{\mathrm{d}'_c}{\mathrm{d}t}\frac{\partial g^a}{\partial\dot{q}^s}\right),\quad \bar{B}_{sr}=-\frac{\partial^2\bar{L}}{\partial\dot{q}^s\partial\dot{q}^r}+\bar{L}_a\frac{\partial^2 g^a}{\partial\dot{q}^s\partial\dot{q}^r}\,,$$

or, via functions A_σ and $B_{\sigma\nu}$ (3),

$$\begin{aligned}\bar{A}_s=\Bigg[&A_s+\sum_{a=1}^{k}A_{m-k+a}\frac{\partial g^a}{\partial\dot{q}^s}\\&+\sum_{a=1}^{k}\left(B_{s,m-k+a}+\sum_{b=1}^{k}B_{m-k+b,m-k+a}\frac{\partial g^b}{\partial\dot{q}^s}\right)\left(\frac{\partial g^a}{\partial t}+\frac{\partial g^a}{\partial q^\sigma}\dot{q}^\sigma\right)\Bigg]\circ\iota\end{aligned}$$

$$\begin{aligned}\bar{B}_{sr}=\Bigg[&B_{sr}+\sum_{a=1}^{k}\left(B_{s,m-k+a}\frac{\partial g^a}{\partial\dot{q}^r}+B_{m-k+a,r}\frac{\partial g^a}{\partial\dot{q}^s}\right)\\&+\sum_{a,b=1}^{k}B_{m-k+b,m-k+a}\frac{\partial g^b}{\partial\dot{q}^s}\frac{\partial g^a}{\partial\dot{q}^r}\Bigg]\circ\iota\,.\end{aligned}$$

The last relations are universal in the following sense: They hold for both types of equations of motion of an initial unconstrained mechanical system, i.e. variational as well as non-variational ones.

We obtained $m-k$ *reduced equations* of a nonholonomic system. These equations together with k equations of the constraint form a complete set of equations of motion of the system for its trajectories $\gamma\colon t \to \gamma(t) = (t,\, q^\sigma\gamma(t)) \in Y$, $1 \le \sigma \le m$.

2.4 Chetaev equations

In the framework of the geometrical theory of nonholonomic systems the well known Chetaev equations of motion can be derived. We present them for completeness. These equations are obtained by introducing the *Chetaev constraint force* into equations of motion. Suppose that $A_\sigma + B_{\sigma\nu}\ddot{q}^\nu = 0$, $1 \le \sigma,\, \nu \le m$, are equations of motion of an unconstrained system. The Chetaev force is defined as the form $\phi = \mu^a \frac{\partial f^a}{\partial \dot{q}^\sigma}\, \omega^\sigma \wedge \mathrm{d}t$. The coefficients μ^a, $1 \le a \le k$, on J^1Y are *Lagrange multipliers*. The Chetaev equations read

$$\left(A_\sigma + B_{\sigma\nu}\ddot{q}^\nu - \mu^a \frac{\partial f^a}{\partial \dot{q}^\sigma} \right) \circ J^2\gamma = 0\,. \tag{10}$$

Together with the equations of the constraint $f^a = 0$, $1 \le a \le k$, we obtain $m+k$ equations for trajectories and Lagrange multipliers. Knowing the Lagrange multipliers we can determine the constraint force ϕ which is important for interpretation of results from the point of view of physics.

3 Nonholonomic constraint symmetries

In this section we present the definition of a (nonholonomic) constraint symmetry and derive general equations for symmetries of a constrained mechanical system arising from an initially unconstrained first order Lagrangian structure.

3.1 The concept of constraint symmetries

Let Z be a Chetaev vector field, i.e. $Z \in \mathcal{C}$. The chart expression of Z is

$$Z = Z^0 \frac{\partial}{\partial t} + Z^l \frac{\partial}{\partial q^l} + Z^{m-k+a} \frac{\partial}{\partial q^{m-k+a}} + \tilde{Z}^l \frac{\partial}{\partial \dot{q}^l}\,,$$

$$Z^{m-k+a} = Z^0 g^a + (Z^s - \dot{q}^s Z^0) \frac{\partial g^a}{\partial \dot{q}^s}\,. \tag{11}$$

The condition for components Z^{m-k+a} follows from the assumption that Z belongs to the canonical distribution, i.e. $i_Z \varphi^a = 0$ for $1 \le a \le k$. We say that Z is a *constraint symmetry* of the nonholonomic mechanical system arising from a primarily unconstrained Lagrangean structure $(\pi,\, \lambda)$ subjected to nonholonomic constraints $\dot{q}^{m-k+a} = g^a(t,\, q^\sigma,\, \dot{q}^l)$ if the constrained system $\iota^*\theta_\lambda$ on Q defined by λ remains invariant under transformations given by the one-parameter group of the vector field Z up to a constraint form. This means that

$$\partial_Z \iota^*\theta_\lambda = i_Z\, \mathrm{d}\iota^*\theta_\lambda + \mathrm{d} i_Z \iota^*\theta_\lambda = F_a \varphi^a, \tag{12}$$

where F_a are some functions on Q. Relation (12) represents the *constraint Noether equation*. From the nonholonomic variation formula (7) we can see that if Z is a constraint symmetry of a nonholonomic mechanical system and γ is a solution of the corresponding reduced equations together with constraints, then $\mathrm{d}J^1\gamma^* i_Z \iota^* \theta_\lambda = 0$, i.e. $(i_Z \iota^* \theta_\lambda) \circ J^1\gamma = \text{const}$. This means that the quantities $\Phi = i_Z \iota^* \theta_\lambda$ are constant along solutions. We obtain

$$\Phi = \left(\bar{L} - \dot{q}^l \frac{\partial \bar{L}}{\partial \dot{q}^l} \right) Z^0 + \frac{\partial \bar{L}}{\partial \dot{q}^l} Z^l. \tag{13}$$

The quantities Φ are called *Noether-type currents* and the conditions $\Phi = \text{const.}$ are the corresponding *conservation laws*.

3.2 Equations for constraint symmetries

Using the definition of constraint symmetries and relations (9) we obtain after some tedious calculations the following set of partial differential equations for $(2(m-k)+1)$ components of these symmetries:

$$\begin{aligned} Z^0 \left[\frac{\mathrm{d}'_c \bar{L}}{\mathrm{d}t} - \left(\frac{\partial_c \bar{L}}{\partial q^l} - \bar{L}_a \varepsilon'_l(g^a) \right) \dot{q}^l \right] &+ Z^l \left(\frac{\partial_c \bar{L}}{\partial q^l} - \bar{L}_a \varepsilon'_l(g^a) \right) \\ &+ \frac{\mathrm{d}'_c Z^0}{\mathrm{d}t} \left(\bar{L} - \frac{\partial \bar{L}}{\partial \dot{q}^l} \dot{q}^l \right) + \frac{\mathrm{d}'_c Z^l}{\mathrm{d}t} \frac{\partial \bar{L}}{\partial \dot{q}^l} = 0 \,, \end{aligned} \tag{14}$$

$$\begin{aligned} Z^0 \Bigg[& \frac{\mathrm{d}'_c}{\mathrm{d}t} \left(\frac{\partial \bar{L}}{\partial \dot{q}^l} \right) - \frac{\partial_c}{\partial q^s} \left(\frac{\partial \bar{L}}{\partial \dot{q}^l} \right) \dot{q}^s + \bar{L}_a \varepsilon'_l(g^a) \\ & - \bar{L}_a \dot{q}^s \left(\frac{\partial_c}{\partial q^l} \right) \left(\frac{\partial g^a}{\partial \dot{q}^s} - \frac{\partial_c}{\partial q^s} \left(\frac{\partial g^a}{\partial \dot{q}^l} \right) \right) \Bigg] \\ & + Z^s \left[\frac{\partial_c}{\partial q^s} \left(\frac{\partial \bar{L}}{\partial \dot{q}^l} \right) + \bar{L}_a \left(\frac{\partial_c}{\partial q^l} \left(\frac{\partial g^a}{\partial \dot{q}^s} \right) - \frac{\partial_c}{\partial q^s} \left(\frac{\partial g^a}{\partial \dot{q}^l} \right) \right) \right] \\ & + \tilde{Z}^s \left(\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} - \bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s} \right) + \frac{\partial_c Z^0}{\partial q^l} \left(\bar{L} - \frac{\partial \bar{L}}{\partial \dot{q}^s} \dot{q}^s \right) + \frac{\partial_c Z^s}{\partial q^l} \frac{\partial \bar{L}}{\partial \dot{q}^s} = 0 \,, \end{aligned} \tag{15}$$

$$\bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s} (Z^s - \dot{q}^s Z^0) + \left(\bar{L} - \frac{\partial \bar{L}}{\partial \dot{q}^s} \dot{q}^s \right) \frac{\partial Z^0}{\partial \dot{q}^l} + \frac{\partial \bar{L}}{\partial \dot{q}^s} \frac{\partial Z^s}{\partial \dot{q}^l} = 0 \,, \tag{16}$$

for $1 \le l \le m-k$. The following expression represents the coefficients F_a of the constraint form $F_a \varphi^a$ (we present them for completeness):

$$\begin{aligned} F_a = i_Z \, \mathrm{d}\bar{L}_a + \bar{L}_b & \left(\frac{\partial g^b}{\partial q^{m-k+a}} Z^0 + \frac{\partial^2 g^b}{\partial q^{m-k+a} \partial \dot{q}^s} (Z^s - \dot{q}^s Z^0) \right) \\ & + \left(\bar{L} - \frac{\partial \bar{L}}{\partial \dot{q}^s} \dot{q}^s \right) \frac{\partial Z^0}{\partial q^{m-k+a}} + \frac{\partial \bar{L}}{\partial \dot{q}^s} \frac{\partial Z^s}{\partial q^{m-k+a}} \,. \end{aligned}$$

For a special but in practical situations frequent case of a semiholonomic constraint (linear constraint with $\varepsilon'_l(g^a) = 0$, $1 \le l \le m-k$, $1 \le a \le k$) the equations for symmetries take a simplified form

$$
\begin{aligned}
&\frac{\mathrm{d}'_c \bar{L}}{\mathrm{d}t} Z^0 + \frac{\partial_c \bar{L}}{\partial q^l}(Z^l - \dot{q}^l Z^0) + \left(\bar{L} - \dot{q}^l \frac{\partial \bar{L}}{\partial \dot{q}^l}\right) \frac{\mathrm{d}'_c Z^0}{\mathrm{d}t} + \frac{\partial \bar{L}}{\partial \dot{q}^l} \frac{\mathrm{d}'_c Z^l}{\mathrm{d}t} = 0\,, \\
&\frac{\mathrm{d}'}{\mathrm{d}t}\left(\frac{\partial \bar{L}}{\partial \dot{q}^l}\right) Z^0 + \frac{\partial_c}{\partial q^s}\left(\frac{\partial \bar{L}}{\partial \dot{q}^l}\right)(Z^s - \dot{q}^s Z^0) \\
&\qquad + \tilde{Z}^s \frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} + \frac{\partial \bar{L}}{\partial \dot{q}^s} \frac{\partial_c Z^s}{\partial q^l} + \frac{\partial_c Z^0}{\partial q^l}\left(\bar{L} - \dot{q}^s \frac{\partial \bar{L}}{\partial \dot{q}^s}\right) = 0\,, \\
&\qquad \left(\bar{L} - \dot{q}^s \frac{\partial \bar{L}}{\partial \dot{q}^s}\right) \frac{\partial Z^0}{\partial \dot{q}^l} + \frac{\partial \bar{L}}{\partial \dot{q}^s} \frac{\partial Z^s}{\partial \dot{q}^l} = 0\,,
\end{aligned}
$$

$1 \leq l,\ s \leq m-k$. These relations are fully consistent with equations for symmetries of Poincaré-Cartan form of unconstrained systems, $\partial_{J^1\xi}\theta_\lambda = 0$, taking into account that for unconstrained systems ξ is a π-projectable vector field on Y, i.e. $\xi^0 = \xi^0(t)$, $\xi^\sigma = \xi^\sigma(t,\, q^\nu)$, $1 \leq \sigma,\ \nu \leq m$, and components $\tilde{\xi}^\sigma$ are uniquely given by ξ^0 and ξ^ν (see relations in Section 2.1). It is obvious that for a nonholonomic case the constraint differential operators are used instead of the usual ones.

Using the expressions for currents and for coefficients of reduced equations $\bar{A}_l$ and $\bar{B}_{ls}$ given by (9) we obtain a more suitable form of equations (14)–(16):

$$
\frac{\mathrm{d}'_c \Phi}{\mathrm{d}t} + \bar{A}_l (Z^l - \dot{q}^l Z^0) = 0\,, \tag{17}
$$

$$
\frac{\partial_c \Phi}{\partial q^l} - \bar{A}_l Z^0 + \left\{\frac{\partial \bar{A}_s}{\partial \dot{q}^l} + \frac{\partial \bar{L}_a}{\partial \dot{q}^l}\varepsilon'_s(g^a)\right\}_{\mathrm{alt}(l,s)} (Z^s - \dot{q}^s Z^0) - \bar{B}_{ls}\tilde{Z}^s = 0\,, \tag{18}
$$

$$
\frac{\partial \Phi}{\partial \dot{q}^l} + \bar{B}_{ls}(Z^s - \dot{q}^s Z^0) = 0\,, \tag{19}
$$

where $1 \leq l,\ s \leq m-k$. (The equations are expressed via currents, for clarity. Nevertheless, the constraint derivatives of the current Φ depend on symmetry components and their derivatives. There arises, of course, the problem of solution of these equations for concrete situations.)

Equations (17)–(19) enable us to obtain symmetries of the mechanical system via currents: For a regular matrix B denote $\mathcal{B} = \bar{B}^{-1}$. Multiplying the system of equations (19) by the matrix $\mathcal{B}$ we get

$$
Z^l - \dot{q}^l Z^0 = -\mathcal{B}^{ls}\frac{\partial \Phi}{\partial \dot{q}^s}\,.
$$

Putting the obtained expressions for $Z^l - \dot{q}^l Z^0$ into (13) we can express the component Z^0 explicitly. Putting the result into (17)–(19) we finally obtain explicit expressions for the components of symmetries:

$$
\begin{aligned}
Z^0 &= \frac{1}{\bar{L}}\left(\Phi + \mathcal{B}^{ls}\frac{\partial \bar{L}}{\partial \dot{q}^l}\frac{\partial \Phi}{\partial \dot{q}^s}\right), \\
Z^l &= \dot{q}^l Z^0 - \mathcal{B}^{ls}\frac{\partial \Phi}{\partial \dot{q}^s}, \\
\tilde{Z}^l &= \mathcal{B}^{ls}\left(\frac{\partial_c \Phi}{\partial q^s} - \bar{A}_s Z^0 + \left\{\frac{\partial \bar{A}_r}{\partial \dot{q}^s} + \frac{\partial \bar{L}_a}{\partial \dot{q}^s}\varepsilon'_r(g^a)\right\}_{\mathrm{alt}(r,s)}(Z^r - \dot{q}^r Z^0)\right).
\end{aligned} \tag{20}
$$

The problem of computing symmetries simplifies if we know the currents (constants of motion). This might happen during the process of solving the equations of motion. It is obvious that for vector fields obtained by such a way the verification of conditions (14)–(16) should be made. In particular we take advantage of this simplification in the example presented in Section 4.

3.3 Classification of constraint symmetries

There is a possibility to classify the constraint Noether-type symmetries in the context of constraint equations of motion. For a regular matrix $\bar{B}$ the equations of motion (9) can be written in the explicit form $\ddot{q}^l = -\mathcal{B}^{ls}\bar{A}_s$ (recall that $\mathcal{B} = \bar{B}^{-1}$). The holonomic paths of these equations in Q are integral sections of local vector field belonging to the canonical distribution $\mathcal{C}$ called *constraint semispray* (see [14])

$$
\Gamma = \frac{\partial}{\partial t} + \dot{q}^l\frac{\partial}{\partial q^l} + g^a\frac{\partial}{\partial q^{m-k+a}} + \tilde{\Gamma}^l\frac{\partial}{\partial \dot{q}^l}, \qquad \tilde{\Gamma}^l = -\mathcal{B}^{ls}\bar{A}_s. \tag{21}
$$

The constraint semispray Γ spans a distribution $\mathcal{D}_\Gamma$ of rank one called a *constraint connection*. Let Z be a vector field on Q. It is a *symmetry of equations of motion* of the corresponding nonholonomic mechanical system if $[\Gamma, Z] = f\Gamma$, where $[\Gamma, Z]$ is the Lie bracket of vector fields Γ and Z and $f = f(t, q^\sigma, \dot{q}^l)$ is a function on Q. Let Φ be a current, i.e. quantity conserved along trajectories of the nonholonomic system (not necessarily a Noether-type current). Then $\Gamma(\Phi) = \partial_\Gamma\Phi = 0$. If Z is a symmetry of equations of motion then $[\Gamma, Z](\Phi) = f\Gamma(\Phi) = 0$. On the other hand, $[\Gamma, Z](\Phi) = \partial_\Gamma\partial_Z\Phi - \partial_Z\partial_\Gamma\Phi = \partial_\Gamma(\partial_Z\Phi)$. This means that $\partial_Z\Phi$ is the current as well.

Let Z be a constraint symmetry of a nonholonomic system. Let us discuss possible relationship between distributions spanned by vector fields Γ, Z and $[\Gamma, Z]$. First of all let us answer the question whether and under what conditions a vector field belonging to the distribution $\mathcal{D}_\Gamma$ can be a constraint symmetry. Putting components of the vector field $f\Gamma$, $f = f(t, q^\sigma, \dot{q}^l)$ being a function on Q, into conditions (14)–(16) we obtain

$$
\frac{\mathrm{d}'_c\bar{L}}{\mathrm{d}t} = 0, \quad \frac{\partial_c\bar{L}}{\partial q^l} = 0, \quad \frac{\partial \bar{L}}{\partial \dot{q}^l} = 0, \quad 1 \le l \le m-k.
$$

Because of the relation

$$
\mathrm{d}F = \frac{\mathrm{d}'_c F}{\mathrm{d}t}\,\mathrm{d}t + \frac{\partial_c F}{\partial q^l}\,\omega^l + \frac{\partial F}{\partial \dot{q}^l}\,\mathrm{d}\dot{q}^l + \frac{\partial F}{\partial q^{m-k+a}}\,\varphi^a
$$

for every function $F = F(t, q^\sigma, \dot{q}^l)$ on Q this means that $\mathrm{d}\bar{L} \in \operatorname{annih} \mathcal{C}$ and $\bar{L}$ is constant along the distribution $\mathcal{D}_\Gamma$. In the following considerations we exclude this trivial situation.

Another question is whether and under what conditions the Lie bracket $[\Gamma, Z]$ belongs to the canonical distribution. For general vector fields $\xi, \zeta \in \mathcal{C}$ it holds $i_{[\xi,\zeta]}\varphi^a = -\mathrm{d}\varphi^a(\xi, \zeta)$. As $\mathrm{d}\varphi^a$ need not belong to the constraint ideal $\mathcal{I}_\mathcal{C}$, it is evident that $[\xi, \zeta]$ need not belong to $\mathcal{C}$. For $\mathrm{d}\varphi^a$ we obtain from (6)

$$\begin{aligned}\mathrm{d}\varphi^a = &- \varepsilon'_s(g^a)\, \bar{\omega}^s \wedge \mathrm{d}t + \frac{\partial_c}{\partial q^r}\left(\frac{\partial g^a}{\partial \dot{q}^s}\right) \bar{\omega}^s \wedge \bar{\omega}^r + \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s}\, \bar{\omega}^s \wedge \mathrm{d}\dot{q}^r \\ &- \frac{\partial g^a}{\partial q^{m-k+b}}\, \varphi^b \wedge \mathrm{d}t - \frac{\partial}{\partial q^{m-k+b}}\left(\frac{\partial g^a}{\partial \dot{q}^s}\right) \varphi^b \wedge \bar{\omega}^s .\end{aligned}$$

Calculating the Lie bracket $[\Gamma, Z]$ using relations (20) and (21) we obtain after some technical calculations

$$\begin{aligned} i_{[\Gamma,Z]}\, \varphi^a &= -\mathrm{d}\varphi^a(\Gamma, Z) \\ &= \mathcal{B}^{ls}\Bigg[\varepsilon'_l(g^a)\frac{\partial \Phi}{\partial \dot{q}^s} \\ &\qquad + \dot{q}^p \frac{\partial^2 g^a}{\partial \dot{q}^p \partial \dot{q}^l}\left(\frac{\partial_c \Phi}{\partial q^s} - \frac{\partial \Phi}{\partial \dot{q}^v}\mathcal{B}^{rv}\left\{\frac{\partial \bar{A}_r}{\partial \dot{q}^s} + \varepsilon'_r(g^a)\frac{\partial \bar{L}_a}{\partial \dot{q}^s}\right\}_{\mathrm{alt}(r,s)}\right)\Bigg], \end{aligned} \tag{22}$$

where Φ is the Noether-type current corresponding to the constraint symmetry Z. We can see that for a semiholonomic constraint this condition is fulfilled and thus $[\Gamma, Z] \in \mathcal{C}$. For a general linear constraint this conditions reduces to

$$i_{[\Gamma,Z]}\, \varphi^a = \mathcal{B}^{ls}\varepsilon'_l(g^a)\frac{\partial \Phi}{\partial \dot{q}^s} . \tag{23}$$

There can be, of course, special cases with a general constraint for which the condition is fulfilled too. We shall see various situations in the example presented in Section 4.

