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On the diophantine equation $x^2 + 5^k 17^l = y^n$

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Abstract. Consider the equation in the title in unknown integers (x, y, k, l, n) with $x \geq 1$, $y > 1$, $n \geq 3$, $k \geq 0$, $l \geq 0$ and $\gcd(x, y) = 1$. Under the above conditions we give all solutions of the title equation (see Theorem 1).

1 Introduction

There are many results concerning the generalized Ramanujan-Nagell equation

$$x^2 + D = y^n, \tag{1}$$

where $D > 0$ is a given integer and x, y, n are positive integer unknowns with $n \geq 3$. Results obtained for general superelliptic equations clearly provide effective finiteness results for this equation, too (see for example [9], [45], [47], and the references given there).

The first result concerning the above equation was due to V. A. Lebesque [28] who proved that there are no solutions for $D = 1$. Ljunggren [29] solved (1) for $D = 2$, and Nagell [39], [40] solved it for $D = 3, 4$ and 5. In his elegant paper [21], Cohn gave a fine summary of the earlier results on equation (1). Further, he developed a method by which he found all solutions of the above equation for 77 positive values of $D \leq 100$. For $D = 74$ and $D = 86$, equation (1) was solved by Mignotte and de Weger [35]. By using the theory of Galois representations and modular forms Bennett and Skinner [8] solved (1) for $D = 55$ and $D = 95$. On combining the theory of linear forms in logarithms with Bennett and Skinner's method and with several additional ideas, Bugeaud, Mignotte and Siksek [13] gave all the solutions of (1) for the remaining 19 values of $D \leq 100$.

Let $S = \{p_1, \ldots, p_s\}$ denote a set of distinct primes and $\mathbf{S}$ the set of non-zero integers composed only of primes from S. Put $P := \max\{p_1, \ldots, p_s\}$ and denote by Q the product of the primes of S. In recent years, equation (1) has been considered also in the more general case when D is no longer fixed but $D \in \mathbf{S}$ with $D > 0$. It follows from Theorem 2 of [46] that in (1) n can be bounded from above by an

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effectively computable constant depending only on P and s. In [25] an effective upper bound was derived for n which depends only on Q. Cohn [20] showed that if $D = 2^{2k+1}$ then equation (1) has solutions only when $n = 3$ and in this case there are three families of solutions. The case $D = 2^{2k}$ were considered by Arif and Abu Muriefah [2]. They conjectured that the only solutions are given by $(x, y) = (2^k, 2^{2k+1})$ and $(x, y) = (11 \cdot 2^{k-1}, 5 \cdot 2^{2(k-1)/3})$, with the latter solution existing only when $(k, n) = (3M + 1, 3)$ for some integer $M \geq 0$. Partial results towards this conjecture were obtained in [2] and [19] and it was finally proved by Arif and Abu Muriefah [5]. Arif and Abu Muriefah [3] proved that if $D = 3^{2k+1}$ then (1) has exactly one infinite family of solutions. The case $D = 3^{2k}$ has been solved by Luca [31] under the additional hypothesis that x and y are coprime. In fact in [32] Luca solved completely equation (1) if $D = 2^a 3^b$ and $\gcd(x, y) = 1$. Abu Muriefah [1] established that equation (1) with $D = 5^{2k}$ may have a solution only if 5 divides x and p does not divide k for any odd prime p dividing n. The case $D = 2^a 3^b 5^c 7^d$ with $\gcd(x, y) = 1$, where a, b, c, d are non-negative integers was studied by Pink [41]. The cases when $D = 7^{2k}$ and $D = 2^a 5^b$ were also considered by Luca and Togbe [33], [34]. For the case $D = 2^a 5^b 13^c$, see Goins, Luca and Togbe [24], while if $D = 5^a 13^b$, see [38]. The cases $D = 2^a 11^b$ and $D = 5^a 11^b$ have been recently considered in [17] and [16], respectively. Let $p \geq 5$ be an odd prime with $p \not\equiv 7 \pmod 8$. Arif and Abu Muriefah [6] determined all solutions of the equation $x^2 + p^{2k+1} = y^n$, where $\gcd(n, 3h_0) = 1$ and $n \geq 3$. Here h_0 denotes the class number of the field $\mathbb{Q}(\sqrt{-p})$. They also obtained partial results [4] if $D = p^{2k}$, where p is an odd prime. In the particular case when $\gcd(x, y) = 1$, $D = p^2$, p prime with $3 \leq p < 100$, Le [27] gave all the solutions of equation (1). The case $D = p^{2k}$ with $2 \leq p < 100$ prime and $\gcd(x, y) = 1$ was considered by Bérczes and Pink [10]. If in (1) $D = a^2$ with $3 \leq a \leq 501$ and a is odd then Tengely [48] solved completely equation (1) under the assumption $(x, y) \in \mathbb{N}^2$, $\gcd(x, y) = 1$. The equation $A^4 + B^2 = C^n$ for $AB \neq 0$ and $n \geq 4$ was completely solved by Bennett, Ellenberg and Nathan [7] (see also Ellenberg [23]). For related results concerning equation (1) see [43], [44] and the references given there. For a survey concerning equation (1) see [14].

2 Results

Consider the following equation

$$x^2 + 5^k 17^l = y^n \tag{2}$$

in integer unknowns x, y, k, l, n satisfying

$$x \geq 1\,, \quad y > 1\,, \quad n \geq 3\,, \quad k \geq 0\,, \quad l \geq 0 \quad \text{and} \quad \gcd(x, y) = 1\,. \tag{3}$$

Theorem 1. *Consider equation* (2) *satisfying* (3)*. Then all solutions of equation* (2) *are:*

$$(x, y, k, l, n) \in \{(94, 21, 2, 1, 3), (2034, 161, 3, 2, 3), (8, 3, 0, 1, 4)\}\,.$$

Remark 1. We may assume without loss of generality that in (2) $n \geq 5$ prime or $n \in \{3, 4\}$. The proof of our Theorem 1 is organized as follows. If $n \geq 5$ prime we use some properties of Lucas sequences, to derive a sharp upper bound for n (see also Pink [41], Theorem 2). Then we apply the result of Bilu, Hanrot and Voutier [11] concerning the existence of primitive prime divisors in Lucas sequences.

If $n \in \{3, 4\}$ there is a general method for giving all solutions of equations of the form $x^2 + p^k q^l = y^n$. Namely the problem is reduced to finding S-integral points on several elliptic curves, where $S = \{p, q\}$. This works well, but in some cases the computation of the rank and the Mordell-Weil group becomes very time consuming so we need another approach. By using the parametrization provided by Lemma 1 we get several equations of the form

$$X \pm Y = 3u^2,$$

where X, Y are S-units and $S = \{p, q\}$. These equations are considered locally to get a contradiction or are transformed to Ljunggren-type equations. In fact, we have to give all S-integral points on the resulting Ljunggren-type curves. Then, using MAGMA we solve completely the equations under consideration.

3 Auxiliary results

Let $S = \{p_1, \ldots, p_s\}$ be a set of distinct primes and denote by $\mathbf{S}$ the set of non-zero integers composed only of primes from S. Equation (2) is a special case of an equation of the type

$$X^2 + D = Y^n, \tag{4}$$

where

$$\gcd(X, Y) = 1 \tag{5}$$

and

$$D \in \mathbf{S}, \quad D > 0, \quad X \geq 1, \quad Y > 1, \quad n \geq 3. \tag{6}$$

The next lemma provides a parametrization for the solutions of equation (4).

Lemma 1. *Suppose that equation* (4) *has a solution under the assumptions* (5) *and* (6) *with* $n \geq 3$ *prime. Denote by* $d > 0$ *the square-free part of* $D = dc^2$ *and let* h *be the class number of the field* $\mathbb{Q}(\sqrt{-d})$. *Then equation* (5) *has a solution with* $d \not\equiv 7 \pmod 8$ *in one of the following cases:*

(a) *there exist* $u, v \in \mathbb{Z}$ *such that* $x + c\sqrt{-d} = (u + v\sqrt{-d})^n$ *and* $y = u^2 + dv^2$.

(b) $d \equiv 3 \pmod 8$ *and there exist* $U, V \in \mathbb{Z}$ *with* $U \equiv V \equiv 1 \pmod 2$ *such that* $x + c\sqrt{-d} = \left(\frac{U+V\sqrt{-d}}{2}\right)^3$ *and* $y = \frac{U^2+dV^2}{4}$.

(c) $n = 3$ *if* $D = 3u^2 \pm 8$ *or if* $D = 3u^2 \pm 1$ *for some* $u \in \mathbb{Z}$.

(d) $n = 5$ *if* $D \in \{19, 341\}$.

(e) $p \mid h$.

Proof. This is a theorem of Cohn [22]. □

Recall that a *Lucas pair* is a pair (α, β) of algebraic integers such that $\alpha+\beta$ and $\alpha\beta$ are non-zero coprime rational integers and α/β is not a root of unity. Given a Lucas pair (α, β) one defines the corresponding sequence of *Lucas numbers* by

$$L_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad (n = 0, 1, 2, \dots).$$

A prime number p is called a *primitive divisor* of L_n if p divides L_n but does not divide $(\alpha - \beta)^2 L_1 \cdots L_{n-1}$.

The next lemma gives a necessary condition for an odd prime p to be a primitive prime divisor of the n-th term of a Lucas sequence if n is an odd prime. Namely we have the following.

Lemma 2. *Let $L_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ be a Lucas sequence and suppose that n is an odd prime. Further, let $A = (\alpha - \beta)^2$. If p is a primitive prime divisor of L_n then $n \mid p - \left(\frac{A}{p}\right)$, where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol with respect to the prime p.*

Proof. See Carmichael [18]. □

The next lemma is a deep result of Bilu, Hanrot and Voutier [11] concerning the existence of primitive prime divisors in a Lucas sequence.

Lemma 3. *Let $L_n = L_n(\alpha, \beta)$ be a Lucas sequence. If $n \geq 5$ is a prime then L_n has a primitive prime divisor except for finitely many pairs (α, β) which are explicitly determined in Table 1 of* [11].

Proof. This follows from Theorem 1.4 of [11] and Theorem 1 of [49]. □

The following lemma of Holzer gives a criterium for the existence of solutions of ternary quadratic equations.

Lemma 4. *Let a, b, c be coprime integers, and consider the equation*

$$ax^2 + by^2 + cz^2 = 0 \tag{7}$$

where x, y, z are unknown integers. If there is a non-trivial solution for (7)*, then there is one satisfying*

$$|x| \leq \sqrt{|bc|}, \quad |y| \leq \sqrt{|ac|}, \quad |z| \leq \sqrt{|ab|}.$$

Proof. See [37]. □

4 Proof of the Theorem

We introduce some notations which will be used in the course of the proof of our Theorem. Consider equation (2) satisfying the assumptions (3). Denote by $d > 0$ the square-free part of $5^k 17^l$ that is $5^k 17^l = d(5^a 17^b)^2$ where $d \in \{1, 5, 17, 85\}$ and $a, b \in \mathbb{Z}_{\geq 0}$. Further, let $\mathbb{K}$ be the imaginary quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ and denote by h the class number of $\mathbb{K}$. As was mentioned in Remark 1, we have to distinguish essentially three cases without loss of generality. Namely, we may assume that in equation (2) $n \geq 5$ prime or $n \in \{3, 4\}$.

Case 1: $n \geq 5$ prime. Suppose first that (2) holds satisfying (3) with $n \geq 5$ prime. If in (2) $y > 1$ is even we obviously have that x is odd. Since for any odd integer t we have $t^2 \equiv 1 \pmod 8$ we get that $1 + d \equiv 0 \pmod 8$ by reducing (2) modulo 8. This leads to $d \equiv 7 \pmod 8$ for $d \in \{1, 5, 17, 85\}$ which is clearly a contradiction. Hence in what follows we may assume that in (2) $y > 1$ is odd (and hence $x \geq 1$ is even). Since for $d \in \{1, 5, 17, 85\}$ the class number of the field $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ is 1 or $2^m, (m \geq 1)$ we get by Lemma 1 that equation (2) can have a solution under assumption (3) with $n \geq 5$ prime only in the cases (a) and (d). Since $k \geq 1$ and $l \geq 1$ we see that in (2) $D = 19$ cannot occur. Further, if $D = 341 = 11 \cdot 31$ then since $D = 5^k \cdot 17^l$ this choice for D is impossible, too. Hence equation (2) can have a solution only in case (a) of Lemma 1. Namely, using the parametrization provided by Lemma 1 and taking complex conjugation, we get

$$(x + 5^a 17^b \sqrt{-d}) = (u + v\sqrt{-d})^n \quad \text{and} \quad (x - 5^a 17^b \sqrt{-d}) = (u - v\sqrt{-d})^n \tag{8}$$

for some $u, v \in \mathbb{Z}$. Further, we also have $y = u^2 + dv^2$. By (9) we see that $u \mid x$ and since $y > 1$ is odd and $\gcd(x, y) = 1$ we get that $\gcd(2u, y) = 1$. Let $\alpha = u + v\sqrt{-d}$ and $\beta = u - v\sqrt{-d}$. Then $\gcd(\alpha\beta, \alpha + \beta) = \gcd(y, 2u) = 1$. If α/β is a root of unity then since $n \geq 5$ is prime we have $\alpha/\beta \in \{\pm 1, \pm i\}$ if $d = 1$. This leads to $u = 0$ or $u = \pm v$. Now $u = 0$ yields $x = 0$ which is a contradiction by (4). If $u = \pm v$ then $2 \mid y = u^2 + v^2$ which contradicts the fact that y is odd. If $d \in \{5, 17, 85\}$, then α/β is a root of unity if $\alpha/\beta \in \{\pm 1\}$, which leads to either $u = 1, v = 0$ or $u = 0, v = 1$. If $u = 1, v = 0$, then we get a contradiction with $y \geq 3$. If $u = 0, v = 1$, then $y = d$ holds, which leads to a contradiction with $\gcd(x, y) = 1$. Thus

$$L_n := \frac{(u + v\sqrt{-d})^n - (u - v\sqrt{-d})^n}{2v\sqrt{-d}} \tag{9}$$

is a Lucas sequence.

Further, by (9) we have

$$L_n = \frac{5^a 17^b}{v}$$

for some non-negative integers a, b. By Lemma 3 we get that L_n has a primitive divisor for $n \geq 5$ prime. Also the only prime divisors of L_n can be 5 or 17. By Lemma 2 we get that if p is a primitive divisor of L_n, then $p \equiv \pm 1 \pmod n$, so $n \mid p \pm 1$ holds. Since $p \in \{5, 17\}$, we have that one of the following cases holds:

$$n \mid 4 = 2^2\,, \quad n \mid 6 = 2 \cdot 3\,, \quad n \mid 16 = 2^4\,, \quad n \mid 18 = 2 \cdot 3^2$$

Since $n \geq 5$ we get a contradiction for all cases, which implies that (2) does not have a solution for $n \geq 5$.

Case 2: $n = 3$. At first, we point out that the usual method concerning the search for S-integral points on certain elliptic curves proves to be time consuming in this case, so we show a different approach.

By Lemma 1, we see that

$$x + 5^a 17^b \sqrt{-d} = (u + v\sqrt{-d})^3 \tag{10}$$

holds, where $d \in \{1, 5, 17, 85\}$ and $u, v \in \mathbb{Z}$. After expanding the right handside of equation (10), and comparing the imaginary parts, we get that

$$5^a 17^b = v(3u^2 - dv^2)\,. \tag{11}$$

In (11) $\gcd(v, 3u^2 - dv^2) = 1$ holds, since otherwise we would get $\gcd(u, v) \neq 1$, which implies $\gcd(x, y) \neq 1$, which is clearly a contradiction. From this, we get the following type of equations:

$$\begin{cases} 3u^2 - dv^2 = f \\ v = g \end{cases} \tag{12}$$

where

$$(f, g) \in \{(\pm 1, \pm 5^a 17^b), (\pm 5^a, \pm 17^b), (\pm 17^b, \pm 5^a), (\pm 5^a 17^b, \pm 1)\}\,.$$

Since $d \in \{1, 5, 17, 85\}$, we get a total of 16 cases, we have to deal with. We will illustrate the method in one of the more interesting cases, all the others can be done in the same way. Let $d = 5$, $f = \pm 17^b$, $g = \pm 5^a$. From this, we get that

$$3u^2 - 5^{2a+1} = \pm 17^b \tag{13}$$

holds. Our main goal is to transform this to Ljunggren-type curves. To reduce the number of curves, and so the time of the computation we write (13) to the form of $Ax^2 + By^2 + Cz^2 = 0$. Now using Holzer's theorem (see Lemma 4) we get, that (13) has a nontrivial solution if and only if b is odd and $3u^2 - 5^{2a+1} = -17^b$ holds. Now we transform this to the following type.

$$3\left(\frac{u}{17^{2b_1}}\right)^2 = 5^{i+1}\left(\frac{5^{a_1}}{17^{b_1}}\right)^4 - 17^{j+1} \tag{14}$$

where $i, j \in \{0, 2\}$, and $a = 4a_1 + i + 1$, $b = 4b_1 + j + 1$. So, the problem is reduced to finding all the $\{17\}$-integral points on quartics of the form of

$$3Y^2 = 5^{i+1}X^4 - 17^{j+1}\,, \quad i, j \in \{0, 2\}\,, \quad \text{where } X = \frac{5^{a_2}}{17^{b_2}} \text{ and } Y = \frac{u}{17^{2b_2}}\,.$$

Now, we can use MAGMA to determine all the solutions of the above equations. Repeating this for all the 16 cases we get that all the solutions of (2) with $n = 3$ are:

$$(x, y, k, l, n) \in \{(94, 21, 2, 1, 3), (2034, 161, 3, 2, 3)\}\,.$$

We point out, that, in many of the above cases the method used can be combined with local methods to simplify the computations. AMdemo

Case 3: $n = 4$. If $n = 4$ holds, then we can write the following:

$$y^4 - x^2 = 5^k 17^l$$

which can be factored as

$$(y^2 - x)(y^2 + x) = 5^k 17^l. \tag{15}$$

In (15) $\gcd(y^2 - x, y^2 + x) = 1$ holds, else we would get a contradiction with $\gcd(x, y) = 1$. So, we get that

$$\begin{cases} y^2 - x = f \\ y^2 + x = g \end{cases}$$

where $(f, g) \in \{(1, 5^k 17^l), (5^k, 17^l), (17^l, 5^k), (5^k 17^l, 1)\}$. Now, by adding the first equation to the second, we get, that

$$2y^2 = f + g$$

holds. Now using the same method as in the $n = 3$ case we get that with $n = 4$ all the solutions of (2) are

$$(x, y, k, l, n) \in \{(8, 3, 0, 1, 4)\}.$$

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General theory of Lie derivatives for Lorentz tensors

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Abstract. We show how the *ad hoc* prescriptions appearing in 2001 for the Lie derivative of Lorentz tensors are a direct consequence of the Kosmann lift defined earlier, in a much more general setting encompassing older results of Y. Kosmann about Lie derivatives of spinors.

1 Introduction

The geometric theory of Lie derivatives of spinor fields is an old and intriguing issue that is relevant in many contexts, among which we quote the applications in Supersymmetry (see [5], [22]) and the problem of separation of variables of Dirac equation (see [10]). It is as well essential for the understanding of the general foundations of the theory of spinor fields and, eventually, of General Relativity as a whole. We stress that despite spinor fields can be endowed with a correct physical interpretation only in a quantum framework, this quantum field theory is obtained by quantization procedures from a classical variational problem. Hence even if a classical field theory describing spinors is not endowed with a direct physical interpretation its variational issues (field equations and conserved quantities) are mathematically interesting on their own as well as they have important consequences on the corresponding quantum field theory.

The situation in Minkowski spacetime (as well as on other maximally symmetric spaces) is pretty well established and it is based on the existence of sufficiently many Killing vectors ξ. The problem of Lie derivatives arises when one wants to generalize these arguments to more general spacetimes, i.e. when Killing vectors are less than enough, or when coupling with gravity, i.e. when the metric background cannot be regarded as being fixed *a priori* but it has to be determined dynamically by field equations. A definition for Lie derivatives of spinors along generic spacetime vector fields, not necessarily Killing ones, on a general curved spacetime was already proposed in 1971 by Y. Kosmann (see [16], [17], [18], [19]) by an *ad hoc* prescription. In 1996 we and coauthors (see also [12]) provided a geometric framework which

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justifies the *ad hoc* prescription within the general framework of Lie derivatives on fiber bundles (see also [24], [23] and [2]) in the explicit context of gauge natural bundles [15] which turn out to be the most appropriate arena for (gauge-covariant) field theories [6].

The key point is the construction of the (generalized) *Kosmann lift* (so-called by us in honour of the original *ad hoc* prescription) which is induced by any spacetime frame. This lift is defined on any principal bundle Σ having the special orthogonal group as structure group in any dimension and signature. According to this prescription a spacetime vector field ξ is uniquely lifted to a bundle vector field $\hat{\xi}_\Sigma$.

This lift $\hat{\xi}_\Sigma$ on the principal bundle Σ defines in turn the Lie derivative operator on sections of *any* fiber bundle associated to Σ, where objects like spinors or spin-connections are defined as sections. Unfortunately, this Lie derivative is not *natural*, in the sense that it does not preserve the commutator unless it is restricted to Killing vectors only. However, we stress that an advantage of this framework consists in showing and definitely explaining why there cannot be and in fact there is no possible natural prescription for the Lie derivative of spinors. As a consequence, one has to choose whether to restrict artificially to Killing vectors (which is certainly physically impossible unless under extremely special conditions) or to learn how to cope with the fact that spinors are non-natural objects. The gauge natural formalism is a possible escape (see [3]). In any case unless restricting to very special situation, one has to define Lie derivatives with respect to arbitrary spacetime vector fields. Furthermore, even in special situations one can *a posteriori* restrict the vector field to be Killing one (if any exists) in order to obtain a unifying view on the matter, in which all Lie derivatives are obtained as a specialization of a general notion.

The very same framework introduced for spinors provides a suitable arena to deal with *Lorentz tensors* in GR. Similar approaches can be found in the literature (see [27]) as well as more recently (see [21]). In GR there are many objects which are endowed with specific transformation rules with respect to Lorentz transformations, even though, of course, in GR these transformations cannot be implemented in general by a subgroup of the whole group of all diffeomorphisms. Let us mention e.g. tetrads and spin connections in a Cartan framework, where pointwise Lorentz transformations act as a gauge group. This framework is also the kinematical arena to define the self-dual formulation of GR that is the starting point of LQG approach.

We shall here review the general theory of Lorentz tensors and their Lie derivative and compare with the direct and *ad hoc* method based on Killing vectors appeared in [22]. The key issue consists in recognizing that Lorentz tensors are, *by definition*, sections of some bundle associated to a suitable principal bundle Σ by means of the appropriate tensorial representation of the appropriate special orthogonal structure group.

2 The Kosmann lift

Let M be a m-dimensional manifold (which will be required to allow global metrics of signature $\eta = (r, s)$, with $m = r + s$). Let us denote by x^μ local coordinates on M, which induce a basis ∂_μ of tangent spaces; let $L(M)$ denote the *general* *frame bundle* of M and set (x^μ, V^μ_a) for fibered coordinates on $L(M)$. We can define a right-invariant basis for vertical vectors on $L(M)$

$$\rho^\mu_\nu = V^\mu_a \frac{\partial}{\partial V^\nu_a}$$

The general frame bundle is natural (see [15]), hence any spacetime vector field $\xi = \xi^\mu \partial_\mu$ defines a natural lift on $L(M)$

$$\hat{\xi} = \xi^\mu \,\partial_\mu + \partial_\mu \xi^\nu \,\rho^\mu_\nu$$

We stress that the lift vector field $\hat{\xi}$ is global whenever ξ is global.

A connection on $L(M)$ is denoted by $\Gamma^\alpha_{\beta\mu}$ and it defines a lift

$$\Gamma : TM \to TL(M) : \xi^\mu \partial_\mu \mapsto \xi^\mu \left(\partial_\mu - \Gamma^\alpha_{\beta\mu} \rho^\beta_\alpha\right)$$

This lift does not in general preserve commutators, unless the connection is flat.

Ordinary tensors are sections of bundles associated to $L(M)$. The connection $\Gamma^\alpha_{\beta\mu}$ induces connections on associated bundles and defines in turn the covariant derivatives of ordinary tensors.

Example 1. For example, tensors of rank $(1,1)$ are sections of the bundle $T^1_1(M)$ associated to $L(M)$ using the appropriate tensor representations, namely

$$\lambda : \mathrm{GL}(m) \times V \to V : (J^\mu_\nu, t^\mu_\nu) \mapsto t'^\mu_\nu = J^\mu_\alpha t^\alpha_\beta \bar{J}^\beta_\nu$$

where the bar denotes the inverse in $\mathrm{GL}(n, \mathbb{R})$.

The connection Γ on $L(M)$ induces on this associated bundle the connection

$$T^1_1(\Gamma) = dx^\mu \otimes \left(\partial_\mu - \left(\Gamma^\alpha_{\gamma\mu} t^\gamma_\beta - \Gamma^\gamma_{\beta\mu} t^\alpha_\gamma\right) \frac{\partial}{\partial t^\alpha_\beta}\right)$$

which in turn defines the standard covariant derivative of such tensors:

$$\nabla_\xi t = Tt(\xi) - T^1_1(\Gamma)(\xi) = \xi^\mu \left(d_\mu t^\alpha_\beta + \Gamma^\alpha_{\gamma\mu} t^\gamma_\beta - \Gamma^\gamma_{\beta\mu} t^\alpha_\gamma\right) \frac{\partial}{\partial t^\alpha_\beta}$$

If a metric $g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$ is given on M then its Christoffel symbols define the Levi-Civita connection of the metric. Such a connection is torsionless (i.e. symmetric in lower indices) and *compatible with the metric*, i.e. such that $\nabla_\mu g_{\alpha\beta} = 0$.

Let now $(\Sigma, M, \pi, \mathrm{SO}(\eta))$ be a principal bundle over the manifold M and let (x^μ, S^a_b) be (overdetermined) fibered "coordinates" on the principal bundle Σ. We can define a right-invariant pointwise basis σ_{ab} for vertical vectors on Σ by setting

$$\sigma_{ab} = \eta_{d[a} \rho^d_{b]} \qquad\qquad \rho^d_b = S^d_c \frac{\partial}{\partial S^b_c}$$

where η_{ab} is the canonical diagonal matrix of signature $\eta = (r, s)$ and square brackets denote skew-symmetrization over indices.

A connection on Σ is in the form

$$\omega = dx^\mu \otimes \left(\partial_\mu - \omega^{ab}_\mu \sigma_{ab}\right)$$

Also in this case the connection on Σ induces connections on any associated bundle and there defines covariant derivatives of sections.

A *frame* is a bundle map $e : \Sigma \to L(M)$ which preserves the right action, i.e. such

$$\begin{array}{ccc} \Sigma & \xrightarrow{\;e\;} & L(M) \\ \downarrow & & \downarrow \\ M & = & M \end{array} \qquad\qquad \begin{array}{ccc} \Sigma & \xrightarrow{\;e\;} & L(M) \\ \downarrow{\scriptstyle R_S} & & \downarrow{\scriptstyle R_{i(S)}} \\ \Sigma & \xrightarrow{\;e\;} & L(M) \end{array}$$

i.e. $e \circ R_S = R_{i(S)} \circ e$, where R denotes the relevant canonical right actions defined on the principal bundles Σ and $L(M)$ and where $i : \mathrm{SO}(\eta) \to \mathrm{GL}(m)$ is the canonical group inclusion. We stress that on any M which allows global metrics of signature η the bundle Σ can always be chosen so that there exist global frames; see [7]. Locally the frame is represented by invertible matrices e^μ_a and it defines a spacetime metric $g_{\mu\nu} = e^\mu_a \, \eta_{ab} \, e^\nu_b$ which is called the *induced metric*.

