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Newton transformations on null hypersurfaces

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Abstract. Any rigged null hypersurface is provided with two shape operators: with respect to the rigging and the rigged vector fields respectively. The present paper deals with the Newton transformations built on both of them and establishes related curvature properties. The laters are used to derive necessary and sufficient conditions for higher-order umbilicity and maximality we introduced in passing, and develop general Minkowski-type formulas for the null hypersurface, supported by some physical models in perfect-fluid space-times.

1 Introduction

It is a well-known fact that null hypersurfaces are exclusive objects of pseudo--Riemannian manifolds in the sense that they have no Riemannian counterpart and hence are interesting on their own from a (differential) geometric point of view. They also play an important role in general relativity namely in the study of black hole horizons (regions of space-time which contains a huge amount of mass compacted into an extremely small volume). From a more technical aspect, they are hypersurfaces having (induced) metrics with (pointwise) vanishing determinants and this degeneracy leads to several difficulties. In pseudo-Riemannian case, due to the causal character of three categories of vector fields (namely, spacelike, timelike and null), the induced metric on a hypersurface is a non-degenerate metric tensor field or a degenerate symmetric tensor field depending on whether the normal vector field is of the first two types or the third one. On non-degenerate hypersurfaces one can consider all the fundamental intrinsic and extrinsic geometric notions. In particular, a well defined (up to sign) notion of the unit orthogonal vector field is known to lead to a canonical splitting of the ambient tangent space into two factors: a tangent and an orthogonal one. Therefore, by respective projections, one has fundamental equations such as the Gauss, the Codazzi, the Weingarten equations, ... along with the second fundamental form, shape operator, induced connection, etc.

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The null hypersurface case is precisely when the normal vector field is null (also called lightlike) and since (contrary to the non-degenerate counterpart) the normal vector bundle intersects (non trivially) with the tangent bundle, one cannot find natural projector (and hence there is no preferred induced connection such as Levi-Civita) to define induced geometric objects as usual. This degeneracy of the induced metric makes it impossible to study them as part of standard submanifold theory, forcing to develop specific techniques and tools. For the most part, these tools are specific to a given problem, or sometimes with auxiliary non-canonical choices on which, unfortunately, depends the constructed null geometry. Indeed, Duggal and Bejancu in [12] introduced a non-degenerate screen distribution (or equivalently a null transversal line vector bundle as we may see below) so as to get a three factors splitting of the ambient tangent space and derive the main induced geometric objects such as second fundamental forms, shape operators, induced connections, curvature, etc. Unfortunately, the screen distribution is not unique and there is no preferred one in general, unless some specific geometric conditions are formulated to select and ensure uniqueness in exceptional cases [6], [5], [8], [7]. From above mentioned difficulties and compared to extensive research on global Riemannian and Lorentzian geometries we find out that considerable works are needed in null geometry to fill the gap.

One of the most important and central tools which have been extremely useful in addressing issues on higher-order r-th mean curvature and related topics in Riemannian geometry are Newton transformations [1], [2], [3], [4], [10], [15]. Since any null hypersurface with a fixed rigging do carry two shape operators: with respect to the rigging and the rigged vector fields respectively, we reasonably expect a role of those transformations in the study of null hypersurfaces. Recently in [11], the authors used above transformations of first type (thus, by duality considering the screen structure but not the null hypersurface structure) and examine conditions under which compact null hypersurfaces are totally umbilical in Robertson-Walker (RW) space-times. In the present paper we consider Newton transformations built on both of the two shape operators and establish related curvature properties and derive necessary and sufficient conditions for higher-order umbilicity and maximality, along with general Minkowski-type formulas for null hypersurfaces. The paper is organized as follows. Section 2 sets notations and definitions on riggings (normalizations) and review basic properties on null hypersurfaces, followed by some technical lemmas. Section 3 starts with introducing Newton transformations with respect to the rigged vector field and establishes their basic properties and some characterization results. The behaviour with respect to change in rigging is then examined on these transformations and the section ends with establishing some Minkowski-type integral formulas. In Section 4 we present some physical models in perfect-fluid space-times. The last section is concerned with the Newton transformations with respect to the rigging vector field.

2 Preliminaries

Let $(\bar{M}, \bar{g})$ be an $(n+2)$-dimensional Lorentzian manifold and M a null hypersurface in M. This means that at each $p \in M$, the restriction $\bar{g}_{p|T_pM}$ is degenerate, that is there exists a non-zero vector $U \in T_pM$ such that $\bar{g}(U, X) = 0$ for all

$X \in T_pM$. Hence, in null setting, the normal bundle $TM^{\perp}$ of the null hypersurface M^{n+1} is a rank 1 vector subbundle of the tangent bundle TM, contrary to the classical theory of non-degenerate hypersurfaces for which the normal bundle has trivial intersection $\{0\}$ with the tangent one and plays an important role in the introduction of the main induced geometric objects on M. Let us start with the usual tools involved in the study of such hypersurfaces according to [12]. They consist in fixing on the null hypersurface a geometric data formed by a lightlike section and a screen distribution. By *screen distribution* on M^{n+1}, we mean a complementary bundle of $TM^{\perp}$ in TM. It is then a rank n non-degenerate distribution over M. In fact, there are infinitely many possibilities of choices for such a distribution provided the hypersurface M be paracompact, but each of them is canonically isomorphic to the factor vector bundle $TM/TM^{\perp}$. For reasons that will become obvious in few lines below, let denote such a distribution by $\mathscr{S}(N)$. We then have

$$TM = \mathscr{S}(N) \oplus_{\text{Orth}} TM^{\perp}, \tag{1}$$

where $\oplus_{\text{Orth}}$ denotes the orthogonal direct sum. From [12], it is known that for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $\text{tr}(TM)$ of $T\bar{M}$ over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighbourhood $\mathscr{U} \subset M$, there exists a unique section N of $\text{tr}(TM)$ on $\mathscr{U}$ satisfying

$$\bar{g}(N,\xi) = 1, \quad \bar{g}(N,N) = \bar{g}(N,W) = 0, \quad \forall W \in \mathscr{S}(N)|_{\mathscr{U}}. \tag{2}$$

Then $T\bar{M}$ is decomposed as follows:

$$T\bar{M}|_M = TM \oplus \text{tr}(TM) = \{TM^{\perp} \oplus \text{tr}(TM)\} \oplus_{\text{Orth}} \mathscr{S}(N). \tag{3}$$

We call $\text{tr}(TM)$ a *(null) transversal vector bundle* along M. In fact, from (2) and (3) one shows that, conversely, a choice of a transversal bundle $\text{tr}(TM)$ determines uniquely the screen distribution $\mathscr{S}(N)$. A vector field N as in (2) is called a *null transversal vector field* of M. It is then noteworthy that the choice of a null transversal vector field N along M determines both the null transversal vector bundle, the screen distribution $\mathscr{S}(N)$ and a unique radical vector field, say ξ, satisfying (2). Tangent vector fields to $\mathscr{S}(N)$ (resp. to $TM^{\perp}$) are called horizontal (resp. vertical). Now, to continue our discussion, we need to clarify the concept of rigging for our null hypersurface.

Definition 1. Let M be a null hypersurface of a Lorentzian manifold. A *rigging* for M is a vector field L defined on some open set containing M such that $L_p \notin T_pM$ for each $p \in M$.