Now let us discuss the relation of the Lie bracket $[\Gamma, Z]$ with respect to distributions spanned by vector fields Γ and Z. Let Φ be again the Noether-type current corresponding to the constraint symmetry Z (not belonging to $\mathcal{D}_\Gamma$). Then $\partial_Z \Phi = 0$ and thus $\partial_{[\Gamma,Z]}\Phi = \partial_\Gamma \partial_Z \Phi - \partial_Z \partial_\Gamma \Phi = 0$. This means that the quantity Φ is conserved along the vector field $[\Gamma, Z]$. On the other hand, let ζ be a vector field belonging to the distribution $\mathcal{D}_{(\Gamma,\, Z)}$ spanned by vector fields Γ and Z, i.e. $\zeta = a\Gamma + bZ$, where $a = a(t, q^\sigma, \dot{q}^l)$ and $b = b(t, q^\sigma, \dot{q}^l)$ are functions on Q. Then $\zeta(\Phi) = \partial_\zeta \Phi = a\, \partial_\Gamma \Phi + b\, \partial_Z \Phi = 0$ and Φ is conserved along the distribution $\mathcal{D}_{(\Gamma,Z)}$. Moreover, because of the relation $[\Gamma, Z](\Phi) = 0$ it is conserved along the distribution $\mathcal{D}$ spanned by vector fields Γ, Z and $[\Gamma, Z]$. There are three possibilities for the relation of a symmetry Z to the vector field Γ:

1) Z is a symmetry of equations of motion, i.e. $[\Gamma, Z] = a\Gamma$, $a = a(t, q^\sigma, \dot{q}^l)$.

2) The Lie bracket of vector fields Γ and Z belongs to the distribution spanned by these vector fields, i.e. $[\Gamma, Z] = a\Gamma + bZ$, where $a = a(t, q^\sigma, \dot{q}^l)$ and $b = b(t, q^\sigma, \dot{q}^l)$ are functions on Q.

3) There is no specific relation of the symmetry Z to the vector field Γ.

In the cases 1) and 2) the the distribution spanned by vector fields Γ, Z and $[\Gamma, Z]$ has the rank two, in the case 3) its rank is three. (Recall that this distribution need not be a subdistribution of the canonical distribution $\mathcal{C}$, because $[\Gamma, Z]$ need not belong to $\mathcal{C}$.) We shall derive the conditions under which situations 1) take place. After some tedious technical calculations we obtain components of the vector field $\Xi = [\Gamma, Z]$ for a vector field Z belonging to the canonical distribution (i.e. relations (11) are considered). It holds

$$\Xi = \Xi^0 \frac{\partial}{\partial t} + \Xi^l \frac{\partial}{\partial q^l} + \Xi^{m-k+a} \frac{\partial}{\partial q^{m-k+a}} + \widetilde{\Xi}^l \frac{\partial}{\partial \dot{q}^l} \,, \tag{24}$$

$$\Xi^0 = -\frac{\mathrm{d}'_c Z^0}{\mathrm{d}t} + \mathcal{B}^{sr} \bar{A}_r \frac{\partial Z^0}{\partial \dot{q}^s} \,,$$

$$\Xi^l = -\frac{\mathrm{d}'_c Z^l}{\mathrm{d}t} + \mathcal{B}^{sr} \bar{A}_r \frac{\partial Z^l}{\partial \dot{q}^s} + \widetilde{Z}^l,$$

$$\Xi^{m-k+a} = \Xi^0 g^a + (\Xi^l - \dot{q}^l \Xi^0) \frac{\partial g^a}{\partial \dot{q}^l} + \left[(Z^l - \dot{q}^l Z^0) \varepsilon'_l(g^a) - Z^0 \mathcal{B}^{sr} \bar{A}_r \dot{q}^l \frac{\partial g^a}{\partial \dot{q}^l \partial \dot{q}^s} \right] ,$$

$$\widetilde{\Xi}^l = Z^0 \frac{\mathrm{d}'_c}{\mathrm{d}t}(-\mathcal{B}^{lr} \bar{A}_r) + (Z^s - \dot{q}^s Z^0) \frac{\partial_c}{\partial q^s}(-\mathcal{B}^{lr} \bar{A}_r)$$
$$+ \widetilde{Z}^s \frac{\partial}{\partial \dot{q}^s}(-\mathcal{B}^{lr} \bar{A}_r) - \frac{\mathrm{d}'_c \widetilde{Z}^l}{\mathrm{d}t} + \frac{\partial \widetilde{Z}^l}{\partial \dot{q}^s} \mathcal{B}^{sr} \bar{A}_r \,.$$

The requirement $[\Gamma,\, Z] = a\Gamma$ (in such a case the constraint symmetry Z is a symmetry of equations of motion as well) means that there exists a function $a = a(t,\, q^\sigma,\, \dot{q}^l)$ on the constraint submanifold Q such that $\Xi^0 = a$, $\Xi^l = a\dot{q}^l$, $\Xi^{m-k+a} = ag^a$, $\widetilde{\Xi}^l = -\mathcal{B}^{ls} \bar{A}_s$, $1 \leq l,\, s \leq m-k$, $1 \leq a \leq k$. This leads to conditions

$$\left(\frac{\mathrm{d}'_c}{\mathrm{d}t} - \mathcal{B}^{sr} \bar{A}_r \frac{\partial}{\partial \dot{q}^s} \right) (Z^l - \dot{q}^l Z^0) - \mathcal{B}^{lr} \bar{A}_r Z^0 - \tilde{Z}^l = 0 \,, \tag{25}$$

$$(Z^l - \dot{q}^l Z^0) \varepsilon'_l(g^a) - \dot{q}^l Z^0 \mathcal{B}^{sr} \bar{A}_r \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s} = 0 \,, \tag{26}$$

$$-\frac{\mathrm{d}'_c}{\mathrm{d}t}(\mathcal{B}^{lr} \bar{A}_r Z^0) - (Z^s - \dot{q}^s Z^0) \frac{\partial_c}{\partial q^s}(\mathcal{B}^{lr} \bar{A}_r) + \tilde{Z}^s \frac{\partial}{\partial \dot{q}^s}(\mathcal{B}^{lr} \bar{A}_r)$$
$$-\frac{\mathrm{d}'_c \tilde{Z}^l}{\mathrm{d}t} + \frac{\partial \tilde{Z}^l}{\partial \dot{q}^s}(\mathcal{B}^{sr} \bar{A}_r) + \mathcal{B}^{lr} \mathcal{B}^{sp} \bar{A}_r \bar{A}_p \frac{\partial Z^0}{\partial \dot{q}^s} = 0 \,. \tag{27}$$

It is evident that the condition (26) is automatically satisfied if the constraint is semiholonomic. The constraint symmetries (vector fields $Z \in \mathcal{C}$ which are solutions of equations (14)–(16)) are simultaneously symmetries of constraint equations of motion iff they obey the above derived conditions (25)–(27).

4 Example: Chaplygin sleigh

In this section we use the geometrical theory for solving the motion of so called Chaplygin sleigh. This example is exposed in [26], where the motion of Chaplygin sleigh is described in another way without considering the problem of symmetries and conservations laws. We study this problem using our results obtained in Section 3.

4.1 Chaplygin sleigh and its motion

The sleigh consists of a rigid body sliding on the horizontal plane without friction (see the figure 1). The constraint is imposed by a sharp blade placed at a point A

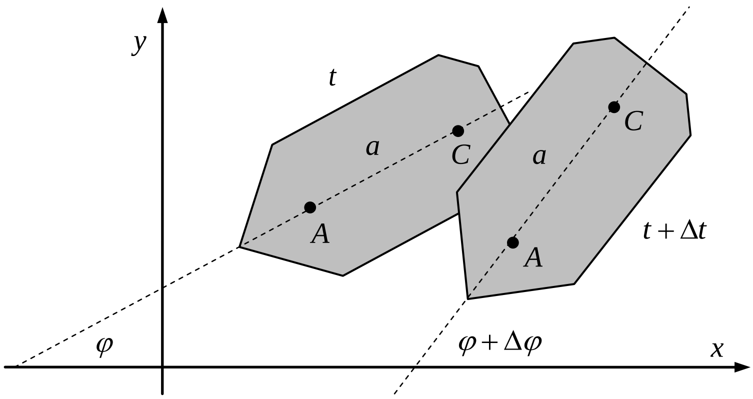

Figure 1: Chaplygin sleigh.

such that the distance between this point and the center of mass of the body C is $AC = a$. The blade prevents the sleigh to move in the direction perpendicular to the straight line AC. The constraint defining the constraint submanifold Q in the fibred chart with coordinates $(t,\,\varphi,\,x,\,y,\,\dot\varphi,\,\dot x,\,\dot y)$, i.e. $m = 3$, reads

$$\dot y\cos\varphi - \dot x\sin\varphi = 0 \implies \dot y = \dot x\tan\varphi\,. \tag{28}$$

The canonical embedding $\iota : Q \to J^1Y$ has the form

$$\iota : Q \ni (t,\,\varphi,\,x,\,y,\,\dot\varphi,\,\dot x) \to (t,\,\varphi,\,x,\,y,\,\dot\varphi,\,\dot x,\,\dot x\tan\varphi) \in J^1Y\,.$$

The canonical distribution is annihilated by the form φ^1 obtained by putting the constraint equation into the general expression (6). We obtain

$$\varphi^1 = \mathrm{d}y - \tan\varphi\,\mathrm{d}x\,.$$

The unconstrained Lagrangian is $\lambda = L\,\mathrm{d}t$, with

$$L = \frac{1}{2}m\left[(\dot x - a\dot\varphi\sin\varphi)^2 + (\dot y + a\dot\varphi\cos\varphi)^2\right] + \frac{1}{2}J\dot\varphi^2,$$

where m and J are the mass and inertia (with respect to the axis perpendicular to the coordinate plane xy and going through C) of the sleigh, respectively. Constraint

Lagrangian functions are

$$\bar{L} = L \circ \iota = \frac{m}{2}\left(\frac{\dot{x}^2}{\cos^2\varphi} + a^2k^2\dot{\varphi}^2\right), \tag{29}$$

$$\bar{L}_1 = \frac{\partial L}{\partial \dot{y}} \circ \iota = m(\dot{x}\tan\varphi + a\dot{\varphi}\cos\varphi)\,, \quad k^2 = 1 + \frac{J}{ma^2}.$$

Putting this into (9) we obtain the matrices $\bar{A}$, $\bar{B}$ and $\mathcal{B} = \bar{B}^{-1}$,

$$\bar{A} = \begin{pmatrix} -\frac{ma\dot{\varphi}\dot{x}}{\cos\varphi} & , & \frac{ma\dot{\varphi}^2}{\cos\varphi} - \frac{m\dot{\varphi}\dot{x}\sin\varphi}{\cos^3\varphi} \end{pmatrix}, \tag{30}$$

$$\bar{B} = \begin{pmatrix} -ma^2k^2 & 0 \\ 0 & -\frac{m}{\cos^2\varphi} \end{pmatrix} \quad \mathcal{B} = \begin{pmatrix} -\frac{1}{ma^2k^2} & 0 \\ 0 & -\frac{\cos^2\varphi}{m} \end{pmatrix},$$

and the equations of motion

$$0 = -ma^2k^2\ddot{\varphi} - \frac{ma}{\cos\varphi}\dot{\varphi}\dot{x} \quad \Longrightarrow \quad \ddot{\varphi} + \frac{\dot{\varphi}\dot{x}}{ak^2\cos\varphi} = 0\,, \tag{31}$$

$$0 = -\frac{m}{\cos^2\varphi}\ddot{\varphi} - \frac{m\sin\varphi}{\cos^3\varphi}\dot{\varphi}\dot{x} \quad \Longrightarrow \quad \ddot{x} - a\dot{\varphi}^2\cos\varphi + \dot{\varphi}\dot{x}\tan\varphi = 0\,. \tag{32}$$

Solutions of these equations take the following form:

$$\varphi(t) = k \arcsin\tanh\left(\frac{C_1}{k^2}(t - C_2)\right) + C_3\,, \quad \varphi = k\psi + C_3\,,$$

$$x(t) = ak^2 \int \cos\left(k\psi + C_3\right)\tan\psi\,\mathrm{d}\psi\,,$$

$$y(t) = ak^2 \int \sin\left(k\psi + C_3\right)\tan\psi\,\mathrm{d}\psi\,,$$

where C_1, C_2 and C_3 are integration constants. Using the initial conditions $\varphi(0) = 0$, $\dot{\varphi}(0) = \omega_0 > 0$, $\dot{x}(0) = 0$ we obtain constants $C_1 = k\omega_0$, $C_2 = 0$, $C_3 = 0$ and the corresponding particular solution

$$\begin{aligned} \varphi(t) &= k \arcsin\tanh\frac{\omega_0 t}{k}\,, \quad \tan\psi \;=\; \sinh\frac{\omega_0 t}{k}\,, \\ x(t) &= ak^2 \int \cos k\psi \tan\psi\,\mathrm{d}\psi\,, \\ y(t) &= ak^2 \int \sin k\psi \tan\psi\,\mathrm{d}\psi\,. \end{aligned} \tag{33}$$

The graphical outputs for some special situations ($a = 1$, $\frac{\omega_0}{k} = 1$, $m = 2$, values $k = 1, 2, 3, 4$) are presented in figures 2–5 for illustration.
Notice that in [26] equivalent equations of motion are obtained for variables u and v representing components of the sleigh velocity with respect to non-inertial reference frame connected with the sleigh, and the variable ω representing the angular velocity $\dot{\varphi}$. The equations of motion are obtained by formulating the second Newton's law in the above mentioned non-inertial reference frame. Thus

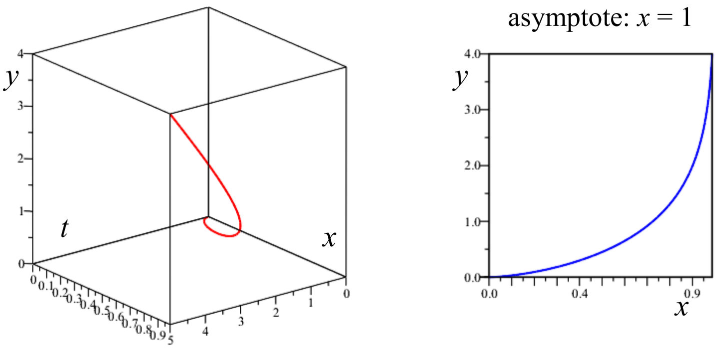

Figure 2: Chaplygin sleigh motion: $k = 1$.

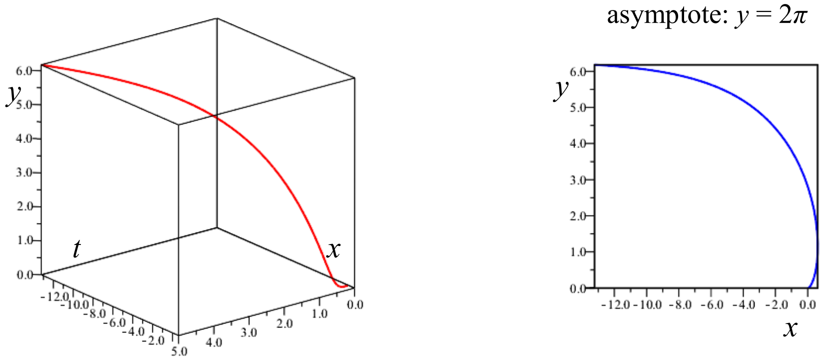

Figure 3: Chaplygin sleigh motion: $k = 2$.

they contain the "fictive" forces $\vec{F}^*$. Moreover, the "reaction" force $\vec{R}$ normal to the straight line AC and representing the constraint is included. Its magnitude is considered as an unknown quantity and it is obtained by solving the equations of motion as well. The solution of these equations of motion is then transformed into the inertial reference frame. Our solution is the same as the last cited one. Recall that in [26] the conservation laws are not discussed.

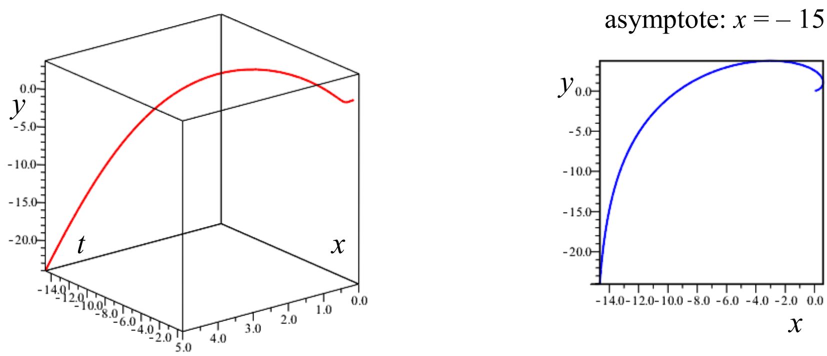

Figure 4: Chaplygin sleigh motion: $k = 3$.

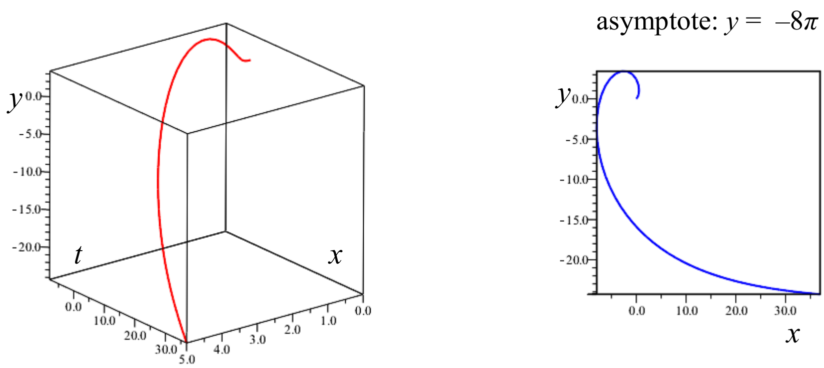

Figure 5: Chaplygin sleigh motion: $k = 4$.

4.2 Constraint symmetries and currents

Putting the expressions (29) for constraint Lagrange functions into equations (17), (18), (19) we obtain

$$\begin{aligned}\frac{\mathrm{d}'_c\Phi}{\mathrm{d}t} - \frac{ma\dot\varphi\dot x}{\cos\varphi}(Z^\varphi - \dot\varphi Z^0) + m\left(\frac{a\dot\varphi^2}{\cos\varphi} - \frac{\dot\varphi\dot x\tan\varphi}{\cos^2\varphi}\right)(Z^x - \dot x Z^0) &= 0\,,\\ \frac{\partial_c\Phi}{\partial\varphi} + \frac{m\dot x^2\tan\varphi}{\cos^2\varphi}Z^0 - m\left(\frac{\dot x\tan\varphi}{\cos^2\varphi} - \frac{a\dot\varphi}{\cos\varphi}\right)Z^x + ma^2k^2\tilde Z^\varphi &= 0\,,\\ \frac{\partial_c\Phi}{\partial x} + m\left(\frac{\dot x\tan\varphi}{\cos^2\varphi} - \frac{a\dot\varphi}{\cos\varphi}\right)Z^\varphi + \frac{m}{\cos^2\varphi}\tilde Z^x &= 0\,,\\ \frac{\partial\Phi}{\partial\dot\varphi} - ma^2k^2(Z^\varphi - \dot\varphi Z^0) &= 0\,,\\ \frac{\partial\Phi}{\partial\dot x} - \frac{m}{\cos^2\varphi}(Z^x - \dot x Z^0) &= 0\,.\end{aligned} \tag{34}$$

Expressing the components $(Z^\varphi - \dot\varphi Z^0)$ and $(Z^x - \dot x Z^0)$ from the last two of these equations, putting them into the first equation and substituting $v = \frac{\dot x}{\cos\varphi}$ we obtain

$$\left(\frac{\partial}{\partial t} + \dot\varphi\frac{\partial}{\partial\varphi} + v\cos\varphi\frac{\partial}{\partial x} + v\sin\varphi\frac{\partial}{\partial y} - \frac{\dot\varphi v}{ak^2}\frac{\partial}{\partial\dot\varphi} + a\dot\varphi^2\frac{\partial}{\partial v}\right)\Phi = 0\,. \tag{35}$$

So, we have the characteristics ODE's

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}\varphi}{\dot\varphi} = \frac{\mathrm{d}x}{v\cos\varphi} = \frac{\mathrm{d}y}{v\sin\varphi} = -ak^2\frac{\mathrm{d}\dot\varphi}{\dot\varphi v} = \frac{\mathrm{d}v}{a\dot\varphi^2}.$$

Integrating the last equation we obtain

$$\frac{1}{2}v^2 + \frac{1}{2}a^2k^2\dot\varphi^2 = \text{const.},\quad \text{i.e.}\quad \frac{1}{2}\frac{\dot x^2}{\cos^2\varphi} + \frac{1}{2}a^2k^2\dot\varphi^2 = \text{const.}$$

This quantity multiplied by the sleigh mass m represents the total mechanical energy E_0 of the sleigh which is the sum of the translational energy $E_T = \frac{1}{2}\frac{m\dot x^2}{\cos^2\varphi}$ and the rotational energy $E_R = \frac{1}{2}(J + ma^2)\dot\varphi^2$ with respect to the vertical axis going through the point A. Recall that due to the Steiner theorem $J + ma^2$ is the inertia of the sleigh with respect to this axis. More precisely, the total mechanical energy of the sleigh expressed via the components of the velocity of the center od mass (x_C, y_C) is

$$E = \frac{m}{2}\left(\dot x_C^2 + \dot y_C^2\right) + \frac{1}{2}J\dot\varphi^2.$$

Taking into account that $x_C = x + a\cos\varphi$, $y_C = y + a\sin\varphi$ and considering the constraint we can immediately see that $E = E_0$. For the particular solution of equations of motion presented in the previous section we have

$$E_0 = \frac{1}{2}ma^2k^2\omega_0^2,\quad C_1 = k\omega_0 = \sqrt{\frac{2E_0}{ma^2}}\,.$$

The corresponding conserved current can be obtained using the equations of motion and the fact that the constrained Lagrange function $\bar{L}$ does not depend on time explicitly,

$$\Phi_1 = -\frac{m}{2}\left(\frac{\dot{x}^2}{\cos^2\varphi} + a^2k^2\dot{\varphi}^2\right). \tag{36}$$

Putting this expression into equations (20) we can verify that the corresponding symmetry is $Z = \frac{\partial}{\partial t}$. Taking into account the solution of equations of motion (section 4.2) we obtain the following expressions for the translational and rotational energy and the angle φ as functions of time (see also figure 6):

$$E_T = E_0 \tanh^2\frac{\omega_0 t}{k}, \quad E_R = E_0 \cosh^{-2}\frac{\omega_0 t}{k}, \quad \sin\frac{\varphi}{k} = \tanh\frac{\omega_0 t}{k}. \tag{37}$$

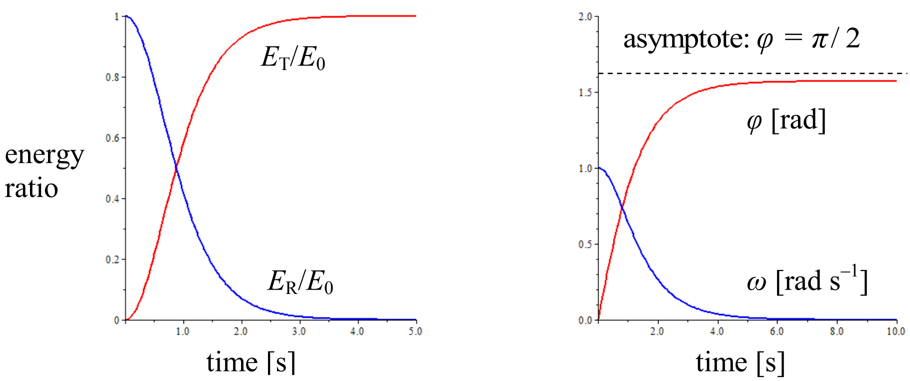

Figure 6: Conservation of energy, damping of rotation.