As for the Levi-Civita connection, a frame defines a connection on Σ (called the spin-connection of the frame) given by

$$\omega^{ab}_\mu = e^a_\alpha \left(\Gamma^\alpha_{\beta\mu} e^{b\beta} + d_\mu e^{b\alpha}\right) \tag{1}$$

where $\Gamma^\alpha_{\beta\mu}$ denote Christoffel symbols of the induced metric. The spin-connection is compatible with the frame in the sense that

$$\nabla_\mu e^\nu_a = d_\mu e^\nu_a + \Gamma^\nu_{\lambda\mu} e^\lambda_a - \omega^c{}_{a\mu} e^\nu_c \equiv 0$$

In general the (natural) lift $\hat{\xi}$ of a spacetime vector field ξ to $L(M)$ is not adapted to the image $e(\Sigma) \subset L(M)$ and thence it does not define any vector field on Σ. With this notation the Kosmann lift of $\xi = \xi^\mu \partial_\mu$ is defined by $\hat{\xi}_K = \xi^\mu \partial_\mu + \hat{\xi}^{ab} \sigma_{ab}$ (see [4]) where we set:

$$\hat{\xi}^{ab} = e^{[a}_\nu \nabla_\mu \xi^\nu e^{b]\mu} - \omega^{ab}_\mu \xi^\mu \tag{2}$$

and where $e^{a\mu} = \eta^{ac} e^\mu_c$ and e^b_ν denote the inverse frame matrix.

Let us stress that despite appearing so, the Kosmann lift (2) does not in fact depend on the connection, but just on the frame and its first derivatives. The same lift can be written as $\hat{\xi}^{ab} = \nabla^{[b} \xi^{a]} - \omega^{ab}_\mu \xi^\mu$ where we set $\xi^a = \xi^\mu e^a_\mu$ since one can prove that

$$\nabla_b \xi^a = e^a_\nu \nabla_\mu \xi^\nu e^\mu_b$$

Another useful equivalent expression for the Kosmann lift is giving the vertical part of the lift with respect to the spin connection (see [6], pages 288–290), namely

$$\hat{\xi}^{ab}_{(V)} := \hat{\xi}^{ab} + \omega^{ab}_\mu \xi^\mu = e^{[a}_\nu \nabla_\mu \xi^\nu e^{b]\mu} = \nabla^{[b} \xi^{a]} \tag{3}$$

This last expression is useful since it expresses a manifestly covariant quantity.

We have to stress that the Kosmann lift does not preserve commutators. In fact if one considers two spacetime vectors ξ and ζ and computes the Kosmann lift of the commutator $[\xi, \zeta]$ one can easily prove that

$$[\xi,\zeta]\hat{}_K = [\hat{\xi}_K, \hat{\zeta}_K] + \frac{1}{2} e^a_\alpha \pounds_\zeta g^{\alpha\lambda} \pounds_\xi g_{\lambda\beta} e^{b\beta} \sigma_{ab}$$

Thence only if one restricts to Killing vectors (i.e. $\pounds_\xi g = 0$) one recovers that the lift preserves commutators.

3 The Lie Derivative of Lorentz Tensors

Let λ be a representation (of rank (p,q)) of $\mathrm{SO}(\eta)$ over a suitable vector space V. Let E_A be a basis of V so that a point $t \in V$ is given by $t = t^A E_A$ and $\lambda(J,t) = \lambda^A_B(J) t^B$.

Example 2. For example, if $V = T^1_1(\mathbb{R}^m) \sim \mathbb{R}^m \otimes \mathbb{R}^m$ with coordinates t^a_b we may have

$$\lambda : \mathrm{SO}(\eta) \times V \to V : (J,t) \mapsto J^a_c t^c_d \bar{J}^d_b$$

the bar denoting now the inverse in $\mathrm{SO}(\eta)$. This is the tensor representation of rank $(1,1)$.

Then, by definition, a *Lorentz tensor* is a section of the bundle $\Sigma_\lambda = \Sigma \times_\lambda V$ associated to λ through the representation λ. Fibered coordinates on Σ_λ are in the form (x^μ, t^A) and transition functions of Σ act on Σ_λ through the representation λ.

If we consider a global infinitesimal generator of automorphisms over Σ (also called a *Lorentz transformation*) locally expressed as

$$\Xi = \xi^\mu(x)\partial_\mu + \xi^{ab}(x)\sigma_{ab}$$

(which projects over the spacetime vector field $\xi = \xi^\mu \partial_\mu$) this induces a global vector field over Σ_λ locally given by

$$\Xi_\lambda = \xi^\mu(x)\partial_\mu + \xi^A \frac{\partial}{\partial t_A} \qquad \xi^A = \xi^{ab}\partial_{ab}\lambda^A_B(\mathbb{I}) t^B$$

Let us remark that this vector field is linear in ξ.

Example 3. For example, if λ is the tensor representation of rank $(1,1)$ given above, then the induced vector field is

$$\Xi_\lambda = \xi^\mu \partial_\mu + \left(\xi^{a\cdot}{}_{c}\, t^c_b - t^a_d\, \xi^{d\cdot}{}_{b}\right) \frac{\partial}{\partial t^a_b}$$

where indices are lowered and raised by η_{ab}.

According to the general framework for Lie derivatives (see [24]) for a section $t : M \to \Sigma_\lambda : x^\mu \mapsto (x, t^A(x))$ of the bundle Σ_λ with respect to the (infinitesimal) Lorentz tranformation Ξ, we find

$$\pounds_\Xi t = Tt(\xi) - \Xi_\lambda \circ t = \left(\xi^\mu d_\mu t^A - \xi^{ab}\partial_{ab}\lambda^A_B(\mathbb{I}) t^B\right) \frac{\partial}{\partial t^A} \tag{4}$$

Example 4. For example, if λ is the tensor representation of rank $(1,1)$ given above the Lie derivative of a section reads as

$$\pounds_\Xi t = \left(\xi^\mu d_\mu t^a_b - \xi^{a\cdot}_{\ \ c}\, t^c_b + t^a_d\, \xi^{d\cdot}_{\ \ b}\right) \frac{\partial}{\partial t^a_b} = \left(\xi^\mu \nabla_\mu t^a_b - (\xi_{(V)})^{a\cdot}_{\ \ c}\, t^c_b + t^a_d\, (\xi_{(V)})^{d\cdot}_{\ \ b}\right) \frac{\partial}{\partial t^a_b}$$

where $(\xi_{(V)})^{a\cdot}_{\ \ c} = \xi^{a\cdot}_{\ \ c} + \omega^a{}_{c\mu}\xi^\mu$ denotes the vertical part of Ξ with respect to the same connection used for the covariant derivative $\nabla_\mu t^a_b = d_\mu t^a_b + \omega^a{}_{c\mu} t^c_b - \omega^c{}_{b\mu} t^a_c$. Let us stress that in spite of its convenient connection-dependent expressions the Lie derivative does not eventually depend on any connection (as it may seem from our second expression).

Notice that this definition of Lie derivatives is natural, i.e. it preserves commutators, namely

$$[\pounds_{\Xi_1}, \pounds_{\Xi_2}]\sigma = \pounds_{[\Xi_1,\Xi_2]}\sigma \tag{5}$$

Unfortunately, Lorentz tranformations as introduced above have nothing to do with coordinate transformations (or spacetime diffeomorphisms). They have been introduced as gauge transformations acting pointwise and completely unrelated to spacetime diffeomorphisms. Indeed the Lie derivative (4) can be performed with respect to bundle vector fields Ξ instead of spacetime vector fields and this is completely counterintuitive if compared with what expected for spacetime objects like, for example, spinors. These objects are in fact expected to react to spacetime transformations; on the other hand, on a general spacetime there is nothing like Lorentz transformations.

We shall hence define Lie derivatives of Lorentz tensors with respect to any spacetime vector field and then show that in Minkowski spacetime, where Lorentz trasformations are defined, these reproduce and extend the standard notion. The price to be paid is loosing naturality like (5) (which will be retained only for Killing vectors if Killing vectors exist on M).

Let us restrict to vector fields $\hat{\xi}_K$ of Σ which are the Kosmann lift of a spacetime vector field ξ and define the Lie derivative of the Lorentz tensor t with respect to the spacetime vector field ξ to be

$$\pounds_\xi t \equiv \pounds_{\hat{\xi}_K} t = (\xi^\mu d_\mu t^A - \hat{\xi}^{ab}\partial_{ab}\lambda^A_B(\mathbb{I})t^B)\frac{\partial}{\partial t^A}$$

where $\hat{\xi}^{ab}$ is expressed in terms of the derivatives of ξ^μ (and the frame) as in (2).

Example 5. For example, for Lorentz tensors of rank $(1,1)$ we have

$$\begin{aligned}
\pounds_\xi t \equiv \pounds_{\hat{\xi}} t &= \left(\xi^\mu d_\mu t^a_b - \hat{\xi}^{a\cdot}_{\ \ c}\, t^c_b + t^a_d\, \hat{\xi}^{d\cdot}_{\ \ b}\right) \frac{\partial}{\partial t^a_b} = \\
&= \left(\xi^\mu \nabla_\mu t^a_b - (\hat{\xi}_{(V)})^{a\cdot}_{\ \ c}\, t^c_b + t^a_d\, (\hat{\xi}_{(V)})^{d\cdot}_{\ \ b}\right) \frac{\partial}{\partial t^a_b} = \\
&= \left(\xi^\mu \nabla_\mu t^a_b - \nabla_c \xi^a\, t^c_b + t^a_d\, \nabla_b \xi^d\right) \frac{\partial}{\partial t^a_b} = \\
&= \left(\nabla_d \left(\xi^d t^a_b\right) - \nabla_c \xi^a\, t^c_b\right) \frac{\partial}{\partial t^a_b}
\end{aligned}$$

For a generic Lorentz tensor of any rank, similar terms arise one for each Lorentz index.

Now since the Kosmann lift on Σ does not preserve commutators these Lie derivatives are not natural unless one artificially restricts ξ to be a Killing vector (of course provided M allows Killing vectors!). In fact, one has generically

$$\pounds_{[\xi,\zeta]} t \equiv \pounds_{[\xi,\zeta]\hat{}_K} t \neq \pounds_{[\hat{\xi}_K,\hat{\zeta}_K]} t = [\pounds_{\hat{\xi}_K}, \pounds_{\hat{\zeta}_K}] t \equiv [\pounds_\xi, \pounds_\zeta] t$$

Example 6. One can try to specialize this to simple cases in order to make non-naturality manifest. For example, if one considers a Lorentz vector v^a and two spacetime vector fields ξ and ζ one can easily check that

$$\pounds_{[\xi,\zeta]} v^a = [\pounds_\xi, \pounds_\zeta] v^a + \frac{1}{4}\left(v^\alpha g^{\beta\rho} e^{a\sigma} - v^\rho g^{\beta\sigma} e^{a\alpha}\right) \pounds_\xi g_{\rho\sigma} \pounds_\zeta g_{\alpha\beta}$$

Let us remark that according to this expression when ξ or ζ are Killing vectors of the metric g commutators are preserved. Moreover, the extra term does not vanish in general.

Of course, there are degenerate cases (e.g. setting $\xi = \zeta$) in which the extra terms vanishes due to coefficients without requiring Killing vectors. However, in this case also the other terms vanish.

4 Properties of Lie Derivatives of Lorentz Tensors

We shall prove here two important properties of Lie derivatives as defined above (see, for example, [11], [14], [25], [26] and references quoted therein)

For the Lie derivative of a frame one has

$$\pounds_\xi e^a_\mu = \xi^\lambda \nabla_\lambda e^a_\mu - \nabla_\mu \xi^\lambda e^a_\lambda + (\hat{\xi}_{(V)})^a_b e^b_\mu$$

If we are using, as we can always choose to do, the spin and the Levi-Civita connections for the relevant covariant derivatives, then $\nabla_\lambda e^a_\mu = 0$. By using the Kosmann lift (3) one easily obtains

$$\pounds_\xi e^a_\mu = -\nabla_\mu \xi^\lambda e^a_\lambda + \nabla^{[b}\xi^{a]} e_{b\mu} = -\nabla_\mu \xi^\lambda e^a_\lambda + \nabla_{[\mu}\xi_{\lambda]} e^{a\lambda} = -\nabla_{(\mu}\xi_{\lambda)} e^{a\lambda} =$$
$$= \frac{1}{2}\pounds_\xi g_{\mu\lambda} e^{a\lambda}$$

This expression holds true for any spacetime vector ξ and of course it proves that the Lie derivative vanishes along Killing vectors.

Let us stress that this last expression, obtained here from the general prescription for the Lie derivative of Lorentz tensors, is trivial in view of the expression on the induced metric as a function of the frame; in fact,

$$\frac{1}{2}\pounds_\xi g_{\mu\lambda} e^{a\lambda} = \pounds_\xi e^c_\mu e_{c\lambda} e^{a\lambda} = \pounds_\xi e^a_\mu$$

For the second property we wish to prove let us first notice that the frame induces an isomorphism between TM (on which one considers (x^μ, v^μ) as fibered coordinates) and the bundle of Lorentz vectors $\Sigma \times_\lambda \mathbb{R}^m$ (on which (x^μ, v^a) are considered as fibered coordinates) by

$$\Phi : TM \to \Sigma \times_\lambda \mathbb{R}^m : v^\mu \mapsto v^a = e^a_\mu v^\mu$$

We can thence express the Lie derivative of a section v of $\Sigma \times_\lambda \mathbb{R}^m$ (i.e. a Lorentz vector) in terms of the Lie derivative of the corresponding section of TM. In fact one has:

$$\begin{aligned}\pounds_\xi v^a &= \xi^\mu \nabla_\mu v^a - (\hat{\xi}_{(V)})^a_b v^b = \xi^b \nabla_b v^a - \nabla^{[b}\xi^{a]} v_b = \pounds_\xi v^\mu e^a_\mu + \nabla^{(b}\xi^{a)} v_b = \\ &= \pounds_\xi v^\mu e^a_\mu - \frac{1}{2} e^a_\mu \pounds_\xi g^{\mu\nu} e^b_\nu v_b = \pounds_\xi v^\mu e^a_\mu + \pounds_\xi e^a_\mu v^\mu \end{aligned} \tag{6}$$

Let us stress that these two properties hold true for any spacetime vector field ξ and they specialize to the ones discussed in [22] for Killing vectors.

The origin and meaning of the Lie derivative (6) can be easily understood: one has to take into account that if one drags ξ^a along a vector field the overall change of the object receives a contribution from how the vector changes but also a contribution from how the frame changes.

Similar properties can be easily found for Lorentz tensors of any rank since the frame transforms ordinary tensors into Lorentz tensors; e.g. one has

$$\Phi : t^\mu_\nu \mapsto t^a_b = e^a_\alpha t^\alpha_\beta e^\beta_b$$

5 Transformation of Lorentz Vectors in Minkowski Spacetime

Let us consider Minkowski spacetime $M = \mathbb{R}^4$ with the metric η; being it contractible any bundle over it is trivial. As a consequence we are forced to choose $\Sigma = \mathbb{R}^4 \times \mathrm{SO}(3,1)$. Since $M \equiv \mathbb{R}^4$ is parallelizable, its frame bundle is trivial, i.e. $L(\mathbb{R}^4) = \mathbb{R}^4 \times \mathrm{GL}(4)$. Let us fix Cartesian coordinates x^μ on $M \equiv \mathbb{R}^4$ and let us fix a frame $e_a = \delta^\mu_a \partial_\mu$; such a frame induces the Minkowski metric $\eta_{\mu\nu}$.

In such notation the Levi-Civita connection vanishes, $\Gamma^\alpha_{\beta\mu} = 0$ and the spin connection too, $\omega^{ab}_\mu = 0$; the Kosman lift hence specializes to

$$\hat{\xi}^{ab}_{(V)} = e^{[b\beta} \nabla_\beta \xi^\alpha e^{a]}_\alpha$$

Let us now consider a vector field ξ the flow which is made of Lorentz coordinate tranformations $x'^\mu = \Lambda^\mu_\nu x^\nu$; since ξ is of course a Killing vector, then the Lie derivative of a Lorentz vector is

$$\pounds_\xi v^a = \pounds_\xi v^\mu e^a_\mu = \left(\xi^\alpha \partial_\alpha v^\mu - v^\alpha \dot{\Lambda}^\mu_\alpha \right) \delta^a_\mu \tag{7}$$

Such a Lie derivative corresponds to the trasformation rules

$$v'^a = \Lambda^a_b v^b \tag{8}$$

which is exactly as a vector is expected to trasform under a Lorentz coordinate transformation.

A similar result can be easily extended to covectors, tensors and, with slight though obvious changes, to spinors. When ξ is not Killing, however, the Lie derivative may not be the infinitesimal counterpart of a finite transformation rule as in (7) and (8); in this case the traditional interpretation of Lie derivatives as a measure of changing of objects dragged along spacetime vector fields fails to hold true. One should however wonder whether such an interpretation is really fundamental to many common uses of Lie derivatives. Our answer is in the negative as one can argue by a detailed analysis of physical quantities containing Lie derivatives.

Lie derivatives appear, e.g., in Noether theorem; in this case they appear naturally as a by-product of variational techniques. Here Noether currents turn out to be expressed in terms of Lie derivatives expressed as in equation (4). The interpretation of such Lie derivatives as measuring infinitesimal changes along symmetry transformations is important since, based on that, one can relate Noether currents to symmetries.

Now the essential point is that there is no reason to expect spacetime vector fields to be the most general (infinitesimal) symmetries in Physics. Fundamentally speaking, symmetries encode the observers' freedom to set their conventions to describe Physical world. While coordinates are certainly necessary conventions for any observer (and hence general covariance principle is a fundamental symmetry that should be expected in any physical system), special systems might need further conventions which might result in independent class of symmetries (as it happens in gauge theories, e.g. electromagnetism).

Of course, since these further conventions are independent of spacetime coordinate fixing, gauge transformations cannot be expressed as spacetime diffeomorphisms, but they are expressed as field transformations. As such they are vector fields on the configuration bundle, not on spacetime. It is hence reasonable and important to have a notion of Lie derivative of fields along bundle vectors, as in (4). It is only in GR where symmetries come from spacetime vector fields that one should expect Lie derivatives along spacetime vector fields and their interpretation as quantities related to the spacetime geometry.

This more general situation, i.e. when the quantities entering Noether theorem are interpreted as Lie derivatives of fields along bundle vectors, can be simply discussed by considering a very well-known physical situation, i.e. covariant electromagnetism.

The electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curvature of a field A_μ which is usually known as a *quadripotential* and, as it is well known, is a connection on a principal bundle P for the group $U(1)$. This is the standard gauge approach to electromagnetism. The Maxwell Lagrangian is

$$L_M = -\frac{1}{4}\sqrt{g} F_{\mu\nu} F^{\mu\nu}_{\cdot\cdot} \tag{9}$$

By variation one obtains

$$\delta L_M = -\frac{1}{2}\sqrt{g} H_{\alpha\beta}\delta g^{\alpha\beta} + \nabla_\mu \left(\sqrt{g} F^{\mu\nu}\right) \delta A_\nu - \nabla_\mu \left(\sqrt{g} F^{\mu\nu} \delta A_\nu\right) \tag{10}$$

where we set $H_{\alpha\beta} = F_{\mu\alpha} F^{\mu}_{\cdot\,\beta} - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma}_{\cdot\cdot} g_{\mu\nu}$ for the standard energy-momentum tensor of the electromagnetic field. The second term in (10) produces Maxwell equations, namely $\nabla_\mu\left(\sqrt{g}F^{\mu\nu}\right)=0$. The third term relates to conservation laws (see [6]).

The Lagrangian (9) is covariant with respect to the infinitesimal transformations

$$\Xi=\xi^\mu\frac{\partial}{\partial x^\mu}+2\partial_\alpha\xi^\mu g^{\alpha\nu}\frac{\partial}{\partial g^{\mu\nu}}+\left(\partial_\mu\xi-\partial_\mu\xi^\nu A_\nu\right)\frac{\partial}{\partial A_\mu}$$

which correspond to 1-parameter families of gauge transformations

$$\begin{cases} x'^\mu = x'^\mu_{(\epsilon)}(x)\\ g'^{\mu\nu} = \dfrac{\partial x'^\mu_{(\epsilon)}}{\partial x'^\alpha}g^{\alpha\beta}\dfrac{\partial x'^\nu_{(\epsilon)}}{\partial x'^\beta}\\ A'_\mu = \dfrac{\partial x^\nu}{\partial x'^\mu_{(\epsilon)}}\left(A_\nu+\partial_\nu\alpha_{(\epsilon)}\right)\end{cases}$$

Here the generator ξ^μ is related to the coordinate change $x'^\mu=x'^\mu_{(\epsilon)}(x)$ while the generator ξ is related to the gauge transformation $\alpha_{(\epsilon)}$.

Let us remark that Ξ is a vector field on the configuration bundle (that is a manifold with coordinates $(x^\mu, g^{\mu\nu}, A_\mu)$), not on spacetime. In a general situation (namely unless the principal bundle P is assumed to be trivial) there is no way of either lifting a spacetime vector field to the configuration bundle or globally setting $\xi=0$ so to split the vector Ξ into a spacetime vector and a *"gauge generator"*. In a physical language one usually says that the condition $\xi=0$ is not gauge covariant and hence local, unless there exist global gauges. (By the way, also when global gauges exist, the condition is not gauge covariant and hence unphysical, from a fundamental viewpoint.)

The Lie derivative of the field A_μ along the symmetry generator Ξ is in this case (see (4))

$$\pounds_\Xi A_\mu=\xi^\lambda F_{\lambda\mu}-\nabla_\mu\left(\xi-\xi^\lambda A_\lambda\right)$$

Noether theorem in this case shows (see again [6]) on-shell conservation of the following Noether current

$$\mathcal{E}^\mu=-\sqrt{g}\left(F^{\mu\nu}\pounds_\Xi A_\nu+\xi^\mu L_M\right)$$

In the special case when $\xi^\mu=0$ one has

$$\mathcal{E}^\mu=\sqrt{g}\left(F^{\mu\nu}\nabla_\mu\xi\right)=\nabla_\mu\left(\sqrt{g}F^{\mu\nu}\xi\right)-\nabla_\mu\left(\sqrt{g}F^{\mu\nu}\right)\xi$$

The second term vanishes on-shell, thus one obtains

$$\mathcal{E}^\mu=\nabla_\mu\left(\sqrt{g}F^{\mu\nu}\xi\right)$$

The corresponding conserved quantity is

$$Q(\xi)=\frac{1}{2}\int_{\partial\Omega}\sqrt{g}F^{\mu\nu}\xi\,ds_{\mu\nu}$$

where $ds_{\mu\nu}$ is the area element on the boundary of the 3-region Ω of spacetime. This is the electric charge defined *à la* Gauss.

This example shows clearly what happens in general when gauge transformations are allowed and symmetry generators live at bundle level: also in this case Noether theorem involves Lie derivatives, though in the generalized sense introduced above. In this case we are not dealing with Lorentz objects so one cannot introduce Kosmann lift (or similar lifts) and reduce everything to spacetime vector fields.

6 Applications

In order to provide an example of concrete aplication of our formalism here introduced in action we shall here consider the application to the so called *Holst's action principle* (see [13]) which is used as an equivalent formulation of GR suitable for developing LQG through the use of the Barbero-Immirzi connection (see [1], [20], [8], [9] as well as references quoted therein).

Let us first consider tetrad-affine formulation of GR: the fundamental fields are a Lorentz connection Γ^{ab}_μ and a vielbein $e^a = e^a_\mu\, dx^\mu$. The connection defines the curvature form $R^{ab} = \frac{1}{2} R^{ab}{}_{\mu\nu} dx^\mu \wedge dx^\nu$. Let us also set $e = \det |e^a_\mu|$, $R^a{}_\mu = R^{ab}{}_{\mu\nu} e^\nu_b$ and $R = R^{ab}{}_{\mu\nu} e^\mu_a e^\nu_b$; here e^ν_b denotes the inverse frame matrix of e^b_ν. The frame also defines a metric $g_{\mu\nu} = e^a_\mu \eta_{ab} e^b_\nu$ which in turn defines its Levi-Civita spacetime connection $\Gamma^\alpha_{\beta\mu}$.

On a spacetime of dimension 4, let us consider the Lagrangian

$$L_{tA} = R^{ab} \wedge e^c \wedge e^d\, \epsilon_{abcd}$$

By variation we obtain

$$\begin{aligned}\delta L_{tA} = &-2 e e^\sigma_a \left(R^a{}_\mu - \frac{1}{2} R e^a_\mu \right) e^\mu_d\, \delta e^d_\mu - \epsilon_{abcd} \nabla_\mu \left(e^c_\rho e^d_\sigma \right) \epsilon^{\mu\nu\rho\sigma} \delta \Gamma^{ab}_\mu + \\ &+ \epsilon_{abcd} \nabla_\mu \left(e^c_\rho e^d_\sigma \delta \Gamma^{ab}_\mu \right) \epsilon^{\mu\nu\rho\sigma}\end{aligned}$$

Thus one obtains field equations

$$\begin{cases} R^a{}_\mu - \dfrac{1}{2} R e^a_\mu = 0 \\ \nabla_{[\mu} \left(e^{[c}_\rho e^{d]}_{\sigma]} \right) = 0 \end{cases}$$

The second field equation forces the connection to be the connection induced by the frame $\Gamma^{ab}_\mu = \omega^{ab}_\mu$ (see eq. (1)); then the first equation forces the induced metric to obey Einstein equations.

This field theory is dynamically equivalent to standard GR, in the sense that it obeys equivalent field equations. However, the theory is in fact richer in its physical interpretation, since the use of different variables and action principles generate larger symmetry and extra conservation laws. In fact, this theory has a bigger symmetry group being generally covariant and Lorentz covariant.

Noether theorem implies then conservation of the current

$$\mathcal{E}^\mu = 4 e e^\mu_a e^\nu_b \pounds_\Xi \Gamma^{ab}_\nu - \xi^\mu L_{tA}$$

along any Lorentz gauge generator $\Xi = \xi^\mu \partial_\mu + \xi^{ab}\sigma_{ab}$. The Lie derivative of a connection is given by

$$\pounds_\Xi \Gamma^{ab}_\nu = \xi^\lambda R^{ab}{}_{\lambda\nu} + \nabla_\nu \hat{\xi}^{ab}$$

where we set $\hat{\xi}^{ab} = \xi^{ab} + \xi^\lambda \Gamma^{ab}_\lambda$.

Hence one obtains

$$\mathcal{E}^\mu = 4ee^\mu_a \left(R^a{}_\mu - \frac{1}{2} R e^a_\mu \right) \xi^\lambda - 4\nabla_\nu \left(ee^\mu_a e^\nu_b \right) \hat{\xi}^{ab} + 4\nabla_\nu \left(ee^\mu_a e^\nu_b \hat{\xi}^{ab} \right)$$

The first and second terms vanish on-shell; hence one obtains

$$\mathcal{E}^\mu = 4\nabla_\nu \left(ee^\mu_a e^\nu_b \hat{\xi}^{ab} \right) \tag{11}$$

Let us stress that this current depends only on the Lorentz generator $\hat{\xi}^{ab}$.