An outstanding property of a rigging is that it allows definition of geometric objects globally on M. We say that we have a *null rigging* in case the restriction of L to the null hypersurface is a null vector field. From now on we fix a null rigging N for M. In particular this rigging fixes a unique null vector field $\xi \in \Gamma(TM^{\perp})$ called the *rigged* vector field, all of them defined in an open set containing M (hence globally on M) such that (1), (2) and (3) hold. Whence, from now on, by a *normalized (or rigged) null hypersurface* we mean a triplet (M, g, N) where $g = \bar{g}_{|M}$ is the induced metric on M and N a null rigging for M. In fact, in case the ambient manifold $\bar{M}$ has Lorentzian signature, at an arbitrary point p in M, a real null cone C_p is invariantly defined in the (ambient) tangent space $T_p\bar{M}$ and is tangent to M along a generator emanating from p. This generator is exactly the radical fibre $\Delta_p = T_pM^{\perp}$ and for each null rigging N for M and each $p \in M$ we have $N_p \in C_p \setminus \Delta_p$. Actually, a lightlike hypersurface M of a Lorentzian manifold is a hypersurface which is tangent to the lightlike cone C_p at each point $p \in M$. Recall that a space-time $(\bar{M}, \bar{g})$ is a connected Lorentzian manifold which is "time-oriented", i.e. a causal cone at each $T_p\bar{M}$, $p \in \bar{M}$ (the "future" causal cone) has been continuously chosen. Hence, null hypersurfaces in space-times can be naturally given an orientation by such a continuous districution of causal cones C_p.

Let N be a null rigging of a null hypersurface of a Lorentzian manifold $(\bar{M}, \bar{g})$ and $\theta = \bar{g}(N, \cdot)$ the 1-form metrically equivalent to N defined on $\bar{M}$. Then, take

$$\eta = i^{\star}\theta$$

to be its restriction to M, the map $i\colon M \hookrightarrow \bar{M}$ being the inclusion map. The normalization (M, g, N) will be said to be *closed* if the 1-form η is closed on M. It is easy to check that $\mathscr{S}(N) = \ker(\eta)$ and that the screen distribution $\mathscr{S}(N)$ is integrable whenever η is closed. On a normalized null hypersurface (M, g, N), the Gauss and Weingarten formulas are given by

$$\begin{aligned}
\bar{\nabla}_X Y &= \nabla_X Y + B^N(X, Y)N, \\
\bar{\nabla}_X N &= -A_N X + \tau^N(X)N, \\
\nabla_X PY &= \overset{\star}{\nabla}_X PY + C^N(X, PY)\xi, \\
\nabla_X \xi &= -\overset{\star}{A}_\xi X - \tau^N(X)\xi,
\end{aligned}$$

for any $X, Y \in \Gamma(TM)$, where $\bar{\nabla}$ denotes the Levi-Civita connection on $(\bar{M}, \bar{g})$, ∇ denotes the connection on M induced from $\bar{\nabla}$ through the projection along the rigging N and $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $\mathscr{S}(N)$ induced from ∇ through the projection morphism P of $\Gamma(TM)$ onto $\Gamma\big(\mathscr{S}(N)\big)$ with respect to the decomposition (1). Now the $(0, 2)$ tensors B^N and C^N are the second fundamental forms on TM and $\mathscr{S}(N)$ respectively, A_N and $\overset{\star}{A}_\xi$ are the shape operators on TM and $\mathscr{S}(N)$ respectively and τ^N a 1-form on TM defined by

$$\tau^N(X) = \bar{g}(\bar{\nabla}_X N, \xi).$$

For the second fundamental forms B^N and C^N the following holds

$$B^N(X, Y) = g(\overset{\star}{A}_\xi X, Y), \quad C^N(X, PY) = g(A_N X, Y) \quad \forall X, Y \in \Gamma(TM), \tag{4}$$

and

$$B^N(X, \xi) = 0, \quad \overset{\star}{A}_\xi \xi = 0. \tag{5}$$

It follows from (5) that integral curves of ξ are pregeodesics in both $\bar{M}$ and M, as $\bar{\nabla}_\xi \xi = \nabla_\xi \xi = -\tau^N(\xi)\xi$. Throughout the paper, and without explicit mention, we consider these integral curves to be geodesics which means that

$$\tau^N(\xi) = 0.$$

A null hypersurface M is said to be *totally umbilical* (resp. *totally geodesic*) if there exists a smooth function ρ on M such that at each $p \in M$ and for all $u, v \in T_pM$, $B^N(p)(u,v) = \rho(p)g(u,v)$ (resp. B^N vanishes identically on M). These are intrinsic notions on any null hypersurface in the following way. Note that N being a null rigging for M, a vector field $\widetilde{N} \in \Gamma(T\bar{M})$ is a null rigging for M if and only if it is defined in an open set containing M and there exist a function ψ on $\bar{M}$ and a section ζ of TM such that $\widetilde{N} \circ i = (\psi N) \circ i + \zeta$ with the properties that $\phi = \psi \circ i$ is nowhere vanishing, being i the inclusion map, and $2\phi\eta(\zeta) + \|\zeta\|^2 = 0$ along M. Then we have (see [7] for details on changes in normalizations) $B^{\widetilde{N}} = \frac{1}{\psi \circ i} B^N$ which shows that total umbilicity and total geodesibility are intrinsic properties for M. The total umbilicity and the total geodesibility conditions for M can also be written respectively as $\overset{\star}{A}_\xi = \rho P$ and $\overset{\star}{A}_\xi = 0$. Also, the screen distribution $\mathscr{S}(N)$ is *totally umbilical* (resp. *totally geodesic*) if $C^N(X, PY) = \lambda g(X,Y)$ for all $X, Y \in \Gamma(TM)$ (resp. $C^N = 0$), which is equivalent to $A_N = \lambda P$ (resp. $A_N = 0$). It is noteworthy to mention that the shape operators $\overset{\star}{A}_\xi$ and A_N are $\mathscr{S}(N)$-valued.

The induced connection ∇ is torsion-free, but not necessarily g-metric unless M is totally geodesic. In fact we have for all tangent vector fields X, Y and Z in TM,

$$(\nabla_X g)(Y,Z) = B^N(X,Y)\eta(Z) + B^N(X,Z)\eta(Y). \tag{6}$$

Denote by $\bar{R}$ and R the Riemann curvature tensors of $\bar{\nabla}$ and ∇, respectively. Then the following are the Gauss-Codazzi equations [12, p. 93].

$$\begin{aligned}
\langle \bar{R}(X,Y)Z, \xi \rangle &= (\nabla_X B^N)(Y,Z) - (\nabla_Y B^N)(X,Z) \\
&\quad + \tau^N(X) B^N(Y,Z) - \tau^N(Y) B^N(X,Z), \qquad (7)\\
\langle \bar{R}(X,Y)Z, PW \rangle &= \langle R(X,Y)Z, PW \rangle + B^N(X,Z) C^N(Y, PW) \\
&\quad - B^N(Y,Z) C^N(X, PW), \\
\langle \bar{R}(X,Y)\xi, N \rangle &= \langle R(X,Y)\xi, N \rangle = C^N(Y, \overset{\star}{A}_\xi X) - C^N(X, \overset{\star}{A}_\xi Y) \\
&\quad - 2\mathrm{d}\tau^N(X,Y), \\
\langle \bar{R}(X,Y)PZ, N \rangle &= \langle (\nabla_X A_N)Y, PZ \rangle - \langle (\nabla_Y A_N)X, PZ \rangle \\
&\quad + \tau^N(Y) \langle A_N X, PZ \rangle - \tau^N(X) \langle A_N Y, PZ \rangle \qquad (8)
\end{aligned}$$

for all $X, Y, Z, W \in \Gamma(TM|_{\mathscr{U}})$. The (shape) operator $\overset{\star}{A}_\xi$ is self-adjoint as the second fundamental form B^N is symmetric. However, this is not the case for the operator A_N as shown in the following lemma.