The graphs show the asymptotic behavior of the sleigh motion: the translational motion accelerates at the expense of the rotational motion which is asymptotically damped.

The decomposition of the energy into the term corresponding to translational motion of the point A and the energy corresponding to the rotation of the sleigh around the axis going through this point is "induced" by the formulation of the problem itself (the constraint concerns the motion of the point A). On the other hand, more correct from the point of view of physics is the energy decomposition into the translational energy of the center of mass C, $E_{T,C} = \frac{1}{2}m(\dot{x}_C^2+\dot{y}_C^2)$, and the rotational energy of the sleigh with respect to the center of mass, $E_{R,C} = \frac{1}{2}J\dot{\varphi}^2$.

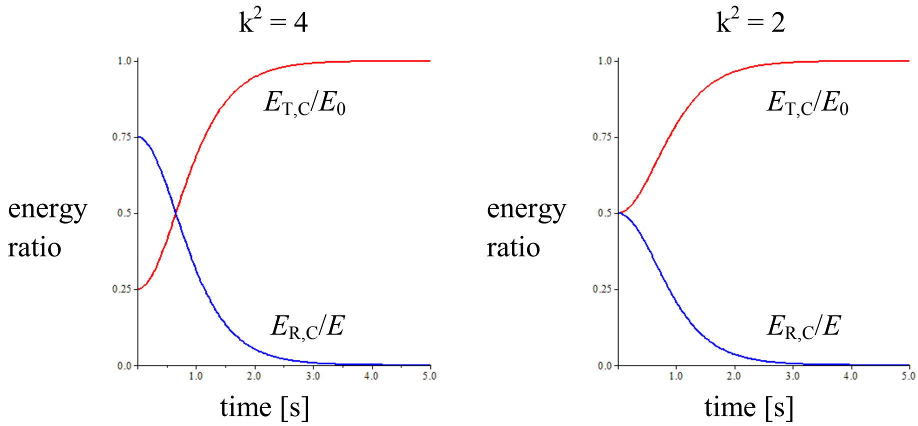

Figure 7: Energy decomposition with respect to the center of mass.

Considering the solution of equations of motion we obtain

$$E_{T,C} = E_0 \left(\tanh^2 \frac{\omega_0 t}{k} + \frac{1}{k^2 \cosh^2 \frac{\omega_0 t}{k}} \right), \; E_{R,C} = \frac{E_0}{\cosh^2 \frac{\omega_0 t}{k}} \left(1 - \frac{1}{k^2} \right). \qquad (38)$$

Figure 7 shows the behavior of both types of kinetic energy during the time for two different values k. Relations (37) represent the limit case of (38) for $J \gg ma^2$, i.e. $k \to \infty$, as expected. Notice that for $k = 1$ (zero inertia with respect to the center of mass, or, more exactly, $J \ll ma^2$) we have $E_{T,C} = E_0$ and $E_{R,C} = 0$. This result is not in contradiction with the initial conditions. $E_{R,C}$ vanishes because of zero inertia, even though $\omega_0 \neq 0$. (Figures 6 and 7 are drawn for $\omega_0/k = 1$ for simplicity.)

Expressing the quantities C_2 and C_3 (the fact that they are zeros for the chosen initial conditions does not affect their general meaning of integration constants) we obtain the following currents

$$\Phi_2 = \frac{m\dot{x}}{\cos\varphi} \sin\left(\frac{\varphi}{k}\right) + mak\dot{\varphi}\cos\left(\frac{\varphi}{k}\right)$$

$$\Phi_3 = \frac{1}{2} ma^2 k^2 \ln \left(\frac{\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2} + \frac{\dot{x}}{a\cos\varphi}}{\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2} - \frac{\dot{x}}{a\cos\varphi}} \right) - ma^2 t \sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2},$$

and in shortened notation with help of energies

$$\Phi_3 = \frac{1}{2} ma^2 k^2 \ln \frac{\sqrt{E_0} + \sqrt{E_T}}{\sqrt{E_0} - \sqrt{E_T}} - at\sqrt{2mE_0}$$

For the special case of zero inertia J, i.e. $k = 1$, the current Φ_2 represents the y-component of the impulse of the sleigh, $p_{C,y} = m\dot{y}_C = m(\dot{x}\tan\varphi + a\dot{\varphi}\cos\varphi)$.

(We shall see later that in such a case the component $p_{C,x}$ must be conserved as well.)

The corresponding symmetries are (denoting $\psi = \frac{\varphi}{k}$ as above)

$$
\begin{aligned}
Z(\Phi_2) = {} & \frac{1}{ak}\cos\psi\frac{\partial}{\partial\varphi} + \cos\varphi\sin\psi\frac{\partial}{\partial x} + \sin\varphi\sin\psi\frac{\partial}{\partial y} \\
& - \frac{\dot{x}\cos\psi}{a^2k^3\cos\varphi}\frac{\partial}{\partial\dot{\varphi}} + \frac{1}{ak}\left(a\dot{\varphi}\cos\varphi - \dot{x}\tan\varphi\right)\cos\psi\frac{\partial}{\partial\dot{x}}\,, \\
Z(\Phi_3) = {} & \frac{k^2}{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2}\ln\left(\frac{\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2} + \frac{\dot{x}}{a\cos\varphi}}{\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2} - \frac{\dot{x}}{a\cos\varphi}}\right)\Gamma \\
& - \frac{\dot{\varphi}t + \frac{\dot{x}}{a\dot{\varphi}\cos\varphi}}{\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2}}\left(\frac{\partial}{\partial\varphi} - \left(a\dot{\varphi}\cos\varphi - \dot{x}\tan\varphi\right)\frac{\partial}{\partial\dot{x}}\right) \\
& + \frac{ak^2\cos\varphi - \dot{x}t}{\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2}}\left(\frac{\partial}{\partial x} + \tan\varphi\frac{\partial}{\partial y} + \frac{\dot{\varphi}}{ak^2\cos\varphi}\frac{\partial}{\partial\dot{\varphi}}\right)\,,
\end{aligned}
$$

or, in shortened notation via energies

$$
\begin{aligned}
Z(\Phi_3) = {} & \frac{ma^2k^2}{2E_0}\ln\left(\frac{\sqrt{E_0} + \sqrt{E_T}}{\sqrt{E_0} - \sqrt{E_T}}\right)\Gamma \\
& - \sqrt{\frac{ma^2}{2E_0}}\left(\dot{\varphi}t + \frac{\dot{x}}{a\dot{\varphi}\cos\varphi}\right)\left(\frac{\partial}{\partial\varphi} - \left(a\dot{\varphi}\cos\varphi - \dot{x}\tan\varphi\right)\frac{\partial}{\partial\dot{x}}\right) \\
& + \sqrt{\frac{ma^2}{2E_0}}\left(ak^2\cos\varphi - \dot{x}t\right)\left(\frac{\partial}{\partial x} + \tan\varphi\frac{\partial}{\partial y} + \frac{\dot{\varphi}}{ak^2\cos\varphi}\frac{\partial}{\partial\dot{\varphi}}\right)
\end{aligned}
$$

where the vector field Γ reads

$$
\Gamma = \frac{\partial}{\partial t} + \dot{\varphi}\frac{\partial}{\partial\varphi} + \dot{x}\frac{\partial}{\partial x} + \dot{x}\tan\varphi\frac{\partial}{\partial y} - \frac{\dot{\varphi}\dot{x}}{ak^2\cos\varphi}\frac{\partial}{\partial\dot{\varphi}} + \left(a\dot{\varphi}^2\cos\varphi - \dot{\varphi}\dot{x}\tan\varphi\right)\frac{\partial}{\partial\dot{x}}\,,
$$

which is the vector field representing the equations of motion on the submanifold Q. Keep in mind that the above presented shortened notation via energies is given only for better clarity. For eventual further calculations the full expression in coordinates $(t,\ \varphi,\ x,\ y,\ \dot{\varphi},\ \dot{x})$ on Q must be used, i.e. it is necessary to put

$$
E_0 = \frac{ma^2}{2}\left(\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2\right), \quad E_T = \frac{m\dot{x}^2}{2\cos^2\varphi}, \quad \psi = \frac{\varphi}{k}
$$

into corresponding expressions.

The equations (14)–(16) take the form

$$
\begin{aligned}
& - \frac{\dot{x}\dot{\varphi}}{a\cos\varphi}Z^\varphi + \left(\frac{\dot{x}\dot{\varphi}\sin\varphi}{a^2\cos^3\varphi} + \frac{\dot{\varphi}^2}{a\cos\varphi}\right)Z^x \\
& \quad - \frac{1}{2}\left(\frac{\dot{x}^2}{a^2\cos^2\varphi} + k^2\dot{\varphi}^2\right)\frac{\mathrm{d}'_c Z^0}{\mathrm{d}t} + k^2\dot{\varphi}\frac{\mathrm{d}'_c Z^\varphi}{\mathrm{d}t} + \frac{\dot{x}}{a^2\cos^2\varphi}\frac{\mathrm{d}'_c Z^x}{\mathrm{d}t} = 0\,, \qquad (39)
\end{aligned}
$$

$$\left(\frac{\dot{x}\sin\varphi}{a^2\cos^3\varphi}+\frac{\dot{\varphi}}{a\cos\varphi}\right)Z^x+k^2\tilde{Z}^\varphi$$
$$-\frac{1}{2}\left(\frac{\dot{x}^2}{a^2\cos\varphi}+k^2\dot{\varphi}^2\right)\frac{\partial_c Z^0}{\partial\varphi}+k^2\dot{\varphi}\frac{\partial_c Z^\varphi}{\partial\varphi}+\frac{\dot{x}}{a^2\cos^2\varphi}\frac{\partial_c Z^x}{\partial\varphi}=0\,, \qquad (40)$$
$$\left(\frac{\dot{x}\sin\varphi}{a^2\cos^3\varphi}-\frac{\dot{\varphi}}{a\cos\varphi}\right)Z^\varphi+\frac{1}{a^2\cos^2\varphi}\tilde{Z}^x$$
$$-\frac{1}{2}\left(\frac{\dot{x}^2}{a^2\cos^2\varphi}+k^2\dot{\varphi}^2\right)\frac{\partial_c Z^0}{\partial x}+k^2\dot{\varphi}\,\frac{\partial_c Z^\varphi}{\partial x}+\frac{\dot{x}}{a^2\cos^2\varphi}\,\frac{\partial_c Z^x}{\partial x}=0\,, \qquad (41)$$
$$-\frac{1}{2}\left(\frac{\dot{x}^2}{a^2\cos^2\varphi}+k^2\dot{\varphi}^2\right)\frac{\partial Z^0}{\partial\dot{\varphi}}+k^2\dot{\varphi}\frac{\partial Z^\varphi}{\partial\dot{\varphi}}+\frac{\dot{x}}{a^2\cos^2\varphi}\,\frac{\partial Z^x}{\partial\dot{\varphi}}=0\,, \qquad (42)$$
$$-\frac{1}{2}\left(\frac{\dot{x}^2}{a^2\cos^2\varphi}+k^2\dot{\varphi}^2\right)\frac{\partial Z^0}{\partial\dot{x}}+k^2\dot{\varphi}\,\frac{\partial Z^\varphi}{\partial\dot{x}}+\frac{\dot{x}}{a^2\cos^2\varphi}\,\frac{\partial Z^x}{\partial\dot{x}}=0\,. \qquad (43)$$

Putting the components of vector fields $Z(\Phi_1)$, $Z(\Phi_2)$ and $Z(\Phi_3)$ into equations (39)–(43) we can verify that they are constraint symmetries. Thus Φ_1, Φ_2 and Φ_3 are Noether-type currents. Nevertheless, the physical interpretation of symmetries $Z(\Phi_2)$ and $Z(\Phi_3)$ and their currents is not completely clear in a general situation.

The relation for the current Φ_2 is linear in variables velocity and angular velocity. This enables us to conclude that for a general description of the sleigh motion it is satisfactory to consider special initial conditions $\dot{x}(0)=0$ and $\dot{\varphi}(0)=\omega(0)\neq 0$. If $v(0)\neq 0$ and $\dot{\varphi}=\omega_0$, then $v(\tau)=0$ and $\dot{\varphi}(\tau)=\Omega_0\neq\omega_0$ at some other time τ.

Calculating $[\Gamma,\, Z]$ for all three obtained symmetries $Z(\Phi_1)$, $Z(\Phi_2)$ and $Z(\Phi_3)$ we can see that only the symmetry $Z(\Phi_1)=\frac{\partial}{\partial t}$ is simultaneously the symmetry of constrained (reduced) equations of motion. Concretely, it is evident that $[\Gamma,\, \frac{\partial}{\partial t}]=0$. Moreover, using the condition (23) we can check that it holds

$$i_{[\Gamma,Z_1]}\varphi^1=0\,,\qquad i_{[\Gamma,Z_2]}\varphi^1=-\frac{\dot{x}}{ak\cos^2\varphi}\cos\frac{\varphi}{k}+\frac{\dot{\varphi}}{\cos\varphi}\sin\frac{\varphi}{k}\,,$$

$$i_{[\Gamma,Z_3]}\varphi^1=-\frac{a}{\dot{\varphi}\cos\varphi}\sqrt{\frac{\dot{x}^2}{a^2\cos^2\varphi}+k^2\dot{\varphi}^2}.$$

This means that for the symmetry Z_1 the vector field $[\Gamma,Z_1]$ belongs to the canonical distribution unlike the vector fields $[\Gamma,Z_2]$ and $[\Gamma,Z_3]$.

4.3 Chetaev equations and constraint forces

Finally, let us express Chetaev equations of motion and the constraint forces as exposed in section 2.4 (equations (10)). Rewriting the constraint as

$$f(t,\,\varphi,\,x,\,y,\,\dot{\varphi},\,\dot{x},\,\dot{y})\equiv\dot{y}-\dot{x}\tan\varphi=0$$

we obtain the following equations:

$$\begin{aligned}
-mak^2\ddot{\varphi} + m\ddot{x}\sin\varphi - m\ddot{y}\cos\varphi &= \frac{\mu}{a}\frac{\partial f}{\partial\dot{\varphi}}\,, & \frac{\partial f}{\partial\dot{\varphi}} &= 0\,,\\
ma\ddot{\varphi}\sin\varphi - m\ddot{x} + ma\dot{\varphi}^2\cos\varphi &= \mu\frac{\partial f}{\partial\dot{x}}\,, & \frac{\partial f}{\partial\dot{x}} &= -\tan\varphi\,,\\
-ma\ddot{\varphi}\cos\varphi - m\ddot{y} + ma\dot{\varphi}^2\sin\varphi &= \mu\frac{\partial f}{\partial\dot{y}}\,, & \frac{\partial f}{\partial\dot{y}} &= 1\,,
\end{aligned} \tag{44}$$

μ being a Lagrange multiplier. The constraint force is

$$\phi = \mu\left(\frac{1}{a}\frac{\partial f}{\partial\dot{\varphi}}, \frac{\partial f}{\partial\dot{x}}, \frac{\partial f}{\partial\dot{y}}\right) = \mu(0, -\tan\varphi, 1)\,. \tag{45}$$

It has a clear physical meaning in the reference frame connected with the point A and rotating with the sleight: Denote $\vec{r}' = (0,\, a\cos\varphi,\, a\sin\varphi)$, $\vec{\omega} = (\dot{\varphi},\, 0,\, 0)$, $\vec{\varepsilon} = (\ddot{\varphi},\, 0,\, 0)$, $\vec{A}(0,\, \ddot{x},\, \ddot{y})$. (Note that $\vec{r}'$ determines the position of the center of mass C of the sleigh with respect to the point A.) Denoting ϕ as $\vec{F}^*$ as it is usual in physics, we obtain

$$\begin{gathered}
\vec{F}^* = \left(ma\ddot{\varphi}\sin\varphi - m\ddot{x} + ma\dot{\varphi}^2\cos\varphi,\ -ma\ddot{\varphi}\cos\varphi - m\ddot{y} + ma\dot{\varphi}^2\sin\varphi,\ 0\right)\,,\\
\vec{F}^* = -m\vec{\varepsilon}\times\vec{r}' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}') - m\vec{A}\,.
\end{gathered} \tag{46}$$

This force is the sum of three terms: the Euler force, the centrifugal force and the translational force. The Coriolis force is missing because the velocity of the center of mass with respect to the reference system connected with the point A is zero.

Using the constraint to write $\ddot{y} = \ddot{x}\tan\varphi + \frac{\dot{\varphi}\dot{x}}{\cos^2\varphi}$ and substituting into (44) we obtain after some calculations the Lagrange multiplier μ and the constraint force ϕ:

$$\mu = -\frac{mJ}{J + ma^2}\dot{\varphi}\dot{x}, \quad \phi = \frac{mJ}{J + ma^2}\left(0,\, \dot{\varphi}\dot{x}\tan\varphi,\, -\dot{\varphi}\dot{x}\right). \tag{47}$$

Notice that these forces are not variational in the sense of e.g. [17], [25], [27], [32]. Thus the Chaplygin sleigh cannot be alternatively described as an unconstrained variational system with an appropriately modified Lagrangian. For $k = 1$ the constraint force vanishes. This is consistent with the (non-realistic, of course) limit case $J \to 0$ in relations (38): The motion of the center of mass is uniform and straightforward (both components of the impulse of the center of mass are conserved), while the sleigh rotates around it with the initial angular velocity ω_0 but with zero energy due to $J = 0$.

5 Acknowledgements

Research supported by grant No. 14-02476S Variations, Geometry and Physics of the Czech Science Foundation. The authors would like to acknowledge Prof. Olga Rossi for helpful discussions.

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Authors' address:
Institute of Theoretical Physics and Astrophysics, Faculty of Science, Masaryk University, Brno, Czech Republic

E-mail: 174054@mail.muni.cz, janam@physics.muni.cz

Received: 10 November, 2014
Accepted for publication: 25 November, 2014
Communicated by: Olga Rossi

Communications in Mathematics 22 (2014) 185–211

Super Wilson Loops and Holonomy on Supermanifolds

Josua Groeger

Abstract. The classical Wilson loop is the gauge-invariant trace of the parallel transport around a closed path with respect to a connection on a vector bundle over a smooth manifold. We build a precise mathematical model of the super Wilson loop, an extension introduced by Mason-Skinner and Caron-Huot, by endowing the objects occurring with auxiliary Graßmann generators coming from S-points. A key feature of our model is a supergeometric parallel transport, which allows for a natural notion of holonomy on a supermanifold as a Lie group valued functor. Our main results for that theory comprise an Ambrose-Singer theorem as well as a natural analogon of the holonomy principle. Finally, we compare our holonomy functor with the holonomy supergroup introduced by Galaev in the common situation of a topological point. It turns out that both theories are different, yet related in a sense made precise.