Here is the issue with physical interpretation: we have two equivalent formulations of Einstein GR where Noether currents in one case depend on spacetime vector fields while in tetrad-affine formulation Noether currents depend on Lorentz generator which *a priori* has nothing to do with spacetime transformations. Let us stress of course that unless the spacetime is Minkowski, there is no class of spacetime diffeomorphisms representing *Lorentz transformations*.

Considering the dynamical equivalence at level of field equations and solution space, one would like this equivalence to be extended at level of conservation laws. Moreover, some of the conserved quantities in standard GR are known to be related to physical quantities such as energy, momentum and angular momentum, while one would wish to be able to identify the corresponding quantities in the second formulation. Kosmann lift is in fact essential to relate Lorentz generators to spacetime diffeomorphisms and the corresponding conservation laws.

The Noether current (11) can be restricted setting $\Xi = \hat{\xi}_K$ so that one obtains

$$\mathcal{E}^\mu_{tA} = 4\nabla_\nu \left(e \nabla^\mu \xi^\nu \right)$$

which corresponds to the standard conserved quantity associated to spacetime diffeomorphisms in GR written in terms of Komar superpotential. This (and only this) restores the equivalence between standard GR and tetrad-affine formulation at level of conservation laws.

As a further example let us consider the covariant Lagrangian:

$$L_H = L_{tA} + \beta R^{ab} \wedge e_a \wedge e_b$$

which is known as Holst's Lagrangian.

By variations one obtains equations

$$\begin{cases} e^\mu_d \left(R^a_\mu - \frac{1}{2} R e^a_\mu \right) e^\sigma_a - \beta R_{d\rho\mu\nu} \epsilon^{\mu\nu\rho\sigma} = 0 \\ \nabla_{[\mu} \left(e^{[c}_\rho e^{d]}_{\sigma]} \right) = 0 \end{cases}$$

The second equation still imposes $\Gamma^{ab}_{\mu} = \omega^{ab}_{\mu}$; this in turns implies $R^a{}_{[\rho\mu\nu]} = 0$ (first Bianchi identity) and hence Einstein equations. This shows how also Holst's Lagrangian provides an equivalent formulation of standard GR.

It is interesting to check if also in this case the equivalence is preserved also at level of conservation laws. The Noether current is

$$\mathcal{E}^{\mu}_{H} = 4ee^{\mu}_{a}e^{\nu}_{b}\pounds_{\Xi}\Gamma^{ab}_{\nu} + ee^{\mu}_{c}e^{\nu}_{d}\epsilon^{cd\cdot\cdot}{}_{ab}\pounds_{\Xi}\Gamma^{ab}_{\nu} - \xi^{\mu}L_{H}$$

As in the previous case this can be recasted modulo terms vanishing on-shell as follows

$$\mathcal{E}^{\mu}_{H} - \mathcal{E}^{\mu}_{tA} = \nabla_{\nu}\left(ee^{\mu}_{c}e^{\nu}_{d}\epsilon^{cd\cdot\cdot}{}_{ab}\hat{\xi}^{ab}\right)$$

Again this has nothing to do with spacetimes symmetries and in general would affect conserved quantities. When Kosmann lift is again inserted into these conservation laws one obtains

$$\mathcal{E}^{\mu}_{H} - \mathcal{E}^{\mu}_{tA} = \nabla_{\nu}\left(\nabla^{\rho}\xi^{\sigma}\epsilon^{\mu\nu}{}_{\rho\sigma}\right)$$

which vanishes being the divergence of a divergence. Hence once again the correspondence at level of conservation laws is preserved when the Kosmann lift is used.

7 Conclusion

We presented a framework to deal with Lorentz objects and showed how it applies to tetrad-affine formulation and Holst's formulation of GR. In particular we showed that equivalence can be extended at the level of conservation laws if one introduces the Kosmann lift which establishes a correspondence among symmetry generators in different formulations.

One could argue whether the Lie derivatives defined above could be physically interpreted in a correct way. Of course, one could always restrict to situations in which enough Killing vectors exist (or even to Minkowski spacetime $(\mathbb{R}^4, \eta)$); in these cases the standard results are obtained in particular.

However, in a generic spacetime (M, g) one has no Killing vectors and at the end one has to decide whether a physical interpretation of these objects along generic spacetime vector field makes any sense.

The framerwork we introduced for Lorentz tensors provides a rigorous way of investigating formal properties which in our opinion are the only necessary basis for a physical intepretation of Lie derivatives of Lorentz tensors themselves.

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Several examples of nonholonomic mechanical systems

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Abstract. A unified geometric approach to nonholonomic constrained mechanical systems is applied to several concrete problems from the classical mechanics of particles and rigid bodies. In every of these examples the given constraint conditions are analysed, a corresponding constraint submanifold in the phase space is considered, the corresponding constrained mechanical system is modelled on the constraint submanifold, the reduced equations of motion of this system (i.e. equations of motion defined on the constraint submanifold) are presented. Finally, solvability of these equations is discussed and general solutions in explicit form are found.

1 Introduction

In some mechanical and engineering problems one encounters different kinds of additional conditions, constraining and restricting motions of mechanical systems. Such conditions are called *constraints*. Constraints may be given by algebraic equations connecting coordinates (holonomic or geometric constraints), or by differential equations, which restrict coordinates and components of velocities (kinematic constraints). Nonintegrable kinematic constraints, which cannot be reduced to holonomic ones, are called *nonholonomic constraints*.

Classical theoretical mechanics deals with nonholonomic constraints only marginally, mostly in a form of short remarks about the existence of such constraints, or mentioning some problems where simple nonholonomic constraints occur. Only rarely, for example, in textbook [2] one can find sections where nonholonomic constraints are discussed in more detail and a few examples of simple mechanical systems subjected to a nonholonomic constraint are solved. However, these books deal only with semiholonomic or linear nonholonomic constraints (constraints linear in components of velocities), arising for example in the connection with rolling

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of rigid bodies. Discussion is usually concluded by a remark that more complicated nonholonomic constraints (when the dependence on velocities is nonlinear) are not mastered by means of classical methods and motion equations of mechanical systems subjected to such constraints are not known.

A significant contribution to the study of problems of nonholonomic mechanics represents an extensive monograph [22] which contains various application problems, mostly problems concerning rolling of rigid bodies on a horizontal plane or on an absolutely rough surface where typically nonholonomic constraints linear in velocities occur. This monograph serves as a classical collection of solved problems of nonholonomic dynamics. However, it does not give a unified and consistent approach applicable to arbitrary nonholonomic mechanical systems. Equations of motion of the considered nonholonomic systems are mostly derived on the basis of a heuristic analogy with holonomic systems. On the other hand their solutions agree with experience and experiments.

During the last 20 years the problems of nonholonomic mechanics have been intensively studied in many papers, e.g. [3], [4], [5], [7], [8], [9], [10], [13], [14], [20], [21], [23] and there have been proposed several alternative geometric concepts, appropriate in different situations, applicable to Lagrangian systems in tangent bundles or in jet bundles. Equations of motion of nonholonomic systems are investigated also in the monographs [1], [6], where a number of concrete application problems is discussed and numerical aspects of solutions are presented. However, it should be stressed, that almost all the work on nonholonomic systems is concerned with the case of constraints linear in components of velocities.

A geometric theory covering general nonholonomic systems has been proposed and developed by Krupková in [14], [15], [16], [17] (see also [18] for review). Her approach is suitable for study of all kinds of mechanical systems – without restricting to Lagrangian, time-independent, or regular ones, and is applicable to arbitrary constraints (holonomic, semiholonomic, linear, nonlinear or general nonholonomic). The theory gives motion equations for constrained mechanical systems in a form of reduced equations defined on the constraint submanifold (without Lagrange multipliers), provides a nonholonomic variational principle [17], [24] from which one can obtain reduced equations as corresponding "nonholonomic Euler-Lagrange equations", enables one to study constraint symmetries and the corresponding conservation laws, etc. In particular, a new treatment of concrete examples of nonholonomic systems is at hand, suitable for either systems with linear constraints [11], [12], [25], [26], [27], or even with nonlinear constraints [19], [25] and providing new methods for explicit studies and solutions.

The aim of this paper is to apply Krupková's geometric theory of nonholonomic mechanical systems to study concrete problems in both linear and nonlinear nonholonomic dynamics. In all the cases we analyse the given constraint conditions, consider the corresponding constraint submanifold in the phase space, we construct the corresponding constrained mechanical system on the constraint submanifold, present the reduced equations of motion of this system, and finally discuss the solvability of these equations. In most cases we are able to obtain general solutions in an explicit form. It turns out that reduced equations indeed represent an effective method for solving concrete mechanical and engineering problems of nonholonomic mechanics.

The paper contains complete and comprehensive solutions of seven problems from the classical mechanics of particles and rigid bodies where nonholonomic constraints appear. Three of them (5.1, 5.4 and 5.5) concern dynamics of a free particle or a particle in a homogeneous gravitational field subject to a *nonlinear nonholonomic constraint*. We find general solutions in an explicit form, with respect to appropriate initial conditions. Problem 5.2 (a dog pursues a man) is formulated in [2]; we study it as a mechanical system modelled on a nonholonomic submanifold and provide the reduced equation of motion. A solution in an explicit form is found by eliminating the time parameter from Chetaev equations. The next problem (5.3) is then a generalization of the previous one. The last two problems belong to the mechanics of rigid bodies (a disc rolling without sliding on a horizontal plane and a ball rolling without sliding on a horizontal plane) and as examples of nonholonomic systems are discussed in the monograph [22]. We study them in a different way, again using the geometric model leading to reduced equations. In particular, compared with [22] where a solution of the last problem 5.7 for the case of constant angular velocity of rotation of the horizontal plane is given, dealing with reduced equations we provide a procedure of solution applicable in the case of constant angular velocity as well as of nonconstant angular velocity.

2 Lagrangian systems on fibered manifolds

Throughout the paper we consider a fibered manifold $\pi\colon Y \to X$ with a one-dimensional base space X and $(m+1)$-dimensional total space Y. We use jet prolongations $\pi_1\colon J^1Y \to X$ and $\pi_2\colon J^2Y \to X$ and jet projections $\pi_{1,0}\colon J^1Y \to Y$ and $\pi_{2,1}\colon J^2Y \to J^1Y$. Configuration space at a fixed time is represented by a fiber of the fibered manifold π and a corresponding phase space is then a fiber of the fibered manifold π_1. Local fibered coordinates on Y are denoted by (t, q^σ), where $1 \le \sigma \le m$. The associated coordinates on J^1Y and J^2Y are denoted by $(t, q^\sigma, \dot q^\sigma)$ and $(t, q^\sigma, \dot q^\sigma, \ddot q^\sigma)$, respectively. In calculations we use either a canonical basis of one forms on J^1Y, $(dt, dq^\sigma, d\dot q^\sigma)$, or a basis adapted to the contact structure, $(dt, \omega^\sigma, d\dot q^\sigma)$, where

$$\omega^\sigma = dq^\sigma - \dot q^\sigma \, dt, \quad 1 \le \sigma \le m.$$

Whenever possible, the summation convention is used. If $f(t, q^\sigma, \dot q^\sigma)$ is a function defined on an open set of J^1Y we write

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma}\,\dot q^\sigma + \frac{\partial f}{\partial \dot q^\sigma}\,\ddot q^\sigma, \quad \frac{\bar d f}{\bar d t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\sigma}\,\dot q^\sigma.$$

A (local) section δ of π_1 is called *holonomic* if $\delta = J^1\gamma$ for a section γ of π.

A vector field ξ defined on J^1Y is called π_1-*vertical* (or simply *vertical*) if $T\pi_1 \cdot \xi = 0$, where T is the tangent functor. Similarly, a vector field ξ is called $\pi_{1,0}$-*vertical* if $T\pi_{1,0} \cdot \xi = 0$.

A differential form ρ is called *contact* if $J^1\gamma^*\rho = 0$ for every section γ of π. A differential form ρ is called *horizontal* if $i_\xi \rho = 0$ for every vertical vector field ξ. We denote by h the operator assigning to ρ its horizontal part. Every 2-form on J^1Y is contact and admits a *unique decomposition* $\pi^*_{2,1}\rho = \rho_1 + \rho_2$, where ρ_1 is a 1-*contact* form on J^2Y (i.e. for every vertical vector field ξ, $i_\xi\rho_1$ is a horizontal form), and ρ_2 is a 2-*contact* form (i.e. for every vertical vector field ξ, $i_\xi\rho_2$ is a 1-contact form). We denote by p_1, and p_2 operators assigning to ρ its 1-contact and 2-contact part, respectively.

By a *distribution* on J^1Y we shall mean a mapping D assigning to every point $z \in J^1Y$ a vector subspace $D(z)$ of the vector space T_zJ^1Y. A distribution can be spanned by a system of (local) vector fields. If D is a distribution, we denote by D^0 its annihilator, i.e. the set of all 1-forms η_κ on J^1Y such that $i_{\xi_\iota}\eta_\kappa = 0$ for every vector field ξ_ι belonging to D. In this sense, every distribution can be defined by a system of (local) 1-forms. For a distributions of a constant rank, i.e. that $\dim D(z)$ does not depend on z, the description by means of vector fields is completely equivalent with that by means of 1-forms. Recall that a section δ of π_1 is called an *integral section* of D if $\delta^*\eta = 0$ for every 1-form η belonging to D^0.

If λ is a Lagrangian on J^1Y, we denote by θ_λ its *Lepage equivalent* or *Cartan form* and E_λ its *Euler-Lagrange form*, respectively. Recall that $E_\lambda = p_1\, d\theta_\lambda$. In fibered coordinates where $\lambda = L(t, q^\sigma, \dot{q}^\sigma)\, dt$, we have

$$\theta_\lambda = L\, dt + \frac{\partial L}{\partial \dot{q}^\sigma}\, \omega^\sigma, \tag{1}$$

and $E_\lambda = E_\sigma(L)\omega^\sigma \wedge dt$, where the components

$$E_\sigma(L) = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma} \tag{2}$$

are the *Euler-Lagrange expressions.* Since the functions E_σ are affine in the second derivatives we write

$$E_\sigma = A_\sigma + B_{\sigma\nu}\, \ddot{q}^\nu,$$

where

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma}\dot{q}^\nu, \quad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}. \tag{3}$$

A section γ of π is called a *path* of the Euler-Lagrange form E_λ if

$$E_\lambda \circ J^2\gamma = 0. \tag{4}$$

In fibered coordinates this equation represents a system of m second-order ordinary differential equations

$$A_\sigma\Big(t, \gamma^\nu, \frac{d\gamma^\nu}{dt}\Big) + B_{\sigma\rho}\Big(t, \gamma^\nu, \frac{d\gamma^\nu}{dt}\Big)\frac{d^2\gamma^\rho}{dt^2} = 0 \tag{5}$$

for components $\gamma^\nu(t)$ of a section γ, where $1 \le \nu \le m$. These equations are called *Euler-Lagrange equations* or *motion equations* and their solutions are called *paths.*

Euler-Lagrange equations (4) or (5) can be written either in an intrinsic form as follows

$$J^1\gamma^* i_\xi d\theta_\lambda = 0,$$

where ξ runs over all π_1-vertical vector fields on J^1Y, or equivalently in the form

$$J^1\gamma^* i_\xi \alpha = 0,$$

where α is any 2-form defined on an open subset $W \subset J^1Y$, such that $p_1\alpha = E_\lambda$. Apparently $\alpha = d\theta_\lambda + F$, where F runs over $\pi_{1,0}$-horizontal 2-contact 2-forms. In fibered coordinates we have $F = F_{\sigma\nu}\, \omega^\sigma \wedge \omega^\nu$, where $F_{\sigma\nu}(t, q^\rho, \dot{q}^\rho)$ are arbitrary functions. Recall from [14] that the family of all such (local) 2-forms:

$$\alpha = d\theta_\lambda + F = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot{q}^\nu + F$$

is called a *first order Lagrangian system*, and is denoted by $[\alpha]$.

It is important to note that motion equations (5) of a Lagrangian system $[\alpha]$ need not be affine with respect to the second derivatives. If they posses this property, i.e. if

$$\det(B_{\sigma\rho}) = \det\left(\frac{\partial^2 L}{\partial\dot{q}^\sigma \partial\dot{q}^\nu}\right) \neq 0,$$

then the Lagrangian system $[\alpha]$ is called *regular*.

3 Constraints

From the physical point of view, constraints on a mechanical system are conditions restricting possible geometrical positions of the mechanical system or limiting its motion. We distinguish between geometric and kinematic constraints.

Constraints are called *geometric* or *holonomic* if they are expressed by equations of the form

$$f^i(t, q^1, \ldots, q^m) = 0, \quad 1 \leq i \leq k,$$

where m is a dimension of the configuration space and k is a given number (the number of constraint equations). Functions f^i are defined on the configuration space. Holonomic constraints are called *skleronomic* if they do not depend explicitly on time

$$f^i(q^1, \ldots, q^m) = 0, \quad 1 \leq i \leq k.$$

From the geometric point of view holonomic constraints represent submanifolds in the configuration space-time Y.

Constraints are called *kinematic* if they are expressed by

$$f^i(t, q^1, \ldots, q^m, \dot{q}^1, \ldots, \dot{q}^m) = 0, \quad 1 \leq i \leq k. \tag{6}$$

Now f^i are functions on the "phase space" J^1Y. Kinematic constraints are said to be *integrable* if the corresponding system of differential equations (6) is integrable. Integrable kinematic constraints are geometric constraints, since after integration they represent a restriction in the configuration space. Nonintegrable kinematic constraints (6), which cannot be reduced to geometric ones are called *nonholonomic* constraints.

Holonomic or nonholonomic constraints which depend explicitly on time are called *rheonomic*.

Nonholonomic constraints (6) are called *affine* or *linear in velocities* if they can be expressed by

$$\mathcal{A}_i(t,q^\nu) + \mathcal{B}_{i\sigma}(t,q^\nu)\,\dot q^\sigma = 0, \quad 1 \le \sigma,\nu \le m,\ 1 \le i \le k. \tag{7}$$

In particular, if the left-hand sides of (7) can be written in the form of total time derivatives of some functions defined on the configuration space, say $\frac{d\psi^i(t,q^\nu)}{dt} = 0$, then instead of equations (7) we write

$$\psi^i(t,q^\nu) - C^i = 0, \quad 1 \le i \le k,$$

where C^i are constants determined by initial conditions. In this case constraints (7) are called *linear integrable* or *semiholonomic* and the following identities hold

$$\mathcal{A}_i = \frac{\partial \psi^i}{\partial t}, \quad \mathcal{B}_{i\sigma} = \frac{\partial \psi^i}{\partial q^\sigma}.$$

Nonholonomic constraints (6) are called *affine of degree n in velocities* if they can be expressed by

$$f^i \equiv \mathcal{A}_i(t,q^\nu) + \mathcal{B}_{i\sigma}(t,q^\nu)\,(\dot q^\sigma)^n = 0, \quad 1 \le \sigma,\nu \le m,\ 1 \le i \le k.$$

For example, a relativistic particle in space-time $\mathbb{R}^4$ with Minkowski metric can be considered as mechanical system subjected to one nonholonomic constraint

$$-(\dot q^1)^2 - (\dot q^2)^2 - (\dot q^3)^2 + (\dot q^4)^2 - 1 = 0,$$

see [19], which is simple affine of degree 2 in velocities.

A geometric meaning of nonholonomic constraints is such that they represent submanifolds in the jet space J^1Y.

4 Nonholonomic Lagrangian systems

Following [14] we introduce general nonholonomic constraints (6) as submanifolds of J^1Y canonically endowed with a distribution.

Let $k < m$ be an integer. By a *constraint submanifold* in J^1Y we mean a fibered submanifold $\pi_{1,0}|_Q\colon Q \to Y$ of the fibered manifold $\pi_{1,0}\colon J^1Y \to Y$. We denote by ι the canonical embedding of Q into J^1Y, and suppose $\operatorname{codim} Q = k < m$ (cf. for example [14], [15], [21], [23]). Locally, Q can be given by equations

$$f^i(t,q^1,\dots,q^m,\dot q^1,\dots,\dot q^m) = 0, \quad 1 \le i \le k,$$

where

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot q^\sigma}\right) = k, \tag{8}$$

or, equivalently in an explicit form

$$\dot q^{m-k+i} = g^i(t,q^\sigma,\dot q^1,\dot q^2,\dots,\dot q^{m-k}), \quad 1 \le i \le k. \tag{9}$$

Equations (9) are called a *system of k nonholonomic constraints in normal form.*

The presence of a constraint submanifold in J^1Y gives rise to a concept of a *constrained section* as a local section $\bar{\delta}$ of the fibered manifold π_1 such that $\bar{\delta}(x) \in Q$ for every $x \in \operatorname{dom} \bar{\delta}$ and a *Q-admissible* section as a section $\bar{\gamma}$ of the fibered manifold π such that $J^1\bar{\gamma}(x) \in Q$ for every $x \in \operatorname{dom} \bar{\gamma}$.

The submanifold Q is naturally endowed with a distribution, called the *canonical distribution* [14], or *Chetaev bundle* [21], and denoted by C. It is annihilated by a system of k linearly independent (local) 1-forms

$$\varphi^i = \iota^* \phi^i, \quad \text{where} \quad \phi^i = f^i dt + \frac{\partial f^i}{\partial \dot{q}^\sigma} \omega^\sigma, \quad 1 \le i \le k,$$

called *canonical constraint 1-forms*. More frequently we shall use equations of a constraint submanifold Q in the form (9), i.e. $f^i = \dot{q}^{m-k+i} - g^i$. In this case canonical contact 1-forms $\bar{\omega}^\sigma = \iota^* \omega^\sigma$, $1 \le \sigma \le m$, restricted on Q split into two kinds of forms $\bar{\omega}^l = dq^l - \dot{q}^l dt$, $1 \le l \le m-k$, and $\bar{\omega}^{m-k+i} = dq^{m-k+i} - g^i dt$, $1 \le i \le k$, and we obtain the following local coordinate representation of canonical constraint 1-forms

$$\varphi^i = - \sum_{l=1}^{m-k} \frac{\partial g^i}{\partial \dot{q}^l} \, \bar{\omega}^l + \bar{\omega}^{m-k+i}, \quad 1 \le i \le k. \tag{10}$$

The ideal in the exterior algebra of forms on Q generated by canonical constraint 1-forms is called the *constraint ideal*, and denoted by I; its elements are called *constraint forms*. The pair (Q, C) is then called a *(nonholonomic) constraint structure* on the fibered manifold π [14], [15].

Remark 1. *From the point of view of physics, the rank of the canonical distribution C has the meaning of the number of (generalized, or "phase space") degrees of freedom of systems constrained to Q, and the canonical distribution itself represents possible (generalized) displacements. Its π_1-vertical and $\pi_{1,0}$-vertical subdistribution then has the meaning of virtual (generalized) displacements and virtual velocities, respectively.*

Now we will recall the concept of a nonholonomic Lagrangian system. Consider on J^1Y an unconstrained Lagrangian system $[\alpha] = [d\theta_\lambda]$. With help of the nonholonomic constraint structure (Q, C) one can construct a new mechanical system directly on the constraint submanifold Q of J^1Y. In keeping with [14], [15], by a related *(nonholonomic) constrained system* we shall mean an equivalence class of 2-forms on Q elements of which are of the form

$$\alpha_Q = \iota^* d\theta_\lambda + \bar{F} + \varphi_{(2)},$$

where $\bar{F}$ and $\varphi_{(2)}$ run over all 2-contact $\pi_{1,0}$-horizontal 2-forms and constraint 2-forms defined on Q, respectively. For the constrained system we use notation $[\alpha_Q]$. Equations of motion of the constrained system $[\alpha_Q]$, then have the following intrinsic form:

$$J^1\bar{\gamma}^* i_\xi \iota^* d\theta_\lambda = 0 \quad \text{for every vertical vector field } \xi \in C, \tag{11}$$

where $\bar{\gamma}$ is a Q-admissible section of π. These equations are sometimes called *reduced equations of motion* of the constrained system $[\alpha_Q]$, since they are restricted to the constraint submanifold Q.

Let us find a coordinate expression of a representative of the class $[\alpha_Q]$ and an explicit expression of reduced equations of motion of the constrained system $[\alpha_Q]$ arising from the Lagrangian system $[\alpha]$ and a nonholonomic constraint structure (Q, C). Let $\lambda = L(t, q^\sigma, \dot{q}^\sigma)\, dt$ be a (local) Lagrangian for an unconstrained Lagrangian system $[\alpha] = [d\theta_\lambda]$, where θ_λ is its Cartan form coordinate representation of which is given by (1), and consider the constraint submanifold Q locally given by equations (9) in normal form. We introduce Lagrange function $\bar{L}$ on the constraint submanifold Q as the restriction of the original unconstrained Lagrange function L on Q, i.e. $\bar{L} = L \circ \iota$, thus $\bar{L}(t, q^\sigma, \dot{q}^l) = L\left(t, q^\sigma, \dot{q}^l, g^i(t, q^\sigma, \dot{q}^l)\right)$. Computing the coordinate expression of $\iota^* d\theta_\lambda$ we get that a representative of the class $[\alpha_Q]$ takes the form

$$\alpha_Q = \sum_{l=1}^{m-k} A'_l \omega^l \wedge dt + \sum_{l,s=1}^{m-k} B'_{ls} \omega^l \wedge d\dot{q}^s + \bar{F} + \varphi_{(2)},$$

where the components A'_l are given by

$$A'_l = \frac{\partial \bar{L}}{\partial q^l} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} - \frac{\bar{d}_c}{dt} \frac{\partial \bar{L}}{\partial \dot{q}^l} + \\ + \left(\frac{\partial L}{\partial \dot{q}^{m-k+j}}\right)_\iota \left[\frac{\bar{d}_c}{dt}\left(\frac{\partial g^j}{\partial \dot{q}^l}\right) - \frac{\partial g^j}{\partial q^l} - \frac{\partial g^j}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l}\right], \quad (12)$$

where

$$\frac{\bar{d}_c}{dt} = \frac{\partial}{\partial t} + \dot{q}^s \frac{\partial}{\partial q^s} + g^i \frac{\partial}{\partial q^{m-k+i}}.$$

Components $B'_{l,s}$ are of the form

$$B'_{ls} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} + \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}}\right)_\iota \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s}. \quad (13)$$

Finally, reduced equations of motion of the constrained system $[\alpha_Q]$ (11) in fibered coordinates take the form

$$\frac{\partial \bar{L}}{\partial q^l} + \frac{\partial \bar{L}}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l} - \frac{d_c}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}^l}\right) + \\ + \left(\frac{\partial L}{\partial \dot{q}^{m-k+j}}\right)_\iota \left[\frac{d_c}{dt}\left(\frac{\partial g^j}{\partial \dot{q}^l}\right) - \frac{\partial g^j}{\partial q^l} - \frac{\partial g^j}{\partial q^{m-k+i}} \frac{\partial g^i}{\partial \dot{q}^l}\right] = 0\,,$$

where

$$\frac{d_c}{dt} = \frac{\bar{d}_c}{dt} + \ddot{q}^s \frac{\partial}{\partial \dot{q}^s}.$$

Notice that the above system of equations can be viewed as 2nd order equations

$$\left(A'_l + \sum_{s=1}^{m-k} B'_{ls} \ddot{q}^s\right) \circ J^2 \bar{\gamma} = 0, \quad (14)$$

for components $\gamma^1(t), \gamma^2(t), \dots, \gamma^{m-k}(t)$ of a Q-admissible section $\bar{\gamma}$ dependent on time t and parameters $q^{m-k+1}, q^{m-k+2}, \dots, q^m$, which have to be determined as functions $\gamma^{m-k+1}(t), \gamma^{m-k+2}(t), \dots, \gamma^m(t)$ from the equations (9) of the constraint

$$\frac{dq^{m-k+i}}{dt} = g^i\left(t, q^\sigma, \frac{dq^1}{dt}, \frac{dq^2}{dt}, \dots, \frac{dq^{m-k}}{dt}\right), \quad 1 \le i \le k.$$

A nonholonomic constraint system $[\alpha_Q]$ is called *regular* if the matrix $(B'_{l,s})$ is regular, i.e.

$$\det\left(\frac{\partial \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} - \left(\frac{\partial L}{\partial \dot{q}^{m-k+i}}\right)_\iota \frac{\partial^2 g^i}{\partial \dot{q}^l \partial \dot{q}^s}\right) \neq 0.$$

For more details on concepts and results in this section the reader is referred e.g. to the survey article [18].