Lemma 1. *For all $X, Y \in \Gamma(TM)$,*

$$\langle A_N X, Y \rangle - \langle A_N Y, X \rangle = \tau^N(X)\eta(Y) - \tau^N(Y)\eta(X) - 2\mathrm{d}\eta(X,Y),$$

where (throughout) $\langle \cdot, \cdot \rangle = \bar{g}$ stands for the Lorentzian metric.

Proof. Recall that $\eta = i^\star\theta$ where $\theta = \langle N, \cdot\rangle$. Taking the differential of θ and using the Weingarten formula, we have for all $X, Y \in \Gamma(TM)$,

$$\begin{aligned} 2\mathrm{d}\eta(X,Y) = 2\mathrm{d}\theta(X,Y) &= \big\langle \bar\nabla_X N, Y\big\rangle - \big\langle \bar\nabla_Y N, X\big\rangle \\ &= -\big\langle A_N X, Y\big\rangle + \tau^N(X)\eta(Y) + \big\langle A_N Y, X\big\rangle - \tau^N(Y)\eta(X). \end{aligned}$$

Hence,

$$\big\langle A_N X, Y\big\rangle - \big\langle A_N Y, X\big\rangle = \tau^N(X)\eta(Y) - \tau^N(Y)\eta(X) - 2\mathrm{d}\eta(X,Y)$$

as announced. □

In case the normalization is closed the (connection) 1-form τ^N is related to the shape operator of M as follows.

Lemma 2. *Let (M, g, N) be a closed normalization of a null hypersurface M in a Lorentzian manifold such that $\tau^N(\xi) = 0$. Then*

$$\tau^N = -\big\langle A_N\xi, \cdot\big\rangle.$$

Proof. Assume $\eta = i^\star\theta$ closed and let X, Y be tangent vector fields to M. The condition $X\cdot\eta(Y) - Y\cdot\eta(X) - \eta([X,Y]) = 0$ is equivalent to $\big\langle \bar\nabla_X N, Y\big\rangle = \big\langle \bar\nabla_Y N, X\big\rangle$. Then by the Weingarten formula, we get

$$\big\langle -A_N X, Y\big\rangle + \tau^N(X)\eta(Y) = \big\langle -A_N Y, X\big\rangle + \tau^N(Y)\eta(X).$$

In this relation, take $Y = \xi$ to get

$$\tau^N(X) = -\big\langle A_N\xi, X\big\rangle + \tau^N(\xi)\eta(X)$$

which gives the desired formula as $\tau^N(\xi) = 0$. □

The following relations (see a detailed proof in [6]) account for effects of the rigging change $N \longrightarrow \widetilde{N}_{|_M} = \phi N + \zeta$ on the induced geometric objects described in Section 2. Throughout, items with the symbol $\sim$ apply to $\widetilde{N}$.

$$\widetilde{\xi} = \frac{1}{\phi}\xi, \quad B^{\widetilde{N}}(X,Y) = \frac{1}{\phi}B^N(X,Y), \quad \widetilde{P} = P - \frac{1}{\phi}g(\zeta,\cdot)\xi$$

$$\begin{aligned} C^{\widetilde{N}}(X,\widetilde{P}Y) &= \phi C^N(X,PY) - g(\nabla_X\zeta, PY) \\ &\quad + \left[\tau^N(X) + \frac{X\cdot\phi}{\phi} + \frac{1}{\phi}B^N(\zeta,X)\right]g(\zeta,Y) \end{aligned} \tag{9}$$

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{\phi}B^N(X,Y)\zeta, \quad \overset{\star}{A}_{\widetilde{\xi}} = \frac{1}{\phi}\overset{\star}{A}_\xi - \frac{1}{\phi^2}B^N(\zeta,\cdot)\xi \tag{10}$$

$$A_{\widetilde{N}} = \phi A_N - \nabla_{\cdot}\zeta + \left[\tau^N + d\ln|\phi| + \frac{1}{\phi}B^N(\zeta,\cdot)\right]\zeta$$

for all tangent vector fields X and Y. Throughout the following ranges of indices is used: $i, j, l = 1, \dots, n, \quad \alpha, \beta = 0, 1, \dots, n, \quad a, b = 0, 1, \dots, n+1.$

3 Newton transformations and Minkowski integral formulas with respect to the rigged section

Due to the first relation in (4), it is noteworthy that among the two shape operators carried out by the rigged null hypersurface M, $\overset{\star}{A}_\xi$ is actually the one that encodes at best its null geometry. We introduce in this section the Newton transformations corresponding to it. The second one A_N is instead more concerned with the screen structure $\mathscr{S}(N)$ and will be considered subsequently.

3.1 Newton transformations of $\overset{\star}{A}_\xi$

Let (M, g, N) be an $(n+1)$-dimensional normalized null hypersurface with rigged vector field ξ. Relation (4) shows that $\overset{\star}{A}_\xi$ is a self-adjoint linear operator on each fibre T_pM ($p \in M$) and $\overset{\star}{A}_\xi \xi = 0$. Then, $\overset{\star}{A}_\xi$ is diagonalizable and have $(n+1)$ real-valued eigenfunctions $\overset{\star}{k}_0 = 0$, $\overset{\star}{k}_1, \dots, \overset{\star}{k}_n$ called principal curvatures of the null hypersurface with respect the shape operator $\overset{\star}{A}_\xi$. With respect to a quasi-orthonormal frame field $\{\overset{\star}{E}_0 = \xi, \overset{\star}{E}_1, \dots, \overset{\star}{E}_n\}$ of corresponding eigenvector fields the matrix of $\overset{\star}{A}_\xi$ take the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \overset{\star}{k}_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overset{\star}{k}_n \end{pmatrix}.$$

The function $\overset{\star}{H}_1 = \frac{1}{n+1} \operatorname{tr}(\overset{\star}{A}_\xi)$ is the mean curvature function of the null hypersurface and is a member of a familly of $n+1$ similar invariants $(\overset{\star}{H}_r)_{0 \le r \le n}$ called r-th mean curvature given by

$$\overset{\star}{H}_r = \binom{n+1}{r}^{-1} \sigma_r(\overset{\star}{k}_0, \dots, \overset{\star}{k}_n) \text{ and } \overset{\star}{H}_0 = 1 \text{ (constant function 1)},$$

where for $1 \le r \le n$, the algebraic invariant σ_r is the r-th elementary symmetric polynomial given by

$$\sigma_r(\overset{\star}{k}_0, \dots, \overset{\star}{k}_n) = \sum_{0 \le i_1 < \cdots < i_r \le n} \overset{\star}{k}_{i_1} \cdots \overset{\star}{k}_{i_r}.$$

It follows that the characteristic polynomial of $\overset{\star}{A}_\xi$ is given by

$$P(t) = \det(\overset{\star}{A}_\xi - tI) = \sum_{a=0}^{n+1} (-1)^a \binom{n+1}{a} \overset{\star}{H}_r t^{n+1-a}.$$

Set $\overset{\star}{S}_r = \sigma_r(\overset{\star}{k}_0, \dots, \overset{\star}{k}_n)$ and $\overset{\star}{S}{}_r^\alpha = \sigma_r(\overset{\star}{k}_0, \dots, \overset{\star}{k}_{\alpha-1}, \overset{\star}{k}_{\alpha+1}, \dots, \overset{\star}{k}_n)$.