1 Introduction

Gluon scattering amplitudes have been known to be dual to Wilson loops along lightlike polygons [1], [2], [7], [11]. While these quantum expectation values, which are formally calculated by means of the path integral, remain problematic from a mathematical point of view, the underlying classical theory has been well understood. In fact, a Wilson line refers to parallel transport with respect to a connection on a vector bundle along a path in the underlying smooth manifold. In the usual context of flat spacetime (Minkowski space) with a single global coordinate chart, the corresponding solution operator can be written in terms of a path-ordered exponential.

Recently, a similar duality (at weak coupling) between the full superamplitude of $\mathcal{N} = 4$ super Yang-Mills theory and two variants of a supersymmetric extension of the Wilson loop has been claimed. The first approach [21] originates in momentum twistor space and translates into the integral over a superconnection in spacetime, while the second [9] attaches to lightlike polygons certain edge and

2010 MSC: 58A50, 53C29, 18F05
Key words: supermanifolds, holonomy, group functor

vertex operators, whose shape is determined by supersymmetry constraints [15]. Both approaches agree, in the common domain of definition, up to a term depending on the equations of motion [6] and indeed satisfy the conjectured duality upon subtracting an anomalous contribution [5].

The first purpose of the present article is to build a supergeometric model of super Wilson loops that leads to the same characteristic formulas as summarised in Section 2.2 of [6]. The main idea is to give the objects occurring an inner structure through auxiliary Graßmann generators coming from S-points. While the resulting additional degrees of freedom come without physical significance, this approach is well-justified mathematically and has been performed successfully in modelling other aspects of superfield theory. Notably, consider "maps with flesh" as introduced by Hélein in [18] as models for superfields including bosons and fermions. See also [10], [16], [19] for the same concept under different terminology and [14] for their differential calculus.

A key feature of our model is the supergeometric parallel transport introduced in Section 2, which allows for a natural notion of holonomy at an S-point of a supermanifold as a Lie group valued functor. A different notion of holonomy on supermanifolds was introduced by Galaev in [12] by taking a suitable generalisation of the Ambrose-Singer theorem as the definition of a super Lie algebra and endowing this to a Harish-Chandra pair, thus obtaining a super Lie group for every topological point of the manifold. Developing a new holonomy theory by means of our parallel transport, and comparing it to Galaev's, is the second objective of this article.

In Section 3, we establish two main results generalising properties of classical holonomy. The first is an Ambrose-Singer theorem, which describes the holonomy Lie algebra in terms of curvature, while the second formulates a natural analogon of the holonomy principle relating parallel sections to holonomy-invariant vectors.

Our Ambrose-Singer theorem facilitates the comparison of our holonomy functor with Galaev's theory, which is the subject matter of Section 4. Since this functor is, in general, not representable, both theories are different in the common situation of a topological point. Nevertheless, we show that they are related in that the generators of Galaev's holonomy algebra can be extracted as certain coefficients by considering special S-points. This construction is based on the knowledge of the geometric significance of the elements and, in this sense, is not algebraic.

2 Super Wilson Loops and Parallel Transport

The super Wilson loop described in [6] and [21] is constructed as follows. Consider n "superpoints" (x_i, θ_i) in chiral superspace, which are symbolic quantities in that their exact mathematical type is not important, only their calculation rules such as

$$x_i^\mu \cdot x_j^\nu = x_j^\nu \cdot x_i^\mu \,, \qquad \theta_i^{\alpha A} \cdot \theta_j^{\beta B} = -\theta_j^{\beta B} \cdot \theta_i^{\alpha A} \tag{1}$$

These superpoints are connected by "straight lines"

$$x(t_i) = x_i - t_i x_{i,i+1} \,, \qquad \theta(t_i) = \theta_i - t_i \theta_{i,i+1} \tag{2}$$

thus yielding a closed "superpath" γ parametrised by one bosonic variable t, which enters the Wilson loop via

$$W_\gamma = \operatorname{tr}\left(X \mapsto \mathcal{P}\exp\left(\int_0^1 ig\mathcal{B}(t)\,dt\right)[X]\right)\,, \qquad \mathcal{B}(t) = \mathcal{B}_\xi \cdot \dot{\gamma}^\xi(t) \tag{3}$$

where $\mathcal{P}\exp$ denotes a path-ordered exponential, and $\mathcal{B}_\xi$ is a connection one-form in coordinates ξ. This connection has a very specific form due to supersymmetry conditions which, however, is not relevant for our purposes.

As a mathematical model for a more general situation, let M be a supermanifold of dimension $\dim M = (\dim M)_{\overline{0}}|(\dim M)_{\overline{1}}$ (such as chiral superspace), and let S be another supermanifold which should be thought of as auxiliary. Throughout, we employ the definitions of Berezin-Kostant-Leites [20]. A supermanifold M is thus, in particular, a ringed space $M = (M_0, \mathcal{O}_M)$, and a morphism $\varphi\colon M \to N$ consists of two parts $\varphi = (\varphi_0, \varphi^\sharp)$ with $\varphi_0\colon M_0 \to N_0$ a smooth map and $\varphi^\sharp$ a generalised pullback of superfunctions $f \in \mathcal{O}_N$. Modern monographs on the general theory of supermanifolds include [8] and [27].

Definition 1. An S-point of M is a morphism $x = (x_0, x^\sharp)\colon S \to M$. A (smooth) S-path γ connecting S-points x and y is a morphism

$$\gamma = (\gamma_0, \gamma^\sharp)\colon S \times [0,1] \to M \qquad \text{such that} \qquad \mathrm{ev}|_{t=0}\gamma^\sharp = x^\sharp\,, \quad \mathrm{ev}|_{t=1}\gamma^\sharp = y^\sharp$$

which we shall denote, by a slight abuse of notation, by $\gamma\colon x \to y$. It is called closed (or an S-loop) if $x = y$.

In the following, we will exclusively consider superpoints

$$S = \mathbb{R}^{0|L} = \left(\{0\}, \bigwedge \mathbb{R}^L\right)\,, \qquad \bigwedge \mathbb{R}^L = \langle \eta^1, \dots, \eta^L \rangle\,, \qquad L \in \mathbb{N}\,. \tag{4}$$

Although most of our results should continue to hold accordingly for general S, this restriction will turn out to suffice for reproducing the characteristic formulas of super Wilson loops as well as allowing for a powerful holonomy theory. This significance of superpoints does not come unexpected. According to [25], an inner Hom object $\underline{\mathrm{Hom}}(M,N)$ in the category of supermanifolds is determined by its $\bigwedge \mathbb{R}^L$-points

$$\underline{\mathrm{Hom}}(M,N)\left(\bigwedge \mathbb{R}^L\right) \cong \mathrm{Hom}_{\mathrm{SMan}}\left(\mathbb{R}^{0|L} \times M,\, N\right)$$

in the sense of Molotkov-Sachse theory [22], [24]. The morphisms on the right are the aforementioned "maps with flesh" [18].

Definition 2. Let $x, y, z\colon S \to M$ be S-points and $\gamma\colon x \to y$ and $\delta\colon y \to z$ be S-paths. For fixed $t_0 \in [0,1]$, we prescribe

$$\mathrm{ev}|_{t=t_0}(\delta \star \gamma)^\sharp := \begin{cases} \mathrm{ev}|_{t=2t_0}\gamma^\sharp & t_0 \le 1/2\,, \\ \mathrm{ev}|_{t=2\left(t_0 - \frac{1}{2}\right)}\delta^\sharp & t_0 \ge 1/2\,. \end{cases}$$

This defines an S-point which coincides with x,y,z for $t_0 = 0, \frac{1}{2}, 1$, respectively. Similarly, we define

$$\mathrm{ev}|_{t=t_0}(\gamma^{-1})^\sharp := \mathrm{ev}|_{t=(1-t_0)}\gamma^\sharp\,.$$

Considering all $t_0 \in [0,1]$ at a time, the previous definition yields S-paths $\delta \star \gamma\colon x \to z$ and $\gamma^{-1}\colon y \to x$, referred to as the concatenation of γ and δ and the inverse of γ, respectively. The concatenation is, however, only piecewise smooth in the sense of the following definition.

Definition 3. Let x, y be S-points. A piecewise smooth S-path $\gamma\colon x \to y$ connecting x with y is a tuple $(\gamma_j\colon S\times[t_j, t_{j+1}] \to M)_{j=0}^{l}$ with $t_0 = 0$, $t_l = 1$ and $t_j < t_{j+1}$ such that $\mathrm{ev}|_{t=t_{j+1}}\gamma_j^\sharp = \mathrm{ev}|_{t=t_{j+1}}\gamma_{j+1}^\sharp$ and $\mathrm{ev}|_{t=0}\gamma_0^\sharp = x^\sharp$ and $\mathrm{ev}|_{t=1}\gamma_l^\sharp = y^\sharp$, and such that $\gamma_j|_{S\times[t_j,t_{j+1}]}$ is a morphism.

The concatenation $(\delta\star\gamma)$ and inverse γ^{-1} of piecewise smooth paths δ and γ are defined analogously. The construction is such that the underlying path $(\delta\star\gamma)_0$ is the classical concatenation of δ_0 and γ_0, and $(\gamma^{-1})_0 = (\gamma_0)^{-1}$.

Example 1. Comparing with the objects in [6], we state the following dictionary. Let $(x^\mu, \theta^{\alpha A})$ denote (global) coordinates on $M \cong \mathbb{R}^{n|m}$ (using spacetime indices μ rather than spinor indices $\dot\alpha\alpha$). Then a superpoint is an S-point $\xi = (\xi_0, \xi^\sharp)\colon S \to M$, identified with $(\xi^\sharp(x^\mu), \xi^\sharp(\theta^{\alpha A})) \in (\mathcal{O}_S)^{n|m}$. The latter tuple is then abbreviated $(x,\theta) = (x^\mu, \theta^{\alpha A})$, for which (1) is satisfied. The straight line connecting superpoints (x_i,θ_i) and (x_{i+1},θ_{i+1}) is the S-path $\xi_{i,i+1}\colon S\times[0,1] \to M$ defined as follows.

$$\begin{aligned}&\big(\xi^\sharp_{i,i+1}(x^\mu), \xi^\sharp_{i,i+1}(\theta^{\alpha A})\big)\\ &\quad := \Big(\xi_i^\sharp(x^\mu) - t\big(\xi_i^\sharp(x^\mu) - \xi_{i+1}^\sharp(x^\mu)\big),\, \xi_i^\sharp(\theta^{\alpha A}) - t\big(\xi_i^\sharp(\theta^{\alpha A}) - \xi_{i+1}^\sharp(\theta^{\alpha A})\big)\Big)\\ &\quad \in (\mathcal{O}_{S\times[0,1]})^{n|m}.\end{aligned}$$

In this sense, we can understand (2). The last line is $\xi_{n,0}$. Concatenation thus yields a loop.

2.1 Super Vector Bundles and Connections

A super vector bundle $\mathcal{E}$ over a supermanifold M is a sheaf of locally free $\mathcal{O}_M$ supermodules on M. We shall denote its even and odd parts by $\mathcal{E}_{\bar 0}$ and $\mathcal{E}_{\bar 1}$, respectively. An important example is the super tangent bundle $\mathcal{S}M := \mathrm{Der}(\mathcal{O}_M)$, which is the sheaf of $\mathcal{O}_M$-superderivations. $\mathcal{E}(U)$ is, for $U \subseteq M_0$ sufficiently small, by definition isomorphic to $\mathcal{O}_M(U)^{\mathrm{rk}\,\mathcal{E}}$ with $\mathrm{rk}\,\mathcal{E} = (\mathrm{rk}\,\mathcal{E})_{\bar 0}|(\mathrm{rk}\,\mathcal{E})_{\bar 1}$ the rank of $\mathcal{E}$. Let $(T^j)_{j=1}^{\mathrm{rk}\,\mathcal{E}}$ be an adapted local basis such that $X \in \mathcal{E}(U)$ is identified with the tuple $(X^j)_{j=1}^{\mathrm{rk}\,\mathcal{E}}$ of functions $X^j \in \mathcal{O}_M(U)$ with respect to right coefficients $X = T^j \cdot X^j$ (sum convention). In general, it is preferable to consider right coordinates on supermodules over supercommutative superalgebras, for then superlinear maps can be identified with matrices. For example, the matrix of the differential $d\varphi[X] := X \circ \varphi^\sharp$ for

$X = \partial_{\xi^k} \cdot X^k \in \mathcal{S}M$ of a map $\varphi\colon M \to N$ with respect to coordinates (ξ^j) and (ζ^j) is given by

$$d\varphi^i{}_k := (-1)^{(|\xi^k|+|\zeta^i|)\cdot|\zeta^i|} \frac{\partial \varphi^\sharp(\zeta^i)}{\partial \xi^k} \quad \text{s.th.} \quad d\varphi[X] = \sum_{i,k} \left(\varphi^\sharp \circ \frac{\partial}{\partial \zeta^i}\right) \cdot d\varphi^i{}_k \cdot X^k \tag{5}$$

Lemma 1 (Chain Rule). *Let $\varphi\colon M \to N$ and $\psi\colon N \to P$ be morphisms. Then*

$$d(\psi \circ \varphi)[X] = \left(\varphi^\sharp \circ \psi^\sharp \circ \frac{\partial}{\partial \pi^l}\right) \cdot \varphi^\sharp(d\psi^l{}_i) \cdot d\varphi^i{}_k \cdot X^k$$

with (π^l) coordinates on P and indices k, i referring to (unlabelled) coordinates on M and N, respectively.

Proof. This is proved by a straightforward calculation in local coordinates. □

Definition 4. For a super vector bundle $\mathcal{E}$, and S as in (4), we define

$$\mathcal{E}_S := \mathcal{E} \otimes_{\mathcal{O}_M} \mathcal{O}_{S\times M}\,.$$

An S-connection on M is an even $\mathbb{R}$-linear sheaf morphism

$$\nabla\colon \mathcal{E}_S \to \mathcal{S}M_S^* \otimes_{\mathcal{O}_{S\times M}} \mathcal{E}_S\ , \quad \nabla(fe) = df \otimes_{\mathcal{O}_{S\times M}} e + f \cdot \nabla e \quad \text{for} \quad f \in \mathcal{O}_{S\times M}\,.$$

In particular, $\mathcal{E}_S$ can be considered as a super vector bundle on $S \times M$ and, in this sense, ∇ is an ordinary superconnection. The local picture is as follows. Let $\xi = (x, \theta)$ be coordinates on M and (T^j) be an $\mathcal{E}$-basis. Then $X \in \mathcal{E}_S$ can be expaned as $X = T^j \cdot X^j$ with $X^j \in \mathcal{O}_{S\times M}(\{0\} \times U)$, and

$$\nabla_{\partial_{\xi^i}} X = (-1)^{|\xi^i||T^j|} T^j \partial_{\xi^i}(X^j) + \Gamma_{\xi^i}[T^j] \cdot X^j\ , \qquad \Gamma_{\xi^i}[T^j] := \nabla_{\partial_{\xi^i}} T^j \tag{6}$$

where $\Gamma_{\xi^i} \in \mathrm{Mat}_{\mathrm{rk}\,\mathcal{E}\times\mathrm{rk}\,\mathcal{E}}(\mathcal{O}_{S\times M}(\{0\} \times U))$, which has an expansion

$$\Gamma_{\xi^i} = \sum_{I=(i_1,\dots,i_{|I|})} \theta^I \cdot (\Gamma_{\xi^i})_I\ , \qquad (\Gamma_{\xi^i})_I \in \mathrm{Mat}_{\mathrm{rk}\,\mathcal{E}\times\mathrm{rk}\,\mathcal{E}}(\mathcal{O}_{S\times M_0}(\{0\} \times U))\,.$$

Example 2. Consider the trivial vector bundle $\mathcal{E} := \mathfrak{su}(N) \otimes_{\mathbb{R}} \mathcal{O}_M$ with $N \in \mathbb{N}$ of rank $\mathrm{rk}\,\mathcal{E} = \dim \mathfrak{su}(N)|0$ over flat superspace with global coordinates $\xi = (x^\mu, \theta^{\alpha A})$. Define $\mathcal{A}_\mu := \Gamma_{x^\mu}$ and $\mathcal{F}_{\alpha A} := \Gamma_{\theta^{\alpha A}}$. With this notation, the θ-expansion assumes the form

$$\begin{aligned}\mathcal{A}_\mu &= (\mathcal{A}_\mu)_0 + \theta^{\beta B}(\mathcal{A}_\mu)_{\beta B} + \theta^{\beta B}\theta^{\gamma C}(\mathcal{A}_\mu)_{\beta B\,\gamma C} + \dots\\ \mathcal{F}_{\alpha A} &= (\mathcal{F}_{\alpha A})_0 + \theta^{\beta B}(\mathcal{F}_{\alpha A})_{\beta B} + \theta^{\beta B}\theta^{\gamma C}(\mathcal{F}_{\alpha A})_{\beta B\,\gamma C} + \dots\end{aligned}$$

Since ∇ is, by definition, even it follows that $\mathcal{A}_\mu$ and $\mathcal{F}_{\alpha A}$ are even respectively odd. The parity of the θ-coefficients in the expansion is thus alternating. This is the situation considered in [6]. In case of a plain connection on $\mathcal{E}$, the odd coefficients in the $\mathcal{A}_\mu$-expansion would be missing, and analogous for $\mathcal{F}_{\alpha A}$.

Let $\mathcal{E} \to N$ be a super vector bundle over N and $\varphi\colon M \to N$ be a morphism of supermanifolds. The pullback of $\mathcal{E}$ under φ is defined as

$$\varphi^*\mathcal{E}(U) := \mathcal{O}_M(U) \otimes_\varphi (\varphi_0^*\mathcal{E})(U)\,, \qquad U \subseteq M_0 \quad \text{open}\,. \tag{7}$$

Here, $\varphi_0^*\mathcal{E}$ is the pullback of the sheaf $\mathcal{E}$ under the continuous map φ_0 which, in terms of its sheaf space, is the bundle of stalks $\mathcal{E}_{\varphi_0(x)}$ attached to $x \in M_0$. In this context, one can define the pullback $\varphi_0^*X \in \varphi_0^*\mathcal{E}$ of $X \in \mathcal{E}$. (7) indeed yields a super vector bundle on M of rank $\mathrm{rk}\,\mathcal{E}$. For details, consult [16] and [26].

A local frame (T^k) of $\mathcal{E}$ gives rise to a local frame $(\varphi_0^*T^k)$ of $\varphi_0^*\mathcal{E}$ and a local frame $(\varphi^*T^k := 1 \otimes_\varphi \varphi_0^*T^k)$ of $\varphi^*\mathcal{E}$ such that, locally, every section $X \in \varphi^*\mathcal{E}$ can be written $X = \varphi^*T^k \cdot X^k$ with $X^k \in \mathcal{O}_M(U)$. For $Y = T^kY^k \in \mathcal{E}$, we find

$$\varphi^*Y = \varphi^*(T^kY^k) = \varphi^*T^k \cdot \varphi^\sharp(Y^k)\,.$$

Definition 5. In the following, we shall identify maps $\varphi\colon S \times M \to N$ with maps $\hat{\varphi}\colon S \times M \to S \times N$ by composing φ with the canonical inclusion $N \hookrightarrow S \times N$.

In particular, we will use this identification for S-points $x\colon S \to M$ and S-paths $\gamma\colon S \times [0,1] \to M$. In terms of generators η^j as in (4), the construction is such that $\hat{\varphi}^\sharp(\eta^j) = \eta^j$.

Lemma 2. *Let $\varphi\colon S \times M \to N$ and $\mathcal{E} \to N$ be a super vector bundle. Then $\varphi^*\mathcal{E} \cong \hat{\varphi}^*\mathcal{E}_S$.*

Locally, this isomorphism is such that $X = (\hat{\varphi}^*T^k)\cdot X^k \in \hat{\varphi}^*\mathcal{E}_S$ is identified with $X = \varphi^*T^k \cdot X^k \in \varphi^*\mathcal{E}$. We define the pullback of $X \in \mathcal{E}_S$ under $\varphi\colon S \times M \to N$ by

$$\varphi^*X := \hat{\varphi}^*X \in \hat{\varphi}^*\mathcal{E}_S \cong \varphi^*\mathcal{E}\,. \tag{8}$$

Similarly, an endomorphism $E \in \mathrm{End}_{\mathcal{O}_{S\times N}}(\mathcal{E}_S)$ is pulled back under φ to an endomorphism along φ as follows.

$$E_\varphi \in \mathrm{End}_{\mathcal{O}_{S\times M}}(\hat{\varphi}^*\mathcal{E}_S)\,, \qquad E_\varphi(\varphi^*Y) := \varphi^*E(Y) \tag{9}$$

and analogous for other tensors.