5 Examples of nonholonomic mechanical systems

5.1 Decelerated motion of a free particle

Consider a "free particle" in $\mathbb{R}^3$ moving in such a way, that the square of its speed decreases proportionally to the reciprocal value of time passed from the beginning of the motion. (See [14], p. 5123, Example 1.)

We denote by (t) the coordinate on $X = \mathbb{R}$, by (t, q^1, q^2, q^3) fibered coordinates on $Y = \mathbb{R} \times \mathbb{R}^3$, and $(t, q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \dot{q}^3)$ the associated coordinates on $J^1Y = \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3$.

Lagrangian of a free particle has the standard form

$$\lambda = L\,dt = \frac{1}{2} m \left((\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2\right)\,dt,$$

where m is the mass of the particle. We consider a first order mechanical system $[\alpha]$

$$\alpha = d\theta_\lambda + F = -m\left(\omega^1 \wedge d\dot{q}^1 + \omega^2 \wedge d\dot{q}^2 + \omega^3 \wedge d\dot{q}^3\right) + F \tag{15}$$

on the fibered manifold $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, related with the Euler–Lagrange form

$$E = \sum_{\sigma=1}^{3} -m\ddot{q}^\sigma\, dq^\sigma \wedge dt.$$

The motion of the mechanical system $[\alpha]$ is for $t > 0$ subject to the following nonholonomic constraint Q

$$f(t, q^\sigma, \dot{q}^\sigma) \equiv \left[\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2 + \left(\dot{q}^3\right)^2\right] - 1/t = 0, \tag{16}$$

meaning that the particle's speed decreases proportionally to $1/\sqrt{t}$. This nonholonomic constraint is rheonomic and is affine of degree 2 in components of velocity. In a neighbourhood of the submanifold Q

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = 2t(\dot{q}^1, \dot{q}^2, \dot{q}^3) = 1,$$

i.e. condition (8) is satisfied.

Let $U \subset J^1Y$ be the set of all points, where $\dot{q}^3 > 0$, and consider on U canonical coordinates and the adapted coordinates $(t, q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \bar{f})$, where $\bar{f} = \dot{q}^3 - g$, $g = \sqrt{1/t - (\dot{q}^1)^2 - (\dot{q}^2)^2}$ is the equation of the constraint (16) in normal form. Notice that $g > 0$ on U.

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ (15) and the constraint Q (16) is the equivalence class of the 2-form

$$\alpha_Q = \sum_{l=1,2} A'_l\, \omega^l \wedge dt + \sum_{l,s=1,2} B'_{ls}\, \omega^l \wedge d\dot{q}^s + \bar{F} + \varphi_{(2)}$$

on Q, where

$$\begin{aligned} A'_l &= \left[-\frac{m\dot{q}^l}{2t(\dot{q}^3)^2} \left(\left(\dot{q}^1\right)^2 + \left(\dot{q}^2\right)^2 + \left(\dot{q}^3\right)^2 \right) \right]_\iota = -\frac{m\dot{q}^l}{2t^2 g^2}, && 1 \le l \le 2, \\ B'_{ls} &= \left[-m \left(\delta_{ls} + \frac{\dot{q}^l \dot{q}^s}{(\dot{q}^3)^2} \right) \right]_\iota = -m \left(\delta_{ls} + \frac{\dot{q}^l \dot{q}^s}{g^2} \right), && 1 \le l, s \le 2, \end{aligned}$$

and $\bar{F}$ is any 2-contact 2-form and $\varphi_{(2)}$ is any constraint 2-form defined on Q. The matrix $(-B'_{ls})$ is on $Q \cap U$ equivalent to the matrix

$$\begin{pmatrix} g^2 + (\dot{q}^1)^2 & \dot{q}^1 \dot{q}^2 \\ \dot{q}^1 \dot{q}^2 & g^2 + (\dot{q}^2)^2 \end{pmatrix},$$

hence

$$\begin{pmatrix} g^2 + (\dot{q}^1)^2 & \dot{q}^1 \dot{q}^2 \\ 0 & \frac{g^2}{t} \end{pmatrix},$$

which is obviously regular at each point of $Q \cap U$. This means that the constrained system $[\alpha_Q]$ is regular on $Q \cap U$.

Reduced equations of motion of the constrained system are as follows

$$\begin{aligned} \left[\frac{m\dot{q}^1}{2t^2 g^2} + m \left(1 + \frac{(\dot{q}^1)^2}{g^2} \right) \ddot{q}^1 + m \frac{\dot{q}^1 \dot{q}^2}{g^2}\, \ddot{q}^2 \right] \circ J^2 \bar{\gamma} = 0\,, \\ \left[\frac{m\dot{q}^2}{2t^2 g^2} + m \left(1 + \frac{(\dot{q}^2)^2}{g^2} \right) \ddot{q}^2 + m \frac{\dot{q}^1 \dot{q}^2}{g^2}\, \ddot{q}^1 \right] \circ J^2 \bar{\gamma} = 0\,, \end{aligned}$$

where $\bar{\gamma} = (t, q^1(t), q^2(t), q^3(t))$ is a Q-admissible section, i.e. a section satisfying the constraint equation $f \circ J^1\gamma = 0$. After arrangements we obtain equations of motion of the constrained system in the following simple form:

$$\begin{aligned} \ddot{q}^1(t) &= -\frac{1}{2t} \dot{q}^1(t)\,, \\ \ddot{q}^2(t) &= -\frac{1}{2t} \dot{q}^2(t)\,, \\ \dot{q}^3(t) &= \sqrt{\frac{1}{t} - \left(\dot{q}^1\right)^2 - \left(\dot{q}^2\right)^2}\,. \end{aligned}$$

Solution of these equations is

$$q^1(t) = C_1^1\sqrt{t} + C_2^1\,,$$
$$q^2(t) = C_1^2\sqrt{t} + C_2^2\,,$$
$$q^3(t) = C_1^3\sqrt{t} + C_2^3\,,$$

where C_j^i are constants connected by the relation $C_1^3 = \sqrt{4 - (C_1^1)^2 + (C_1^2)^2}$. Analogous results are obtained if one considers the other adapted charts belonging to an atlas covering Q.

5.2 A dog pursuing a man

Consider a man and a dog moving in the plane. The man starts from the origin O of the coordinate system Oxy and moves along the y-axis with a constant velocity c. His dog starts at the same moment from the point $[x_0, y_0]$, $x_0 \geq 0$, $y_0 \neq 0$ and runs in such a way, that its velocity at each moment is given by the line connecting its instantaneous position and the instantaneous position of the man. We shall find the trajectory of the dog. (See [2], pp. 236–239.)

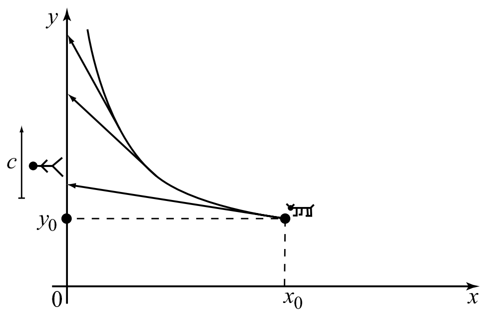

Figure 1

We denote by (t) the coordinate on $X = \mathbb{R}$, by (t, x, y) the canonical coordinates on $Y = \mathbb{R}\times\mathbb{R}^2$ and by $(t, x, y, \dot{x}, \dot{y})$ the associated coordinates on $J^1Y = \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$.

The Lagrangian of this problem is

$$\lambda = L\,dt = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\,dt$$

and defines a first order mechanical system $[\alpha]$ on the fibered manifold $\mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ represented by the Lepage 2-form

$$\alpha = d\theta_\lambda + F = -\,m\,\omega^1 \wedge d\dot{x} - m\,\omega^2 \wedge d\dot{y} + F, \tag{17}$$

where m denotes the mass of the dog, $\omega^1 = dx - \dot{x}\,dt$, $\omega^2 = dy - \dot{y}\,dt$ are corresponding contact 1-forms and F is any 2-contact 2-form. This mechanical system is related to the dynamical form

$$E = -m\ddot{x}\,dx \wedge dt - m\ddot{y}\,dy \wedge dt.$$

The constraint is given by the requirement that at each moment the direction of the motion of the dog is known. For the angular coefficient of the dog's trajectory it holds

$$\frac{dy}{dx} = G(t, x, y). \tag{18}$$

This equation can be written in the equivalent form

$$G(t, x, y)\,\dot{x} - \dot{y} = 0 \tag{19}$$

which is a rheonomic nonholonomic constraint affine in components of velocity. On the other hand, the instantaneous direction of the motion of the dog at a time t and at a point $[x, y]$ is given by the line connecting this point with the point $[0, c\,t]$ where the man is at this moment. Hence the angular coefficient of the trajectory at a time t and at a point $[x, y]$ is given by

$$G(t, x, y) = \frac{y - c\,t}{x}, \quad x \neq 0. \tag{20}$$

Consequently, the nonholonomic constraint (19) has the form

$$\dot{y} = \frac{y - c\,t}{x}\,\dot{x}. \tag{21}$$

This equation defines a constraint submanifold $Q \subset J^1Y$, since the rank condition (8)

$$\operatorname{rank}\left(\frac{y - c\,t}{x}, \, -1\right) = 1$$

is satisfied. The canonical constraint 1-form (10) reads

$$\varphi = -(y - c\,t)\,dx + x\,dy.$$

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ (17) and the constraint Q given by (21) is the equivalence class of the 2-form

$$\alpha_Q = A'_1\,\omega^1 \wedge dt + B'_{11}\,\omega^1 \wedge d\dot{x} + \bar{F} + \varphi_{(2)},$$

where

$$A'_1 = \frac{mc\dot{x}\,(y - c\,t)}{x^2}, \quad B'_{11} = -\,m\left(1 + \frac{(y - c\,t)^2}{x^2}\right),$$

and $\bar{F}$ is any 2-contact 2-form and $\varphi_{(2)}$ is any constraint 2-form defined on this constraint submanifold Q. Since

$$\det B'_{11} = -\,m\left(\frac{x^2 + (y - c\,t)^2}{x^2}\right) \neq 0,$$

the constrained system $[\alpha_Q]$ is regular.

The reduced equation of motion of the constrained system is

$$\left[\frac{mc\,(y - c\,t)}{x^2}\,\dot{x} - m\left(\frac{x^2 + (y - c\,t)^2}{x^2}\right)\ddot{x}\right] \circ J^2\bar{\gamma} = 0, \tag{22}$$

where $\bar{\gamma} = (t, x(t), y(t))$ is a Q-admissible section satisfying the constraint equation (21).

In [2] the dynamics is obtained by solving Chetaev equations of motion (equations with Lagrange multipliers), which take a very simple form

$$\ddot{x} = \mu^* G(x, y, t)\,,$$
$$\ddot{y} = -\mu^*\,.$$

The symbol $\mu^* = \mu/m$ denotes a (reduced) Lagrange multiplier and G is the function given by (20). Now, multiplying the first equation by $\dot{x}$ and the second one by $\dot{y}$ and adding these equations we get

$$\frac{d}{dt}\left[\frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\right] = \mu^*[G(x, y, t)\,\dot{x} - \dot{y}].$$

Since the constraint equation (19) holds we obtain a first integral

$$\dot{x}^2 + \dot{y}^2 = v^2 = const. \tag{23}$$

This means that the dog moves with a constant speed. This fact together with equation (18) enables us to determine the trajectory of the dog in an explicit form, i.e. $y = y(x)$. To this end we eliminate time parameter from the equations. First we notice that one can write

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \equiv \dot{x}\,y'. \tag{24}$$

Substituting (20) into (18) we obtain

$$\frac{dy}{dx} \equiv y' = \frac{y - c\,t}{x} \quad \text{resp.} \quad x\,y' = y - c\,t,$$

and after differentiating this equation with respect to x,

$$x\,y'' = -c\,\frac{dt}{dx}.$$

Hence, under appropriate conditions,

$$\dot{x} = -\frac{c}{x\,y''}. \tag{25}$$

Since the motion takes place in the first quadrant, relations $x > 0$, $\dot{x} < 0$ hold, and subsequently $y'' > 0$. Substituting identity (24) to the first integral (23) we get

$$\dot{x}^2\left(1 + (y')^2\right) = v^2,$$

and after extracting the square root we can write

$$-\dot{x} = \frac{v}{\sqrt{1 + (y')^2}}.$$

Finally we compare the last equation with equation (25) and after separation of variables we gain the desired differential equation for the curve of pursuit

$$\frac{y''}{\sqrt{1+(y')^2}} = \frac{c}{v}\frac{1}{x}. \tag{26}$$

The fact that both sides of this equation can be written by means of total derivative with respect to x in the following way

$$\frac{d}{dx}\left[\ln\left(y' + \sqrt{1+(y')^2}\right)\right] = \frac{d}{dx}\left(\frac{c}{v}\ln x\right),$$

enables one a reduction of equation (26) to the following first order implicit differential equation

$$\ln\left(y' + \sqrt{1+(y')^2}\right) = \frac{c}{v}\ln x + \ln A, \tag{27}$$

where $\ln A$ is a constant which can be determined with help of initial conditions. Equation (27) can be written in a simpler form

$$y' + \sqrt{1+(y')^2} = A\,x^{\alpha},$$

where $\alpha = \frac{c}{v}$. Expressing y'

$$y' = \frac{1}{2}\left(A\,x^{\alpha} - \frac{1}{A\,x^{\alpha}}\right),$$

and after integration we obtain for $\alpha \neq 1$ a general solution described by the function

$$y = \frac{1}{2}\left[\frac{A}{1+\alpha}\,x^{1+\alpha} - \frac{1}{A(1-\alpha)}\,x^{1-\alpha}\right] + C,$$

where C is a constant to be determined with help of initial conditions. The final explicit form of the desired curve of pursuit is

$$y = y_0 + \frac{1}{2}\left[\frac{A}{1+\alpha}\,(x^{1+\alpha} - x_0^{1+\alpha}) - \frac{1}{A(1-\alpha)}\,(x^{1-\alpha} - x_0^{1-\alpha})\right],$$

where

$$A = \frac{y_0 + \sqrt{x_0^2 + y_0^2}}{x_0^{1+\alpha}},$$

and x_0, y_0 are coordinates of the initial position of the dog.

5.3 Pursuit of a general motion in a plane

Consider an object moving in a plane along an a-priori given curve described by parametric equations $x = \xi(t)$, $y = \eta(t)$, and consider a dog which starts from a point $[x_0, y_0]$, $x_0 \geq 0$, $y_0 \neq 0$, and pursues this object in the same way as above, i.e. that its velocity at each moment is given by the line connecting its instantaneous position and the instantaneous position of the object. We shall find equations of motion of the dog.

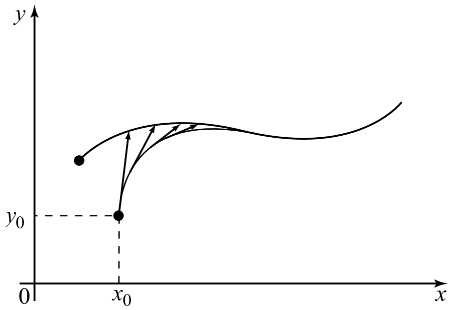

Figure 2

The configuration space Y, the Lagrangian λ and the mechanical system $[\alpha]$ are the same as above, however, restriction of the motion of the dog now is given by the corresponding generalization of the constraint (21) to

$$\dot{y} = G(t,x,y)\,\dot{x} = \frac{y-\eta(t)}{x-\xi(t)}\,\dot{x}. \tag{28}$$

This is again a rheonomic nonholonomic constraint affine in components of velocity, which defines a constraint submanifold Q in the phase space J^1Y. The canonical constraint 1-form (10) now reads

$$\varphi = -(y-\eta(t))\,dx + (x-\xi(t))\,dy.$$

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ (17) and the constraint Q given by (28) is again an equivalence class as follows,

$$\alpha_Q = A'_1\,\omega^1 \wedge dt + B'_{11}\,\omega^1 \wedge d\dot{x} + \bar{F} + \varphi_{(2)},$$

where

$$A'_1 = m\dot{x}\,\frac{\dot{\eta}\,(y-\eta)(x-\xi) - \dot{\xi}\,(y-\eta)^2}{(x-\xi)^3}, \quad B'_{11} = -\,m\left(1 + \frac{(y-\eta)^2}{(x-\xi)^2}\right),$$

and $\bar{F}$ is any 2-contact 2-form and $\varphi_{(2)}$ is any constraint 2-form on Q. Since

$$\det\,B'_{11} = -\,m\,\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2} \neq 0,$$

the constrained system $[\alpha_Q]$ is again regular.

The reduced equation of motion of the constrained system is

$$m\left[\dot{x}\,\frac{(y-\eta)}{(x-\xi)^2}\,\dot{\eta} - \dot{x}\,\frac{(y-\eta)^2}{(x-\xi)^3}\,\dot{\xi} - \ddot{x}\left(1 + \frac{(y-\eta)^2}{(x-\xi)^2}\right)\right] \circ J^2\bar{\gamma} = 0,$$

where $\bar{\gamma} = (t, x(t), y(t))$ is a Q-admissible section satisfying constraint equation (28). In particular, if we put $\xi(t) = 0$, $\eta(t) = c\,t$, i.e. we consider the motion along the y-axis with a constant speed c, we obtain motion equation (22).

In the same way as in the previous example we can write down Chetaev equations of motion, which have the same form as above,

$$\ddot{x} = \mu^* G(x, y, t)\,,$$
$$\ddot{y} = -\mu^*,$$

but now the function G is given by formula

$$G(x, y, t) = \frac{y - \eta(t)}{x - \xi(t)}.$$

Repeating the same procedure we obtain a first integral

$$\dot{x}^2 + \dot{y}^2 = v^2 = const.$$

However, now we cannot eliminate the time parameter from the equations because of the fact that the pursuing object moves along a curve determined by parametric equations $x = \xi(t)$, $y = \eta(t)$, which need not represent a straight motion with a constant velocity as in the previous example.

5.4 Motion of a particle in a homogeneous gravitational field with constant velocity

Consider a particle of mass m moving in a homogeneous gravitational field (the gravitational acceleration is denoted by G) from a point $(q^1(0), q^2(0), q^3(0))$, $q^3(0) > 0$, with the initial velocity given by a vector $(p^1(0), p^2(0), p^3(0))$, where all the components are non-zero and positive. The motion is restricted by the condition that the speed of the particle remains constant. (See [9], pp. 991, Example 4.2.)

This is a problem originally formulated by Leibnitz in 1689 as follows: find a curve along which a particle moves in a homogeneous gravitational field with a constant speed. A solution of the problem was found by Jacob Bernoulli in 1694 as a curve called the *paracentric isochrone*. However the problem was solved only from the kinematic point of view in the framework of differential geometry of curves. For a complete description of dynamics of the problem it is necessary to understand the requirement of the constant speed as a nonholonomic, so called *isotachystonic* constraint, which is nonlinear.

Our aim is to study the dynamics of the Leibnitz particle.

The configuration space is again $Y = \mathbb{R} \times \mathbb{R}^3$, (t, q^σ), $1 \le \sigma \le 3$, are fibered coordinates on Y. The Lagrangian has the form

$$\lambda = L\,dt = \left[\frac{1}{2} m \left((\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2\right) - mGq^3\right] dt.$$

The mechanical system $[\alpha]$ is represented by a Lepage 2-form

$$\alpha = -mG\,\omega^3 \wedge dt - m\left(\omega^1 \wedge d\dot{q}^1 + \omega^2 \wedge d\dot{q}^2 + \omega^3 \wedge d\dot{q}^3\right) + F, \tag{29}$$

where F is a 2-contact 2-form. The corresponding dynamical form is then

$$E = -mG\,dq^3 \wedge dt - \sum_{\sigma=1}^{3} m\ddot{q}^\sigma\, dq^\sigma \wedge dt\,.$$

The constraint on the motion is given by equation

$$f \equiv (\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2 - C = 0, \tag{30}$$

where $C = \big(p^1(0)\big)^2 + \big(p^2(0)\big)^2 + \big(p^3(0)\big)^2$ is the square of the initial speed of the particle. Equation (30) defines a constraint submanifold Q in J^1Y. It is a skleronomic nonholonomic constraint, affine of degree 2 in components of velocity. Let $U \subset J^1Y$ be the set of all points where $\dot{q}^3 > 0$ and consider on U the adapted coordinates $(t, q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \bar{f})$, where $\bar{f} = \dot{q}^3 - g$, $g = \sqrt{C - (\dot{q}^1)^2 - (\dot{q}^2)^2}$ is equation of the constraint (30) in normal form.

The constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ (29) and the constraint Q (30) is the equivalence class of 2-forms

$$\alpha_Q = \sum_{l=1,2} A'_l\, \omega^l \wedge dt + \sum_{l,s=1,2} B'_{ls}\, \omega^l \wedge d\dot{q}^s + \bar{F} + \varphi_{(2)} \tag{31}$$

on Q, where

$$A'_l = \left[-m\,G\,\frac{\dot{q}^l}{\dot{q}^3}\right]_\iota = -m\,G\,\frac{\dot{q}^l}{g}, \qquad 1 \le l \le 2,$$

$$B'_{ls} = \left[-\,m\left(\delta_{ls} + \frac{\dot{q}^l\dot{q}^s}{(\dot{q}^3)^2}\right)\right]_\iota = -\,m\left(\delta_{ls} + \frac{\dot{q}^l\dot{q}^s}{g^2}\right), \qquad 1 \le l, s \le 2,$$

and $\bar{F}$ is a 2-contact 2-form and $\varphi_{(2)}$ is a constraint 2-form defined on the constraint submanifold Q. The constrained system $[\alpha_Q]$ is regular since the matrix $(-B'_{ls})$ is the same in the second example above. The motion of this constrained system is described by two reduced equations

$$\left[m\,G\,\frac{\dot{q}^1}{g} + m\left(1 + \frac{(\dot{q}^1)^2}{g^2}\right)\ddot{q}^1 + m\frac{\dot{q}^1\dot{q}^2}{g^2}\,\ddot{q}^2\right] \circ J^2\bar{\gamma} = 0\,,$$

$$\left[m\,G\,\frac{\dot{q}^2}{g} + m\left(1 + \frac{(\dot{q}^2)^2}{g^2}\right)\ddot{q}^2 + m\frac{\dot{q}^1\dot{q}^2}{g^2}\,\ddot{q}^1\right] \circ J^2\bar{\gamma} = 0\,,$$

where $\bar{\gamma} = (t, x(t), y(t))$ is a Q-admissible section satisfying the constraint equation

$$\dot{q}^3 = \sqrt{C - (\dot{q}^1)^2 - (\dot{q}^2)^2}.$$

After simple computations equations of motion of the constrained system take the form

$$\ddot{q}^1(t) = \frac{G}{C}\,\dot{q}^1\,\sqrt{C - (\dot{q}^1)^2 - (\dot{q}^2)^2}\,,$$

$$\ddot{q}^2(t) = \frac{G}{C}\,\dot{q}^2\,\sqrt{C - (\dot{q}^1)^2 - (\dot{q}^2)^2}\,,$$

$$\dot{q}^3(t) = \sqrt{C - (\dot{q}^1)^2 - (\dot{q}^2)^2}\,.$$

The same equations were obtained in [9] by a different method.

The above system of differential equations can be reduced to the first order system

$$\begin{aligned} \dot{p}^1(t) &= D\, p^1 \sqrt{C-(p^1)^2-(p^2)^2}\,, \\ \dot{p}^2(t) &= D\, p^2 \sqrt{C-(p^1)^2-(p^2)^2}\,, \\ \dot{q}^3(t) &= \sqrt{C-(p^1)^2-(p^2)^2}\,, \end{aligned}$$

where we denoted $D = G/C$. Since $\dot{p}^1 p^2 - p^1 \dot{p}^2 = 0$, and if moreover $p^2 \neq 0$, then $p^1/p^2 = \kappa$ is a first integral of these equations, which has the positive value $\kappa = p^1(0)/p^2(0)$ determined by the given components of the initial velocity. If we suppose that in a certain interval of time the components p^1, p^2 of the instantaneous velocity are not zero, we can separate equations for p^1 and p^2 and integrate

$$\begin{aligned} \int \frac{dp^1}{p^1 \sqrt{C - \left(1+\frac{1}{\kappa^2}\right)(p^1)^2}} &= \int D\, dt \\ \int \frac{dp^2}{p^2 \sqrt{C - (1+\kappa^2)(p^2)^2}} &= \int D\, dt\,. \end{aligned}$$

After integration we can write

$$\begin{aligned} \sqrt{C}\, \ln \left[\frac{\sqrt{\frac{C\kappa^2}{1+\kappa^2}} - \sqrt{\frac{C\kappa^2}{1+\kappa^2} - (p^1)^2}}{p^1} \right] &= \frac{G}{C}\, t + b_1\,, \\ \sqrt{C}\, \ln \left[\frac{\sqrt{\frac{C}{1+\kappa^2}} - \sqrt{\frac{C}{1+\kappa^2} - (p^2)^2}}{p^2} \right] &= \frac{G}{C}\, t + b_2\,, \end{aligned}$$

where

$$\frac{\kappa^2}{1+\kappa^2} = \frac{\left(p^1(0)\right)^2}{\left(p^1(0)\right)^2 + \left(p^2(0)\right)^2}, \qquad \frac{1}{1+\kappa^2} = \frac{\left(p^2(0)\right)^2}{\left(p^1(0)\right)^2 + \left(p^2(0)\right)^2},$$

and b_1, b_2 are some integration constants. Expressing variables p^1, p^2 we obtain

$$\begin{aligned} p^1 &= \frac{dq^1}{dt} = \sqrt{\frac{C\kappa^2}{1+\kappa^2}}\; \frac{2B_1 e^{\frac{G}{\sqrt{C}}t}}{B_1^2 e^{\frac{2G}{\sqrt{C}}t} + 1}\,, \\ p^2 &= \frac{dq^2}{dt} = \sqrt{\frac{C}{1+\kappa^2}}\; \frac{2B_2 e^{\frac{G}{\sqrt{C}}t}}{B_2^2 e^{\frac{2G}{\sqrt{C}}t} + 1}\,, \end{aligned} \tag{32}$$

where B_1, B_2 are constants determined by means of b_1, b_2 by the following relations $B_1 = e^{\sqrt{C}\, b_1}$, $B_2 = e^{\sqrt{C}\, b_2}$. If we take into account given components of the initial velocity $p^1(0), p^2(0), p^3(0)$ which are positive as we assumed, and with respect to the value of the first integral $\kappa = p^1(0)/p^2(0)$ we obtain that

$$B_1 = B_2 = B = \frac{\sqrt{C} - p^3(0)}{\sqrt{\left(p^1(0)\right)^2 + \left(p^2(0)\right)^2}}.$$

We find the primitive function

$$\int \frac{e^{\alpha\, t}}{B^2 e^{2\alpha\, t} + 1} = \frac{1}{\alpha B}\ \arctan\left(B\, e^{\alpha\, t}\right),$$

where $\alpha = G/\sqrt{C}$. Hence the desired functions $q^1(t), q^2(t)$ are

$$q^1(t) = \frac{2C}{G}\sqrt{\frac{\kappa^2}{1+\kappa^2}}\ \arctan\left(B\, e^{\frac{G}{\sqrt{C}}\, t}\right) + A_1\,,$$
$$q^2(t) = \frac{2C}{G}\sqrt{\frac{1}{1+\kappa^2}}\ \arctan\left(B\, e^{\frac{G}{\sqrt{C}}\, t}\right) + A_2\,,$$

and A_1, A_2 are constants, which are determined by the initial position of the particle. After elimination of the parameter t from the last equations we can see, that the particle moves in the plane $q^1 - \kappa q^2 - A_1 + \kappa\, A_2 = 0$, which is parallel to the q^3-axis.