Definition 2. Let r be an integer such that $1 \leq r \leq n$. The null hypersurface M is r-umbilical (resp. r-maximal) if

$$\overset{\star}{S}{}_r^i = \overset{\star}{S}{}_r^j \quad \forall i, j \in \{1, \dots, n\} \quad (\text{resp. } \overset{\star}{H}_r = 0).$$

Remark 1.

1. As we show below (18) both r-maximality and r-total umbilicity are independent of the rigging.
2. The r-total umbilicity (respectively, r-maximality) generalize the totally umbilical (respectively, maximal) obtained when $r = 1$. But, it is easy to check that any totally umbilical hypersurface is r-totally umbilical for all r.
3. For a 4-dimensional null hypersurface (i.e. $n = 3$), total umbilicity and 2-total umbilicity are equivalent.

Example 1. Consider the 6-dimensional space $\bar{M} = \mathbb{R}^6$ endowed with the Lorentzian metric

$$\bar{g} = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + \exp 2x^0[(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2] + \exp 2x^1[(\mathrm{d}x^4)^2 + (\mathrm{d}x^5)^2],$$

$(x^0, \dots, x^5)$ being the usual rectangular coordinates on $\bar{M}$. The only non-zero Christoffel coefficients of the Levi-Civita connection of $\bar{g}$ are

$$\Gamma_{02}^2 = \Gamma_{03}^3 = \Gamma_{14}^4 = \Gamma_{15}^5 = 1, \quad \Gamma_{22}^0 = \Gamma_{33}^0 = -\exp 2x^0, \quad \Gamma_{44}^1 = \Gamma_{55}^1 = \exp 2x^1.$$

Now, consider the hypersurface M of $\bar{M}$ define by

$$M = \{(x^0, \dots, x^5) \in \mathbb{R}^6 \,;\ x^0 + x^1 = 0\}.$$

Then, M is a null hypersurface of $(\bar{M}, \bar{g})$ and the vector field $N = -\frac{1}{2}\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right)$ is a null rigging for M with rigged vector field $\xi = \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}$ and we have $\mathscr{S}(N) = \operatorname{span}\{\overset{\star}{E}_1, \overset{\star}{E}_2, \overset{\star}{E}_3, \overset{\star}{E}_4\}$ with

$$\overset{\star}{E}_1 = e^{-2x^0}\frac{\partial}{\partial x^2}, \quad \overset{\star}{E}_2 = e^{-2x^0}\frac{\partial}{\partial x^3}, \quad \overset{\star}{E}_3 = e^{-2x^1}\frac{\partial}{\partial x^4}, \quad \overset{\star}{E}_4 = e^{-2x^1}\frac{\partial}{\partial x^5}.$$

Then it is easy to check that

$$\begin{aligned}
\nabla_{\overset{\star}{E}_1}\xi &= \overset{\star}{E}_1 \Rightarrow \overset{\star}{k}_1 = -1,\\
\nabla_{\overset{\star}{E}_2}\xi &= \overset{\star}{E}_2 \Rightarrow \overset{\star}{k}_2 = -1,\\
\nabla_{\overset{\star}{E}_3}\xi &= -\overset{\star}{E}_3 \Rightarrow \overset{\star}{k}_3 = 1,\\
\nabla_{\overset{\star}{E}_4}\xi &= -\overset{\star}{E}_4 \Rightarrow \overset{\star}{k}_4 = 1.
\end{aligned}$$

Hence, M is 2-totally umbilical but it is not totally umbilical.

For each $r = 0, \dots, n+1$, the r-th Newton transformation $\overset{\star}{T}_r\colon \Gamma(TM) \to \Gamma(TM)$ of the endomorphism $\overset{\star}{A}_\xi$, is given by

$$\overset{\star}{T}_r = \sum_{a=0}^{r} (-1)^a \overset{\star}{S}_a \overset{\star}{A}{}_\xi^{r-a}.$$

Inductively,

$$\overset{\star}{T}_0 = I \quad \text{and} \quad \overset{\star}{T}_r = (-1)^r \overset{\star}{S}_r I + \overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1},$$

where I denotes the identity map in $\Gamma(TM)$. According to the Cayley-Hamilton theorem, we have $\overset{\star}{T}_{n+1} = 0$. By elementary algebraic computations, the following is straightforward.

Proposition 1. *(a) $\overset{\star}{T}_r$ is self-adjoint and commute with $\overset{\star}{A}_\xi$;*

(b) $\overset{\star}{T}_r \overset{\star}{E}_\alpha = (-1)^r \overset{\star}{S}{}_r^\alpha \overset{\star}{E}_\alpha$;

(c) $\operatorname{tr}(\overset{\star}{T}_r) = (-1)^r (n+1-r) \overset{\star}{S}_r$;

(d) $\operatorname{tr}\left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}\right) = (-1)^{r-1} r \overset{\star}{S}_r$;

(e) $\operatorname{tr}\left(\overset{\star}{A}{}_\xi^2 \circ \overset{\star}{T}_{r-1}\right) = (-1)^r \left(\overset{\star}{S}_1 \overset{\star}{S}_r + (r+1) \overset{\star}{S}_{r+1}\right)$;

(f) $\operatorname{tr}(\overset{\star}{T}_{r-1} \circ \nabla_X \overset{\star}{A}_\xi) = (-1)^r X \cdot \overset{\star}{S}_r$.

Proof. The first item is due to the fact that $\overset{\star}{A}_\xi$ is self-adjoint. We show (b) inductively. In item (b) observe that the equality is trivial for $r = 0$. Assume that (b) holds for $r-1$ and observe that $\overset{\star}{S}{}_r^\alpha = \overset{\star}{S}_r - \overset{\star}{k}_\alpha \overset{\star}{S}{}_{r-1}^\alpha$. Then using the above and the well-known iterative relation characterizing the $\overset{\star}{T}_r$, we get,

$$\begin{aligned}
\overset{\star}{T}_r \overset{\star}{E}_\alpha &= (-1)^r \overset{\star}{S}_r \overset{\star}{E}_\alpha + \overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1} \overset{\star}{E}_\alpha \\
&= (-1)^r \left(\overset{\star}{S}_r - \overset{\star}{k}_\alpha \overset{\star}{S}{}_{r-1}^\alpha\right) \overset{\star}{E}_\alpha \\
&= (-1)^r \overset{\star}{S}{}_r^\alpha \overset{\star}{E}_\alpha
\end{aligned}$$

which shows (b). Through the above proof of b we see that $(-1)^r \overset{\star}{S}{}_r^\alpha$ are eigenfunctions associated to $\overset{\star}{E}_\alpha$ for each α and then we have