Let ∇ be a connection on $\mathcal{E} \to N$ and $\varphi\colon M \to N$ be a morphism. There are two types of pullback connections. With respect to coordinates (ξ^k) of M, we write $X = (\varphi^*\partial_{\xi^i}) \cdot X^i \in \varphi^*\mathcal{S}N$ and prescribe

$$\begin{aligned}&(\varphi^*\nabla)\colon \varphi_0^*\mathcal{E} \to (\varphi^*\mathcal{S}N)^* \otimes_{\mathcal{O}_M} \varphi^*\mathcal{E}\\ &(\varphi^*\nabla)_{(\varphi^*\partial_i)X^i}(\varphi^*Z) := (-1)^{|X^i||\partial_i|}X^i \cdot \varphi^*(\nabla_{\partial_i}Z)\end{aligned} \tag{10}$$

The local representations glue together to a well-defined object satisfying a Leibniz rule. For the second, more common, pullback note that $X \in \varphi^*\mathcal{S}N$ acts naturally on sections $f \in \mathcal{O}_N$ as the superderivation $X(f) := (-1)^{|X^i||f|}(\varphi^\sharp \circ \partial_{\xi^i})(f) \cdot X^i$ along φ. We define

$$\begin{aligned}&(\varphi^*\nabla)\colon \varphi^*\mathcal{E} \to \mathcal{S}M^* \otimes_{\mathcal{O}_M} \varphi^*\mathcal{E}\\ &(\varphi^*\nabla)_X((\varphi^*T^k)Z^k) := (-1)^{|X||T^k|}(\varphi^*T^k) \cdot X(Z^k) + (\varphi^*\nabla)_{d\varphi[X]}\varphi^*T^k \cdot Z^k\end{aligned} \tag{11}$$

using (5) and (10) for the second summand. Again, this prescription is independent of coordinates and $\mathcal{E}$-bases and yields a connection on $\varphi^*\mathcal{E} \to M$.

Let now ∇ be an S-connection on $\mathcal{E}_S$ over N and $\varphi\colon S \times M \to N$. We may consider ∇ as an ordinary connection over $S \times N$ and apply (10) to obtain

$$(\hat{\varphi}^*\nabla)\colon \hat{\varphi}_0^*\mathcal{E}_S \to (\hat{\varphi}^*\mathcal{S}(S \times N))^* \otimes_{\mathcal{O}_{S\times M}} \hat{\varphi}^*\mathcal{E}_S\,.$$

Concatenating this with the adjoint of the inclusion $\mathcal{S}N_S \subseteq \mathcal{S}(S \times N)$, and using $\hat{\varphi}_0^*\mathcal{E}_S = \varphi_0^*\mathcal{E} \otimes \mathcal{O}_S$ as well as $\hat{\varphi}^*\mathcal{S}N_S = \varphi^*\mathcal{S}N$, we yield the first pullback, denoted

$$(\varphi^*\nabla)\colon \varphi_0^*\mathcal{E} \otimes \mathcal{O}_S \to (\varphi^*\mathcal{S}N)^* \otimes_{\mathcal{O}_{S\times M}} \varphi^*\mathcal{E}\,. \tag{12}$$

The second pullback is the connection

$$(\varphi^*\nabla)\colon \varphi^*\mathcal{E} \to \mathcal{S}M_S^* \otimes_{\mathcal{O}_{S\times M}} \varphi^*\mathcal{E} \tag{13}$$

defined verbatim to (11) by means of (12) The local picture is as follows.

$$(\varphi^*\nabla)_X Z = (-1)^{|X||T^k|}(\varphi^*T^k)X(Z^k) + X(\varphi^*(\xi^l))\hat{\varphi}^*(\nabla_{\partial_{\xi^l}}T^k)\cdot Z^k \tag{14}$$

2.2 Parallel Transport

Definition 6. A section $X \in \gamma^*\mathcal{E}$ is called parallel if $(\gamma^*\nabla)_{\partial_t}X \equiv 0$.

The local form is as follows. As above, we write $X = (\gamma^*T^k)\cdot X^k$, thus identifying X with the t-dependent column vector $X(t) \in (\mathcal{O}_S)^{\mathrm{rk}\,\mathcal{E}}$. We further use the notation $\Gamma_{lk}^m \cdot T^m := \Gamma_{\xi^l}[T^k]$ with Γ_{ξ^l} as in (6). By (14), the parallelness condition in local coordinates reads

$$\partial_t X(t) = -\mathcal{B}(t)\cdot X(t)\,, \qquad \mathcal{B}(t)^m{}_k = (-1)^{|T^m|(|T^k|+1)}\partial_t(\gamma^*(\xi^l))\cdot\hat{\gamma}^*(\Gamma_{lk}^m) \tag{15}$$

with $\mathcal{B}(t) \in \mathrm{End}_{\mathcal{O}_S}(\gamma^*\mathcal{E})_{\overline{0}} \cong \mathrm{Mat}_{\mathrm{rk}\,\mathcal{E}\times\mathrm{rk}\,\mathcal{E}}(\mathcal{O}_S)_{\overline{0}}$.

Example 3. In the situation of Example 2, the matrix $\mathcal{B}(t)$ can be written in the form

$$\mathcal{B}(t) = \dot{x}^\mu(t)\mathcal{A}_\mu + \dot{\theta}^{\alpha A}(t)\mathcal{F}_{\alpha A}\,.$$

This is equation (17) of [6].

The next result follows from standard facts on ODEs applied to (15).

Lemma 3. *Let $X_x \in x^*\mathcal{E}$ be a section along an S-point $x\colon S \to M$, and γ be a piecewise smooth S-path with $\mathrm{ev}|_{t=0}\gamma^\sharp = x^\sharp$. Then there exists a unique parallel section $X \in \gamma^*\mathcal{E}$ along γ such that $\mathrm{ev}|_{t=0}X = X_x$.*

Definition 7. Let $\gamma\colon x \to y$ be a smooth S-path and let $X_x \in x^*\mathcal{E}$ be a vector field along $x\colon S \to M$. We define the parallel transport

$$P_\gamma\colon x^*\mathcal{E} \to y^*\mathcal{E}\,, \qquad P_\gamma(X_x) := \mathrm{ev}_{t=1}X$$

where $X \in \gamma^*\mathcal{E}$ denotes the parallel vector field such that $\mathrm{ev}_{t=0}X = X_x$.

For smooth S-paths $\gamma\colon x \to y$ and $\delta\colon y \to z$, we define parallel transport of the concatenation by $P_{\delta\star\gamma} := P_\delta \circ P_\gamma$. If $\delta \star \gamma$ happens to be smooth, this definition agrees with the one from Definition 7 by the following lemma.

Lemma 4. *Let $a < b < c$ and $\gamma\colon S \times [a,c] \to M$ be a smooth S-path. Then*

$$P_{\gamma|_{S\times[b,c]}} \circ P_{\gamma|_{S\times[a,b]}} = P_\gamma\,.$$

Proof. Let $X_x \in x^*\mathcal{E}$. We define $X \in \gamma^*\mathcal{E}$ by setting

$$\begin{aligned} X(t) &:= P_{\gamma|_{S\times[a,t]}}[X_x] \quad (t \in [a,b])\,,\\ X(t) &:= P_{\gamma|_{S\times[b,t]}} \circ P_{\gamma|_{S\times[a,b]}}[X_x] \quad (t \in [b,c])\,. \end{aligned}$$

Then $X(t)$ satisfies $(\gamma^*\nabla)_{\partial_t} X = 0$ for every $t \in [a,c]$ and has the initial condition $X(0) = X_x$. By uniqueness of the solution, we thus conclude that $X(t) = P_{\gamma|_{S\times[a,t]}}[X_x]$. □

Lemma 5. *P_γ is even (i.e. parity-preserving), $\mathcal{O}_S$-superlinear and invertible such that $(P_\gamma)^{-1} = P_{\gamma^{-1}}$.*

Proof. This is shown by standard ODE arguments as follows. Parallel transport is even since the matrix $\mathcal{B}(t)$ in (15) is even. From the same equation, $\mathcal{O}_S$-linearity is clear. It is invertible since both P_γ and $P_{\gamma^{-1}}$ satisfy the same equation (15) at t and $1-t$, respectively. □

By restriction, an S-connection ∇ on $\mathcal{E}_S$ induces a connection

$$\nabla^{\mathcal{E}}\colon \mathcal{E} \to \mathcal{S}M^* \otimes_{\mathcal{O}_M} \mathcal{E}\,.$$

By further restriction, we obtain a classical connection

$$\nabla^0\colon \Gamma(E) \to \Gamma(TM_0) \otimes \Gamma(E)$$

on the vector bundle $E := \bigcup_{x\in M_0} \mathcal{E}_x \to M_0$ (denoted $\tilde{\nabla}$ in [12]). Let $P_{\gamma_0}\colon E_{\gamma_0(0)} \to E_{\gamma_0(1)}$ denote parallel transport along a path $\gamma_0\colon [0,1] \to M_0$ (denoted τ_γ in [12]). On the other hand, let $(P_\gamma)^0\colon E_{\gamma_0(0)} \to E_{\gamma_0(1)}$ denote the restriction of ∇-parallel transport along $\gamma\colon S \times [0,1] \to M$.

Lemma 6. *Let $\gamma\colon x \to y$ be an S-path. Then $(P_\gamma)^0 = P_{\gamma_0}$.*

Proof. This follows immediately from (15). Note that $\partial_t(\gamma^*(\xi^l))$ is odd, for ξ^l an odd coordinate, and thus projected to zero, leaving only even indices l in $\gamma^*(\Gamma^m_{lk})$. □

By Lemma 5, P_γ is an isomorphism from $x^*\mathcal{E}$ to $y^*\mathcal{E}$. With respect to local bases (T^k) and $(\tilde{T}^k)$ of $\mathcal{E}$ around $\gamma_0(0)$ and $\gamma_0(1)$, respectively, it can thus be identified with a matrix in $GL_{\mathrm{rk}\,\mathcal{E}}(\mathcal{O}_S)$.

Lemma 7. *The solution to (15) is given by*

$$X(t) = \mathcal{P}\exp\left(-\int_0^t \mathcal{B}(\tau)d\tau\right)[X_x]$$
$$:= \sum_{j=0}^{\infty}(-1)^j \int_0^t d\tau_j \ldots \int_0^{\tau_2} d\tau_1 \mathcal{B}(\tau_j)\cdot \ldots \cdot \mathcal{B}(\tau_1)X_x$$

where $X_x \in x^*\mathcal{E}$ *and* $x^\sharp = \mathrm{ev}|_{t=0}\gamma^\sharp$.

Proof. By assumption, γ_0 takes values in $U_0 \subseteq M_0$ such that both $M|_{U_0}$ and $\mathcal{E}|_{U_0}$ are trivial. We may thus identify (as vector spaces), for every $t \in [0,1]$, $\mathrm{ev}|_t\gamma^*\mathcal{E}$ with $\mathbb{R}^{\mathrm{rk}\,\mathcal{E}} \otimes \bigwedge \mathbb{R}^L \cong \mathbb{R}^M$ for some $M \in \mathbb{N}$. With this identification, the $\mathbb{R}$-linear operator $\mathcal{B}(t)$ becomes a matrix in $\mathrm{Mat}(M\times M,\mathbb{R})$, and $\partial_t X(t) = -\mathcal{B}(t)\cdot X(t)$ can be considered as a classical first order linear ordinary differential equation. It remains to show that the series stated converges absolutely in the Banach space $C^1([0,1],\mathrm{Mat}(M\times M,\mathbb{R}))$. Then, differentiating termwise, it follows that it is indeed the solution operator. These steps are standard. See Lemma 2.6.7 of [4] for a similar treatment. □

Remark 1. Redefining (6) as $\Gamma_{\xi^i}[T^j] := \frac{i}{g}\nabla_{\partial_{\xi^i}}T^j$, we get the parallelness equation $\partial_t X(t) = ig\mathcal{B}(t)\cdot X(t)$, and thus the solution operator

$$X(t) = \mathcal{P}\exp\left(ig\int_0^t \mathcal{B}(\tau)d\tau\right)[X_x]$$
$$:= \sum_{j=0}^{\infty}(ig)^j \int_0^t d\tau_j \ldots \int_0^{\tau_2} d\tau_1 \mathcal{B}(\tau_j)\cdot \ldots \mathcal{B}(\tau_1)X_x$$

as in (3). This convention is more usual in the physical literature.

An important property of the Wilson loop is its gauge-invariance. We close this chapter showing that the trace of parallel transport around an S-loop is gauge--invariant, thus qualifying as a model for the super Wilson loop. We restrict attention to local gauge transformations in a coordinate chart $U \subseteq M$, which is sufficient for the situation $M \cong \mathbb{R}^{n|m}$ considered in [6] and avoids the theory of super principal bundles.

Definition 8. A (local) gauge transformation is a morphism of supermanifolds

$$V\colon S\times U \to GL_{\mathrm{rk}\,\mathcal{E}} \quad \text{identified with} \quad \big(V^\sharp(\zeta^{kl})\big)_{kl} \in GL_{\mathrm{rk}\,\mathcal{E}}\big(\mathcal{O}_{S\times M}(U)\big)$$

where ζ^{kl} denote the global standard coordinates of the super Lie group $GL_{\mathrm{rk}\,\mathcal{E}}$. It acts on sections $\psi \in \mathcal{E}_S(U)$ and connections ∇ via

$$\psi \mapsto V\cdot\psi\,, \qquad \Gamma_{\xi^i} \mapsto V\cdot\Gamma_{\xi^i}\cdot V^{-1} - (\partial_{\xi^i}V)V^{-1}$$

where Γ_{ξ^i} is as in (6).

Consider an S-path $\gamma\colon S\times[0,1]\to U$ and the concatenation

$$V_\gamma := V\circ\hat{\gamma}\colon S\times[0,1]\to GL_{\mathrm{rk}\,\mathcal{E}}\,.$$

Let $\mathcal{B}(t)$ be as in (15) with respect to the original connection ∇ (and γ) and $\tilde{\mathcal{B}}(t)$ be its gauge transformed counterpart. Then

$$\begin{aligned}\tilde{\mathcal{B}}(t) &= \partial_t(\hat{\gamma}^*(\xi^l))\cdot\hat{\gamma}^*\left(V\Gamma_{\xi^l}V^{-1}-(\partial_{\xi^l}V)V^{-1}\right)\\ &= V_\gamma\cdot\mathcal{B}(t)\cdot V_\gamma^{-1}-(\partial_t V_\gamma)\cdot V_\gamma^{-1}\end{aligned}$$

It follows that

$$(\gamma^*\tilde{\nabla})_{\partial_t}(V_\gamma\cdot X)=\left(\partial_t+V_\gamma\cdot\mathcal{B}(t)\cdot V_\gamma^{-1}-(\partial_t V_\gamma)\cdot V_\gamma^{-1}\right)V_\gamma\cdot X=V_\gamma\cdot(\gamma^*\nabla)_{\partial_t}X$$

In particular, $X\in\gamma^*\mathcal{E}$ is ∇-parallel if and only if $V_\gamma\cdot X\in\gamma^*\mathcal{E}$ is $\tilde{\nabla}$-parallel.

Now let $X_x\in x^*\mathcal{E}$, and let $\gamma\colon x\to y$ connect the S-points x and y. Then, $V_x\cdot X_x$ with $V_x:=V\circ\hat{x}$ is moved by $\tilde{\nabla}$-parallel transport to V_y times ∇-parallel transport of X_x. We thus arrive at the following result.

Proposition 1. *Let $\tilde{P}$ denote parallel transport with respect to the gauge transformed connection $\tilde{\nabla}$. Then $\tilde{P}=V_y\cdot P\cdot V_x^{-1}$. In particular, if $\gamma\colon x\to x$ is closed,*

$$\tilde{P}=V_x\cdot P\cdot V_x^{-1}$$

and the trace $\mathrm{tr}\,P=\mathrm{tr}\,\tilde{P}$ is a gauge invariant quantity.

By now, we have achieved the first aim of this article of constructing a mathematical model of super Wilson loops. Superpoints are S-points, and a super Wilson loop is the gauge-invariant trace of parallel transport around an S-loop. The exact choice of $S=\mathbb{R}^{0|L}$ is not important, except that L should be sufficiently large to make calculations consistent. By means of S, the super Wilson loop acquires an (unphysical) inner structure.

3 The Holonomy of an S-Point

Let $\mathcal{E}$ continue to denote a super vector bundle over a supermanifold M and ∇ be an S-connection on $\mathcal{E}_S$ with S a superpoint (4). In this section, we define the holonomy group of an S-point $x\colon S\to M$ and prove an analogon of the Ambrose--Singer theorem. After endowing the holonomy group to a functor, we establish a holonomy principle in this context, whose proof makes use of at least $(\dim M)_{\bar{1}}$ additional Graßmann generators.

Definition 9. A piecewise smooth S-homotopy is a map

$$\Xi\colon S\times[0,1]\setminus\{t_0,\ldots,t_l\}\times[0,1]\to M$$

such that, denoting the real coordinates by t and s, respectively,

(i) the prescription $\Xi^\sharp_{s_0}:=\mathrm{ev}|_{s=s_0}\Xi^\sharp$ yields a piecewise smooth S-path Ξ_{s_0} for every $s_0\in[0,1]$, and

(ii) $\Xi^\sharp(f)$ is smooth in s for every $f \in \mathcal{O}_M$.

Ξ is called proper if $\mathrm{ev}_{s,t=0}\Xi^\sharp = p^\sharp$ and $\mathrm{ev}_{s,t=1}\Xi^\sharp = q^\sharp$ for all $s \in [0,1]$ and S-points p and q.

Definition 10 (S-Holonomy). Let $x\colon S \to M$ be an S-point. We set

$$\mathrm{Hol}_x := \{P_\gamma \mid \gamma\colon x \to x \text{ piecewise smooth}\} \subseteq \mathrm{End}_{\mathcal{O}_S}(x^*\mathcal{E})$$
$$\mathrm{Hol}_x^0 := \{P_\gamma \mid \gamma\colon x \to x \text{ piecewise smooth and contractible}\}$$

with contractible in the sense that there exists a piecewise smooth proper homotopy Ξ such that $\Xi_0 = x$ and $\Xi_1 = \gamma$.

By Lemma 5, Hol_x is a group which can be identified with a subgroup of $GL_{\mathrm{rk}\,\mathcal{E}}(\mathcal{O}_S)$ with respect to a local basis (T^k) of $\mathcal{E}$. By Theorem 1 below, it is indeed a Lie group. For $S = \mathbb{R}^{0|0}$, it follows by Lemma 6 that $\mathrm{Hol}_x = \mathrm{Hol}_{\nabla^0}(x_0(0))$ is the holonomy group with respect to the underlying connection ∇^0.

We call M path-connected if, for any two S-points x, y, there is an S-path $\gamma\colon x \to y$. By the following result this, as well as contractability, is determined by the classical counterparts such that, in particular, Hol_x does not depend on the restriction of M to any connected component of M_0 different from that of $x_0(0)$.

Lemma 8. *M is path-connected if and only if M_0 is. Moreover, a piecewise smooth S-loop $\gamma\colon x \to x$ is contractible to x if and only if γ_0 is contractible to x_0.*

Proof. It is clear that path-connectedness of M implies that of M_0. Conversely, let $x, y\colon S \to M$ and $\gamma_0\colon x_0(0) \to y_0(0)$ be a connecting classical path. Let $t_j \in [0,1]$ be such that $\gamma_0|_{[t_j,t_{j+1}]}$ is smooth and its image is contained in the open set U_0 for a coordinate chart $U \subseteq M$ with coordinates (ξ^k). Any (smooth) morphism $\gamma^j\colon S \times [t_j, t_{j+1}] \to U$ can be identified with $(\dim M)_{\bar 0} + (\dim M)_{\bar 1}$ smooth maps $\gamma^{j\sharp}(\xi^k)\colon [t_j, t_{j+1}] \to \bigwedge \mathbb{R}^L$ or, equivalently, with a single smooth map $\tilde\gamma^j\colon [t_j, t_{j+1}] \to \mathbb{R}^M$ for some $M \in \mathbb{N}$. An S-path $\gamma\colon x \to y$ with underlying path γ_0 can then be constructed by glueing together suitable maps $\tilde\gamma^j$. The details are standard and thus omitted. The proof of the second statement is similar. □

3.1 An Ambrose-Singer Theorem

The classical Ambrose-Singer theorem characterises the holonomy Lie algebra in terms of the curvature of the connection considered. In this section, we show that this theorem continues to hold in the more general situation of S-holonomy in the sense of Definition 10. Our proof is modelled on a classical proof due to Levi-Civita as presented in [3]. We define the curvature of ∇ as usual by

$$R(X, Y)\,Z := \nabla_X\nabla_Y Z - (-1)^{|X||Y|}\nabla_Y\nabla_X Z - \nabla_{[X,\,Y]}Z$$

for $X, Y \in \mathcal{S}M_S$ and $Z \in \mathcal{E}_S$, where $[X, Y] := XY - (-1)^{|X||Y|}YX$ is the supercommutator. This definition is such that

$$R \in \mathrm{Hom}_{\mathcal{O}_{S\times M}}\left(\mathcal{S}M_S \otimes_{\mathcal{O}_{S\times M}} \mathcal{S}M_S \otimes_{\mathcal{O}_{S\times M}} \mathcal{E}_S,\, \mathcal{E}_S\right)_{\bar 0}.$$

The curvature is skew-symmetric

$$R\,(Y,\,X) = -(-1)^{|X||Y|}R\,(X,\,Y)$$

which is inherited to the pullback. Let $\varphi\colon S \times N \to M$ be a supermanifold morphism. Then

$$R_\varphi\,(A,\,B) = -(-1)^{|A||B|}R_\varphi\,(B,\,A) \tag{16}$$

for $A, B \in \varphi^*\mathcal{S}M$. This is shown by a straightforward calculation in coordinates, writing $A = (\varphi^*\partial_{\xi^k}) \cdot A^k$ etc.