Now we can substitute the functions $p^1(t)$, $p^2(t)$ given by (32) into the constraint condition $\dot q^3 = \sqrt{C - (p^1)^2 - (p^2)^2}$:

$$\dot q^3 = \sqrt{C}\ \frac{|B^2\, e^{\frac{2G}{\sqrt{C}}\, t} - 1|}{B^2\, e^{\frac{2G}{\sqrt{C}}\, t} + 1}. \tag{33}$$

Indeed, for $t = 0$ we obtain $\dot q^3(0) = p^3(0)$.

We notice the fact that

$$B^2 = 1 - \frac{2\, p^3(0)\left(\sqrt{C} - p^3(0)\right)}{\left(p^1(0)\right)^2 + \left(p^2(0)\right)^2} < 1,$$

since all the components of the initial velocity are non-zero.

As a consequence of the above property and due to the physical reason that potential energy of a homogeneous gravitational field increases proportionally to q^3, it turns out that in some time $T = -\frac{\sqrt{C}}{G}\ \ln B$ the motion in the vertical direction stops, i.e. $\dot q^3(T) = 0$, and then it proceeds with $\dot q^3(t) < 0$. Hence for the time $t > T$ one has to consider the constraint condition in the form $\dot q^3 = -\sqrt{C - (p^1)^2 - (p^2)^2}$.

Integrating equation (33) we get that in the time interval $(0,\, T)$ the solution $q^3(t)$ is described by the function

$$q^3(t) = \frac{C}{2G}\ \ln\left[\frac{B^2\, e^{\frac{2G}{\sqrt{C}}\, t}}{\left(B^2\, e^{\frac{2G}{\sqrt{C}}\, t} + 1\right)^2}\right] + A_3 = -\frac{C}{G}\ \ln\left[2\cosh\left(\frac{G\, t + b\, C}{\sqrt{C}}\right)\right] + A_3,$$

where the relationship between constants B and b is given by $b = 1/\sqrt{C}\ \ln B$, and A_3 is a constant, which can be determined by means of $q^3(0)$.

It is worth notice properties of the "nonholonomic fall" in a homogeneous gravitational field: One could expect that the motion will turn to the vertical direction and the particle will fall down with increasing acceleration. However, the constraint condition keeps the speed constant, therefore the components $\dot q^1(t)$, $\dot q^2(t)$ of the instantaneous velocity have to decrease proportionally, and after some time the motion will proceed in the vertical direction with a constant velocity determined by the vector $(0, 0, \sqrt{C})$.

5.5 Motion of a particle in a homogeneous gravitational field subject to a nonlinear constraint

Consider a particle of mass m in a homogeneous gravitational field (the same as in the previous example). The motion of the particle is now subjected to a nonholonomic condition $b^2\left((\dot q^1)^2+(\dot q^2)^2\right)-(\dot q^3)^2=0$, where b is a constant. (See [9], pp. 992, Example 4.3.)

This mechanical system is the same as above, i.e. it is represented by the Lepage form (29). However the constraint condition

$$f \equiv b^2\left((\dot q^1)^2+(\dot q^2)^2\right)-(\dot q^3)^2=0, \tag{34}$$

or equivalently in normal form

$$\dot q^3 = g = b\sqrt{(\dot q^1)^2+(\dot q^2)^2} \tag{35}$$

is different. The constraint (34) is again a skleronomic nonholonomic constraint, which is affine of degree 2 in components of velocity.

The corresponding constrained mechanical system is given by the equivalence class $[\alpha_Q]$ of 2-forms (31), where

$$\begin{aligned}
A'_l &= \left[-m\,G\,\frac{b^2\dot q^l}{\dot q^3}\right]_\iota = -m\,G\,\frac{b\,\dot q^l}{\sqrt{(\dot q^1)^2+(\dot q^2)^2}} && 1\le l\le 2,\\
B'_{ls} &= \left[-m\left(\delta_{ls}+b^4\frac{\dot q^l\dot q^s}{(\dot q^3)^2}\right)\right]_\iota = -m\left(\delta_{ls}+b^2\frac{\dot q^l\dot q^s}{(\dot q^1)^2+(\dot q^2)^2}\right) && 1\le l,s\le 2.
\end{aligned}$$

Reduced equations of motion become the following system of second order ODE's

$$\begin{aligned}
&\left[\frac{Gb\,\dot q^1}{\sqrt{(\dot q^1)^2+(\dot q^2)^2}}+\left(1+b^2\frac{(\dot q^1)^2}{(\dot q^1)^2+(\dot q^2)^2}\right)\ddot q^1+b^2\frac{\dot q^1\dot q^2}{(\dot q^1)^2+(\dot q^2)^2}\,\ddot q^2\right]\circ J^2\bar\gamma=0\,,\\
&\left[\frac{Gb\,\dot q^2}{\sqrt{(\dot q^1)^2+(\dot q^2)^2}}+\left(1+b^2\frac{(\dot q^2)^2}{(\dot q^1)^2+(\dot q^2)^2}\right)\ddot q^2+b^2\frac{\dot q^1\dot q^2}{(\dot q^1)^2+(\dot q^2)^2}\,\ddot q^1\right]\circ J^2\bar\gamma=0\,,
\end{aligned}$$

where $\bar\gamma=(t,x(t),y(t))$ is a Q-admissible section satisfying constraint equation (35). Expressing the second derivatives we obtain

$$\begin{aligned}
\ddot q^1(t) &= -\,b\,G\,\frac{\dot q^1}{(1+b^2)\sqrt{(\dot q^1)^2+(\dot q^2)^2}}\,,\\
\ddot q^2(t) &= -\,b\,G\,\frac{\dot q^2}{(1+b^2)\sqrt{(\dot q^1)^2+(\dot q^2)^2}}\,.
\end{aligned} \tag{36}$$

The same equations are derived in [9] by a different method.

We shall solve the reduced equations. First we differentiate constraint equation (35)

$$\ddot q^3 = \frac{b^2}{\dot q^3}\,(\dot q^1\,\ddot q^1+\dot q^2\,\ddot q^2).$$

Substituting reduced equations (36) we obtain the equality

$$\ddot{q}^3 = -\frac{G\,b^2}{1+b^2},$$

which can be simply integrated

$$\dot{q}^3 \equiv b\sqrt{(\dot{q}^1)^2 + (\dot{q}^2)^2} = -\frac{G\,b^2}{1+b^2}\,t + K_1^3.$$

Finally we substitute the last equality back to (36), and we obtain simple differential equations, which can be reduced to first order equations with separable variables. A complete solution of the problem is obtained in the form

$$\begin{aligned} q^1(t) &= -\frac{1}{2}\frac{G\,b^2}{1+b^2}\,K_1^1\,t^2 + K_1^1 K_1^3\,t + K_2^1\,, \\ q^2(t) &= -\frac{1}{2}\frac{G\,b^2}{1+b^2}\,K_1^2\,t^2 + K_1^2 K_1^3\,t + K_2^2\,, \\ q^3(t) &= -\frac{1}{2}\frac{G\,b^2}{1+b^2}\,t^2 + K_1^3\,t + K_2^3\,, \end{aligned}$$

where K_j^i are constants, and the identity $(K_1^1)^2 + (K_1^2)^2 = 1/b^2$ holds true.

5.6 A rolling disc on a horizontal plane

Consider a disc of radius R rolling without sliding on a horizontal plane. Let $Oxyz$ be a fixed orthogonal system of coordinates with the x and y-axis in the horizontal plane and the z-axis directed vertically upwards. Then the position of the disc on the plane may be given by five generalized coordinates $x, y, \psi, \varphi, \vartheta$, where x and y are the coordinates of the point P of contact of the disc and the horizontal plane, ψ is the angle of proper rotation of the disc, φ is the angle between the tangent to the disc at the point P and the x-axis, and ϑ is the angle between the rotating axis of the disc and the parallel line to the z-axis which is going through the point P (i.e. $\pi/2 - \vartheta$ is the angle of inclination between the plane of the disc and the horizontal plane). (See [22], pp. 55.)

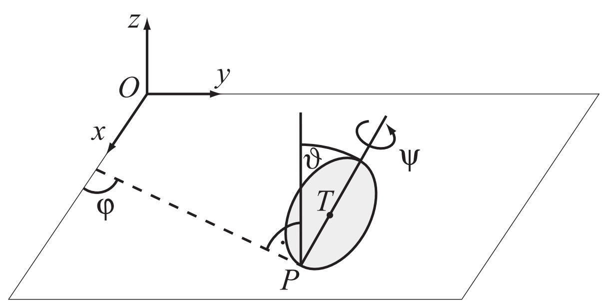

Figure 3

So the base space $X = \mathbb{R}$, the configuration space is $Y = \mathbb{R} \times \mathbb{R}^2 \times S^1 \times S^1 \times S^1$ and phase space is $J^1Y = \mathbb{R} \times \mathbb{R}^2 \times S^1 \times S^1 \times S^1 \times \mathbb{R}^2 \times S^1 \times S^1 \times S^1$. Hence fibered coordinates on Y are $(t, x, y, \psi, \varphi, \vartheta)$ and the associated coordinates on J^1Y are $(t, x, y, \psi, \varphi, \vartheta, \dot{x}, \dot{y}, \dot{\psi}, \dot{\varphi}, \dot{\vartheta})$.

The Lagrange function of this mechanical system is given by relation $L = T - V$. The kinetic energy T is given by the sum of the energy of translation and rotation of the disc:

$$\begin{aligned} T = {} & \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + R^2 \dot{\vartheta}^2 + R^2 \dot{\varphi}^2 \sin^2 \vartheta \right) - \\ & - mR \left(\dot{\vartheta} \cos \vartheta \, (\dot{x} \sin \varphi - \dot{y} \cos \varphi) + \dot{\varphi} \sin \vartheta \, (\dot{x} \cos \varphi + \dot{y} \sin \varphi) \right) + \\ & + \frac{1}{2} I_1 \left(\dot{\vartheta}^2 + \dot{\varphi}^2 \cos^2 \vartheta \right) + \frac{1}{2} I_2 \left(\dot{\psi} + \dot{\varphi} \sin \vartheta \right)^2 , \end{aligned} \tag{37}$$

where m is the mass, and I_1, I_2 are the principal moments of inertia of the disc. The potential energy of the disc is $V = mgR \cos \vartheta$. Formula (37) for kinetic energy of this problem is presented in [22] and is derived in detail in [27].

If we compute motion equation (5) of this Lagrangian system according to (2) and (3), where $1 \leq \sigma, \rho \leq 5$ and coordinates $(q^1, q^2, q^3, q^4, q^5)$ are substituted by corresponding coordinates $(x, y, \psi, \varphi, \vartheta)$, we obtain the following five Euler--Lagrange equations:

$$\begin{aligned} -m\ddot{x} + mR \left((\cos \varphi \sin \vartheta) \ddot{\varphi} + (\sin \varphi \cos \vartheta) \ddot{\vartheta} \right) - \\ - mR \left((\sin \varphi \sin \vartheta)(\dot{\varphi}^2 + \dot{\vartheta}^2) - (2 \cos \varphi \cos \vartheta) \dot{\varphi} \dot{\vartheta} \right) = 0 \, , \\ -m\ddot{y} + mR \left((\sin \varphi \sin \vartheta) \ddot{\varphi} - (\cos \varphi \cos \vartheta) \ddot{\vartheta} \right) + \\ + mR \left((\cos \varphi \sin \vartheta)(\dot{\varphi}^2 + \dot{\vartheta}^2) + (2 \sin \varphi \cos \vartheta) \dot{\varphi} \dot{\vartheta} \right) = 0 \, , \\ I_2 (\ddot{\psi} + \sin \vartheta \ddot{\varphi}) + (I_2 \cos \vartheta) \dot{\varphi} \dot{\vartheta} = 0 \, , \\ mR \left((\cos \varphi \sin \vartheta) \ddot{x} + (\sin \varphi \sin \vartheta) \ddot{y} \right) - (I_2 \sin \vartheta) \ddot{\psi} - \\ - \left((mR^2 + I_2) \sin^2 \vartheta + I_1 \cos^2 \vartheta \right) \ddot{\varphi} - \\ - (I_2 \cos \vartheta) \dot{\psi} \dot{\vartheta} - 2(mR^2 - I_1 + I_2)(\sin \vartheta \cos \vartheta) \dot{\varphi} \dot{\vartheta} = 0 \, , \\ mR \left((\sin \varphi \cos \vartheta) \ddot{x} - (\cos \varphi \cos \vartheta) \ddot{y} \right) - (mR^2 + I_1) \ddot{\vartheta} + \\ + (mR^2 - I_1 + I_2)(\sin \vartheta \cos \vartheta) \dot{\varphi}^2 + (I_2 \cos \vartheta) \dot{\psi} \dot{\varphi} + mgR \sin \vartheta = 0 \, . \end{aligned}$$

The condition that the disc rolls without sliding on the horizontal plane means, that the instantaneous velocity of the point of contact of the disc is equal to zero at all times. This gives rise to the following nonholonomic constraints

$$f^1 \equiv \dot{x} - R \cos \varphi \dot{\psi} = 0, \qquad f^2 \equiv \dot{y} - R \sin \varphi \dot{\psi} = 0, \tag{38}$$

or in normal form

$$\dot{x} = g^1 \equiv R \cos \varphi \dot{\psi}, \qquad \dot{y} = g^2 \equiv R \sin \varphi \dot{\psi}.$$

One can see that constraints above are linear, or more precisely affine in components of velocities. Equations (38) define a constraint submanifold $Q \subset J^1Y$, since the condition (8) is satisfied, i.e.

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = \operatorname{rank}\begin{pmatrix} 1 & 0 & -R\cos\varphi & 0 & 0 \\ 0 & 1 & -R\sin\varphi & 0 & 0 \end{pmatrix} = 2.$$

Thus $\dim Q = \dim J^1 Y - 2 = 9$. Constraint 1-forms (10) are in this case the following two forms

$$\varphi^1 = dx - R\cos\varphi d\psi, \qquad \varphi^2 = dy - R\sin\varphi d\psi.$$

Now one can construct the constrained system $[\alpha_Q]$ related to the mechanical system $[\alpha]$ and the constraint Q as the equivalence class of the 2-form

$$\alpha_Q = A'_1\,\omega^1 \wedge dt + A'_2\,\omega^2 \wedge dt + A'_3\,\omega^3 \wedge dt + \\ + \sum_{l=1}^{3} B'_{l1}\,\omega^l \wedge d\dot{\psi} + B'_{l2}\,\omega^l \wedge d\dot{\phi} + B'_{l3}\,\omega^l \wedge d\dot{\vartheta} + \bar{F} + \varphi_{(2)}$$

on Q, where $\omega^1 = d\psi - \dot{\psi}dt$, $\omega^2 = d\varphi - \dot{\varphi}dt$, $\omega^3 = d\vartheta - \dot{\vartheta}dt$ are the corresponding contact 1-forms, and where $\bar{F}$ is a 2-contact 2-form and $\varphi_{(2)}$ is a constraint 2-form defined on Q. Computing the coefficients A'_l according to (12) we obtain the following expressions:

$$A'_1 = (2mR^2 - I_2)(\cos\vartheta)\dot{\varphi}\dot{\vartheta}\,,$$
$$A'_2 = -I_2\cos\vartheta\dot{\psi}\dot{\vartheta} - 2(mR^2 - I_1 + I_2)(\sin\vartheta\cos\vartheta)\dot{\varphi}\dot{\vartheta}\,,$$
$$A'_3 = (I_2 - mR^2)\cos\vartheta\dot{\psi}\dot{\varphi} + (mR^2 - I_1 + I_2)(\sin\vartheta\cos\vartheta)\dot{\varphi}^2 + mgR\sin\vartheta\,,$$

and coefficients B'_{ls} according to (13) are

$$B'_{11} = -(mR^2 + I_2)\,, \qquad B'_{12} = B'_{21} = (mR^2 - I_2)\sin\vartheta\,,$$
$$B'_{22} = -(mR^2 + I_2)\sin^2\vartheta - I_1\cos^2\vartheta\,, \qquad B'_{23} = B'_{32} = 0\,,$$
$$B'_{33} = -(mR^2 + I_1)\,, \qquad B'_{31} = B'_{13} = 0\,.$$

Hence, reduced equations of motion (14) of the constrained system $[\alpha_Q]$ take the form (see also [26]):

$$(mR^2 + I_2)\ddot{\psi} + (I_2 - mR^2)(\sin\vartheta)\ddot{\varphi} + (I_2 - 2mR^2)(\cos\vartheta)\dot{\varphi}\dot{\vartheta} = 0\,,$$
$$(mR^2 - I_2)(\sin\vartheta)\ddot{\psi} - \left((mR^2 + I_2)\sin^2\vartheta + I_1\cos^2\vartheta\right)\ddot{\varphi} - \\ - I_2(\cos\vartheta)\dot{\psi}\dot{\vartheta} - 2(mR^2 - I_1 + I_2)(\sin\vartheta\cos\vartheta)\dot{\varphi}\dot{\vartheta} = 0\,,$$
$$-(mR^2 + I_1)\ddot{\vartheta} + (mR^2 - I_1 + I_2)(\sin\vartheta\cos\vartheta)\dot{\varphi}^2 + \\ + (I_2 - mR^2)(\cos\vartheta)\dot{\psi}\dot{\varphi} + mgR\sin\vartheta = 0\,.$$

These equations can be solved numerically; it turns out that solutions are unstable with respect to a small change of initial conditions.

5.7 A homogeneous ball on a rotating table

Consider a homogeneous ball of radius R rolling without sliding on a horizontal plane which rotates with a nonconstant angular velocity $\Omega(t)$ around the vertical axis. We assume that except the constant gravitational force, no other external forces act on the ball. (See [22], pp. 131, Example 3.)

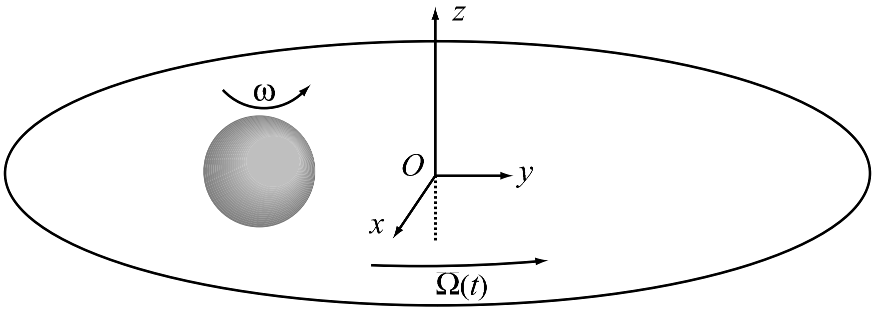

Figure 4

Let the z-axis of the fixed system of coordinates $Oxyz$ coincide with the axis of rotation. Let (x, y) denote the position of contact of the ball with the plane and ϑ, φ, ψ denote Euler angles of the rotating ball. The angle ϑ is the angle of inclination, the φ is the rotating angle and ψ is the angle of precession. Hence $(t, x, y, \vartheta, \varphi, \psi)$ are fibered coordinates on the configuration space $Y = \mathbb{R} \times \mathbb{R}^2 \times SO(3)$, where $SO(3)$ is the special orthogonal group parametrized by Euler angles, and $(t, x, y, \vartheta, \varphi, \psi, \dot{x}, \dot{y}, \dot{\vartheta}, \dot{\varphi}, \dot{\psi})$ are associated coordinates on $J^1Y = \mathbb{R} \times \mathbb{R}^2 \times SO(2) \times \mathbb{R}^2 \times SO(2)$.

The potential energy is constant, so without loss of generality we put $V = 0$. In addition, since we do not consider external forces, the Lagrange function is given by the kinetic energy of the rotating ball

$$L = T = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + k^2(\dot{\vartheta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\vartheta)\right), \tag{39}$$

where k is the radius of gyration and the mass of the ball is $m = 1$.

The motion equations of this Lagrangian system in coordinates $(q^1, \dots, q^5) = (x, y, \vartheta, \varphi, \psi)$ become:

$$\begin{aligned}
\ddot{x} &= 0\,,\\
\ddot{y} &= 0\,,\\
k^2(\ddot{\vartheta} + \sin\vartheta\,\dot{\varphi}\dot{\psi}) &= 0\,,\\
k^2(\ddot{\varphi} + \cos\vartheta\,\ddot{\psi} - \sin\vartheta\,\dot{\vartheta}\dot{\psi}) &= 0\,,\\
k^2(\cos\vartheta\,\ddot{\varphi} + \ddot{\psi} - \sin\vartheta\,\dot{\vartheta}\dot{\varphi}) &= 0\,.
\end{aligned}$$

Denoting by ω the instantaneous angular velocity of the ball, we write down the condition of rolling without sliding of the ball on the rotating plane

$$\dot{x} - R\omega_y + \Omega(t)\,y = 0, \qquad \dot{y} + R\omega_x - \Omega(t)\,x = 0, \tag{40}$$

or, using the Euler angles we obtain the following two equations

$$f^1 \equiv \dot{x} - R\sin\psi\,\dot{\vartheta} + R\sin\vartheta\cos\psi\,\dot{\varphi} + \Omega(t)\,y = 0\,,$$
$$f^2 \equiv \dot{y} + R\cos\psi\,\dot{\vartheta} + R\sin\vartheta\sin\psi\,\dot{\varphi} - \Omega(t)\,x = 0\,,$$

which represent two nonholonomic constraints affine in components of velocities. These equations evidently satisfy condition (8),

$$\operatorname{rank}\left(\frac{\partial f^i}{\partial \dot{q}^\sigma}\right) = \operatorname{rank}\begin{pmatrix} 1 & 0 & -R\sin\psi & R\sin\vartheta\cos\psi & 0 \\ 0 & 1 & -R\sin\varphi & R\sin\vartheta\cos\psi & 0 \end{pmatrix} = 2,$$

thus $\dim Q = \dim J^1Y - 2 = 9$. Constraint 1-forms (10) take the form

$$\varphi^1 = dx + \Omega(t)ydt - R\sin\psi d\vartheta + R\sin\vartheta\cos\psi d\varphi\,,$$
$$\varphi^2 = dy - \Omega(t)xdt + R\cos\psi d\vartheta + R\sin\vartheta\sin\psi d\varphi\,.$$

The constrained system $[\alpha_Q]$ is in this case represented by the equivalence class of a 2-form

$$\alpha_Q = A'_1\,\omega^1\wedge dt + A'_2\,\omega^2\wedge dt + A'_3\,\omega^3\wedge dt + \\ + \sum_{l=1}^{3} B'_{l1}\,\omega^l\wedge d\dot{\vartheta} + B'_{l2}\,\omega^l\wedge d\dot{\varphi} + B'_{l3}\,\omega^l\wedge d\dot{\psi} + \bar{F} + \varphi_{(2)}$$

on Q, where $\omega^1 = d\vartheta - \dot{\vartheta}dt$, $\omega^2 = d\varphi - \dot{\varphi}dt$, $\omega^3 = d\psi - \dot{\psi}dt$, and where for the coefficients A'_l we obtain

$$A'_1 = -(R^2+k^2)\dot{\varphi}\dot{\psi}\sin\vartheta + \\ + R\Omega(t)(\dot{x}\cos\psi + \dot{y}\sin\psi) + R\dot{\Omega}(t)(x\cos\psi + y\sin\psi)\,,$$
$$A'_2 = -R^2\dot{\vartheta}\dot{\varphi}\sin\vartheta\cos\vartheta + (R^2+k^2)\dot{\vartheta}\dot{\psi}\sin\vartheta + \\ + R\dot{\Omega}(t)\sin\vartheta(x\sin\psi - y\cos\psi) + R\Omega(t)\sin\vartheta(\dot{x}\sin\psi - \dot{y}\cos\psi)\,,$$
$$A'_3 = k^2\dot{\vartheta}\dot{\varphi}\sin\vartheta\,,$$

and for the coefficients B'_{ls} we have

$$B'_{11} = -(R^2+k^2)\,, \qquad B'_{12} = 0\,, \qquad B'_{13} = 0\,,$$
$$B'_{21} = 0\,, \qquad B'_{22} = -(R^2\sin^2\vartheta + k^2)\,, \qquad B'_{23} = -k^2\cos\vartheta\,,$$
$$B'_{31} = 0\,, \qquad B'_{32} = -k^2\cos\vartheta\,, \qquad B'_{33} = -k^2\,.$$

The motion of this constrained system is described by the following three reduced equations (see [26]):

$$(R^2+k^2)\,\ddot{\vartheta} + (R^2+k^2)\,\dot{\varphi}\,\dot{\psi}\sin\vartheta - \\ - R\,\Omega(t)(\dot{x}\cos\psi + \dot{y}\sin\psi) - R\,\dot{\Omega}(t)(x\cos\psi + y\sin\psi) = 0\,,$$
$$(R^2\sin^2\vartheta + k^2)\,\ddot{\varphi} + k^2\cos\vartheta\,\ddot{\psi} + \\ + R^2\dot{\vartheta}\,\dot{\varphi}\sin\vartheta\cos\vartheta - (R^2+k^2)\,\dot{\vartheta}\,\dot{\psi}\sin\vartheta - \\ - R\,\Omega(t)\sin\vartheta\,(\dot{x}\sin\psi - \dot{y}\cos\psi) - R\,\dot{\Omega}(t)\sin\vartheta\,(x\sin\psi - y\cos\psi) = 0\,,$$
$$k^2\cos\vartheta\ddot{\phi} + k^2\ddot{\psi} - k^2\dot{\vartheta}\,\dot{\varphi}\sin\vartheta = 0\,.$$

To simplify these equations we can use other coordinates, so called *quasicoordinates*. Recall that $\omega_x, \omega_y, \omega_z$ denote the components of the instantaneous angular velocity, which are determined by means of the Euler angles

$$\begin{aligned} \omega_x &= \dot{\vartheta}\cos\psi + \dot{\varphi}\sin\vartheta\sin\psi\,, \\ \omega_y &= \dot{\vartheta}\sin\psi - \dot{\varphi}\sin\vartheta\cos\psi\,, \\ \omega_z &= \dot{\psi} + \dot{\varphi}\cos\vartheta\,. \end{aligned} \tag{41}$$

Consider now "quasicoordinates" q^1, q^2, q^3 on the configuration space defined byt $\dot{q}^1 = \omega_x, \dot{q}^2 = \omega_y, \dot{q}^3 = \omega_z$. Denote by $(t, x, y, q^1, q^2, q^3, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z)$ associated coordinates on J^1Y. Then the expression of Lagrangian (39) in quasicoordinates is as follows:

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2)\right),$$

and equations of the constrained submanifold take the form (40). Reduced equations of motion of the constrained mechanical system in the quasicoordinates have the form

$$\begin{aligned} \left(R^2 + k^2\right)\ddot{q}^1 - R^2\Omega(t)\,\dot{q}^2 - R\dot{\Omega}(t)\,x + R\Omega^2(t)\,y &= 0\,, \\ \left(R^2 + k^2\right)\ddot{q}^2 + R^2\Omega(t)\,\dot{q}^1 - R\dot{\Omega}(t)\,y - R\Omega^2(t)\,x &= 0\,, \\ -\,k\,\ddot{q}^3 &= 0\,. \end{aligned} \tag{42}$$

Using the definition of the quasicoordinates q^1, q^2, q^3 we obtain that

$$\dot{q}^3 = \omega_z = C_3 = \mathit{const},$$

and the first two equations of the system (42) can be reduced to a system of first order linear differential equations

$$\begin{aligned} \left(R^2 + k^2\right)\dot{\omega}_x - R^2\Omega(t)\,\omega_y - R\,\dot{\Omega}(t)\,x + R\,\Omega^2(t)\,y &= 0\,, \\ \left(R^2 + k^2\right)\dot{\omega}_y + R^2\Omega(t)\,\omega_x - R\,\dot{\Omega}(t)\,y - R\,\Omega^2(t)\,x &= 0\,. \end{aligned} \tag{43}$$

Substituting constraint equations (40) into equations (42) we get two first integrals:

$$\begin{aligned} \left(R^2 + k^2\right)\omega_x - R\,\Omega(t)\,x &= D_1\left(R^2 + k^2\right)\,, \\ \left(R^2 + k^2\right)\omega_y - R\,\Omega(t)\,y &= D_2\left(R^2 + k^2\right)\,, \end{aligned} \tag{44}$$

where D_1, D_2 are arbitrary constants. Comparing the expressions for ω_x, ω_y from the constraint equations (40) and from (44) we obtain

$$\dot{x} + \frac{k^2\Omega(t)}{R^2 + k^2}\,y + R\,D_1 = 0, \qquad \dot{y} - \frac{k^2\Omega(t)}{R^2 + k^2}\,x - R\,D_2 = 0. \tag{45}$$

Differentiating the last two equations we get the following system of second order differential equations

$$\ddot{x} + \frac{k^2\Omega(t)}{R^2 + k^2}\,\dot{y} + \frac{k^2\dot{\Omega}(t)}{R^2 + k^2}\,y = 0, \qquad \ddot{y} - \frac{k^2\Omega(t)}{R^2 + k^2}\,\dot{x} + \frac{k^2\dot{\Omega}(t)}{R^2 + k^2}\,x = 0 \tag{46}$$

for unknown functions $x(t), y(t)$, which describe the motion of the point of contact of the ball with the plane.