$$\operatorname{tr}(\overset{\star}{T}_r) = (-1)^r \sum_{\alpha=0}^{n} \overset{\star}{S}{}_r^\alpha$$

and each of the $\binom{n+1}{r}$ degree r monomials of $\overset{\star}{S}_r$ can be counted $(n+1)\binom{n}{r}$ times in the above summation. Thus

$$\sum_{\alpha=0}^{n} \overset{\star}{S}{}_r^\alpha = \frac{(n+1)\binom{n}{r}}{\binom{n+1}{r}} \overset{\star}{S}_r = (n+1-r) \overset{\star}{S}_r$$

and (c) is proved. By using the iterative formula of $\overset{\star}{T}_r$,

$$\begin{aligned}\operatorname{tr}\left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}\right) &= \operatorname{tr}(\overset{\star}{T}_r) + (-1)^r \overset{\star}{S}_r \operatorname{tr}(I)\\ &= (-1)^r \left((n+1-r)\overset{\star}{S}_r + (n+1)\overset{\star}{S}_r\right)\\ &= (-1)^{r-1} r \overset{\star}{S}_r,\end{aligned}$$

that is (d). Item (e) is immediate as

$$\begin{aligned}\operatorname{tr}\left(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}\right) &= \operatorname{tr}(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r) + (-1)^r \overset{\star}{S}_r \operatorname{tr}(\overset{\star}{A}_\xi)\\ &= (-1)^r \left(\overset{\star}{S}_1 \overset{\star}{S}_r + (r+1)\overset{\star}{S}_{r+1}\right).\end{aligned}$$

Finally,

$$\begin{aligned}g(\overset{\star}{T}_{r-1}(\nabla_X \overset{\star}{A}_\xi)\overset{\star}{E}_i, \overset{\star}{E}_i) &= g(\overset{\star}{T}_{r-1}\nabla_X \overset{\star}{k}_i \overset{\star}{E}_i, \overset{\star}{E}_i) - g(\overset{\star}{T}_{r-1} \circ \overset{\star}{A}_\xi \nabla_X \overset{\star}{E}_i, \overset{\star}{E}_i)\\ &= X(\overset{\star}{k}_i) g(\overset{\star}{T}_{r-1}\overset{\star}{E}_i, \overset{\star}{E}_i)\\ &= (-1)^{r-1} X(\overset{\star}{k}_i)\overset{\star}{S}{}^i_{r-1}\end{aligned}$$

and $\eta(\overset{\star}{T}_{r-1}(\nabla_X \overset{\star}{A}_\xi)\xi) = 0$. Hence

$$\operatorname{tr}(\overset{\star}{T}_{r-1} \circ \nabla_X \overset{\star}{A}_\xi) = (-1)^{r-1} \sum_{i=1}^{n} X(\overset{\star}{k}_i)\overset{\star}{S}{}^i_{r-1} = (-1)^{r-1} X(\overset{\star}{S}_r),$$

which completes the proof. □

Now, we get the following.

Proposition 2. *Let r be an integer such that $1 \le r \le n$. A non-maximal point $p \in M$ is r-umbilical if and only if*

$$\forall i \in \{1, \dots, n\}, \quad \overset{\star}{S}{}^i_r(p) = (r+1)\frac{\overset{\star}{S}_{r+1}(p)}{\overset{\star}{S}_1(p)}.$$

Proof. Just observe that $\overset{\star}{S}_{r+1} = \overset{\star}{S}{}^i_{r+1} + \overset{\star}{k}_i \overset{\star}{S}{}^i_r$. □

Remark 2. For a large class of null hypersurfaces, namely closed null hypersurfaces, the above proposition cannot be applied globally as they do admit (at least) one maximal point [14, Remark 10, page 7].

From now on, only Lorentzian ambient manifolds will be in consideration. Recall that to a normalized null hypersurface (M^{n+1}, g, N) is associated a (nondegenerate) metric $g_\eta = g + \eta \otimes \eta$ [9]. The ambient manifold being Lorentzian, the induced metric g on M has signature $(0, n)$. It follows that the hypersurface M equipped with the associated metric g_η is a Riemannian manifold. Let $(e_0 = \xi, e_1, \dots, e_n)$

be a g_η-orthonormal basis of $\Gamma(TM)$ with $\mathscr{S}(N) = \mathrm{span}\{e_1, \ldots, e_n\}$. The divergence of the operator $\overset{\star}{T}_r\colon \Gamma(TM) \to \Gamma(TM)$ is the vector field $\mathrm{div}^\nabla(\overset{\star}{T}_r) \in \Gamma(TM)$ defined as the trace of the $\mathrm{End}(TM)$-valued operator $\nabla\overset{\star}{T}_r$ and given by

$$\mathrm{div}^\nabla(\overset{\star}{T}_r) = \mathrm{tr}(\nabla\overset{\star}{T}_r) = \sum_{\alpha,\beta=0}^{n} g_\eta^{\alpha,\beta}(\nabla\overset{\star}{T}_r)(e_\alpha, e_\beta) = \sum_{\alpha=0}^{n}(\nabla_{e_\alpha}\overset{\star}{T}_r)e_\alpha.$$

By using the definition of the covariant derivative of a tensor and using (6),

$$g((\nabla_{e_\alpha}\overset{\star}{A}_\xi)\overset{\star}{T}_{r-1}e_\alpha, X) = g(\overset{\star}{T}_{r-1}e_\alpha, (\nabla_{e_\alpha}\overset{\star}{A}_\xi)X) - \eta(X)B^N(e_\alpha, \overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}e_\alpha).$$

Hence

$$\begin{aligned}\sum_{\alpha=0}^{n} g((\nabla_{e_\alpha}\overset{\star}{A}_\xi)\overset{\star}{T}_{r-1}e_\alpha, X) = \sum_{\alpha=0}^{n} g(\overset{\star}{T}_{r-1}e_\alpha, (\nabla_{e_\alpha}\overset{\star}{A}_\xi)X)& \\ - \eta(X)\,\mathrm{tr}\left(\overset{\star}{A}{}_\xi^2 \circ \overset{\star}{T}_{r-1}\right).&\end{aligned} \tag{11}$$

Proposition 3. *For all $X \in \Gamma(TM)$,*

$$\begin{aligned} g(\mathrm{div}\,\overset{\star}{T}_r, X) = {}& \sum_{a=0}^{r-1}\sum_{i=1}^{n} \bar{g}\left(\bar{R}(e_i, \xi)\overset{\star}{T}_a e_i, \overset{\star}{A}{}_\xi^{r-1-a}X\right) \\ &+ \sum_{a=0}^{r-1}\left(\tau^N(\overset{\star}{A}{}_\xi^{r-1-a}X)\,\mathrm{tr}(\overset{\star}{A}_\xi \circ \overset{\star}{T}_a) - \tau^N(P(\overset{\star}{A}_\xi \circ \overset{\star}{T}_a X))\right) \\ &+ (-1)^r\eta(X)\left(\sum_{i=1}^{n}\overset{\star}{S}{}_{r-1}^{i}\overset{\star}{k}{}_i^2 - \xi(\overset{\star}{S}_r)\right)\end{aligned} \tag{12}$$