Definition 11. Let $x\colon S \to M$ be an S-point. Let $\mathfrak{g}_x$ denote the Lie subalgebra of $(\mathfrak{gl}_{\mathrm{rk}\,\mathcal{E}}(\bigwedge\mathbb{R}^L))_{\bar{0}}$ which is generated by the following set of endomorphisms.

$$\{P_\gamma^{-1} \circ R_y\,(u,\,v) \circ P_\gamma \;\big|\; y\colon S \to M\,,\ \gamma\colon x \to y \text{ piecewise smooth}\,,\ u, v \in (y^*\mathcal{S}M)_{\bar{0}}\}$$

Hol_x is contained in $GL_{\mathrm{rk}\,\mathcal{E}}(\bigwedge\mathbb{R}^L)$. By the following lemma, this is a Lie group. In general, every Lie subalgebra of the Lie algebra of a Lie group is the Lie algebra of a unique immersed connected Lie subgroup (see Chapter 2 of [13]). Let $G_x \subseteq GL_{\mathrm{rk}\,\mathcal{E}}(\bigwedge\mathbb{R}^L)$ denote this Lie subgroup corresponding to $\mathfrak{g}_x \subseteq (\mathfrak{gl}_{\mathrm{rk}\,\mathcal{E}}(\bigwedge\mathbb{R}^L))_{\bar{0}}$.

Lemma 9. *$GL_{n|m}(\bigwedge\mathbb{R}^L)$ is a real Lie group with Lie algebra $(\mathfrak{gl}_{n|m}(\bigwedge\mathbb{R}^L))_{\bar{0}}$.*

Proof. $M \in (\mathfrak{gl}_{n|m}(\bigwedge\mathbb{R}^L))_0$ is invertible if and only if its image under the canonical projection to $\mathfrak{gl}_{n|m}(\mathbb{R})$ is (Lemma 3.6.1 in [27]). Therefore

$$GL_{n|m}\left(\bigwedge\mathbb{R}^L\right) = (GL_n(\mathbb{R}) \times GL_m(\mathbb{R})) \oplus \left(\mathfrak{gl}_{n|m}\left(\bigwedge(\mathbb{R}^L)_{\mathrm{nilpotent}}\right)\right)_{\bar{0}}$$

which is open in $(\mathfrak{gl}_{n|m}(\bigwedge\mathbb{R}^L))_{\bar{0}}$ and as such a submanifold with a group structure such that the tangent space at 1 can be identified with $(\mathfrak{gl}_{n|m}(\bigwedge\mathbb{R}^L))_{\bar{0}}$. Writing the matrix entries of a product $M \cdot L$ in terms of real coefficients of odd generators, it is clear that multiplication is smooth, and similar for inversion. One further shows that the Lie algebra commutator coincides with the commutator $[X,\,Y] = XY - YX$. $\square$

Theorem 1 (Ambrose-Singer Theorem). *The Lie groups $G_x = \mathrm{Hol}_x^0$ coincide. In particular, Hol_x is a Lie group with identity component Hol_x^0 and Lie algebra $\mathrm{hol}_x = \mathfrak{g}_x$.*

We defer the proof of the theorem to the end of the present section. It is based on Proposition 2 and Proposition 3 below. The following two lemmas are needed in the proof of the first proposition.

Lemma 10. *Let $f\colon S \times [a,b] \times [b,c] \to M$ be a morphism and $X \in f^*\mathcal{E}$ be a section along f. Then*

$$(f^*\nabla)_{\partial_s}(f^*\nabla)_{\partial_t}X - (f^*\nabla)_{\partial_t}(f^*\nabla)_{\partial_s}X = R_f\,(df[\partial_s],\,df[\partial_t])\,X$$

where (s,t) denote the standard coordinates on $[a,b] \times [b,c]$.

Proof. This is shown by a direct calculation in local coordinates (ξ^k) of M and a trivialisation (T^k) of $\mathcal{E}$, writing $X = (\varphi^* T^l) \cdot X^l$ with $X^l \in \mathcal{O}(S \times [a,b] \times [b,c])$. $\square$

Let $x, y \colon S \to M$ and $\gamma \colon x \to y$. A tuple $(e^1, \ldots, e^k)$ of sections $e^j \in \gamma^*\mathcal{E}$ is a basis of $\gamma^*\mathcal{E}$ if and only if $(\mathrm{ev}|_{t=t_0} e^1, \ldots, \mathrm{ev}|_{t=t_0} e^k)$ is a basis of $\mathrm{ev}|_{t=t_0}\gamma^*\mathcal{E}$ for every $t_0 \in [0,1]$. It is called parallel if all e^i are parallel. Such a basis is determined by its evaluation at $t = 0$ via $\mathrm{ev}|_{t=t_0} e^j = P_{\gamma|_{S\times[0,t_0]}}(\mathrm{ev}|_{t=0} e^j)$. In particular, a parallel basis, as used in the proof of the following lemma, exists.

Lemma 11. *Let $X \in \gamma^*\mathcal{E}$ be a section along γ. Let $P_t := P^{-1}_{\gamma|_{S\times[0,t]}}$ be the parallel displacement from $\mathrm{ev}|_t\gamma^\sharp$ to $x^\sharp = \mathrm{ev}|_{t=0}\gamma^\sharp$. Then*

$$P_t \mathrm{ev}|_t (\gamma^*\nabla)_{\partial_t} X = \partial_t P_t(\mathrm{ev}|_t X) \in x^*\mathcal{E}$$

Proof. Let (e^j) be a parallel basis along γ. Writing $X = e^i \cdot X^i$ with $X^i \in \mathcal{O}_{S\times[0,1]}$, it follows that $P_t(\mathrm{ev}|_t X) = \mathrm{ev}_{t=0} e^i \cdot \mathrm{ev}|_t X^i$, and

$$\partial_t P_t(\mathrm{ev}|_t X) = \mathrm{ev}|_{t=0} e^i \cdot \mathrm{ev}|_t(\partial_t X^i)$$

On the other hand, $(\gamma^*\nabla)_{\partial_t} X = e^i \cdot \partial_t(X^i)$ implies

$$P_t \mathrm{ev}|_t (\gamma^*\nabla)_{\partial_t} X = P_t(\mathrm{ev}|_t e^i \cdot \mathrm{ev}|_t(\partial_t X^i)) = \mathrm{ev}|_{t=0} e^i \cdot \mathrm{ev}|_t(\partial_t X^i)$$

such that both sides agree. $\square$

For the following proposition note that, for a proper S-homotopy Ξ, we may identify $\mathrm{ev}|_{t=0}\Xi^\sharp$ and $\mathrm{ev}|_{t=1}\Xi^\sharp$ with single S-points $x, y \colon S \to M$, respectively.

Proposition 2. *Let Ξ be a proper S-homotopy, and let $P_{s,t} := P_{\Xi_s|_{S\times[t,1]}}$ denote parallel transport along the restriction of the S-path Ξ_s to $S \times [t,1]$. Then*

$$\partial_s P_{s,0} = \left(\int_0^1 R_{s,t} dt \right) P_{s,0} \in \mathrm{Hom}_{\mathcal{O}_{S\times[0,1]}}(x^*\mathcal{E},\, y^*\mathcal{E})$$

with $R_{s,t} := P_{s,t} \mathrm{ev}|_{s,t} R_\Xi\left(d\Xi[\partial_t],\, d\Xi[\partial_s]\right) P^{-1}_{s,t}$

Proof. Let $Z \in \Xi^*\mathcal{E}$. For Ξ proper, the term $\partial_s(\Xi_s^*(\xi^l))$ in (15) vanishes for $t = 0$ as well as $t = 1$, such that

$$\mathrm{ev}|_{s,t=0}(\Xi^*\nabla)_{\partial_s} Z = \partial_s \mathrm{ev}|_{s,t=0} Z\ , \qquad \mathrm{ev}|_{s,t=1}(\Xi^*\nabla)_{\partial_s} Z = \partial_s \mathrm{ev}|_{s,t=1} Z$$

Consider Z such that the first term vanishes and, moreover, $(\Xi^*\nabla)_{\partial_t} Z \equiv 0$. By Lemma 11 and Lemma 10, we yield

$$\begin{aligned} \partial_t P_{s,t} \mathrm{ev}|_{s,t} (\Xi^*\nabla)_{\partial_s} Z &= P_{s,t} \mathrm{ev}_{s,t} (\Xi^*\nabla)_{\partial_t} (\Xi^*\nabla)_{\partial_s} Z \\ &= P_{s,t} \mathrm{ev}|_{s,t} R_\Xi\left(d\Xi[\partial_t],\, d\Xi[\partial_s]\right) Z \\ &= R_{s,t} \mathrm{ev}_{s,t=1} Z \end{aligned}$$

since $\mathrm{ev}_{s,t}Z = P_{s,t}^{-1}\mathrm{ev}_{s,t=1}Z$ by assumption. This, together with the assumptions on Z and $P_{s,1} = \mathrm{id}$, implies the following.

$$\begin{aligned}\partial_s P_{s,0}\mathrm{ev}|_{s,t=0}Z &= \partial_s \mathrm{ev}|_{s,t=1}Z \\ &= P_{s,1}\mathrm{ev}|_{s,t=1}(\Xi^*\nabla)_{\partial_s}Z - P_{s,0}\mathrm{ev}|_{s,t=0}(\Xi^*\nabla)_{\partial_s}Z \\ &= \int_0^1 \partial_t \left(P_{s,t}\mathrm{ev}_{s,t}(\Xi^*\nabla)_{\partial_s}Z\right) dt \\ &= \left(\int_0^1 R_{s,t}dt\right) P_{s,0}\mathrm{ev}|_{s,t=0}Z\end{aligned}$$

Let $Z_x \in x^*\mathcal{E}$ .Then, setting $Z(s,t) := P_{\Xi_s|_{S\times[0,t]}}Z_x$, defines a section $Z \in \Xi^*\mathcal{E}$ that satisfies the assumptions made in the beginning of the proof as well as $\mathrm{ev}|_{s,t=0}Z = Z_x$, such that the equation to be proved holds applied to Z_x. Since Z_x was arbitrary, it holds in general. $\square$

Let $a\colon S \to M$ be an S-point and $u, v \in (a^*\mathcal{S}M)_{\overline{0}}$. With respect to local coordinates (ξ^i) on $U \subseteq M$ around $a_0(0)$, we write $u = (a^*\partial_{\xi^i}) \cdot u^i$ with $u^i \in \mathcal{O}_S$ and likewise for v. Let (x, y) denote standard coordinates of $\mathbb{R}^2$. Then the map

$$f\colon S \times \mathbb{R}^2 \to U\ , \qquad f^\sharp(\xi^i) := a^\sharp(\xi^i) + (-1)^{|\xi^i|}u^i \cdot x + (-1)^{|\xi^i|}v^i \cdot y$$

is such that

$$\mathrm{ev}|_{(x,y)=(0,0)}f^\sharp = a^\sharp\ , \qquad \mathrm{ev}|_{(0,0)}df[\partial_x] = u\ , \qquad \mathrm{ev}|_{(0,0)}df[\partial_y] = v \tag{17}$$

Consider also the following piecewise smooth homotopy $g\colon S \times [0,1] \times [0,1] \to \mathbb{R}^2$.

$$g_0(s,t) := \begin{cases} (4st, 0) & 0 \le t \le 1/4 \\ (s, s(4t-1)) & 1/4 \le t \le 1/2 \\ (s(3-4t), s) & 1/2 \le t \le 3/4 \\ (0, 4s(1-t)) & 3/4 \le t \le 1 \end{cases}\ , \qquad \begin{aligned} g^\sharp(x) &:= g_0^*(x) \\ g^\sharp(y) &:= g_0^*(y)\end{aligned}$$

Proposition 3. *Let $a\colon S \to M$ be an S-point and $u, v \in (a^*\mathcal{S}M)_{\overline{0}}$. Let f be such that* (17)*, and let P_s denote parallel translation along Ξ_s for*

$$\Xi := f \circ \hat{g}\colon S \times [0,1] \times [0,1] \to U \subseteq M$$

Then

$$\mathrm{ev}|_{s=0}\partial_s P_s = 0\ , \qquad \mathrm{ev}|_{s=0}\partial_s\partial_s P_s = 2R_a\,(v,\,u)$$

Proof. By Lemma 1, we have $d\Xi[\partial_t] = (\Xi^*\partial_l)g^\sharp(d{f^l}_i)\partial_t g^\sharp(x^i)$ where x^i runs over x and y. For $t \le 1/4$,

$$R_\Xi\left(d\Xi[\partial_t],\, d\Xi[\partial_s]\right) = R_\Xi\left((\Xi^*\partial_l)g^\sharp(d{f^l}_x)4s,\, (\Xi^*\partial_m)g^\sharp(d{f^m}_x)4t\right) = 0$$

vanishes by skew-symmetry (16), and analogous for $t \ge 3/4$. For $1/4 \le t \le 3/4$, we find

$$R_\Xi\left(d\Xi[\partial_t],\, d\Xi[\partial_s]\right) = -R_\Xi\left((\Xi^*\partial_l)g^\sharp(d{f^l}_x),\, (\Xi^*\partial_m)g^\sharp(d{f^m}_y)\right) \cdot 4s\,.$$

Using (17), we further calculate, for $1/4 \le t \le 3/4$,

$$\begin{aligned} R_{s,t} &= P_{s,t}\mathrm{ev}|_{s,t} R_\Xi \left(d\Xi[\partial_t],\, d\Xi[\partial_s]\right) P_{s,t}^{-1} \\ &= 4s \cdot P_{s,t}\mathrm{ev}|_{s,t} R_\Xi \left((\Xi^*\partial_l) g^\sharp(df^l{}_y),\, (\Xi^*\partial_m) g^\sharp(df^m{}_x)\right) P_{s,t}^{-1} \\ &= 4s \cdot P_{s,t} R_a \left((a^*\partial_l)v^l,\, (a^*\partial_m)u^m\right) P_{s,t}^{-1} . \end{aligned}$$

Proposition 2 now yields

$$\partial_s P_s = 4s \left(\int_{\frac{1}{4}}^{\frac{3}{4}} P_{s,t} R_a\left(v,\, u\right) P_{s,t}^{-1} dt \right) P_s$$

which vanishes for $s \to 0$. Likewise

$$\mathrm{ev}_{s=0}\partial_s(\partial_s P_s) = \lim_{s\to 0} \left(\frac{1}{s} 4s \left(\int_{\frac{1}{4}}^{\frac{3}{4}} P_{s,t} R_a\left(v,\, u\right) P_{s,t}^{-1} dt \right) P_s \right) = 2R_a\left(v,\, u\right) . \quad \square$$

Proof. [Proof of Theorem 1] Let $\gamma\colon x \to x$ be piecewise smooth and contractible. We choose a piecewise smooth proper homotopy Ξ such that $\Xi_0 = x$ and $\Xi_1 = \gamma$, and let $P_s := P_{\Xi_s} \in GL_{\mathrm{rk}\,\mathcal{E}}(\bigwedge \mathbb{R}^L)$ denote parallel translation along Ξ_s. By Proposition 2, it satisfies the differential equation

$$\partial_s P_s = g(s) \cdot P_s\ , \qquad g(s) := \left(\int_a^b R_{s,t} dt \right) \in \mathfrak{g}_x$$

By standard Lie group theory (cf. Chapter 2 of [13]), we conclude that $P_s \in G_x$ and, in particular, $P_\gamma = P_1 \in G_x$. Therefore, $\mathrm{Hol}_x^0 \subseteq G_x$ is a path-connected subgroup. By a theorem of Yamabe [28], it is a Lie subgroup.

Let a be an S-point, $\gamma\colon x \to a$ and $u, v \in (a^*\mathcal{S}M)_{\bar{0}}$. Let Ξ be as in Proposition 3, and let $P_s \in \mathrm{Hol}_x^0$ denote parallel translation along $\hat{\Xi}_s := \gamma \star \Xi_s \star \gamma^{-1}$. Then

$$\begin{aligned} \partial_s P_s|_{s=0} &= P_\gamma \circ \partial_s P_{\Xi_s}|_{s=0} \circ P_\gamma^{-1} = 0 \\ \partial_s \partial_s P_s|_0 &= P_\gamma \circ \partial_s \partial_s P_{\Xi_s}|_0 \circ P_\gamma^{-1} = 2P_\gamma \circ R\left(v,\, u\right) \circ P_\gamma^{-1} \end{aligned}$$

by Proposition 3. Hol_x^0 can be identified with a submanifold of some $\mathbb{R}^M$. By the vanishing of the first derivative we can thus conclude that $\partial_s \partial_s P_s|_0 \in \mathrm{hol}_x = T_e(\mathrm{Hol}_x^0)$. Therefore, all generators of $\mathfrak{g}_x$ are contained in hol_x. It follows that $\mathfrak{g}_x = \mathrm{hol}_x$ and $\mathrm{Hol}_x^0 = G_x$. $\square$

3.2 The Holonomy Group Functor

So far, we have considered a fixed superpoint $S = \mathbb{R}^{0|L}$ along with an S-connection ∇ on an S-bundle $\mathcal{E}_S$. In Section 2, it was argued that having S-connections (compared to plain connections in $\mathcal{E}$) is necessary to model superconnections as in [6], whereas the exact value of L cannot have any physical significance. But also for purely mathematical reasons, it is desirable to allow for extending the number of auxiliary Graßmann generators, as will become clear in the proof of the holonomy principle (Theorem 2) below. This extension results in a categorical theory to be described next.

Let ∇ be an S-connection on $\mathcal{E}_S$ with respect to $S = \mathbb{R}^{0|L}$, and let $T = \mathbb{R}^{0|L'}$ be another superpoint. By $\bigwedge \mathbb{R}^{L'}$-linear extension, ∇ can be considered as an $S \times T$-connection on $\mathcal{E}_{S\times T}$. Similarly, an S-point $x\colon S \to M$ canonically induces an $S \times T$-point $x_T\colon S \times T \to M$ by composing x with the canonical projection $S \times T \to S$. For the next proposition, note that a morphism $\varphi\colon T \to T'$ can be identified with a Graßmann algebra morphism φ^* and as such acts naturally on $GL_{\mathrm{rk}\,\mathcal{E}}(\mathcal{O}_{T'})$.

Proposition 4. *The assignment*

$$T \mapsto \mathrm{Hol}_x(T) := \mathrm{Hol}_{x_T}\ , \qquad (\varphi\colon T \to T') \mapsto \big(L \mapsto \varphi^*(L),\, \mathrm{Hol}_{x_{T'}} \to \mathrm{Hol}_{x_T}\big)$$

defines a group-valued functor.

In the following, we will denote both the holonomy with respect to x and the induced holonomy functor by Hol_x. We will also use the notation $\mathrm{hol}_x(T) := \mathrm{hol}_{x_T}$.

Proof. Let $L \in \mathrm{Hol}_{x_{T'}}$. We must show that the pullback $\varphi^*(L)$ is indeed contained in Hol_{x_T}. Then the induced map $\mathrm{Hol}_{x_{T'}} \to \mathrm{Hol}_{x_T}$ is clearly a group homomorphism.

Let $\gamma\colon x_{T'} \to x_{T'}$ be such that $L = P_\gamma$, and prescribe

$$x_\varphi := x_{T'} \circ (\mathrm{id}_S \times \varphi)\colon S \times T \to M\,,$$
$$\gamma_\varphi := \gamma \circ \varphi := \gamma \circ (\mathrm{id}_S \times \varphi \times \mathrm{id}_{[0,1]})\colon x_\varphi \to x_\varphi\,.$$

It is clear that $x_\varphi = x_T$ independent of φ. Let $\mathcal{B}(t)$ be as in (15) with respect to γ. It follows that the local parallelness condition with respect to γ_φ reads

$$\partial_t X(t) = -\big(\varphi^*\mathcal{B}(t)\big) \cdot X(t)\,.$$

We can, therefore, conclude that $X \in \gamma^*\mathcal{E}$ parallel along γ implies that $\varphi^* X \in \gamma_\varphi^*\mathcal{E}$ is parallel along γ_φ. Therefore

$$\varphi^*\left(P_\gamma[X_{x_{T'}}]\right) = P_{\gamma_\varphi}\left[\varphi^*(X_{x_{T'}})\right] \quad \text{for all } X_{x_{T'}} \in x_{T'}^*\mathcal{E}$$

and $\varphi^*(L) = \varphi^* P_\gamma = P_{\gamma_\varphi} \in \mathrm{Hol}_{x_T}$. □

The Molotkov-Sachse theory defines a supermanifold to be a certain functor from the category Gr of Graßmann algebras to that of smooth manifolds [22], [24] such that, in the finite-dimensional case, the resulting category is equivalent to that of Berezin-Kostant-Leites supermanifolds. It is thus natural to conjecture that Hol_x is representable in that it defines such a supermanifold. If this was true, a neighbourhood of 1 in $\mathrm{Hol}_x(T)$ would be isomorphic to $(V \otimes \bigwedge \mathbb{R}^{L'})_{\overline{0}}$ for a fixed finite-dimensional super vector space V. It would follow that

$$\mathrm{hol}_x(T) \cong T_e(\mathrm{Hol}_x(T)) \cong (V \otimes \bigwedge \mathbb{R}^{L'})_{\overline{0}}$$

such that, in particular, $\mathrm{hol}_x(\bigwedge \mathbb{R}^0) = V_{\overline{0}}$. The following example shows that the holonomy functor is, in general, not representable.