Let us suppose, that for a given function $\Omega(t)$ of the angular velocity of the rotating plane we have found a solution $x(t), y(t)$ of (46). If we put

$$A = \left(R^2 + k^2\right), \qquad b(t) = R^2\,\Omega(t),$$

and denote

$$\begin{aligned} F_1\left(t, x(t), y(t)\right) &= R\,\dot{\Omega}(t)\,x - R\,\Omega^2(t)\,y\,, \\ F_2\left(t, x(t), y(t)\right) &= R\,\dot{\Omega}(t)\,y + R\,\Omega^2(t)\,x\,, \end{aligned}$$

then the system (43) can be written in the form

$$\begin{aligned} A\,\dot{\omega}_x - b(t)\,\omega_y &= F_1\left(t, x(t), y(t)\right)\,, \\ A\,\dot{\omega}_y + b(t)\,\omega_x &= F_2\left(t, x(t), y(t)\right)\,. \end{aligned} \tag{47}$$

This is a system of two first order linear non-homogeneous differential equations with nonconstant coefficients. First, we solve the corresponding homogeneous system

$$\dot{\omega}_x = \frac{B(t)}{A}\,\omega_y, \qquad \dot{\omega}_y = -\frac{B(t)}{A}\,\omega_x$$

and obtain the following result

$$\begin{aligned} \omega_x^H(t) &= C_1 \sin\left(\frac{B(t)}{A}\right) + C_2 \cos\left(\frac{B(t)}{A}\right)\,, \\ \omega_y^H(t) &= -C_2 \sin\left(\frac{B(t)}{A}\right) + C_1 \cos\left(\frac{B(t)}{A}\right)\,, \end{aligned}$$

where $B(t) = \int b(t)\,dt$. Next we are looking for a particular solution by the standard procedure of variation of constants

$$\begin{aligned} \omega_x^P(t) &= C_1(t) \sin\left(\frac{B(t)}{A}\right) + C_2(t) \cos\left(\frac{B(t)}{A}\right)\,, \\ \omega_y^P(t) &= C_1(t) \cos\left(\frac{B(t)}{A}\right) - C_2(t) \sin\left(\frac{B(t)}{A}\right)\,, \end{aligned}$$

where $C_1(t), C_2(t)$ are obtained by integrating the following equations

$$\begin{aligned} \dot{C}_1(t) &= F_1\left(t, x(t), y(t)\right) \sin\left(\frac{B(t)}{A}\right) + F_2\left(t, x(t), y(t)\right) \cos\left(\frac{B(t)}{A}\right)\,, \\ \dot{C}_2(t) &= F_1\left(t, x(t), y(t)\right) \cos\left(\frac{B(t)}{A}\right) - F_2\left(t, x(t), y(t)\right) \sin\left(\frac{B(t)}{A}\right)\,. \end{aligned}$$

A general solution of equations (47) is then of the form

$$\begin{pmatrix} \omega_x(t) \\ \omega_y(t) \end{pmatrix} = \begin{pmatrix} \omega_x^H(t) \\ \omega_y^H(t) \end{pmatrix} + \begin{pmatrix} \omega_x^P(t) \\ \omega_y^P(t) \end{pmatrix}.$$

The solution in terms of quasicoordinates is then determined by elementary quadratures

$$q^1(t) = \int \omega_x(t)\,dt, \qquad q^2(t) = \int \omega_y(t)\,dt, \qquad q^3(t) = \int C_3\,dt,$$

and the solution in terms of Euler angles is described by differential equations (41).

In a particular case, when $\Omega(t) = \Omega_0 = const.$, (see [22]) the system (46) takes the form

$$\ddot{x} + \frac{k^2\Omega_0}{R^2+k^2}\,\dot{y} = 0, \qquad \ddot{y} - \frac{k^2\Omega_0}{R^2+k^2}\,\dot{x} = 0.$$

Using first integrals (45) we write:

$$\ddot{x} + \left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 x = -\frac{k^2 R\,\Omega_0}{R^2+k^2}\,D_2\,,$$
$$\ddot{y} + \left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 y = -\frac{k^2 R\,\Omega_0}{R^2+k^2}\,D_1\,.$$

A solution of the corresponding homogeneous system is:

$$x^H(t) = A_1 \sin\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] + A_2 \cos\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right],$$
$$y^H(t) = A_3 \sin\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] + A_4 \cos\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right],$$

where A_1, A_2, A_3, A_4 are arbitrary constants. Using the procedure of variation of constants we get a particular solution:

$$x^P(t) = -R\,D_2\,\frac{R^2+k^2}{k^2\Omega_0}, \qquad y^P(t) = -R\,D_1\,\frac{R^2+k^2}{k^2\Omega_0}.$$

Finally, the general solution takes the form

$$x(t) = A_1 \sin\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] + A_2 \cos\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] - R\,D_2\,\frac{R^2+k^2}{k^2\Omega_0}\,,$$
$$y(t) = A_3 \sin\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] + A_4 \cos\left[\left(\frac{k^2\Omega_0}{R^2+k^2}\right)^2 t\right] - R\,D_1\,\frac{R^2+k^2}{k^2\Omega_0}\,,$$

where D_1, D_2 are constants, which occur in the first integrals (44). Hence the ball on the rotating table moves along ellipses parameters of which depend on initial conditions.

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On D'Alembert's Principle

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Abstract. A formulation of the D'Alembert principle as the orthogonal projection of the acceleration onto an affine plane determined by nonlinear nonholonomic constraints is given. Consequences of this formulation for the equations of motion are discussed in the context of several examples, together with the attendant singular reduction theory.

1 D'Alembert's principle

Let us suppose that we have a Lagrangian or Hamiltonian mechanical system and we wish to impose a constraint. The system has an n-dimensional configuration space Q with local coordinates $\{q^a\}$, velocity phase space TQ with the natural chart $\{q^a, v^a\}$, and momentum phase space $P = T^*Q$ with local coordinates (q^a, p_a). To start, suppose we have a Lagrangian of the classical form kinetic energy minus potential energy,

$$l = \frac{1}{2}g(v, v) - u(q).$$

We want to write down the equations of motion if we impose a (possibly time dependent and nonholonomic) constraint of the form

$$c(q, v, t) = 0.$$

Later on we will discuss what happens if we have Lagrangians not of this simple form, or more constraints, but for now it suffices to just to consider this case.

Differentiating the constraint with respect to the time t gives

$$\frac{d}{dt}c = c_a v^a + c_{\dot{a}}\dot{v}^a + c_t = 0$$

where $c_a = \partial c/\partial q^a$, $c_{\dot{a}} = \partial c/\partial v^a$, and $c_t = \partial c/\partial t$. The acceleration a is given by

$$a^b = \dot{v}^b + \Gamma^b_{kl}v^k v^l$$

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where the Γ^b_{kl} are the Christoffel symbols of the Levi-Civita connection of the metric. Substituting the acceleration into the differentiated constraint yields

$$c_{\dot{k}} a^k + (c_k v^k - c_{\dot{a}} \Gamma^a_{kl} v^k v^l + c_t) = 0.$$

The important point here is that for fixed (q, v, t) this equation represents an *affine relation* for the accelerations in the tangent space $T_q Q$, where we have used the connection to identify the vertical space $V_{(q,v)} TQ$ with $T_q Q$.

Given the acceleration a of the unconstrained problem, it must be modified so that it lies in the affine plane. This is done by subtracting the component $a_\perp$ of a orthogonal to the plane, so the acceleration of the constrained problem is the difference

$$a_{\text{constrained}} = a - a_\perp$$

and lies in the affine plane. Observe that the vector $a_\perp$ has components

$$a_\perp{}^k = -\lambda g^{kr} c_{\dot{r}}$$

for some real number λ because of the form of the affine equation. Since we already have a Lagrangian description of the unconstrained problem, the force covector for the constrained problem may be written in the form

$$\frac{d}{dt}\left(\frac{\partial l}{\partial v^a}\right) - \frac{\partial l}{\partial q^a} = \lambda c_{\dot{a}}.$$

Observe that the specific form of the Lagrangian was not essential, one can equally well work with the velocity Hessian $g_{ab} := l_{\dot{a}\dot{b}}$ as long as the Lagrangian is regular, i.e., the velocity Hessian defines an invertible metric. Furthermore, the argument generalizes to the case of more than one constraint function, say $c^1 = 0$, $\ldots, c^K = 0$, yielding the constrained Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial l}{\partial v^a}\right) - \frac{\partial l}{\partial q^a} = \lambda_A c^A_{\dot{a}}$$

involving the Lagrange multipliers $\lambda_1, \ldots, \lambda_K$.

2 Other formulations

Assuming the regularity of the Lagrangian, we may push everything over to the cotangent bundle and give a Hamiltonian description as well. This may be written in a coordinate free manner for the vector field

$$X = \dot{q}^a \partial_{q^a} + \dot{p}_b \partial^{p_b}$$

by using ϑ_0, the canonical one-form on T^*Q, the symplectic form $\omega = -d\vartheta_0$, the Legendre transform $\mathscr{L}$, as well as $\vartheta^A := Fc^{A*}\vartheta_0$, where Fc^A is the fiber derivative of c^A. Set $\phi^A = \mathscr{L}_* \vartheta^A$. Then the constrained Hamilton's equations may be written in the form

$$X \,\lrcorner\, \omega = dh + \lambda_A \phi^A$$

The time derivative of a function f (with Hamiltonian vector field X_f) along an integral curve of the constrained vector field X is

$$\frac{df}{dt} = \langle df, X\rangle = \omega(X_f, X) = -X_f \lrcorner (X \lrcorner \omega) = -X_f \lrcorner (dh + \lambda_A \phi^A).$$

Setting $\psi := dh + \lambda_A \phi^A$, we may write this derivative in Poisson bracket form as

$$\dot{f} = \{f, \psi\}$$

where we are taking the Poisson bracket of the function f and the one form ψ to be $\Lambda(df, \psi)$, where $\Lambda = \omega^{-1}$ is the structure tensor of the Poisson bracket.

3 Easy consequences

The following is a partial list of easy consequences of the nonlinear formulation so that one may see the similarities and differences with the affine theory. Some of these are known and may be found in earlier work by de Leon et al [5].

3.1 Conservation of energy

Suppose that we have a time independent Lagrangian and impose the constraint of constant energy. Then the Lagrange multiplier is zero, and the problem reduces to Hamilton's equations on a constant energy surface. We may view this as a consistency check for the nonlinear constraint theory.

3.2 Nonconservation of energy

Suppose that we have a time independent Lagrangian and impose a time independent constraint c. Then, letting the energy e be $e = pv - l = l_v v - l$ as usual, we find

$$\dot{e} = \left(\frac{d}{dt}\left(\frac{\partial l}{\partial v^a}\right) - \frac{\partial l}{\partial q^a}\right) v^a,$$

and so by the equation of motion for the constrained problem

$$\dot{e} = \lambda_A \frac{\partial c^A}{\partial v^a} v^a.$$

In general we do not expect this term to vanish, so unlike the case of linear nonholonomic constraints, we do not have energy conservation even in the time independent situation. Since this is not what one would expect from the usual Noether theory, it only goes to show that such problems are really not variational problems in the usual way, even though we have a Lagrangian. However, if the Lagrange multipliers are not zero, there is an important case where this term will vanish, and that is when the constraint functions are each homogeneous of some degree in the velocities. For then, by Euler's theorem on homogeneous functions,

$$\frac{\partial c^A}{\partial v^a} v^a \propto c^A = 0$$

by the constraint equation. Note that this is the case for linear constraints.

3.3 The Lagrange multiplier

Suppose we have just one constraint. The Lagrange multiplier λ is chosen so that the constrained vector field X is tangent to the constraint surface: $\langle dc, X\rangle = 0$. Since the constrained vector field X satisfies

$$X \lrcorner\, \omega = dh + \lambda\phi,$$

we have by symplectic inversion

$$\langle dc, X_h + \lambda\phi^{\#}\rangle = 0,$$

so that

$$\lambda = -\frac{\langle dc, X_h\rangle}{\langle dc, \phi^{\#}\rangle} = -\frac{\{c, h\}}{\langle dc, \phi^{\#}\rangle}$$

where X_h is the Hamiltonian vector field of h, and $\{c, h\}$ is the Poisson bracket of c and h. Note that the solvability of the Lagrange multiplier assumes the independence condition $dc \wedge \phi \neq 0$.

An immediate corollary is that if the constraint function is a first integral of the Hamiltonian, then imposing the integral as a constraint is really no constraint at all, since the multiplier is zero.

If we have multiple constraints, say $c^1, c^2, \ldots, c^K = 0$, then the equations of motion take the form

$$X \lrcorner\, \omega = dh + \lambda_A\phi^A.$$

The Lagrange multipliers λ_A may be found from the K equations

$$\langle dc^A, X_h\rangle + \lambda_B\langle dc^A, \phi^{B\#}\rangle = 0.$$

An evaluation of $\langle dc^A, \phi^{B\#}\rangle$ gives $-M^{AB} = -g^{ab}c^A_{\dot{a}}c^B_{\dot{b}}$, a contravariant metric on the subspace spanned by the one forms $c^A_{\dot{a}}$. As long as this metric is invertible one can uniquely find the λ_A. If g_{ab} has a Euclidean signature this will be the case as long as the one forms $c^A_{\dot{a}}$ are linearly independent. This is not sufficient, however, for Lorentz signature metrics, such as those appearing in our relativistic particle examples below.

4 Examples (1)

4.1 The brachystochrone

A good place to begin is with the brachystochrone. The brachystochrone is the problem where a bead slides down a frictionless wire from rest at $(x, y) = (0, 0)$ to the point $(x, y) = (a, b)$. Here we take the positive y direction to point vertically downwards. The problem is to determine the shape of the wire so as to have a minimum time of descent. For a falling body, from conservation of energy one obtains $v^2 = 2gy$. This can be viewed as a nonholomic constraint. In section 3.1 above we noted that such a constraint does no work—the associated constraint force vanishes. But we can exploit this nonholonomic constraint in another way. Using $dl^2 = dx^2 + dy^2$ we may write the time T of descent to be

$$T = \int dt = \int \frac{dl}{v} = \int \frac{1}{\sqrt{2gy}}\sqrt{\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2}\, d\sigma.$$

where σ is any path parameter. The parameter invariance is connected with the fact that this Lagrangian is degenerate. Consequently a proper Hamiltonian analysis requires the Dirac algorithm. We will develop that shortly and include a description of the results it yields when applied to this problem. But first let us note that for this problem one can avoid the degeneracy of the Legendre transformation. Observe that our Lagrangian can be viewed as the arc length due to the metric

$$ds^2 = \frac{1}{2gy}(dx^2 + dy^2).$$

And so the objective is just to find a certain geodesic path of this metric. Now it well known that there is an "equivalent" alternative to the arc length Lagrangian for geodesics, namely its square:

$$L = \frac{1}{2gy}\left[\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right].$$

This will give the same path, but with a uniform speed parameterization. Moreover, for our problem it is apparent that we specifically want the unit speed geodesics of this metric. Then the parameter is actually the physical time. A virtue of this new Lagrangian is that it is non-degenerate, so there is no complication in passing to the Hamiltonian. The Hamiltonian equations of motion for the associated Hamiltonian $h = gy(p_x^2 + p_y^2)/2$ are (upon setting $g = 1$)

$$\dot{x} = yp_x, \qquad \dot{p}_x = 0, \qquad \dot{y} = yp_y, \qquad \dot{p}_y = -\frac{h}{y}.$$

The translational invariance of the metric in the x direction implies the conservation of p_x, so we may reduce at $\mu = p_x$ along with $h = 1$ to get the reduced equation

$$\dot{y} = yp_y = \sqrt{y - \mu^2 y^2}.$$

This separates and integrates to $s = \frac{1}{\mu}\arccos(2\mu^2 y - 1)$. Inverting yields $y = \frac{1}{2\mu^2}(1 - \cos\mu s)$, and hence $x = \frac{1}{2\mu}(s - \frac{1}{\mu}\sin\mu s)$, from which we recognize the familiar cycloidal solutions. In this example we have finessed the degeneracy of the Legendre transformation. Further on we will reconsider that issue. Other aspects of this problem are discussed in [6].

4.2 Dirac constraints

To deal with the proper dynamical formulation for relativistic particles, which also involves finding an appropriate parametrization (the proper time) we first sketch the Dirac Hamiltonian theory. In order to pass from the Lagrange equations together with the undetermined constraint forces $\lambda_A \partial c^A/\partial v^k$, to the Hamiltonian description, one uses the Legendre transformation. If the Lagrangian is not regular, so the momenta are not independent, then they satisfy some primary constraints $\Phi_\alpha(q,p) = 0$. Following Dirac, one includes these constraints in the Hamiltonian with Lagrange multipliers, so the total Hamiltonian takes the form $h = h_0 + u^\alpha \Phi_\alpha$.

The Hamiltonian evolution equations, including the velocity constraint forces, take the form

$$\frac{dq^k}{dt} = \frac{\partial h_0}{\partial p_k} + u^\alpha \frac{\partial \Phi_\alpha}{\partial p_k},$$
$$\frac{dp_k}{dt} = -\frac{\partial h_0}{\partial p_k} - u^\alpha \frac{\partial \Phi_\alpha}{\partial p_k} + \lambda_A \phi^A_k.$$

These differential equations are to be considered along with the two sets of constraint equations

$$\Phi_\alpha = 0, \qquad c^A = 0.$$

The first possible obstruction is whether the constraint functions c^A can be chosen so that there exist one-forms on phase space ϕ^A_k which are related by the degenerate Legendre transform to $c^A_{\dot{k}}$. Since the dynamical equations are required to preserve the constraints, this leads to additional conditions which may yield new constraints or fix the multipliers. In general, in attempting to determine the unknown multipliers one can expect similar outcomes to the usual Dirac procedure:

1. there may be no solution,
2. there may be additional constraints,
3. the solution may not be unique.

Observe that the constraints $\Phi_\alpha = 0$ are well defined on phase space and their time derivatives are linear in the unknown multipliers. In this case we know from Dirac how to proceed. However, the constraints $c^A = 0$ are velocity constraints, they are not defined on phase space. In general there are no phase space functions which are related to c^A. Fortunately, from the Hamiltonian perspective there is a natural "inverse" to the Legendre transformation, given by the first half of the Hamiltonian evolution equations. Hence the velocity constraint function can be given in terms of the phase space variables as

$$c^A(q^k, v^k) = c^A\left(q^k, \frac{\partial h_0}{\partial p_k} + u^\alpha \frac{\partial \Phi_\alpha}{\partial p_k}\right) = 0.$$

If c^A is linear in v this expression will be linear in the multipliers, and there is no insurmountable difficulty. In the general case the velocity constraints could be *nonlinear* functions of the unknown multipliers. Moreover, preserving these constraints could lead to expressions involving the derivatives of the multipliers. This is the second obstruction.

In the relativistic examples below the velocity constraints are linear in an expression which is homogeneous of degree one in velocity, so they turn out to be linear in the multiplier, so there is no difficulty.

4.3 The relativistic particle

We apply the above procedure to the following relativistic particle Lagrangians with their associated *proper time-constant magnitude* velocity constraints (see for example Gràcia [8] or Krupková [10])

$$l_1 = -mc\sqrt{-g_{\mu\nu}v^\mu v^\nu} - V(x) + qv^\mu A_\mu, \qquad C_1 = c - \sqrt{-g_{\mu\nu}v^\mu v^\nu},$$
$$l_2 = \frac{1}{2}mg_{\mu\nu}v^\mu v^\nu - V(x) + qv^\mu A_\mu, \qquad C_2 = \frac{1}{2c}\big[g_{\mu\nu}v^\mu v^\nu = c^2\big].$$

Here $v^\mu = dx^\mu/d\sigma$, where σ is an *a priori* arbitrary time parameter. Just for this example we have changed our notation for the constraints in order to avoid any possible confusion with the speed of light c. In both cases the dynamical equation of motion can be rearranged to be in the form

$$\frac{d}{d\sigma}(mg_{\mu\nu}u^\nu) = qv^\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) - \partial_\mu V + \lambda g_{\mu\nu}u^\nu,$$

with $u^\mu = dx^\mu/d\tau$ with τ being the proper time. Contracting the equation of motion with the 4-velocity gives the value of the multiplier: $-u^\mu\partial_\mu V - \lambda c^2 = 0$. The final form of the equation of motion is then

$$\frac{d}{d\tau}(mg_{\mu\nu}u^\nu) = qF_{\mu\nu}u^\nu - (\delta^\nu_\mu + c^{-2}u^\nu g_{\mu\gamma}u^\gamma)\partial_\nu V,$$

where, as usual, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell field. The authors know of no textbook that gives a proper treatment of the relativistic velocity constraint. Typically, the equations of motion are obtained and then the constraint is imposed (see for example [7]) without any consideration of the force of constraint. This happens to work if the the dynamics is compatible with the constraint, as is the case of a charged particle interacting with the Maxwell field. Then the velocity constraint does no work, so the multiplier λ vanishes. This is not the case for the scalar potential, where the force must be Lorentz orthogonal to the 4-velocity. There does not seem to be any way to get the proper relativistic force due to a scalar potential directly from a Lagrangian. Nonholonomic constraints play an essential role.

It is also of interest to give the Hamiltonian formulation of these examples. For the Lagrangian l_1, the Legendre transformation is degenerate, and so there is a primary constraint. From

$$p_\mu = \frac{\partial l_1}{\partial v^\mu} = mc\frac{g_{\mu\nu}v^\nu}{\sqrt{-g_{\alpha\beta}v^\alpha v^\beta}} + qA_\mu,$$

set $\mathbf{p}_\mu := p_\mu - qA_\mu$ so the primary constraint has the form

$$g^{\mu\nu}\mathbf{p}_\mu\mathbf{p}_\nu = -(mc)^2.$$

The Hamiltonian $h_0 = p_\mu v^\mu - l_1 = V(x)$ and so

$$h = V + \frac{N}{2m}\big[g^{\mu\nu}\mathbf{p}_\mu\mathbf{p}_\nu + (mc)^2\big],$$

where the primary Dirac constraint Lagrange multiplier N is called the lapse. Hamilton's equations take the form

$$\frac{dx^\mu}{d\sigma} = \frac{N}{m} g^{\mu\nu} \mathbf{p}_\nu,$$
$$\frac{dp_\mu}{d\sigma} = -\partial_\mu V + \frac{N}{m} \mathbf{p}_\alpha g^{\alpha\nu} (q\partial_\mu A_\nu) + \lambda \frac{\mathbf{p}_\mu}{mc}.$$

The velocity constraint $0 = c - \sqrt{-\frac{N^2}{m^2}\mathbf{p}_\mu \mathbf{p}^\mu}$ determines that the Dirac multiplier $N = 1$. The preservation of the primary Dirac constraint determines the velocity constraint multiplier since $0 = -\mathbf{p}^\mu \partial_\mu V - \lambda \mathbf{p}_\mu \mathbf{p}^\mu/(mc)$ implies that $\lambda = (mc)^{-1} \mathbf{p}^\mu \partial_\mu V$.

4.4 The brachystochrone as a constrained system

Now we can briefly return to the brachystochrone as a constrained system. The Lagrangian has a general curve parameter σ and thus a gauge freedom type degeneracy, which leads to a primary constraint

$$2gy(p_x^2 + p_y^2) - 1 = 0.$$

The Lagrangian is homogeneous of degree one in the velocities. Consequently by Euler's theorem the energy function vanishes. Then the Dirac Hamiltonian is just given by a Lagrange multiplier multiple of the primary constraint:

$$H = \frac{u}{4}\big[2gy(p_x^2 + p_y^2) - 1\big].$$

The primary constraint is preserved, and it is first class; the multiplier is an undetermined gauge parameter. The simple choice $u = 1$ gives the same Hamiltonian equations found earlier. One might also consider including the gauge fixing condition $\dot{x}^2 + \dot{y}^2 = 2gy$ as a non-holonomic constraint along with its attendant constraint force. This is essentially just imposing the constant energy as a non-holonomic constraint, and, as mentioned, such a constraint does no work and has vanishing constraint force. This is representative of what happens for other time parameter gauge invariant actions (e.g., Jacobi), and their gauge fixing options.

5 The distributional splitting

In this section we derive the key distributional splitting and look at the special case of homogeneous constraints. A consequence of homogeneity is that there is a distributional formulation of the constrained Hamiltonian equations. The proofs of these results are little changed from the earlier work of Bates and Śniatycki [2] who treated the linear case, but we reproduce them here for the sake of completeness.