Proof. Using iterative formula,

$$\begin{aligned}\mathrm{div}^\nabla(\overset{\star}{T}_r) &= (-1)^r\,\mathrm{div}(\overset{\star}{S}_r I) + \mathrm{div}(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}) \\ &= (-1)^r\sum_{\alpha=0}^{n}\left((e_\alpha \cdot \overset{\star}{S}_r)e_\alpha + (\nabla_{e_\alpha}\overset{\star}{A}_\xi)\overset{\star}{T}_{r-1}e_\alpha\right) + \overset{\star}{A}_\xi(\mathrm{div}\,\overset{\star}{T}_{r-1}).\end{aligned}$$

Hence by using (11) we get

$$\begin{aligned} g(\mathrm{div}^\nabla(\overset{\star}{T}_r), X) = g(\mathrm{div}\,\overset{\star}{T}_{r-1}, \overset{\star}{A}_\xi X) + (-1)^r PX(\overset{\star}{S}_r) - \eta(X)\,\mathrm{tr}\left(\overset{\star}{A}{}_\xi^2 \circ \overset{\star}{T}_{r-1}\right)& \\ + \sum_{\alpha=0}^{n} g\left(\overset{\star}{T}_{r-1}e_\alpha, (\nabla_{e_\alpha}\overset{\star}{A}_\xi)X\right).&\end{aligned} \tag{13}$$

By using the Gauss-Codazzi equation (8) with the substitutions

$$X \longleftarrow e_\alpha, \quad Y \longleftarrow X, \quad Z \longleftarrow \overset{\star}{T}_{r-1}e_\alpha,$$

we get

$$\begin{aligned} g\big(\overset{\star}{T}_{r-1}e_\alpha, (\nabla_{e_\alpha}\overset{\star}{A}_\xi)X\big) &= \bar{g}\big(\bar{R}(e_\alpha, X)\overset{\star}{T}_{r-1}e_\alpha, \xi\big) + g\big(\overset{\star}{T}_{r-1}e_\alpha, (\nabla_X \overset{\star}{A}_\xi)e_\alpha\big) \\ &\quad + B^N(e_\alpha, \overset{\star}{T}_{r-1}e_\alpha)\tau^N(X) \\ &\quad - B^N(X, \overset{\star}{T}_{r-1}e_\alpha)\tau^N(e_\alpha). \end{aligned} \tag{14}$$

Observe that

$$\sum_{\alpha=0}^{n} g\big(\overset{\star}{T}_{r-1}e_\alpha, (\nabla_X \overset{\star}{A}_\xi)e_\alpha\big) = \operatorname{tr}(\overset{\star}{T}_{r-1} \circ \nabla_X \overset{\star}{A}_\xi), \tag{15}$$

and using this along with (13), (14), (15) and Proposition 1, we obtain

$$\begin{aligned} g(\operatorname{div}^\nabla(\overset{\star}{T}_r), X) &= g(\operatorname{div}\overset{\star}{T}_{r-1}, \overset{\star}{A}_\xi X) + (-1)^{r-1}\eta(X)\xi(\overset{\star}{S}_r) \\ &\quad + \sum_{\alpha=0}^{n}\Big(\bar{g}(\bar{R}(e_\alpha, X)\overset{\star}{T}_{r-1}e_\alpha, \xi) - B^N(X, \overset{\star}{T}_{r-1}e_\alpha)\tau^N(e_\alpha)\Big) \\ &\quad + \tau^N(X)\operatorname{tr}\Big(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}\Big) - \eta(X)\operatorname{tr}\Big(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}\Big) \\ &= g(\operatorname{div}\overset{\star}{T}_{r-1}, \overset{\star}{A}_\xi X) + (-1)^{r-1}\eta(X)\xi(\overset{\star}{S}_r) \\ &\quad + \sum_{\alpha=0}^{n}\bar{g}(\bar{R}(e_\alpha, X)\overset{\star}{T}_{r-1}e_\alpha, \xi) \\ &\quad - \tau^N\Big(P(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}X)\Big) \\ &\quad + \tau^N(X)\operatorname{tr}\Big(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}\Big) - \eta(X)\operatorname{tr}\Big(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}\Big). \end{aligned}$$

By using the above iterative formula and Proposition 1, we deduce (12). □

Remark 3. Taking $r = 1$ in (12) and $X = \xi$, we get

$$\overline{\operatorname{Ric}}(\xi) = \xi(\overset{\star}{S}_1) + \tau^N(\xi)\overset{\star}{S}_1 - \sum_{i=1}^{n}\overset{\star}{k}{}_i^2. \tag{16}$$

In case the ambient manifold $\bar{M}$ is a space form and $\tau^N = 0$, the vector field $\operatorname{div}\overset{\star}{T}_r$ is $TM^\perp$-valued, that is $g(\operatorname{div}^\nabla(\overset{\star}{T}_r), X) = 0$ for all $X \in TM$, and

$$\xi(\overset{\star}{S}_r) = (-1)^{r-1}\operatorname{tr}\Big(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}\Big).$$

Also (setting $X = \xi$) the following partial differential equation holds for each $r = 1, \dots, n+1$

$$(-1)^{r-1}\xi(\overset{\star}{S}_r) + \tau^N(\xi)\operatorname{tr}(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}) - \operatorname{tr}(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}) = 0; \tag{17}$$

or equivalently

$$\xi(\overset{\star}{S}_r) + r\overset{\star}{S}_r\tau^N(\xi) - \sum_{i=1}^{n}\overset{\star}{k}{}_i^2\overset{\star}{S}{}^\alpha_{r-1} = 0.$$

From the above equation, we recover the well-known fact that for totally umbilical null hypersurfaces with principal curvature (umbilicity factor) ρ in a space form, the following partial differential equation holds [12, p. 108]:

$$\xi(\rho) + \rho\tau^N(\xi) - \rho^2 = 0.$$

We also derive the following.

Theorem 1. *Let (M^{n+1}, g, N) be a normalized null hypersurface of a Lorentzian space form $(\bar{M}(c)^{n+2}, \bar{g})$ with rigged vector field ξ and $\tau^N = 0$. Then*

(a) For each $r \in \{1, \dots, n\}$, M is r-maximal if and only if the endomorphism $\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}$ is trace-free.

(b) M is maximal if and only if M is totally geodesic.

(c) If M is r-maximal for some $r = 1, \dots, n$, then M is s-maximal for all $s \geq r$.

Proof. From (17),

$$(-1)^{r-1}\xi(\overset{\star}{S}_r) - \operatorname{tr}(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}) = 0,$$

as $\tau^N = 0$. Then the first item is immediate. Now, take $r = 1$ in the same equation (17) to get (b). Finally, if M is r-maximal then by the first item, $\operatorname{tr}\big(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}\big) = 0$. Hence, Proposition 1 leads to

$$\overset{\star}{S}_1 \overset{\star}{S}_r + (r+1)\overset{\star}{S}_{r+1} = 0,$$

which shows that $\overset{\star}{S}_r = 0$ implies $\overset{\star}{S}_{r+1} = 0$ and the proof is complete. □

Recall that a pseudo-Riemannian manifold satisfies the null (resp. the reverse null) convergence condition if $\overline{\operatorname{Ric}}(V) \geq 0$ (resp. $\overline{\operatorname{Ric}}(V) \leq 0$) for any null vector field V.