Example 4. Consider $S := \mathbb{R}^{0|0}$ and $M := \mathbb{R}^{0|1}$ with the (S-)connection defined by $\nabla_{\partial_\theta}\partial_\theta = \theta\partial_\theta$ on $\mathcal{E}_S := \mathcal{S}M_S = \mathcal{S}M$ such that $R\,(\partial_\theta,\,\partial_\theta)\,\partial_\theta = 2\partial_\theta$. Let 0 denote the unique S-point corresponding to $0 \in \mathbb{R}^0$. By Theorem 1, $\mathrm{hol}_0(T)$ is generated by $P_\gamma^{-1} \circ R_y\,(u,\,v) \circ P_\gamma$ for $y\colon T \to M$, $\gamma\colon x \to y$ and $u, v \in (y^*\mathcal{S}M)_{\bar{0}}$. We write $u = (y^*\partial_\theta) \cdot u^\theta$ with $u^\theta \in (\mathcal{O}_T)_{\bar{1}}$ and analogous for v. Let $w \in y^*\mathcal{S}M$. Then a short calculation yields

$$P_\gamma^{-1} \circ R_y\,(u,\,v)\,P_\gamma[w] = -2u^\theta v^\theta \cdot w\,.$$

For $T = \mathbb{R}^{0|0}$, u^θ and v^θ vanish, such that $\mathrm{hol}_0 = \{0\}$ is trivial, while

$$\mathrm{hol}_0(T) = \mathfrak{gl}(0|1) \otimes ((\mathcal{O}_T)_{\bar{1}})^2 \subseteq \mathfrak{gl}(0|1) \otimes (\mathcal{O}_T)_{\bar{0}} = (\mathfrak{gl}(0|1) \otimes \mathcal{O}_T)_{\bar{0}}$$

for $T = \mathbb{R}^{0|L'}$, $L' \geq 2$. By the preceding paragraph, the functor $\mathrm{Hol}_0(T)$ is thus not representable.

By the holonomy principle, to be established next, a parallel section $X \in \mathcal{E}_S$ is uniquely determined by its $\mathrm{Hol}_x(T)$-invariant pullback $x^*X \in x^*\mathcal{E}$ as defined in (8), where the number L' of additional generators must be sufficiently large.

Theorem 2 (Holonomy Principle). *Let M be connected. Let ∇ be an S-connection on $\mathcal{E}_S$, $x\colon S \to M$ be an S-point and $T = \mathbb{R}^{0|L'}$ with $L' \geq (\dim M)_{\bar{1}}$. Then the following holds true.*

(i) Let $X \in \mathcal{E}_S$ be a parallel section $\nabla X \equiv 0$ and define $X_x := x^*X \in x^*\mathcal{E}$. Then, for all $y\colon S \times T \to M$ and $\gamma\colon x \to y$, it holds $y^*X = P_\gamma[X_x]$, where X_x is identified with a section of $x_T^*\mathcal{E}$. In particular, X_x is holonomy invariant $\mathrm{Hol}_x(T) \cdot X_x = X_x$.

(ii) Conversely, let $X_x \in x^*\mathcal{E}$ be a section such that $\mathrm{Hol}_x(T) \cdot X_x = X_x$. Then there exists a unique section $X \in \mathcal{E}_S$ with $x^*X = X_x$, which is parallel $\nabla X \equiv 0$.

Proof. Let $\gamma\colon x \to y$ be a piecewise smooth S-path. The assumption $\nabla X \equiv 0$ implies $\nabla_{\partial_t}(\gamma^*X) = 0$. Parallel transport along γ is thus

$$P_\gamma[X_x] = \mathrm{ev}|_{t=1}\gamma^*X = y^*X$$

which proves the first assertion.

Conversely, let $X_x \in x^*\mathcal{E}$ be such that $\mathrm{Hol}_x(T) \cdot X_x = X_x$. For a superpoint $y\colon S \times T \to M$, we define $X_y := P_\gamma[X_x]$ where $\gamma\colon x \to y$ is an $S \times T$-path. Since X_x is $\mathrm{Hol}_x(T)$-invariant, X_y is well-defined independent of the choice of γ. We aim at constructing X out of the set of X_y inductively over the degree of $\mathcal{O}_S$-monomials. Without loss of generality, we may assume that $M \cong \mathbb{R}^{n|m}$ has global coordinates $\xi = (x, \theta)$. For, assume that the statement is true for M replaced by a neighbourhood $U \subseteq M$ of $x_0(0)$, thus resulting in a parallel section $X \in \mathcal{E}_S(U)$. Then, by the first part of the theorem, X satisfies $\mathrm{Hol}_y(T) \cdot X_y = X_y$ for all $y\colon S \times T \to U$. Repeating the local construction in a neighbourhood $V \subseteq M$

of $y_0(0)$ yields a parallel section $\tilde{X} \in \mathcal{E}_S(V)$ which, by uniqueness, agrees with X on the intersection $U_0 \cap V_0$. Without loss of generality, we may further assume that $\mathcal{E}$ is trivial with a global adapted basis (T^j). We expand

$$X_y = X_y|_{\eta^I} \cdot \eta^I = T^j \cdot X_y^j|_{\eta^I} \cdot \eta^I \ , \qquad X = X|_{\eta^I} \cdot \eta^I = T^j \cdot X^j|_{\eta^I} \cdot \eta^I$$

for multiindices $I = (i_1, \ldots, i_{|I|})$ with $1 \leq i_j \leq L$, such that $X_y^j|_{\eta^I} \in \mathbb{R}$ and $X^j|_{\eta^I} \in \mathcal{O}_M$ and $X|_{\eta^I} \in \mathcal{E}$. Similarly, ∇ is characterised by $\Gamma_{ij}^k = \left(\Gamma_{ij}^k\right)|_{\eta^I} \cdot \eta^I$.

In the first step, we construct $X^0 \in \mathcal{E}$. Letting $q := y_0(0)$, we define its value at q by $X^0(q) := X_y|_{\eta^0} = (P_\gamma[X_x])|_{\eta^0}$. By Lemma 6, it arises by classical parallel transport along γ_0. It is thus independent of y such that $q = y_0(0)$, and $X^0(q)$ depends smoothly on q. By (16) of [12] applied to the induced connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, $X^0(q)$ extends to a section $X^0 \in \mathcal{E}$ such that $0 = \nabla^{\mathcal{E}}_{\partial_{\theta^j}} X^0 = (\nabla_{\partial_{\theta^j}} X^0)|_{\eta^0}$. By construction, X^0 satisfies $(y^* X^0)|_{\eta^0} = X^0(q) = (P_\gamma[X_x])|_{\eta^0}$. Again by Lemma 6, we further note that $(\nabla X^0)|_{\theta^0\eta^0} \equiv 0$.

In the second step, we consider multiindices $I = (i_1, \ldots, i_{|I|})$ with $1 \leq i_j \leq L + (\dim M)_{\bar{1}}$, such that $\eta^I \in \mathcal{O}_{S\times T}$. Assume, by induction, that we have constructed $X^N \in \mathcal{E}_S$ for $N \in \mathbb{N}$ such that

- 0_N X^N has an expansion $X^N = \sum_{|I|\leq N} X|_{\eta^I} \cdot \eta^I$ such that $X|_{\eta^I} = 0$ whenever there is $i_j \in I$ with $i_j \geq L+1$.
- 1_N $(y^* X^N)|_{\eta^I} = (P_\gamma[X_x])|_{\eta^I} = X_y|_{\eta^I}$ for every $y\colon S \times T \to M$, $\gamma\colon x \to y$ and $|I| \leq N$.
- 2_N $(\nabla_{\partial_{\theta^j}} X^N)|_{\eta^I} \equiv 0$ for all $|I| \leq N$.
- 3_N $(\nabla_{\partial_{x^j}} X^N)|_{\theta^A\eta^B} \equiv 0$ for all A,B such that $|A| + |B| \leq N$, where $A = (a_1, \ldots, a_{|A|})$ with $1 \leq a_j \leq (\dim M)_{\bar{1}}$.

Condition 1_{N+1} is equivalent to 1_N together with

$$X_y|_{\eta^J} \stackrel{!}{=} (y^* X^{N+1})|_{\eta^J} = y_0^*(X|_{\eta^J}) + (y^* X^N)|_{\eta^J} \qquad \text{for } |J| = N+1$$

We are thus led to define the value of $X|_{\eta^J}$ at q by

$$X|_{\eta^J}(q) := X_y|_{\eta^J} - (y^* X^N)|_{\eta^J} \qquad \text{for } |J| = N+1 \tag{18}$$

This prescription is independent of $y\colon S \times T \to M$ such that $y_0(0) = q$. Indeed, let y^1,y^2 be two such $S \times T$-points and $\gamma^{1,2}\colon x \to y^{1,2}$ be connecting S-paths. Moreover, let $\delta\colon y^1 \to y^2$ be such that $\delta_0(t) \equiv q$.

$$\begin{tikzcd} & y^2 \\ x \arrow[ur, "\gamma^2"] \arrow[r, "\gamma^1"'] & y^1 \arrow[u, "\delta"'] \end{tikzcd}$$

Since X_x is holonomy invariant, we have $X_{y^2} = P_\delta[X_{y^1}]$. We calculate, using (15),

$$\begin{aligned}
\partial_t\left(X_\delta|_{\eta^J} - (\delta^* X^N)|_{\eta^J}\right) &= \left(\partial_t P_\delta|_{[0,t]}[X_{y^1}] - \partial_t \delta^* X^N\right)\Big|_{\eta^J} \\
&= \Big(-(-1)^{|T^m|(|T^n|+1)}(\delta^* T^m)\partial_t(\delta^*(\xi^l))\cdot\hat{\delta}^*(\Gamma^m_{ln})\cdot P_\delta|_{[0,t]}[X_{y^1}]^n \\
&\qquad - \partial_t\delta^*(\xi^l)(\delta^*\circ\partial_{\xi^l})(X^N)\Big)\Big|_{\eta^J}
\end{aligned}$$

By assumption, the term $\partial_t(\delta^*(\xi^l))$ is nilpotent such that, using induction assumption 1_N, we may replace $P_\delta|_{[0,t]}[X_{y^1}]^n$ by $(\delta^* X^N)^n = \delta^* X^{N\,n}$. Therefore, the right hand side equals

$$\begin{aligned}
&\left((\delta^* T^m)\partial_t(\delta^*(\xi^l))\hat{\delta}^*\left(-(-1)^{|T^m|(|T^n|+1)}\Gamma^m_{ln}X^{N\,n} - \partial_{\xi^l}X^{N\,m}\right)\right)\Big|_{\eta^J} \\
&\quad = -\left((\delta^* T^m)\partial_t(\delta^*(\xi^l))\hat{\delta}^*(\nabla_{\partial_{\xi^l}}X^N)^m\right)\Big|_{\eta^J}
\end{aligned}$$

By 2_N and 3_N (and nilpotency of $\partial_t(\delta^*(\xi^l))$), this expression vanishes, thus showing that $X_\delta|_{\eta^J} - (\delta^* X^N)|_{\eta^J}$ is constant, which proves that (18) is well-defined.

We next endow $X|_{\eta^J}(q)$ to a section $X|_{\eta^J} \in \mathcal{E}$ such that

$$X^{N+1} := \sum_{|J|\le N+1} X|_{\eta^J}\cdot\eta^J$$

satisfies 2_{N+1}. 2_N implies that $(\nabla_{\partial_{\theta^m}}X^{N+1})|_{\eta^I} = 0$ with $|I| \le N$ for any such X^{N+1}. Under this induction hypothesis, 2_{N+1} is thus equivalent to $(\nabla_{\partial_{\theta^m}}X^{N+1})|_{\eta^J} = 0$ for $|J| = N+1$ which, in turn, is equivalent to

$$\left(\partial_{\theta^r}\ldots\partial_{\theta^1}\partial_{\theta^m}X^j|_{\eta^J}\right)\big|_{\theta^0} = -(-1)^{|T^j|(|T^i|+1)}\left(\partial_{\theta^r}\ldots\partial_{\theta^1}\left(\Gamma^j_{mi}X^{i\,N+1}\right)\right)\Big|_{\eta^J\theta^0}$$

for all $r \le (\dim M)_{\bar{1}}$. Similar to the construction of $X^0 \in \mathcal{E}$ above, these equations uniquely determine $X|_{\eta^J}$, for $|J| = N+1$, by X^N and $X|_{\eta^J}(q)$, such that 2_{N+1} holds. If any index $l_j \in J$ satisfies $l_j > L$, the right hand side of (18) vanishes upon considering $y\colon S \to M$, such that 0_{N+1} is satisfied. By construction, also 1_{N+1} holds.

We show that X^{N+1} further satisfies 3_{N+1}. 1_{N+1} implies that $(z^* X^{N+1})|_{\eta^I} = P_\delta[X_y]|_{\eta^I}$ for all z and $\delta\colon y \to z$ and $|I| \le N+1$. In particular, we let $q \in M_0$ and define y and δ as follows.

$$\begin{aligned}
y^\sharp(x^k) &:= q^*(x^k) = q^k\,, & y^\sharp(\theta^k) &:= \eta^{L+k}(\in \mathcal{O}_T)\,, \\
\delta^\sharp(x^k) &:= q^k + t\delta^{kk_0}\,, & \delta^\sharp(\theta^k) &:= \eta^{L+k}
\end{aligned}$$

This is such that $\mathrm{ev}|_{t=0}\delta^\sharp = y^\sharp$. We thus yield

$$0 = \left((\delta^*\nabla)_{\partial_t}(\delta^* X^{N+1})\right)|_{\eta^I} = \hat{\delta}^*(\nabla_{\partial_{x^{k_0}}}X^{N+1})|_{\eta^I}$$

Writing $\nabla_{\partial_{x^{k_0}}}X^{N+1} =: N^{AB}\theta^A\eta^B$ with $\eta^B \in \mathcal{O}_S$, we conclude that

$$0 = \hat{y}^*(\nabla_{\partial_{x^{k_0}}}X^{N+1})|_{\eta^I} = \left(N^{AB}(q)\cdot\eta^{A_L}\eta^B\right)|_{\eta^I}$$

with A_L arising from the multiindex A by shifting all indices by L, such that $\eta^{A_L} \in \mathcal{O}_T$. For $|A|+|B| = |A_L|+|B| = |I| \leq N+1$, this implies that $N^{AB}(q) = 0$.

Proceeding inductively yields a section $X := X^L = X^{L+(\dim M)_{\bar{1}}} \in \mathcal{E}_S$ such that the induction hypotheses hold with respect to $L + (\dim M)_{\bar{1}}$. X is, therefore, parallel. Concerning uniqueness, assume that $\tilde{X} \in \mathcal{E}_S$ is a second such section. Then $y^*(X - \tilde{X}) = 0$ for all $y\colon S \times T \to M$ such that $X - \tilde{X} = 0$ by an argument analogous to that in the previous proof of 3_{N+1}. □

4 Comparison with Galaev's Holonomy Theory

Considering $S = \mathbb{R}^{0|0}$, let ∇ be a connection on a super vector bundle $\mathcal{E} \to M$ and $x \in M_0$ be a (topological) point. In this chapter, we will compare the functor Hol_x with Galaev's holonomy super Lie group $\mathrm{Hol}_x^{\mathrm{Gal}}$, which was introduced in [12] by means of a certain Harish-Chandra pair built around the super Lie algebra $\mathrm{hol}_x^{\mathrm{Gal}}$ generated by endomorphisms

$$P_{\gamma_0}^{-1} \circ \left(\overline{\nabla}^r_{Y_r,\ldots,Y_1} R\right)_y (Y,\, Z) \circ P_{\gamma_0} : x^*\mathcal{E} \to x^*\mathcal{E}$$

with $y \in M_0$, $\gamma_0\colon x \to y$, $r \geq 0$ and $Y_1, \ldots, Y_r, Y, Z \in y^*\mathcal{S}M$, and where $\overline{\nabla}^r_{Y_r,\ldots,Y_1} R$ denotes the r-fold covariant derivative of the curvature R with respect to ∇ and some auxiliary connection $\overline{\nabla}$ on $\mathcal{S}M$ in a neighbourhood of y. This derivative is defined analogous to the classical (non-super) case with appropriate signs. For $r = 1, 2$, it reads as follows.

Definition 12. Let

$$R \in \mathrm{Hom}_{\mathcal{O}_{S\times M}}\left(\mathcal{S}M_S \otimes_{\mathcal{O}_{S\times M}} \mathcal{S}M_S \otimes_{\mathcal{O}_{S\times M}} \mathcal{E}_S,\, \mathcal{E}_S\right)\,,$$

and $u, v \in \mathcal{S}M_S$. For $X, Y \in \mathcal{S}M_S$, we define

$$\begin{aligned}
\overline{\nabla}_X R(u,\, v) :=\; & \nabla_X \circ R(u,\, v) - (-1)^{|R||X|} R\left(\overline{\nabla}_X u,\, v\right) \\
& - (-1)^{|X|(|R|+|u|)} R\left(u,\, \overline{\nabla}_X v\right) \\
& - (-1)^{|X|(|R|+|u|+|v|)} R(u,\, v) \circ \nabla_X \\
\overline{\nabla}^2_{X,Y} R(u,\, v) :=\; & \overline{\nabla}_X \left(\overline{\nabla}_Y R\right)(u,\, v) - \nabla_{\overline{\nabla}_X Y} \circ R(u,\, v) \\
& + (-1)^{(|X|+|Y|)|R|} R\left(\overline{\nabla}_{\overline{\nabla}_X Y} u,\, v\right) \\
& + (-1)^{(|X|+|Y|)(|R|+|u|)} R\left(u,\, \overline{\nabla}_{\overline{\nabla}_X Y} v\right) \\
& + (-1)^{(|X|+|Y|)(|R|+|u|+|v|)} R(u,\, v) \circ \nabla_{\overline{\nabla}_X Y}
\end{aligned}$$

According to Example 4, the functor Hol_x is, in general, not representable such that Galaev's holonomy theory is a priori different from ours. Nevertheless, we will show that the generators of $\mathrm{hol}_x^{\mathrm{Gal}}$ can be extracted in a geometric way, in a sense to be made precise.

4.1 Parallel Transport and Covariant Derivatives

The aforementioned extraction of generators of $\mathrm{hol}_x^{\mathrm{Gal}}$ is based on the following observation. Consider again the more general situation of an S-connection ∇ on $\mathcal{E}_S$ for $S = \mathbb{R}^{0|L}$ and $x\colon S \to M$ an S-point. As shown next, the pullback connection $x^*\nabla$ – along with its induced connections on tensors as well as higher covariant derivatives – arises by means of infinitesimal parallel transport. We will not treat the most general situation here but content ourselves with the following. First, we consider only even vector fields to be differentiated along. The general case is expected to work along the lines of the flow of vector bundles developed in [23]. Second, we consider tensors of the following type: sections, endomorphisms and curvature-type. The general case should be analogous. Third, we consider covariant derivatives up to second order. Analogous results for higher order derivatives are expected to be obtainable by an inductive proof.

For $X \in \mathcal{S}S$ and $Z \in x^*\mathcal{E}$, the pullback $(x^*\nabla)_X Z \in x^*\mathcal{E}$ was defined in (13). Let also $Y \in \mathcal{S}S$ and $\overline{\nabla}$ be an S-connection on $\mathcal{S}M_S$. We define the second covariant derivative of Z, with respect to ∇ and $\overline{\nabla}$, as follows.

$$(x^*\overline{\nabla}^2)_{X,Y} Z := (x^*\nabla)_X (x^*\nabla)_Y Z - (x^*\nabla)_{(x^*\overline{\nabla})_X [dx[Y]]} Z$$

The corresponding first and second covariant derivatives of endomorphisms and tensors of curvature type are defined likewise.

Definition 13. Let $E \in \mathrm{End}_{\mathcal{O}_{S\times M}}(\mathcal{E}_S)$ be an endomorphism and E_x its pullback under x as in (9). For $X, Y \in \mathcal{S}S$, we define

$$\begin{aligned}
(x^*\nabla)_X E_x &:= (x^*\nabla)_X \circ E_x - (-1)^{|X||E|} E_x \circ (x^*\nabla)_X \in \mathrm{End}_{\mathcal{O}_S}(x^*\mathcal{E}) \\
(x^*\overline{\nabla}^2)_{X,Y} E_x &:= (x^*\nabla)_X \left((x^*\nabla)_Y E_x\right) \\
&\quad - (x^*\nabla)_{(x^*\overline{\nabla})_X [dx[Y]]} \circ E_x \\
&\quad + (-1)^{|E|(|X|+|Y|)} E_x \circ (x^*\nabla)_{(x^*\overline{\nabla})_X [dx[Y]]}
\end{aligned}$$

Definition 14. Let

$$R \in \mathrm{Hom}_{\mathcal{O}_{S\times M}} \left(\mathcal{S}M_S \otimes_{\mathcal{O}_{S\times M}} \mathcal{S}M_S \otimes_{\mathcal{O}_{S\times M}} \mathcal{E}_S,\, \mathcal{E}_S\right)\,,$$

and $u, v \in x^*\mathcal{S}M$. For $X, Y \in \mathcal{S}S$, we define

$$\begin{aligned}
(x^*\overline{\nabla})_X R_x\,(u,\,v) &:= (x^*\nabla)_X \circ R_x\,(u,\,v) - (-1)^{|R||X|} R_x\left((x^*\overline{\nabla})_X(u),\,v\right) \\
&\quad - (-1)^{|X|(|R|+|u|)} R_x\left(u,\,(x^*\overline{\nabla})_X(v)\right) \\
&\quad - (-1)^{|X|(|R|+|u|+|v|)} R_x\,(u,\,v) \circ (x^*\nabla)_X \\
(x^*\overline{\nabla}^2)_{X,Y} R_x\,(u,\,v) &:= (x^*\overline{\nabla})_X \left((x^*\overline{\nabla})_Y R_x\right)(u,\,v) - (x^*\nabla)_{(x^*\overline{\nabla})_X [dx[Y]]} \circ R_x\,(u,\,v) \\
&\quad + (-1)^{(|X|+|Y|)|R|} R_x\left((x^*\overline{\nabla})_{(x^*\overline{\nabla})_X [dx[Y]]}(u),\,v\right) \\
&\quad + (-1)^{(|X|+|Y|)(|R|+|u|)} R_x\left(u,\,(x^*\overline{\nabla})_{(x^*\overline{\nabla})_X [dx[Y]]}(v)\right) \\
&\quad + (-1)^{(|X|+|Y|)(|R|+|u|+|v|)} R_x\,(u,\,v) \circ (x^*\nabla)_{(x^*\overline{\nabla})_X [dx[Y]]}
\end{aligned}$$

Our next lemma ensures existence of an S-path as occurring in the subsequent proposition concerning first covariant derivatives.