Let M be the manifold given by the common zeroes of the constraint functions $\{c^1, \ldots, c^K\}$, and let F be the distribution consisting of vectors in the kernel of the forms $\{\phi^1, \ldots, \phi^K\}$,

$$F = \{v \in TP \mid \langle \phi^A, v\rangle = 0,\, a = 1, \ldots, K\}.$$

Set H to be the distribution formed by the intersection

$$H = F \cap TM.$$

So far the only assumption needed on the metric was that it was nondegenerate. However, in order to progress with the theory for indefinite metrics, we need an additional assumption.

Definition 1. The constraint manifold M is said to be g-nondegenerate if the restriction of the metric g to the distribution $\pi_* TM^\omega$ is nondegenerate. Here $\pi : P \to Q$ is the cotangent bundle projection.[1]

It is easy to check that this if M is given locally by the common zeroes of constraint functions $c^1, \ldots, c^K$, then the condition is equivalent to the nondegeneracy of the matrix M^{AB} of section 3.3 whose AB component is $g(\pi_* X_{c^A}, \pi_* X_{c^B})$.

Theorem 1. *On a g-nondegenerate constraint manifold M, the restriction of ω to the distribution H, denoted ω_H, is nondegenerate.*

Proof. Since the forms ϕ^A are assumed independent and semi-basic (since they annihilate the vertical space VTP), we may assert the existence of $n-K$ additional independent semi-basic one-forms $\phi^{K+1}, \ldots, \phi^n$. In the local chart $\{q^1, \ldots, p_n\}$, $\phi^a = \phi^a_i dq^i$. Let ϕ be the matrix with ab component ϕ^a_b. Our assumption implies that the matrix ϕ is invertible. Define forms χ_a by

$$\chi_a = (\phi^{-1})^j_a dp_j, \quad a = 1, \ldots, n.$$

It then follows that $\{\phi^1, \ldots, \phi^n, \chi_1, \ldots, \chi_n\}$ is a symplectic coframe as

$$\phi^a \wedge \chi_a = \phi^a_i (\phi^{-1})^j_a dq^i \wedge dp_j = dq^a \wedge dp_a = \omega.$$

Since the restriction of ω to F is

$$\omega|_F = \phi^{K+1} \wedge \chi_{K+1} + \cdots + \phi^n \wedge \chi_n,$$

it follows that the symplectic perpendicular F^ω is

$$F^\omega = \ker\{\phi^1, \ldots, \phi^n, \chi_{K+1}, \ldots, \chi_n\},$$

and this implies that F is coisotropic. Since M is defined by the common zeroes of $c^1, \ldots, c^K$, tangent vectors to M are defined by the kernel of the forms

$$\psi^A = dc^A = c^A_{,m} dq^m + c^{A,r} dp_r \quad A = 1, \ldots, K.$$

It follows that the intersection of F^ω and TM is given by

$$F^\omega \cap TM = \ker\{\phi^1, \ldots, \phi^n, \chi_{K+1}, \ldots, \chi_n, \psi^1, \ldots, \psi^K\}.$$

[1] That F^ω could intersect the constraint manifold M nontransversally was overlooked in the original proof for linear constraints found in [2], where I thought that the determinental multiplier was always 1.

The conclusion will follow if we can show that

$$\phi^1 \wedge \cdots \wedge \phi^n \wedge \chi_{K+1} \wedge \cdots \wedge \chi_n \wedge \psi^1 \wedge \cdots \wedge \psi^K$$

is a volume, since then it follows that $F^\omega \cap TM = 0$, and so $F^\omega \cap H = 0$. Since $F^\omega \oplus H = F$ by dimension count, and F is coisotropic, we may conclude that ω_H is nondegenerate.

To actually show that the $2n$ form is a volume, first observe that since the result is a pointwise result, we may choose, for a fixed point z, a local symplectic chart such that

$$\phi^1(z) = dq^1, \ldots, \phi^n(z) = dq^n, \chi_1(z) = dp_1, \ldots, \chi_n(z) = dp_n.$$

In other words, $\phi^a_b(z) = \delta^a_b$. This means that the wedge product

$$\phi^1 \wedge \cdots \wedge \phi^n \wedge \chi_{K+1} \wedge \cdots \wedge \chi_n \wedge \psi^1 \wedge \cdots \wedge \psi^K$$

will equal $\det(g_K) dq^1 \wedge \cdots \wedge dq^n \wedge dp_1 \wedge \cdots \wedge dp_n$ where $\det(g_K)$ is the determinant of the upper left $K \times K$ block of the metric g^{ab} in this frame (this is just the earlier defined M^{AB}). This is immediate once one realizes that the only part of the forms ψ^a that survive the wedge product with all of the ϕ^a are the terms $\phi^a_r g^{rs} dp_s$, and all the terms involving $dp_{K+1}, \ldots, dp_n$ are annihilated by being wedged with $\chi_{K+1} \wedge \cdots \wedge \chi_n$. Observe that the inequality $\det(g_K) \neq 0$ is exactly the condition of g-nondegeneracy in our special frame. □

Define the distribution K by $K = TM \cap H^\omega$. Since $TM = H \oplus K$, the constrained vector field X may be decomposed as $X = X^H + X^K$. Two extreme cases of this are when the constraint is the Hamiltonian itself, so the Lagrange multiplier vanishes and $X = X^K$, and when the constraints are homogeneous, and then $X = X^H$. To see this, observe that for $A = 1, \ldots, K$, the pairing

$$\langle \phi^A, X \rangle = \langle \phi^A, \dot{q} \rangle = 0$$

by homogeneity, which implies that X is in H. Evaluating the constrained equation of motion

$$X \lrcorner\, \omega = dh + \lambda_A \phi^A$$

on the distribution H annihilates the terms involving the Lagrange multipliers, and we obtain

$$X \lrcorner\, \omega_H = dh_H.$$

The expression dh_H denotes the restriction of dh to the distribution H, and we may think of the constrained Hamiltonian equations as being in distributional form.

6 Conservation laws

In Hamiltonian mechanics symmetry (and the closely related reduction theory) are usually studied together with conservation laws because of their equivalence, which is the content of the first Noether theorem. In nonholonomic systems, this equivalence is in general broken, so not all symmetries yield conservation laws, and not all conservation laws yield symmetries (see [2] for a simple example.) Looking ahead to the reduction theory, we will say that a vector is *horizontal* if it lies in the distribution H.

Suppose that we have a Lie group G acting in a Hamiltonian way on the phase space P such that it possesses a momentum map $j : P \to \mathfrak{g}^*$. In other words, for each $\zeta \in \mathfrak{g}^*$, the momentum j_ζ corresponding to ζ

$$j_\zeta = \langle j, \zeta \rangle$$

has a Hamiltonian vector field X_ζ satisfying

$$X_\zeta \lrcorner\, \omega = dj_\zeta.$$

If we further assume that G leaves the constraint manifold M, the constraint forms ϕ^a and the Hamiltonian h invariant, then it also leaves the Lagrange multipliers invariant, and thus the structure of the equations of motion

$$X \lrcorner\, \omega = dh + \lambda_A \phi^A$$

is invariant as well. The vector field X_ζ is called an infinitesimal symmetry.

Lemma 1. *The momentum j_ζ associated to ζ is conserved if the vector field X_ζ is horizontal.*

Proof. This is a simple calculation:

$$\langle dj_\zeta, X \rangle = X \lrcorner\, (X_\zeta \lrcorner\, \omega\rangle = -X_\zeta \lrcorner\, (dh + \lambda_A \phi^A) = -X_\zeta \lrcorner\, \lambda_A \phi^A = 0. \qquad \square$$

Observe that the invariance of the constraint manifold M was never used in the proof of the lemma. This implies that the following more general theorem is true.

Theorem 2. *Let f be a function with Hamiltonian vector field X_f with the property that it preserves the Hamiltonian h and lies in the distribution F consisting of the kernel of the constraint forms ϕ^A. Then f is a constant of motion.*

7 Symmetry and reduction

So far the nonlinear constraint theory looks virtually identical to the linear theory. It is in the reduction by symmetry that the nonlinear case differs, and this is because the constrained vector field does not have to lie in the distribution H.

Recall that the time derivative of a function f along an integral curve of the constrained vector field X is

$$\frac{df}{dt} = \langle df, X \rangle = -X_f \lrcorner\, (dh + \lambda_A \phi^A).$$

Set $\psi := dh + \lambda_A \phi^A$, write this derivative in Poisson bracket form as

$$\dot{f} = \{f, \psi\}$$

where we are taking the Poisson bracket of the function f and the one form ψ to be $\Lambda(df, \psi)$, where $\Lambda = \omega^{-1}$ is the structure tensor of the Poisson bracket.

In specific problems it is often more convenient to solve for the Lagrange multipliers explicitly and then use a Dirac bracket-like formulation in which the multipliers are eliminated. Since our bracket convention implies

$$\{f, dh + \lambda_A \phi^A\} = \{f, h\} + \lambda_A \{f, \phi^A\}$$

and the preservation of the constraints by the dynamics implies

$$0 = \dot{c}^A = X_{c^A} \lrcorner (X \lrcorner \omega) = \omega(X_h, X_{c^A}) - \lambda_B \{c^A, \phi^B\},$$

and we already have the matrix $M^{BD} := -\{c^B, \phi^D\}$ with assumed inverse $M_{EF}M^{FD} = \delta_E^D$ from which the Lagrange multiplier λ_A may be found as

$$\lambda_A = -M_{AB}\{h, c^B\}.$$

This implies that the bracket formulation of the equations of motion may be given as

$$\dot{f} = \{f, h\} + \{f, \phi^A\} M_{AB} \{c^B, h\}.$$

It is important to note that this dynamical equation is only defined on the image of the constraint manifold under the Legendre transformation, and not on the entire phase space. In practice however, keeping track of this is not an issue, and we will continue to ignore it as it makes no essential conceptual difference in what follows.

Let G be a symmetry group of the dynamical system, by which we mean that the group G acts symplectically on the phase space, preserves the constraint manifold and constraint forms, and leaves the Hamiltonian invariant. These assumptions imply that the group action also preserves the Poisson bracket, the constraint distributions H and K as well as the Lagrange multipliers.

Now, if f is a function invariant under the action of the symmetry group G, $f \in C^\infty(P)^G$, then the Poisson bracket $\{f, \psi\}$ is invariant as well, since both Λ and ψ are. This implies that the map

$$\{\cdot, \psi\} : C^\infty(P)^G \to C^\infty(P)^G : f \to \{f, \psi\}$$

is an outer Poisson derivation[2] (it is an outer derivation since it involves an invariant one-form and not an invariant function) on the ring of invariant functions, and so if $f \in C^\infty(P)^G$, $\dot{f} = \{f, \psi\}$ may be viewed as a differential equation on the reduced space P/G. In this way a vector field $\bar{X}$ is defined on the reduced space, and this is the projection of X by the quotient map $P \to P/G$ when restricted to the quotient of the constraint manifold M/G. A key benefit to considering reduction in this formulation is that the construction is well-defined even when the quotient space is not a manifold, as long as we assume that the group action is proper, for then the quotient space is a subcartesian differential space, and the invariant functions still separate points on the quotient (a good reference for this material may be found in Śniatycki [13]). In this sense we may view the Poisson bracket-like formulation as providing the singular reduction of nonlinear nonholonomic constraints simply by restriction to the invariant functions.

[2]It is important to note here that there is no statement that the dynamical flow preserves the Poisson bracket, even on the constraint manifold.

8 Examples (2)

8.1 The central force problem with a speed constraint

Consider the motion of a particle in space subject to the action of a central force and and the constraint of constant speed. Then the Hamiltonian may be written as

$$h = \frac{1}{2}|p|^2 + V(r)$$

with $r = |q|$. Take the constraint to be $c = \frac{1}{2}|p|^2$. The equations of motion are

$$\begin{aligned} \dot{q}^a &= p_a \\ \dot{p}_a &= -\frac{q^a}{r}V'(r) - \lambda p_a \end{aligned}$$

Solving for the Lagrange multiplier yields

$$\lambda = -\frac{1}{2c}\frac{q \cdot p}{r}V'(r).$$

Since the problem is rotationally invariant, we introduce the three independent rotation invariants $\sigma_1 = r^2 = |q|^2$, $\sigma_2 = q \cdot p$, and $\sigma_3 = |p|^2 = 2c$. The singularly reduced problem in σ_1-σ_2 space (with the semi-algebraic constraint $\sigma_1 \geq 0$) has equation of motion

$$\begin{aligned} \dot{\sigma}_1 &= 2\sigma_2, \\ \dot{\sigma}_2 &= 2\left(c - \left(\sigma_1 - \frac{1}{2c}\sigma_2^2\right)\frac{dV}{d\sigma_1}\right). \end{aligned}$$

The dynamical interest here is in the destabilization of the circular orbits in Keplerian like potentials with the imposition of the constant speed constraint. If this seems counterintuitive, one may think of it as the imposition of the constant speed constraint, even though it is rotationally invariant, means that the angular momentum is no longer conserved.

8.2 A classical particle with spin

The reduction by symmetry required that the bracket map the invariant functions to invariant functions:

$$\{\cdot, \psi\} : C^\infty(P)^G \to C^\infty(P)^G.$$

However, and this is the crucial point, there is no requirement that the form ψ actually be invariant as well in order to have reduced dynamics[3]. Thus, it is possible to have reduction to invariant functions without the full problem being invariant. From a distributional point of view, this says that reduction exists in the case when the constrained vector field varies under the group action, but only in directions that are parallel to the tangents to the group action. A nontrivial example of this

[3]There *is* such a requirement if one wants to have the reduced dynamics still in Poisson like form, with a corresponding reduced form $\bar{\psi}$.

behaviour may be found in the reduction of a classical spinning rigid body to the classical particle with spin. Consider a uniformly charged symmetric rigid body with moment of inertia I, total charge q, mass m and gyromagnetic ratio g in the presence of a magnetic field B. The equations of motion are

$$\begin{aligned}\dot{x} &= v,\\ \dot{v} &= \frac{q}{m}(v \times B) + \frac{gI}{m} DB^*(x)(k(\mathrm{Ad}_A X))\\ \dot{A} &= A(X - g\,\mathrm{Ad}_{A^{-1}} B(x)),\\ \dot{X} &= 0.\end{aligned}$$

Here $x \in \mathbb{R}^3$ is the position of the centre of mass, $v \in \mathbb{R}^3$ is the velocity of the centre of mass, $A \in SO(3)$ is the orientation of the rigid body, and $X \in so(3)$ is the angular velocity. k is the Killing metric on the rotation group. We have also employed the identification of antisymmetric matrices and 3-vectors as appropriate. The nonlinear nonholonomic constraint of constant length of angular momentum is applied, and the problem is reduced with respect to the action of the rotation group to get Souriau's model of a classical particle with spin

$$\begin{aligned}\dot{x} &= v,\\ \dot{v} &= \frac{q}{m}(v \times B) + \frac{g}{m} DB^*(x)(S)\\ \dot{S} &= g[S, B]\end{aligned}$$

The details of this calculation, but not this point of view, may be found in [4].

9 Notes

1. In the mechanics literature there are differing definitions of just what constitutes D'Alembert's principle, the virtual work of perfect constraints, etc. It seems to us that the D'Alembert principle is at heart the choice to pick the *orthogonal* projection of the acceleration of the unconstrained problem onto the affine plane. One could pick a different projection, but it would in general result in different constrained equations of motion. In coming to this understanding we profited greatly from the thoughtful discussions in Marle [11] and Rosenberg [12], as well as many insightful comments from J. Śniatycki.

2. One may observe a superficial analogy between the constructions of various authors computing a nonholonomic bracket (van der Schaft and Maschke [14], Bates [1], de Leon [5], Koon and Marsden [9] etc) and this paper. However, there is a fundamental difference in that we are using the standard Poisson bracket and putting all of the nonholonomic information into the one-form that drives the dynamics.

3. In various examples one may of course consider what happens to the image of the various distributions H, K under reduction. This is of interest in the special case where the reduced distribution $\bar{K} = 0$, so the invariant part of the dynamics satisfies the distributional Hamiltonian equations. Our results

would indicate that this is not the most fundamental way to view these problems, and this is why we have stressed that the fundamental structure that enables reduction by symmetry is the action of the outer Poisson derivation on the invariant functions. Compare the discussion in Cantrijn et al [3].

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Gradient estimates for a nonlinear equation $\Delta_f u + cu^{-\alpha} = 0$ on complete noncompact manifolds

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Abstract. Let (M, g) be a complete noncompact Riemannian manifold. We consider gradient estimates on positive solutions to the following nonlinear equation

$$\Delta_f u + cu^{-\alpha} = 0 \quad \text{in } M,$$

where α, c are two real constants and $\alpha > 0$, f is a smooth real valued function on M and $\Delta_f = \Delta - \nabla f \nabla$. When N is finite and the N-Bakry-Emery Ricci tensor is bounded from below, we obtain a gradient estimate for positive solutions of the above equation. Moreover, under the assumption that ∞-Bakry-Emery Ricci tensor is bounded from below and $|\nabla f|$ is bounded from above, we also obtain a gradient estimate for positive solutions of the above equation. It extends the results of Yang [16].

1 Introduction

Let (M, g) be a complete noncompact n-dimensional Riemannian manifold. For a smooth real-valued function f on M, the drifting Laplacian (see [11], [12]) is defined by $\Delta_f = \Delta - \nabla f \nabla$. There is a naturally associated measure $d\mu = e^{-f} dV$ on M, which makes the operator Δ_f self-adjoint. The N-Bakry-Emery Ricci tensor is defined by

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \nabla^2 f - \frac{1}{N} df \otimes df$$

for $0 \le N \le \infty$ and $N = 0$ if and only if $f = 0$. Here ∇^2 is the Hessian and Ric is the Ricci tensor. In particular, the ∞-Bakry-Emery Ricci tensor is denoted by

$$\mathrm{Ric}_f := \mathrm{Ric}_f^\infty = \mathrm{Ric} + \nabla^2 f$$

with $\mathrm{Ric}_f = \lambda g$ is called a gradient Ricci soliton which is extensively studied in Ricci flow.

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The author in [16] obtained interesting gradient estimates for positive solutions to the following elliptic equation with singular nonlinearity

$$\Delta u + cu^{-\alpha} = 0 \quad \text{in } M, \tag{1}$$

where α, c are two real constants and $\alpha > 0$. For the importance of equation (1), the authors who are interested in it see [5], [8]. In this paper, we consider the following equation

$$\Delta_f u + cu^{-\alpha} = 0 \quad \text{in } M, \tag{2}$$

where f is a smooth real-valued function on M. For some interesting gradient estimates in this direction, for example, we refer to [2], [3], [6], [7], [9], [10], [15]. When N is finite and the N-Bakry-Emery Ricci tensor is bounded from below, we obtain a gradient estimate for positive solutions of the above equation. Moreover, under the assumption that ∞-Bakry-Emery Ricci tensor is bounded from below and $|\nabla f|$ is bounded from above, we also obtain a gradient estimate for positive solutions of the above equation. Main results of this paper are stated as follows:

Theorem 1. Let (M, g) be a complete noncompact n-dimensional Riemannian manifold with N-Bakry-Emery Ricci tensor bounded from below by the constant $-K := -K(2R)$, where $R > 0$ and $K(2R) \geq 0$, in the metric ball $B_p(2R)$ with radius $2R$ around $p \in M$. Let u be a positive solution of (2) with α, c two real constants and $\alpha > 0$. Then

(1) If $c > 0$, we have

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} + cu^{-\alpha-1} \leq \frac{(n+N)(n+N+2){c_1}^2}{R^2} + \frac{(n+N)[(n+N-1)c_1 + c_2]}{R^2} \\ + \frac{(n+N)\sqrt{(n+N)K}c_1}{R} + 2(n+N)K.\end{aligned} \tag{3}$$

(2) If $c < 0$, we have

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} + cu^{-\alpha-1} &\leq (A + \sqrt{A})|c|(\inf_{B_p(2R)} u)^{-\alpha-1} + \frac{(n+N)[(n+N-1)c_1 + c_2]}{R^2} \\ &+ \frac{(n+N)c_1^2}{R^2}\Big(n+N+2+\frac{n+N}{2\sqrt{A}}\Big) + \frac{(n+N)\sqrt{(n+N)K}c_1}{R} \\ &+ \Big(2 + \frac{1}{\sqrt{A}}\Big)(n+N)K,\end{aligned} \tag{4}$$

where $A = (n+N)(\alpha+1)(\alpha+2)$.

Theorem 2. Let (M, g) be a complete noncompact n-dimensional Riemannian manifold and $f \in C^2(M)$ be a function satisfying $|\nabla f| \leq \theta$. Assume that ∞-Bakry-Emery Ricci tensor bounded from below by the constant $-K := -K(2R)$, where $R > 0$ and $K(2R) \geq 0$, in the metric ball $B_p(2R)$ with radius $2R$ around $p \in M$. Let u be a positive solution of (2) with α, c two real constants and $\alpha > 0$. Then

(1) If $c > 0$, we have

$$\frac{|\nabla u|^2}{u^2} + cu^{-\alpha-1} \le \frac{n[(n+2)c_1^2 + (n-1)c_1 + c_2]}{R^2} + \frac{5nc_1\theta}{R} + 4\theta^2 + \frac{nc_1\sqrt{(n-1)K}}{R} + 2nK. \tag{5}$$

(2) If $c < 0$, we have

$$\frac{|\nabla u|^2}{u^2} + cu^{-\alpha-1} \le (B+\sqrt{B})|c|(\inf_{B_p(2R)} u)^{-\alpha-1} + \frac{n}{R^2}\Big((2+2n+\frac{n}{\sqrt{B}})c_1^2 + (n-1)c_1 + c_2\Big) + \frac{nc_1\theta}{R} + \Big(1+\frac{1}{2\sqrt{B}}\Big)8\theta^2 + \frac{nc_1\sqrt{(n-1)K}}{R} + \Big(2+\frac{1}{\sqrt{B}}\Big)nK, \tag{6}$$

where $B = n(\alpha+1)(\alpha+2)$.

From (1) in Theorem 1, we obtain the following result immediately:

Corollary 1. *Let (M, g) be a complete noncompact n-dimensional Riemannian manifold with nonnegative N-Bakry-Emery Ricci tensor. Assume that two real constants α, c in (2) are positive. Then the equation (2) does not have a positive smooth solution.*

2 Proof of Theorem 1

Let $h = \log u$. Then one has from (2) that

$$\Delta_f h = \frac{1}{u}\Delta_f u - |\nabla h|^2 = -cu^{-\alpha-1} - |\nabla h|^2.$$

Define $F = cu^{-\alpha-1} + |\nabla h|^2$, then we have $\Delta_f h = -F$. It is well known that for the N-Bakry-Emery Ricci tensor, we have the Bochner formula (see [14]):

$$\begin{aligned}\Delta_f|\nabla h|^2 &\ge \frac{2}{n+N}|\Delta_f h|^2 + 2\langle\nabla h, \nabla(\Delta_f h)\rangle - 2K|\nabla h|^2\\ &= \frac{2}{n+N}F^2 - 2\langle\nabla h, \nabla F\rangle - 2K|\nabla h|^2.\end{aligned}$$

Hence, one gets

$$\begin{aligned}\Delta_f F &= c\Delta_f u^{-\alpha-1} + \Delta_f|\nabla h|^2\\ &\ge c(\alpha+1)(\alpha+2)u^{-\alpha-1}|\nabla h|^2 - c(\alpha+1)u^{-\alpha-2}\Delta_f u\\ &\quad + \frac{2}{n+N}F^2 - 2\langle\nabla h, \nabla F\rangle - 2K|\nabla h|^2.\end{aligned} \tag{7}$$

Let ξ be a cut-off function such that $\xi(r) = 1$ for $r \le 1$, $\xi(r) = 0$ for $r \ge 2$, $0 \le \xi(r) \le 1$, and

$$0 \ge \xi^{-\frac{1}{2}}(r)\xi'(r) \ge -c_1$$
$$\xi''(r) \ge -c_2$$

for positive constants c_1 and c_2. Denote ϕ by $\rho(x) = d(x, \rho)$ the distance between x and p in M. Let

$$\phi(x) = \xi\left(\frac{\rho(x)}{R}\right).$$

Using an argument of Calabi [1] (see also Cheng and Yau [4]), we can assume without loss of generality that the function ϕ is smooth in $B_p(2R)$. Then, we have

$$\frac{|\nabla\phi|^2}{\phi} \leq \frac{c_1^2}{R^2}. \tag{8}$$

It has been shown by Qian[13] that

$$\Delta_f(\rho^2) \leq (n+N)\Big(1 + \sqrt{1 + \frac{4K\rho^2}{n+N}}\Big).$$

Hence, we have

$$\begin{aligned}\Delta_f\rho &= \frac{1}{2\rho}[\Delta_f(\rho^2) - 2|\nabla\rho|^2] \\ &\leq \frac{n+N-2}{2\rho} + \frac{n+N}{2\rho}\Big(1 + \sqrt{\frac{4K\rho^2}{n+N}}\Big) \\ &= \frac{n+N-1}{\rho} + \sqrt{(n+N)K}.\end{aligned}$$

It follows that

$$\begin{aligned}\Delta_f\phi &= \frac{\xi''(r)|\nabla\rho|^2}{R^2} + \frac{\xi'(r)\Delta_f\rho}{R} \\ &\geq -\frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2}.\end{aligned} \tag{9}$$

Define $G = \phi F$. We may assume that G achieves its maximal value Q at the point $x \in B_p(2R)$ and assume that Q is positive (otherwise the proof is trivial). Then at the point x,

$$0 = \nabla G = \phi\nabla F + F\nabla\phi$$

and $\Delta_f G \leq 0$. Therefore, at the point x, it holds that

$$\begin{aligned}0 \geq \Delta_f G &= \Delta G - \langle\nabla f, \nabla G\rangle \\ &= \phi\Delta_f F + F\Delta_f\phi + 2\langle\nabla\phi, \nabla F\rangle \\ &= \phi\Delta_f F + F\Delta_f\phi - 2F\frac{|\nabla\phi|^2}{\phi} \\ &\geq \frac{2}{n+N}\phi F^2 - 2\phi\langle\nabla h, \nabla F\rangle - 2K\phi|\nabla h|^2 \\ &\quad - \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2}F \\ &\quad - \frac{2c_1^2}{R^2}F + c(\alpha+1)(\alpha+2)u^{-\alpha-1}\phi|\nabla h|^2 - c(\alpha+1)u^{-\alpha-2}\phi\Delta_f u,\end{aligned}$$

which shows that

$$\begin{aligned} 0 \geq{}& \frac{2}{n+N} G^2 + 2G\langle \nabla h, \nabla \phi \rangle - 2K\phi^2 |\nabla h|^2 \\ &- \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2} G \\ &- \frac{2c_1^2}{R^2} G + c(\alpha+1)(\alpha+2)u^{-\alpha-1}\phi^2|\nabla h|^2 - c(\alpha+1)u^{-\alpha-2}\phi^2 \Delta_f u. \end{aligned} \tag{10}$$

Next, we consider the following two cases: (1) $c > 0$; (2) $c < 0$.