Theorem 2. *Let $(\bar{M}, \bar{g})$ be a Lorentzian manifold. If for $\bar{M}$ the null convergence condition holds, then for any null hypersurface M of $\bar{M}$, M is maximal if and only if M is totally geodesic.*

Proof. Assume M is maximal. From (16) we have

$$\overline{\operatorname{Ric}}(\xi) = -\sum_{i=1}^{n} \overset{\star}{k}{}_i^2 \geq 0 \quad \text{as} \quad \overset{\star}{S}_1 = 0\,.$$

Hence each $\overset{\star}{k}_i$ vanishes and M is totally geodesic. The converse is immediate. □

3.2 Newton transformations and change of rigging

As stated above, N being a null rigging for M, a vector field $\widetilde{N} \in \Gamma(T\bar{M})$ is a null rigging for M if and only if it is defined in an open set containing M and there exist a smooth function ϕ on M and a section ζ of TM such that $\widetilde{N} \circ i = \phi N + \zeta$ with the properties that ϕ is nowhere vanishing, being i the inclusion map, and $2\phi\eta(\zeta) + \|\zeta\|^2 = 0$ along M (see [7] for details on changes in normalizations). For each i, set

$$\overset{\star}{\widetilde{E}}_i = \widetilde{P}\overset{\star}{E}_i = \overset{\star}{E}_i - \frac{1}{\phi}g(\zeta, \overset{\star}{E}_i)\xi \quad \text{and} \quad \overset{\star}{\widetilde{E}}_0 = \widetilde{\xi} := \frac{1}{\phi}\xi.$$

Lemma 3. *$(\overset{\star}{\widetilde{E}}_0, \dots, \overset{\star}{\widetilde{E}}_n)$ is a quasi-orthonormal basis of $\Gamma(TM)$ which diagonalizes $\overset{\star}{A}_{\widetilde{\xi}}$ with eigenfunctions $\overset{\star}{\widetilde{k}}_\alpha = \frac{1}{\phi}\overset{\star}{k}_\alpha$.*

Proof. $g(\overset{\star}{\widetilde{E}}_0, \overset{\star}{\widetilde{E}}_\alpha) = \frac{1}{\phi}g(\xi, \overset{\star}{\widetilde{E}}_\alpha) = 0$ and $g(\overset{\star}{\widetilde{E}}_i, \overset{\star}{\widetilde{E}}_j) = g(\overset{\star}{E}_i, \overset{\star}{E}_j) = \delta_{ij}$, $\overset{\star}{A}_{\widetilde{\xi}}\widetilde{\xi} = 0$ and

$$\begin{aligned}
\overset{\star}{A}_{\widetilde{\xi}}\overset{\star}{\widetilde{E}}_i &= \overset{\star}{A}_\xi \overset{\star}{\widetilde{E}}_i - \frac{1}{\phi^2}B^N(\zeta, \overset{\star}{\widetilde{E}}_i)\xi \\
&= \overset{\star}{A}_\xi \overset{\star}{E}_i - \frac{1}{\phi^2}g(\zeta, \overset{\star}{A}_\xi \overset{\star}{E}_i)\xi \\
&= \frac{1}{\phi}\overset{\star}{k}_i\left(\overset{\star}{E}_i - \frac{1}{\phi}g(\zeta, \overset{\star}{E}_i)\xi\right) \\
&= \frac{1}{\phi}\overset{\star}{k}_i\overset{\star}{\widetilde{E}}_i.
\end{aligned}$$

□

Hence, through the change $\widetilde{N} = \phi N + \zeta$,

$$\overset{\star}{\widetilde{k}}_\alpha = \frac{1}{\phi}\overset{\star}{k}_\alpha, \quad \overset{\star}{\widetilde{H}}_r = \frac{1}{\phi^r}\overset{\star}{H}_r, \quad \overset{\star}{\widetilde{S}}_r = \frac{1}{\phi^r}\overset{\star}{S}_r, \quad \overset{\star}{\widetilde{S}}{}^i_r = \frac{1}{\phi^r}\overset{\star}{S}{}^i_r \tag{18}$$

and we have the next lemma.

Lemma 4. *Let (M^{n+1}, g, N) be a normalized null hypersurface of a Lorentzian manifold $(\bar{M}^{n+2}, \bar{g})$. Consider the change of normalization $\widetilde{N} = \phi N + \zeta$. Then*

$$\overset{\star}{\widetilde{T}}_r = \frac{1}{\phi^r}\overset{\star}{T}_r - \frac{1}{\phi^{r+1}}\sum_{a=0}^{r-1}(-1)^a\overset{\star}{S}_a g\left(\zeta, \overset{\star}{A}{}^{r-a}_{\widetilde{\xi}}\right)\xi.$$

Proof. By use of second relation in (10) we have

$$\begin{aligned}
\overset{\star}{\widetilde{T}}_r &= \sum_{a=0}^{r}(-1)^a\overset{\star}{\widetilde{S}}_a\overset{\star}{A}{}^{r-a}_{\widetilde{\xi}} \\
&= (-1)^r\overset{\star}{\widetilde{S}}_r I + \sum_{a=0}^{r-1}(-1)^a\overset{\star}{\widetilde{S}}_a\left(\frac{1}{\phi}A_\xi - \frac{1}{\phi^2}B^N(\zeta, \cdot)\xi\right)^{r-a}.
\end{aligned}$$

As $r-a\geq 1$ and $\overset{\star}{A}_\xi\xi=0$,

$$\left(\frac{1}{\phi}A_\xi-\frac{1}{\phi^2}B^N(\zeta,\cdot)\xi\right)^{r-a}=\frac{1}{\phi^{r-a}}A_\xi^{r-a}-\frac{1}{\phi^{r+1-a}}B^N(\zeta,\overset{\star}{A}{}_\xi^{r-a-1})\xi.$$

This completes the proof. $\square$

For each i, in view of (9) we get

$$\begin{aligned}\widetilde{\nabla}_{\tilde{\overset{\star}{E}}_i}\tilde{\overset{\star}{E}}_i&=\nabla_{\tilde{\overset{\star}{E}}_i}\tilde{\overset{\star}{E}}_i-\frac{1}{\phi}B^N(\tilde{\overset{\star}{E}}_i,\tilde{\overset{\star}{E}}_i)\zeta\\&=\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i-\overset{\star}{E}_i\left(\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\right)\xi-\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\nabla_{\overset{\star}{E}_i}\xi-\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\nabla_\xi\overset{\star}{E}_i\\&\quad+\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\nabla_\xi\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\xi-\frac{1}{\phi}B^N(\overset{\star}{E}_i,\overset{\star}{E}_i)\zeta\\&=\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i-\frac{\overset{\star}{k}_i}{\phi}g(\zeta,\overset{\star}{E}_i)\overset{\star}{E}_i-\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)P\nabla_\xi\overset{\star}{E}_i-\frac{\overset{\star}{k}_i}{\phi}P\zeta\\&\quad+\left[\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\tau^N(\overset{\star}{E}_i)-\overset{\star}{E}_i\left(\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\right)\right.\\&\qquad\left.+\frac{1}{2}\xi\left(\frac{1}{\phi^2}g(\zeta,\overset{\star}{E}_i)^2\right)-\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\eta(\nabla_\xi\overset{\star}{E}_i)\right]\xi\end{aligned}$$