Lemma 12. *Let $x\colon S \to M$ be an S-point and $\xi \in (x^*\mathcal{S}M)_{\overline{0}}$. We write (in coordinates around x_0) $\xi = (x^*\partial_i) \cdot \xi^i$ and assume that every $\xi^i \in \mathcal{O}_S$ is nilpotent. Then there is an S-path γ (connecting x to some other S-point y) such that $\mathrm{ev}|_0\partial_t \circ \gamma^\sharp = \xi$.*

Proof. Through Definition 5, and setting $x^\sharp(t) := t$, we extend x to a map $x\colon S\times\mathbb{R} \to S \times M \times \mathbb{R}$. In this sense, we define

$$\gamma^\sharp := x^\sharp \circ \sum_{n=0}^{\infty} \frac{\left(\sum_i (t\xi^i\partial_i)\right)^n}{n!} \,.$$

Every $\xi^i\partial_i$ is, by assumption, even and nilpotent such that there are no ordering problems and the sum is finite. Such γ is indeed a morphism by the derivation property of $\sum_i(t\xi^i\partial_i)$ as shown analogous as in the proof of Lemma 1.1 in [17]. A straightforward calculation shows, moreover, that γ indeed satisfies the required initial condition. □

Proposition 5. *Let $x\colon S \to M$ be an S-point, $Y \in \mathcal{E}_S$ and $\xi \in (x^*\mathcal{S}M)_{\overline{0}}$. Let γ be an S-path (connecting x to some y) such that $\mathrm{ev}|_0\partial_t \circ \gamma^\sharp = \xi$. Then*

$$\frac{d}{dt}\Big|_0 \left(P_\gamma|_{[0,t]}^{-1}(\gamma^*Y)\right) = (x^*\nabla)_\xi(x^*Y)$$

In particular, for $\xi = X \circ x^\sharp = dx[X]$ with $X \in (\mathcal{S}S)_{\overline{0}}$, we find

$$\frac{d}{dt}\Big|_0 \left(P_\gamma|_{[0,t]}^{-1}(\gamma^*Y)\right) = (x^*\nabla)_X(x^*Y)$$

Similarly, the first covariant derivatives of E_x and R_x, with E and R as in Definition 13 and Definition 14, arise from parallel transport as

$$\frac{d}{dt}\Big|_0 \left(P_\gamma|_{[0,t]}^{-1} \circ E_\gamma \circ P_\gamma|_{[0,t]}\right) = (x^*\nabla)_X E_x$$

$$\frac{d}{dt}\Big|_0 \left(P_\gamma|_{[0,t]}^{-1} \circ R_\gamma\left(\overline{P}_\gamma|_{[0,t]}(u), \overline{P}_\gamma|_{[0,t]}(v)\right) \circ P_\gamma|_{[0,t]}\right) = (x^*\overline{\nabla})_X R_x\,(u, v)$$

Proof. Let (T^j) be an $\mathcal{E}$-basis in a neighbourhood of $x_0(0) \in M_0$. For t sufficiently small, we identify $P_{\gamma|_{[0,t]}}$ and its inverse with a matrix with respect to bases (x^*T^j) and $(\gamma_t^*T^j)$. By (15), we find that

$$\mathrm{ev}|_{t=0}P_{\gamma|_{[0,t]}} = \mathrm{id}\,, \qquad \partial_t|_0 P_{\gamma|_{[0,t]}} = -\mathcal{B}(0)\,, \qquad \partial_t|_0 P^{-1}_{\gamma|_{[0,t]}} = \mathcal{B}(0)$$

where the sign in the last equation is due to replacing t by $1-t$ in γ^{-1} within the definition of $\mathcal{B}(t)$. The first statement is shown by the following calculation, writing $Y = T^kY^k$.

$$\frac{d}{dt}\Big|_0 \left(P_\gamma|_{[0,t]}^{-1}(\gamma^*Y)\right) = \mathcal{B}(0)\cdot(x^*Y) + (x^*T^k)\partial_t|_0\gamma^*Y^k = (x^*\nabla)_\xi(x^*Y)$$

For the second statement note that, by (9), the matrix of E_γ is the pullback under γ of the matrix of E. For $Y \in x^*\mathcal{E}$, we thus yield

$$\begin{aligned}\frac{d}{dt}\Big|_0 \left(P_\gamma|_{[0,t]}^{-1} \circ E_\gamma \circ P_\gamma|_{[0,t]}\right)(Y) &= \left(\mathcal{B}(0)E_x + \partial_t|_0 E_\gamma - E_x\mathcal{B}(0)\right)(Y)\\ &= \mathcal{B}(0)E_x[Y] + X(E_xY) - E_x[X(Y)] - E_x\mathcal{B}(0)[Y]\\ &= \left((x^*\nabla)_X \circ E_x - E_x \circ (x^*\nabla)_X\right)(Y)\end{aligned}$$

Finally, the third statement is established by an analogous calculation. □

We now come to second covariant derivatives. Let $X, Y \in (\mathcal{S}S)_{\overline{0}}$ and consider a map $\gamma\colon S \times [0,1] \times [0,1] \to M$ such that

$$\mathrm{ev}|_{(0,0)}\partial_t \circ \gamma^* = X \circ x^* \,, \qquad \mathrm{ev}|_{s=0}\partial_s \circ \gamma^\sharp = \overline{P}_{\gamma_{s=0}|_{[0,t]}}(Y \circ x^\sharp) =: Y_t \tag{19}$$

such that $Y_0 = Y \circ x^\sharp$. Such a homotopy indeed exists. First, by Lemma 12, there is $\tilde{\gamma}\colon S \times [0,1] \to M$ (parameter t) such that the first condition in (19) is satisfied. Now fix t. For this t, there is, by the same lemma, an S-path $\gamma_t\colon S \times [0,1] \to M$ (parameter s) such that also the second condition holds true with parallel transport $\overline{P}_{\tilde{\gamma}}|_{[0,t]}$ on the right hand side. By construction, γ_t depends smoothly on t and s, thus yielding γ as required.

Proposition 6. *Let $Z \in \mathcal{E}_S$ and $E \in \mathrm{End}_{\mathcal{O}_N}(\mathcal{E})$. Then*

$$\frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 (P^2_{s,t})^{-1}(\gamma^* Z) = (x^*\overline{\nabla}^2)_{X,Y}(x^*Z)$$

$$\frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 \left((P^2_{s,t})^{-1} \circ E_\gamma \circ P^2_{s,t}\right) = (x^*\overline{\nabla}^2)_{X,Y} E_x$$

$$\frac{d}{dt}\Big|_0 \frac{d}{ds}\Big|_0 \left((P^2_{s,t})^{-1} \circ R_\gamma\left(\overline{P}^2_{s,t}(u),\, \overline{P}^2_{s,t}(v)\right) \circ P^2_{s,t}\right) = (x^*\overline{\nabla}^2)_{X,Y} R_x(u,v)$$

with

$$P^2_{s,t} := P_{\gamma_t|_{[0,s]}} \circ P_{\gamma_{s=0}|_{[0,t]}} \,, \qquad \overline{P}^2_{s,t} := \overline{P}_{\gamma_t|_{[0,s]}} \circ \overline{P}_{\gamma_{s=0}|_{[0,t]}}$$

Proof. Using Proposition 5 and $|Y_t^l| = |dx[Y]^l| = |Y\big(x^*(\xi^l)\big)| = |\xi^l|$, we calculate

$$\begin{aligned}&\partial_s|_0\partial_t|_0 (P_{\gamma_{s=0}|_{[0,t]}})^{-1}(P_{\gamma_t|_{[0,s]}})^{-1}(\gamma^*Z)\\ &\quad= \partial_t|_0 (P_{\gamma_{s=0}|_{[0,t]}})^{-1}\partial_s|_0 (P_{\gamma_t|_{[0,s]}})^{-1}(\gamma^*Z)\\ &\quad= \partial_t|_0 (P_{\gamma_{s=0}|_{[0,t]}})^{-1}\left((\gamma_t^*\nabla)_{Y_t}(\gamma_t^*Z)\right)\\ &\quad= (-1)^{|\xi^l||Z|}\partial_t|_0 (P_{\gamma_{s=0}|_{[0,t]}})^{-1}\gamma_t^*(\nabla_{\partial_l}Z)\cdot dx[Y]^l + (-1)^{|\xi^l||Z|}x^*(\nabla_{\partial_l}Z)\partial_t|_0 Y_t^l\end{aligned}$$

Now we use $\partial_t|_0 (P_{\gamma_{s=0}|_{[0,t]}})^{-1}\gamma_t^*(\nabla_{\partial_l}Z) = (x^*\nabla)_X(x^*\nabla_{\partial_l}Z)$ and

$$\partial_t|_0 Y_t = -\mathcal{B}^X(s=0)dx[Y] = -(-1)^{|n|}dx[Y]^n dx[X](\xi^l)x^*(\overline{\nabla}_{\partial_{\xi^l}}\partial_{\xi^n})$$

to yield the first statement after a straightforward calculation.

The left hand side of the second equation is treated as follows.

$$\begin{aligned}
\mathrm{LHS} &= \frac{d}{dt}\Big|_0 \left(P_\gamma|^{-1}_{s=0,[0,t]}\partial_s|_0 \left(P_\gamma|^{-1}_{t,[0,s]} \circ E_\gamma \circ P_\gamma|_{t,[0,s]}\right) \circ P_\gamma|_{s=0,[0,t]}\right) \\
&= \frac{d}{dt}\Big|_0 \left(P_\gamma|^{-1}_{s=0,[0,t]} \left((\gamma_t^*\nabla)_{Y_t} \circ E_{\gamma_t} - E_{\gamma_t} \circ (\gamma_t^*\nabla)_{Y_t}\right) \circ P_\gamma|_{s=0,[0,t]}\right) \\
&= (x^*\nabla)_X \left((x^*\nabla)_Y E_x\right) - (x^*\nabla)_{(x^*\overline{\nabla})_X[dx[Y]]} \circ E_x + E_x \circ (x^*\nabla)_{(x^*\overline{\nabla})_X[dx[Y]]}
\end{aligned}$$

Here, the second equation follows from Proposition 5 applied to the second condition in (19). For the third equation, we use again Proposition 5 to obtain the first term and find, in addition, two derivative terms with respect to $(\gamma_t^*\nabla)_{Y_t}$ which are obtained as in the previous calculation.

Similarly we yield, for the left hand side of the last equation to be shown,

$$\begin{aligned}
\mathrm{LHS} = \frac{d}{dt}\Big|_0 P_\gamma|^{-1}_{s=0,[0,t]} \Big(& (\gamma_t^*\nabla)_{Y_t} \circ R_{\gamma_t}\left(\overline{P}_\gamma|_{s=0,[0,t]}(u),\, \overline{P}_\gamma|_{s=0,[0,t]}(v)\right) \\
& - R_{\gamma_t}\left((\gamma_t^*\overline{\nabla})_{Y_t}(\overline{P}_\gamma|_{s=0,[0,t]}(u)),\, \overline{P}_\gamma|_{s=0,[0,t]}(v)\right) \\
& - R_{\gamma_t}\left(\overline{P}_\gamma|_{s=0,[0,t]}(u),\, (\gamma_t^*\overline{\nabla})_{Y_t}(\overline{P}_\gamma|_{s=0,[0,t]}(v))\right) \\
& - R_{\gamma_t}\left(\overline{P}_\gamma|_{s=0,[0,t]}(u),\, \overline{P}_\gamma|_{s=0,[0,t]}(v)\right) \circ (\gamma_t^*\nabla)_{Y_t}\Big) P_\gamma|_{s=0,[0,t]}
\end{aligned}$$

Analogously to the previous calculation for the second statement, Proposition 5 together with derivative terms from the first calculation yields the right hand side as claimed. □

4.2 Reconstruction of Galaev's Holonomy Algebra

By means of the previously established relation between covariant derivatives and parallel transport, we will now make contact with Galaev's holonomy algebra $\mathrm{hol}_x^{\mathrm{Gal}}$. Let $S = \mathbb{R}^{0|0}$, ∇ be a connection on $\mathcal{E} \to M$, and $x \in M_0$ be a topological point identified with an S-point. We aim at gaining generating elements of $\mathrm{hol}_x^{\mathrm{Gal}}$ as coefficients of special elements of $\mathrm{hol}_x(T)$ for $T = \mathbb{R}^{0|L'}$ with $L' \geq (\dim M)_{\overline{1}}$. Let $q \in M_0$, and define the $(S\times)T$-point y by prescribing

$$y^\sharp(x^k) := q^*(x^k) = q^k\,, \qquad y^\sharp(\theta^i) := \eta^i \tag{20}$$

with respect to coordinates $\xi = (x, \theta)$ around q. Then, a straightforward calculation using (14) shows that

$$\begin{aligned}
(y^*\nabla)_{\partial_{\eta^j}}(y^*Z) &= \hat{y}^*(\nabla_{\partial_{\theta^j}} Z) \\
(y^*\nabla)_{(y^*\overline{\nabla})_{\partial_{\eta^j}}[dy[\partial_{\eta^k}]]}(y^*Z) &= \hat{y}^*\left(\nabla_{\overline{\nabla}_{\partial_{\theta^j}}\partial_{\theta^k}} Z\right)
\end{aligned}$$

For the curvature terms, it follows that

$$\begin{aligned}
R_y\left(y^*\partial_{\xi^i},\, y^*\partial_{\xi^j}\right) &= \hat{y}^*\left(R\left(\partial_{\xi^i},\, \partial_{\xi^j}\right)\right) \\
\left((y^*\overline{\nabla})_{\partial_{\eta^l}} R_y\right)\left(y^*\partial_{\xi^i},\, y^*\partial_{\xi^j}\right) &= \hat{y}^*\left((\overline{\nabla}_{\partial_{\theta^l}} R)\left(\partial_{\xi^i},\, \partial_{\xi^j}\right)\right) \\
\left((y^*\overline{\nabla}^2)_{\partial_{\eta^l},\partial_{\eta^m}} R_y\right)\left(y^*\partial_{\xi^i},\, y^*\partial_{\xi^j}\right) &= \hat{y}^*\left((\overline{\nabla}^2_{\partial_{\theta^l},\partial_{\theta^m}} R)\left(\partial_{\xi^i},\, \partial_{\xi^j}\right)\right)
\end{aligned} \tag{21}$$

Lemma 13. *Let y be the T-point (20), $\gamma\colon x \to y$ be a connecting T-path and I_k denote a multiindex of parity $|\xi^k|$ such that $\eta^{I_k} \in \mathcal{O}_T$. Then*

$$\eta^{I_{k_1}}\eta^{I_{k_2}} \cdot P_\gamma^{-1} \circ y^* \left(R\left(\partial_{\xi^{k_2}}, \partial_{\xi^{k_1}}\right)\right) \circ P_\gamma \in \mathrm{hol}_x(T)$$

$$\eta^{I_{k_1}}\eta^{I_{k_2}}\eta^{I_{k_3}} \cdot P_\gamma^{-1} \circ y^* \left((\nabla_{\partial_{\theta^{k_3}}} R)\left(\partial_{\xi^{k_2}}, \partial_{\xi^{k_1}}\right)\right) \circ P_\gamma \in \mathrm{hol}_x(T)$$

$$\eta^{I_{k_1}}\eta^{I_{k_2}}\eta^{I_{k_3}}\eta^{I_{k_4}} \cdot P_\gamma^{-1} \circ y^* \left((\nabla^2_{\partial_{\theta^{k_4}}, \partial_{\theta^{k_3}}} R)\left(\partial_{\xi^{k_2}}, \partial_{\xi^{k_1}}\right)\right) \circ P_\gamma \in \mathrm{hol}_x(T)$$

Proof. By Theorem 1, the first term

$$\eta^{I_{k_1}}\eta^{I_{k_2}} \cdot P_\gamma^{-1} \circ y^* \left(R\left(\partial_{k_2}, \partial_{k_1}\right)\right) \circ P_\gamma = P_\gamma^{-1} \circ R_y\left(\eta^{I_{k_2}} \cdot (y^* \circ \partial_{k_2}), \eta^{I_{k_1}} \cdot (y^* \circ \partial_{k_1})\right) \circ P_\gamma$$

is clearly contained in $\mathrm{hol}_x(T)$. For the second, let δ be an S-path connecting y to some S-point z such that $\mathrm{ev}|_0 \partial_t \circ \delta^\sharp = \xi := dy\left[\eta^{I_{k_3}} \cdot \partial_{\eta^{k_3}}\right]$. Using (21), followed by Proposition 5 applied to y, ξ, δ as well as $u := \eta^{I_{k_2}} \cdot (y^* \circ \partial_{k_2})$ and $v := \eta^{I_{k_1}} \cdot (y^* \circ \partial_{k_1})$, we yield

$$\begin{aligned}
&\eta^{I_{k_1}}\eta^{I_{k_2}}\eta^{I_{k_3}} \cdot P_\gamma^{-1} \circ y^* \left((\nabla_{\theta^{k_3}} R)\left(\partial_{k_2}, \partial_{k_1}\right)\right) \circ P_\gamma \\
&\quad = P_\gamma^{-1} \circ \left((y^*\nabla)_{\eta^{I_{k_3}} \cdot \partial_{\eta^{k_3}}} R_y\right)\left(\eta^{I_{k_2}} \cdot (y^* \circ \partial_{k_2}), \eta^{I_{k_1}} \cdot (y^* \circ \partial_{k_1})\right) \circ P_\gamma \\
&\quad = P_\gamma^{-1} \circ \partial_t|_0 \left(P_\delta|_{[0,t]}^{-1} \circ R_\delta\left(\overline{P}_\delta|_{[0,t]}(u), \overline{P}_\delta|_{[0,t]}(v)\right) \circ P_\delta|_{[0,t]}\right) \circ P_\gamma \\
&\quad = \partial_t|_0 \left(P_\gamma^{-1} \circ P_\delta|_{[0,t]}^{-1} \circ R_\delta\left(\overline{P}_\delta|_{[0,t]}(u), \overline{P}_\delta|_{[0,t]}(v)\right) \circ P_\delta|_{[0,t]} \circ P_\gamma\right)
\end{aligned}$$

By Theorem 1, the term in parentheses lies, for every $t \in [0,1]$, in $\mathrm{hol}_x(T)$, which is a vector space. Therefore, the differential is also contained in $\mathrm{hol}_x(T)$.

The second covariant derivative term is treated analogous. □

Consider the zero-derivative term in Lemma 13. For generic choice of $\eta^{I_{k_1}}$ and $\eta^{I_{k_2}}$, we find that

$$\begin{aligned}
&\left(\partial_{\eta^{I_{k_1}}}\partial_{\eta^{I_{k_2}}}\left(\eta^{I_{k_1}}\eta^{I_{k_2}} P_\gamma^{-1} \circ y^*\left(R\left(\partial_{\xi^{k_2}}, \partial_{\xi^{k_1}}\right)\right) \circ P_\gamma\right)\right)_0 \\
&\quad = P_{\gamma_0}^{-1} \circ R_{y_0}\left(\partial_{\xi^{k_2}}, \partial_{\xi^{k_1}}\right) \circ P_{\gamma_0} \in \mathrm{hol}_x^{\mathrm{Gal}}
\end{aligned}$$

and analogous for the first and second derivative terms and, by conjecture, for all higher derivative terms. The generating elements of $\mathrm{hol}_x^{\mathrm{Gal}}$ can thus be extracted out of $\mathrm{hol}_x(T)$ as certain coefficients of special elements in the way made precise by Lemma 13. This construction is based on the knowledge of the geometric significance of the elements. It remains an open question whether $\mathrm{hol}_x^{\mathrm{Gal}}$ can be obtained from $\mathrm{hol}_x(T)$ in a purely algebraic way.

Acknowledgements

I would like to thank Alexander Alldridge, Anton Galaev, Harald Dorn and Jan Plefka for interesting discussions, and again Anton Galaev for spotting a mistake in a previous version of this article.

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Author's address:
Universität zu Köln, Institut für Theoretische Physik, Zülpicher Str. 77, 50937 Köln, Germany

E-mail: `groegerj@thp.uni-koeln.de`

Received: 10 November, 2014
Accepted for publication: 28 December, 2014
Communicated by: Olga Rossi

Communications in Mathematics 22 (2014) 213–216

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Communications in Mathematics, Volume 22, 2014, is the continuation of the journal *Acta Mathematica et Informatica Universitatis Ostraviensis*, ISSN 1211-4774 (1993–2003), and of the journal *Acta Mathematica Universitatis Ostraviensis*, ISSN 1214-8148 (2004–2009).

Published by The University of Ostrava
December 2014

Typeset by TEX

ISSN 1804-1388 (Print), ISSN 2336-1298 (Online)

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