(1) When $c > 0$, then we have $F = |\nabla h|^2 + cu^{-\alpha-1} > 0$ and $|\nabla h| < F^{\frac{1}{2}}$. Since

$$\langle \nabla h, \nabla \phi \rangle \leq |\nabla h||\nabla \phi| \leq \frac{c_1}{R} F^{\frac{1}{2}} \phi^{\frac{1}{2}},$$

$$\frac{2c_1}{R} G^{\frac{3}{2}} \leq \frac{(n+N)c_1^2}{R^2} G + \frac{1}{n+N} G^2,$$

then (10) yields

$$\begin{aligned} 0 \geq{}& \frac{2}{n+N} G^2 - \frac{2c_1}{R} G^{\frac{3}{2}} - 2K\phi|\nabla h|^2 \\ &- \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2} G \\ &- \frac{2c_1^2}{R^2} G + c(\alpha+1)^2 u^{-\alpha-1}\phi^2|\nabla h|^2 + c(\alpha+1)u^{-\alpha-1}\phi^2 F \\ \geq{}& \frac{1}{n+N} G^2 - \frac{(n+N+2)c_1^2}{R^2} G - 2KG \\ &- \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2} G. \end{aligned} \tag{11}$$

From (11), we obtain

$$\begin{aligned} G \leq{}& \frac{(n+N)(n+N+2)c_1^2}{R^2} + \frac{(n+N)[(n+N-1)c_1 + c_2]}{R^2} \\ &+ \frac{(n+N)c_1}{R}\sqrt{(n+N)K} + 2(n+N)K \end{aligned}$$

and hence

$$\begin{aligned} \sup_{B_p(2R)} F \leq G \leq{}& \frac{(n+N)(n+N+2)c_1^2}{R^2} + \frac{(n+N)[(n+N-1)c_1 + c_2]}{R^2} \\ &+ \frac{(n+N)c_1}{R}\sqrt{(n+N)K} + 2(n+N)K. \end{aligned} \tag{12}$$

Now (1) of Theorem 1 follows easily from the inequality above.

(2) When $c < 0$, if $F \leq 0$, then the estimate in (2) of Theorem 1 is trivial. Hence we assume $F > 0$. Under the assumption that $F > 0$, one gets $|\nabla h| > F^{\frac{1}{2}}$. Since

$$2G\langle \nabla h, \nabla \phi \rangle \leq \frac{1}{n+N} G^2 + \frac{(n+N)c_1^2}{R^2} \phi |\nabla h|^2,$$

then (10) yields

$$
\begin{aligned}
0 \geq & \frac{1}{n+N}G^2 - \frac{(n+N)c_1^2}{R^2}\phi|\nabla h|^2 - 2K\phi^2|\nabla h|^2 \\
& - \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2}G \\
& - \frac{2c_1^2}{R^2}G + c(\alpha+1)(\alpha+2)(\inf_{B_p(2R)} u)^{-\alpha-1}\phi^2|\nabla h|^2 \\
& + c^2(\alpha+1)(\sup_{B_p(2R)} u)^{-2\alpha-2}\phi^2 \\
\geq & \frac{1}{n+N}G^2 - \frac{(n+N)c_1^2}{R^2}\phi F - \frac{(n+N)c_1^2}{R^2}\phi|c|(\inf_{B_p(2R)} u)^{-\alpha-1} \\
& - \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2}G \\
& - \frac{2c_1^2}{R^2}G - J(2R)\phi^2 F - L(2R)\phi^2,
\end{aligned}
$$

where

$$
\begin{aligned}
J(2R) &= 2K - c(\alpha+1)(\alpha+2)(\inf_{B_p(2R)} u)^{-\alpha-1}, \\
L(2R) &= |c|J(2R)(\inf_{B_p(2R)} u)^{-\alpha-1} - c^2(\alpha+1)(\sup_{B_p(2R)} u)^{-2\alpha-2}.
\end{aligned}
$$

This shows that

$$
\begin{aligned}
0 \geq & \frac{1}{n+N}G^2 \\
& - \Big(\frac{(n+N+2)c_1^2}{R^2} + \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2} + J(2R)\Big)G \\
& - \frac{(n+N)c_1^2}{R^2}|c|(\inf_{B_p(2R)} u)^{-\alpha-1} - L(2R).
\end{aligned}
$$

Hence

$$
G \leq \frac{b+\sqrt{b^2+4d}}{2} \leq b + \sqrt{d}, \tag{13}
$$

where

$$
\begin{aligned}
b = & (n+N)J(2R) + \frac{(n+N)[(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2]}{R^2} \\
& + \frac{(n+N)(n+N+2)c_1^2}{R^2}, \\
d = & (n+N)L(2R) + \frac{(n+N)^2 c_1^2}{R^2}|c|(\inf_{B_p(2R)} u)^{-\alpha-1}.
\end{aligned}
$$

Let $m = (\inf_{B_p(2R)} u)^{-\alpha-1}$, $M = (\sup_{B_p(2R)} u)^{-\alpha-1}$. We have

$$\begin{aligned}
\sqrt{d} &= \sqrt{(n+N)c^2(\alpha+1)[(\alpha+2)m^2 - M^2] + [\frac{(n+N)c_1^2}{R^2}|c| + 2(n+N)|c|K]m} \\
&\le \sqrt{(n+N)c^2(\alpha+1)(\alpha+2)m^2 + [\frac{(n+N)c_1^2}{R^2}|c| + 2(n+N)|c|K]m} \\
&\le \sqrt{(n+N)(\alpha+1)(\alpha+2)}|c|m + \frac{\frac{(n+N)c_1^2}{R^2} + 2(n+N)K}{2\sqrt{(n+N)(\alpha+1)(\alpha+2)}}.
\end{aligned}$$

It follows from (13) that

$$\begin{aligned}
G &\le 2(n+N)K + A|c|m + \frac{(n+N)[(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2]}{R^2} \\
&\quad + \frac{(n+N)(n+N+2)c_1^2}{R^2} + \sqrt{A}|c|m + \frac{\frac{(n+N)^2c_1^2}{R^2} + 2(n+N)K}{2\sqrt{A}} \\
&= (A+\sqrt{A})|c|m + \frac{(n+N)[(n+N-1)c_1 + c_2]}{R^2} \\
&\quad + \frac{(n+N)c_1^2}{R^2}\left(n+N+2+\frac{n+N}{2\sqrt{A}}\right) \\
&\quad + \frac{(n+N)\sqrt{(n+N)K}c_1}{R} + \left(2+\frac{1}{\sqrt{A}}\right)(n+N)K,
\end{aligned} \tag{14}$$

where

$$A = (n+N)(\alpha+1)(\alpha+2).$$

Therefore, we obtain (2) of Theorem 1. □

3 Proof of Theorem 2

Let $h = \log u$. Then we have

$$\Delta_f h = -cu^{-\alpha-1} - |\nabla h|^2.$$

Denote by $F = cu^{-\alpha-1} + |\nabla h|^2$, then we have $\Delta_f h = -F$. Applying the Bochner formula to h, we get (see [14]):

$$\Delta_f |\nabla h|^2 = 2|D^2h|^2 + 2\langle \nabla h, \nabla(\Delta_f h)\rangle + 2\mathrm{Ric}_f(\nabla h, \nabla h). \tag{15}$$

Since

$$\begin{aligned}
|D^2h|^2 &\ge \frac{1}{n}(\Delta h)^2 \\
&= \frac{1}{n}[F - \langle \nabla h, \nabla f\rangle]^2 \\
&\ge \frac{1}{n}F^2 - \frac{2}{n}F\langle \nabla h, \nabla f\rangle,
\end{aligned}$$

then we derive from (15)

$$\Delta_f |\nabla h|^2 \ge \frac{2}{n}F^2 - \frac{4}{n}F\langle \nabla h, \nabla f\rangle - 2\langle \nabla h, \nabla F\rangle - 2K|\nabla h|^2. \tag{16}$$

Thus we have

$$\begin{aligned}\Delta_f F &= c\Delta_f u^{-\alpha-1} + \Delta_f |\nabla h|^2 \\ &\geq c(\alpha+1)(\alpha+2)u^{-\alpha-1}|\nabla h|^2 - c(\alpha+1)u^{-\alpha-2}\Delta_f u \\ &\quad + \frac{2}{n}F^2 - \frac{4}{n}F\langle\nabla h, \nabla f\rangle - 2\langle\nabla h, \nabla F\rangle - 2K|\nabla h|^2.\end{aligned} \tag{17}$$

Let ξ be a cut-off function such that $\xi(r) = 1$ for $r \leq 1$, $\xi(r) = 0$ for $r \geq 2$, $0 \leq \xi(r) \leq 1$, and

$$0 \geq \xi^{-\frac{1}{2}}(r)\xi^{'}(r) \geq -c_1$$

$$\xi''(r) \geq -c_2$$

for positive constants c_1 and c_2. Denote ϕ by $\rho(x) = d(x, \rho)$ the distance between x and p in M. Let

$$\phi(x) = \xi\left(\frac{\rho(x)}{R}\right).$$

Using an argument of Calabi [1] (see also Cheng and Yau [4]), we can assume without loss of generality that the function ϕ is smooth in $B_{2R}(p)$. Then, we have

$$\frac{|\nabla\phi|^2}{\phi} \leq \frac{c_1^2}{R^2}. \tag{18}$$

Since $\mathrm{Ric}_f \geq -K$ and $|\nabla f| \leq \theta$, we have from the Theorem 1.1 in [14]:

$$\begin{aligned}\Delta_f \rho &\leq \sqrt{(n-1)K}\coth\Big(\sqrt{\frac{K}{n-1}}\rho\Big) + \theta \\ &\leq (n-1)\Big(\frac{1}{\rho} + \sqrt{\frac{K}{n-1}}\Big) + \theta.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\Delta_f \phi &= \frac{\xi''(r)|\nabla\rho|^2}{R^2} + \frac{\xi'(r)\Delta_f\rho}{R} \\ &\geq -\frac{(n-1+\sqrt{(n-1)K}R + \theta R)c_1 + c_2}{R^2}.\end{aligned} \tag{19}$$

Define $G = \phi F$. We assume that G achieves its maximal value Q at the point $x \in B_p(2R)$ and assume that Q is positive (otherwise the proof is trivial). Then at the point x,

$$0 = \nabla G = \phi\nabla F + F\nabla\phi$$

and $\Delta_f G \leq 0$. This shows that

$$\nabla F = -\frac{F}{\phi}\nabla\phi.$$

Therefore, at the point x, it holds that

$$\begin{aligned}
0 \geq \Delta_f G &= \phi\Delta_f F + F\Delta_f\phi + 2\langle\nabla\phi, \nabla F\rangle \\
&= \phi\Delta_f F + F\Delta_f\phi - 2F\frac{|\nabla\phi|^2}{\phi} \\
&\geq \frac{2}{n}\phi F^2 - \frac{4}{n}\phi F\langle\nabla h, \nabla f\rangle - 2\phi\langle\nabla h, \nabla F\rangle - 2K\phi|\nabla h|^2 \\
&\quad - \frac{(n-1+\sqrt{(n-1)K}R + \theta R)c_1 + c_2}{R^2}F - \frac{2c_1^2}{R^2}F \\
&\quad + c(\alpha+1)(\alpha+2)u^{-\alpha-1}\phi|\nabla h|^2 - c(\alpha+1)u^{-\alpha-2}\phi\Delta_f u,
\end{aligned}$$

which means that

$$\begin{aligned}
0 \geq{} & \frac{2}{n}G^2 - \frac{4}{n}\phi G\langle\nabla h, \nabla f\rangle + 2G\langle\nabla h, \nabla\phi\rangle - 2K\phi^2|\nabla h|^2 \\
& - \frac{2c_1^2 + (n-1)c_1 + c_2}{R^2}G - \frac{(\sqrt{(n-1)K} + \theta)c_1}{R}G \\
& + c(\alpha+1)(\alpha+2)u^{-\alpha-1}\phi^2|\nabla h|^2 - c(\alpha+1)u^{-\alpha-2}\phi^2\Delta_f u.
\end{aligned} \tag{20}$$

Next, we consider two cases: (1) $c > 0$; (2)$c < 0$.

(1) When $c > 0$, we have $F = |\nabla h|^2 + cu^{-\alpha-1} > 0$ and $|\nabla h| < F^{\frac{1}{2}}$. Since

$$|\langle\nabla h, \nabla\phi\rangle| \leq |\nabla h||\nabla\phi| \leq \frac{c_1}{R}F^{\frac{1}{2}}\phi^{\frac{1}{2}},$$

$$|\langle\nabla h, \nabla f\rangle| \leq |\nabla h||\nabla f| \leq F^{\frac{1}{2}}|\nabla f|,$$

then from (20) we obtain

$$\begin{aligned}
0 \geq{} & \frac{2}{n}G^2 - \frac{4}{n}|\nabla f|G^{\frac{3}{2}} - \frac{2c_1}{R}G^{\frac{3}{2}} - 2K\phi|\nabla h|^2 - \frac{2c_1^2 + (n-1)c_1 + c_2}{R^2}G \\
& - \frac{(\sqrt{(n-1)K} + \theta)c_1}{R}G + c(\alpha+1)^2u^{-\alpha-1}\phi^2|\nabla h|^2 \\
& + c(\alpha+1)u^{-\alpha-1}\phi^2 F \\
\geq{} & \frac{2}{n}G^2 - \frac{4}{n}|\nabla f|G^{\frac{3}{2}} - \frac{2c_1}{R}G^{\frac{3}{2}} - 2KG - \frac{2c_1^2 + (n-1)c_1 + c_2}{R^2}G \\
& - \frac{(\sqrt{(n-1)K} + \theta)c_1}{R}G.
\end{aligned} \tag{21}$$

Using the Schwarz inequality, one has

$$\begin{aligned}
\Big(\frac{4}{n}|\nabla f| + \frac{2c_1}{R}\Big)G^{\frac{3}{2}} &\leq n\Big(\frac{2}{n}|\nabla f| + \frac{c_1}{R}\Big)^2 G + \frac{1}{n}G^2 \\
&= \Big(\frac{4}{n}|\nabla f|^2 + \frac{4c_1}{R}|\nabla f| + \frac{nc_1^2}{R^2}\Big)G + \frac{1}{n}G^2.
\end{aligned} \tag{22}$$

Inserting (22) into (21) yields

$$\begin{aligned}
0 \geq{} & \frac{1}{n}G^2 - \Big(\frac{4}{n}|\nabla f|^2 + \frac{4c_1}{R}|\nabla f|\Big)G - 2KG \\
& - \frac{(n+2)c_1^2 + (n-1)c_1 + c_2}{R^2}G - \frac{(\sqrt{(n-1)K} + \theta)c_1}{R}G.
\end{aligned}$$

Hence

$$G \le \frac{n[(n+2)c_1^2+(n-1)c_1+c_2]}{R^2}+\frac{5nc_1\theta}{R}+4\theta^2+\frac{nc_1\sqrt{(n-1)K}}{R}+2nK, \quad (23)$$

and

$$\sup_{B_p(2R)} F \le G \le \frac{n[(n+2)c_1^2+(n-1)c_1+c_2]}{R^2} + \frac{5nc_1\theta}{R}+4\theta^2+\frac{nc_1\sqrt{(n-1)K}}{R}+2nK.$$

We complete the proof of (1) in Theorem 2.

(2) When $c<0$, if $F\le 0$, then the estimate in (2) of Theorem 2 is trivial. Hence we assume $F>0$ and hence $|\nabla h|>F^{\frac{1}{2}}$. Noticing

$$2G\langle\nabla h,\nabla\phi\rangle \le 2\frac{c_1}{R}G\phi^{\frac{1}{2}}|\nabla h| \le \frac{1}{2n}G^2+\frac{2nc_1^2}{R^2}\phi|\nabla h|^2,$$
$$\frac{4}{n}\phi G\langle\nabla h,\nabla f\rangle \le \frac{4}{n}\phi G|\nabla h||\nabla f| \le \frac{1}{2n}G^2+\frac{8}{n}|\nabla f|^2\phi^2|\nabla h|^2,$$

we have from (20)

$$\begin{aligned}
0 \ge{}& \frac{1}{n}G^2-\frac{8}{n}|\nabla f|^2\phi^2|\nabla h|^2-\frac{2nc_1^2}{R^2}\phi|\nabla h|^2-2K\phi^2|\nabla h|^2-\frac{2c_1^2+(n-1)c_1+c_2}{R^2}G\\
&-\frac{(\sqrt{(n-1)K}+\theta)c_1}{R}G+c(\alpha+1)(\alpha+2)(\inf_{B_p(2R)} u)^{-\alpha-1}\phi^2|\nabla h|^2\\
&+c^2(\alpha+1)(\sup_{B_p(2R)} u)^{-2\alpha-2}\phi^2\\
\ge{}& \frac{1}{n}G^2-\Big(\frac{8}{n}|\nabla f|^2+\frac{2nc_1^2}{R^2}\Big)\phi F-\Big(\frac{8}{n}|\nabla f|^2+\frac{2nc_1^2}{R^2}\Big)\phi|c|(\inf_{B_p(2R)} u)^{-\alpha-1}\\
&-\frac{2c_1^2+(n-1)c_1+c_2}{R^2}G-\frac{(\sqrt{(n-1)K}+\theta)c_1}{R}G-J(2R)\phi^2F-L(2R)\phi^2,
\end{aligned}$$

where

$$J(2R)=2K-c(\alpha+1)(\alpha+2)(\inf_{B_p(2R)} u)^{-\alpha-1},$$
$$L(2R)=|c|J(2R)(\inf_{B_p(2R)} u)^{-\alpha-1}-c^2(\alpha+1)(\sup_{B_p(2R)} u)^{-2\alpha-2}.$$

This shows that

$$\begin{aligned}
0 \ge{}& \frac{1}{n}G^2\\
&-\Big(\frac{8}{n}|\nabla f|^2+\frac{(2n+2)c_1^2+(n-1)c_1+c_2}{R^2}+\frac{(\sqrt{(n-1)K}+\theta)c_1}{R}+J(2R)\Big)G\\
&-\Big(\frac{8}{n}|\nabla f|^2+\frac{2nc_1^2}{R^2}\Big)|c|(\inf_{B_p(2R)} u)^{-\alpha-1}-L(2R).
\end{aligned}$$

Hence one has

$$G \le \frac{b + \sqrt{b^2 + 4d}}{2} \le b + \sqrt{d}, \tag{24}$$

where

$$b = nJ(2R) + 8|\nabla f|^2 + \frac{n[(2n+2)c_1^2 + (n-1)c_1 + c_2]}{R^2} + \frac{nc_1(\sqrt{(n-1)K} + \theta)}{R},$$

$$d = nL(2R) + \Big(8|\nabla f|^2 + \frac{2n^2c_1^2}{R^2}\Big)|c|(\inf_{B_p(2R)} u)^{-\alpha-1}.$$

Let $m = (\inf_{B_p(2R)} u)^{-\alpha-1}$, $M = (\sup_{B_p(2R)} u)^{-\alpha-1}$. We have

$$\begin{aligned}
\sqrt{d} &= \sqrt{nc^2(\alpha+1)[(\alpha+2)m^2 - M^2] + (2nK + 8|\nabla f|^2 + \frac{2n^2c_1^2}{R^2})|c|m} \\
&\le \sqrt{nc^2(\alpha+1)(\alpha+2)m^2 + (2nK + 8|\nabla f|^2 + \frac{2n^2c_1^2}{R^2})|c|m} \\
&\le \sqrt{n(\alpha+1)(\alpha+2)}|c|m + \frac{nK + 4|\nabla f|^2 + \frac{n^2c_1^2}{R^2}}{\sqrt{n(\alpha+1)(\alpha+2)}}.
\end{aligned}$$

It follows from (24) and $|\nabla f| \le \theta$ that

$$\begin{aligned}
G &\le 2nK + B|c|m + 8\theta^2 + \frac{n[(2n+2)c_1^2 + (n-1)c_1 + c_2]}{R^2} \\
&\quad + \frac{nc_1(\sqrt{(n-1)K} + \theta)}{R} + \sqrt{B}|c|m + \frac{nK + 4\theta^2 + \frac{n^2c_1^2}{R^2}}{\sqrt{B}} \\
&= (B + \sqrt{B})|c|m + \frac{n}{R^2}\Big((2 + 2n + \frac{n}{\sqrt{B}})c_1^2 + (n-1)c_1 + c_2\Big) + \frac{nc_1\theta}{R} \\
&\quad + \Big(1 + \frac{1}{2\sqrt{B}}\Big)8\theta^2 + \frac{nc_1\sqrt{(n-1)K}}{R} + \Big(2 + \frac{1}{\sqrt{B}}\Big)nK,
\end{aligned}$$

where

$$B = n(\alpha+1)(\alpha+2).$$

The proof of (2) in Theorem 2 is completed finally. □

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Book Review

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Geometry of Nonholonomically Constrained Systems by R. Cushman, H. Duistermaat and J. Śniatycki. Advanced Series in Nonlinear Dynamics, Vol 26, World Scientific, 2010.

The admissible configurations and velocities of point masses or rigid bodies in a mechanical system are often observed to be restricted. In many cases these limitations can be handled by introducing constraint equations into the framework. One usually distinguishes between two types of such constraints. Holonomic constraints are restrictions on the position of the system only. A constraint is said to be nonholonomic if the restriction depends also on the velocities of the system, and if by no means it can be integrated to a holonomic constraint. Typical engineering problems that involve nonholonomic constraints arise for example in robotics, where the wheels of a mobile robot are often required to roll without slipping, or where one is interested in guiding the motion of a cutting tool or a skate. Some of the well-studied textbook examples of nonholonomic systems include the rolling disk, the rattleback, the rolling ball in a cylindrical tube, the problem of pursuit and the snakeboard. A classical reference for nonholonomic systems is the book by Neĭmark and Fufaev [3].

Since the second half of last century, tools and techniques from differential geometry (Riemann geometry, contact geometry, symplectic and Poisson geometry, Lie groups, fibre bundles, jet bundles, connections, distributions, etc.) have had an ever growing impact on the analysis of problems in mechanics. The discipline that emerged from the contact between geometry and mechanics is now commonly called 'Geometric Mechanics'. Two fairly recent books that deal specifically with geometric approaches to nonholonomic mechanical systems are e.g. the monographs by Bloch [1] and Cortés [2], and the book under review can be thought off as an new addition to this category. The book can be divided in two parts: while the first four chapters contain the theoretical material, the last three chapters concentrate on three specific examples.

Chapter 1 introduces the category of nonholonomically constrained mechanical systems. As in the bigger part of the literature, only non-holonomic constraints that

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are time-independent and that depend linearly on velocities are considered. The advantage is that they can geometrically be represented in the form of a distribution D on the configuration space Q. The authors only consider Lagrangians of so-called 'mechanical type', that is of the form $k - V$, where k is a kinetic energy function associated to a Riemannian metric on Q, and V is a potential function on Q. The authors work in what they call the 'distributional Hamiltonian formalism'. One should not think here of the adjective 'Hamiltonian' as meaning 'set in (a part of) phase space', but one should rather interpret it as 'set in a symplectic framework'. That is to say, one can pullback, by means of the Legendre transformation, the canonical symplectic two-form on T^*Q to a symplectic two-form on TQ, the so-called Poincaré-Cartan two-form. It can then be shown that the distribution D gives rise to a certain distribution H on the submanifold of TQ determined by D, and that the restriction ϖ of the Poincaré-Cartan form to H is non-degenerate. The object ϖ can therefore be thought off as playing the role of a symplectic form for nonholonomic systems. Analogously, if one assigns to the energy function $h = k+V$ on TQ the role of Hamiltonian, the symplectic-type equation $Y_h \lrcorner\, \varpi = \partial_H h$ defines a unique vector field Y_h on D whose base integral curves are solutions of the nonholonomic dynamics. After a few basic properties of Y_h, the rest of the first chapter deals mainly with the Dirac bracket and the almost Poisson structure that one can define in this context, with the projection principle (which states that the nonholonomic dynamical vector field Y_h is a certain projection of the free dynamical vector field) and with a nonholonomic version of Noether's Theorem (on the relation between infinitesimal symmetries and constants of motion).

A large part of the book deals with the theory of Lie group symmetry reduction for nonholonomic systems. The benefits of exploiting symmetry are self-evident: if a dynamical system exhibits a symmetry, one may hope to reduce the system to one with fewer variables, possibly easier to solve. Throughout Chapters 2, 3 and 4 a special emphasis is put on the singular case, where the action is not necessarily free and proper, and this is precisely what sets this part of the book apart from the existing literature. In Chapter 2 the basic concepts related to Lie group actions on manifolds are reviewed. A free and proper action defines a principal fibre bundle structure on the orbit space, which in particular becomes a smooth manifold. The authors show that even in the singular case it is possible to define a kind of differential structure on the orbit space, by introducing the concept of a differential space. It is clear that when one works with spaces that do not necessarily possess a smooth manifold structure, one needs to rethink a lot of concepts, such as e.g. the definition of a tangent space and a vector field. Chapter 2 therefore mainly deals with re-inventing, in the more general set-up of differential spaces, familiar concepts known for manifolds. Chapter 3 contains the actual descriptions of the reduced distributional Hamiltonian systems, both for singular and regular Lie group symmetry reduction. Special attention is given to the subclass of Chaplygin nonholonomic systems (where the constraint distribution is the horizontal space of a principal connection, and where the reduced equations are of pure second-order type). Further, given that for nonholonomic systems the usual interplay between symmetries and conserved momenta is no longer valid, the authors situate the so-called momentum equations within the context of the reduced equations. Chapter 4 is about reconstruction equations (whose solutions enable one to reconstruct a complete solution from a solution of the reduced equations), about relative equilibria (i.e. equilibria of the reduced equations) and about relative periodic orbits. Again, we find in this chapter a careful analysis of the situation when the action is not necessarily free.

The following citation, taken from the introduction of Chapter 5, fully captures the spirit of the second part of the book: "In this chapter we will discuss the classical nonholonomically constrained system known as Carathéodory's sleigh. In order to illustrate the theory given in chapters 1,2 and 4, we will derive the equations of motion in five different ways, construct the reduced system in three different ways, and carry out reconstruction explicitly.". Indeed, the following three chapters contain a very comprehensive analysis of three famous examples of nonholonomic systems: Carathéodory's sleigh (which is a planar rigid body with a sharp edge in a vertical plane that makes contact with a horizontal plane in its lowest point), the example of a smooth, strongly convex rigid body rolling without slipping on a horizontal plane (under the influence of a constant vertical gravitational force), and the example of the rolling disk (which is not necessarily confined to roll vertically). These chapters do not only contain explicit formulae of reduced equations in many different forms and fashions, but, in particular in the chapter on the rolling disk, also contain a lot of information on the qualitative behaviour of particular solutions to the problems, such as e.g. a stability analysis for the relative equilibria and a study of the limiting behaviour of the disk when it nearly falls flat and then rises up again.

I believe that the book under review will become a standard reference work for people working in the field of geometric mechanics. I enjoyed the clear writing style of the main body of the text and I also appreciated the background information in the 'Notes' section at the end of each chapter. More importantly, in my opinion the book contains a lot of interesting research paths which makes it distinct from other books on this topic. One of them is the analysis of singular actions in the theoretical part, which is barely touched upon in e.g. [1] and [2]. An other major feat is that concrete simple examples are stripped down naked in an instructive manner and that they are shown to be the source of a very rich variety of interesting geometric problems. The new methods and techniques developed in the last three chapters for the specific examples may even become inspiration for future research on nonholonomic systems in general. A bit disappointingly, however, is that, as far as I could check, none of the Lie group actions involved in the last chapters were actually not-free, and that the singular tools from the first chapters were left unused in the last chapters. To my mind, this is a bit a missed opportunity.

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