Hence

$$\begin{aligned}\widetilde{\nabla}_{\tilde{\overset{\star}{E}}_i}\tilde{\overset{\star}{E}}_i&=\frac{1}{\phi}\Big(\overset{\star}{k}_ig(\zeta,\overset{\star}{E}_i)\overset{\star}{E}_i-g(\zeta,\overset{\star}{E}_i)\sum_{j=1}^n g(\nabla_\xi\overset{\star}{E}_i,\overset{\star}{E}_j)\overset{\star}{E}_j-\overset{\star}{k}_i\sum_{j=1}^n g(\zeta,\overset{\star}{E}_j)\overset{\star}{E}_j\Big)\\&\quad+\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i+\eta\big(\widetilde{\nabla}_{\tilde{\overset{\star}{E}}_i}\tilde{\overset{\star}{E}}_i-\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i\big)\xi,\end{aligned}\tag{19}$$

and

$$\begin{aligned}\eta(\widetilde{\nabla}_{\tilde{\overset{\star}{E}}_i}\tilde{\overset{\star}{E}}_i-\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i)&=\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\tau^N(\overset{\star}{E}_i)-\overset{\star}{E}_i\left(\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\right)\\&\quad-\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\eta(\nabla_\xi\overset{\star}{E}_i)+\frac{1}{2}\xi\left(\frac{1}{\phi^2}g(\zeta,\overset{\star}{E}_i)^2\right).\end{aligned}$$

Lemma 5. *Let (M^{n+1},g,N) be a normalized null hypersurface of a Lorentzian manifold $(\bar{M}^{n+2},\bar{g})$ such that for a fixed r, $\xi\cdot\overset{\star}{S}{}_r^i=0$ for $i=1,\dots,n$. Consider the change of normalization $\widetilde{N}=\phi N+\zeta$, $\zeta|_M\in\Gamma(TM)$, $\phi\in\mathbb{R}$. Then*

$$\begin{aligned}\mathrm{div}^{\widetilde{\nabla}}(\tilde{\overset{\star}{T}}_r)&=\frac{1}{\phi^r}\,\mathrm{div}^{\nabla}(\overset{\star}{T}_r)+\eta(\mathrm{div}^{\widetilde{\nabla}}(\tilde{\overset{\star}{T}}_r)-\frac{1}{\phi^r}\,\mathrm{div}^{\nabla}(\overset{\star}{T}_r))\xi\\&\quad+\frac{(-1)^r}{\phi^{r+1}}\sum_{j=1}^n\sum_{i=1}^n(\overset{\star}{S}{}_r^j-\overset{\star}{S}{}_r^i)\left(g(\nabla_\xi\overset{\star}{E}_i,\overset{\star}{E}_j)g(\zeta,\overset{\star}{E}_i)+\overset{\star}{k}_ig(\zeta,\overset{\star}{E}_j)\right)\overset{\star}{E}_j.\end{aligned}\tag{20}$$

In particular for $r = 0,\dots,n$, $\operatorname{div}^{\widetilde{\nabla}}\big(\overset{\star}{\widetilde{T}}_r\big) - \frac{1}{\phi^r}\operatorname{div}^{\nabla}(\overset{\star}{T}_r)$ *is* $TM^{\perp}$*-valued if and only if for each* $j = 1,\dots,n$,

$$\sum_{i=1}^{n}(\overset{\star}{S}{}^{j}_{r} - \overset{\star}{S}{}^{i}_{r})\Big(g(\nabla_{\xi}\overset{\star}{E}_i,\overset{\star}{E}_j)g(\zeta,\overset{\star}{E}_i) + \overset{\star}{k}_i g(\zeta,\overset{\star}{E}_j)\Big) = 0. \tag{21}$$

Proof. Observe that $(\widetilde{\nabla}_{\widetilde{\xi}}\overset{\star}{\widetilde{T}}_r)\widetilde{\xi} = (-1)^r\widetilde{\xi}(\widetilde{S}_r)\widetilde{\xi}$. Then

$$\begin{aligned}\operatorname{div}^{\widetilde{\nabla}}(\overset{\star}{\widetilde{T}}_r) &= \sum_{i=1}^{n}(\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{T}}_r)\overset{\star}{\widetilde{E}}_i\\ &= \sum_{i=1}^{n}(\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{T}}_r\overset{\star}{\widetilde{E}}_i - \overset{\star}{\widetilde{T}}_r\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{E}}_i) + (-1)^r\widetilde{\xi}(\widetilde{S}_r)\widetilde{\xi}.\end{aligned}$$

By the second item in Proposition 1,

$$\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{T}}_r\overset{\star}{\widetilde{E}}_i = (-1)^r\overset{\star}{\widetilde{S}}{}^{i}_{r}\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{E}}_i + (-1)^r\big(\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{S}}{}^{i}_{r}\big)\overset{\star}{\widetilde{E}}_i.$$

Then thanks to (19) and by direct calculation, we get

$$\begin{aligned}\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{T}}_r\overset{\star}{\widetilde{E}}_i = {}&\frac{(-1)^r}{\phi^{r+1}}\overset{\star}{S}{}^{i}_{r}\Big(\overset{\star}{k}_i g(\zeta,\overset{\star}{E}_i)\overset{\star}{E}_i - \sum_{j=1}^{n}\big(g(\zeta,\overset{\star}{E}_i)g(\nabla_{\xi}\overset{\star}{E}_i,\overset{\star}{E}_j) + \overset{\star}{k}_i g(\zeta,\overset{\star}{E}_j)\big)\overset{\star}{E}_j\Big)\\ &+ \frac{1}{\phi^r}\nabla_{\overset{\star}{E}_i}\overset{\star}{T}_r\overset{\star}{E}_i + \eta(\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{T}}_r\overset{\star}{\widetilde{E}}_i - \frac{1}{\phi^r}\nabla_{\overset{\star}{E}_i}\overset{\star}{T}_r\overset{\star}{E}_i)\xi\\ &+ (-1)^r\Big(\overset{\star}{S}{}^{i}_{r}\overset{\star}{E}_i(1/\phi^r) - \frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\xi(\overset{\star}{S}{}^{i}_{r}/\phi^r)\Big)\overset{\star}{E}_i\end{aligned}$$

in which the last term vanishes due to $\xi\cdot\overset{\star}{S}{}^{i}_{r} = 0$ and $\phi\in\mathbb{R}$. Now (19) and Lemma 4 yield

$$\begin{aligned}\overset{\star}{\widetilde{T}}_r\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{E}}_i &= \frac{1}{\phi^r}\overset{\star}{T}_r\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{E}}_i - \frac{1}{\phi^{r+1}}\sum_{a=0}^{r-1}(-1)^a\overset{\star}{S}_a g(\overset{\star}{A}{}^{r-a}_{\xi},\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{E}}_i)\xi\\ &= \frac{(-1)^r}{\phi^{r+1}}\overset{\star}{k}_i\overset{\star}{S}{}^{i}_{r}g(\zeta,\overset{\star}{E}_i)\overset{\star}{E}_i\\ &\quad + \frac{(-1)^{r+1}}{\phi^{r+1}}\sum_{j=1}^{n}\overset{\star}{S}{}^{j}_{r}\big(g(\zeta,\overset{\star}{E}_i)g(\nabla_{\xi}\overset{\star}{E}_i,\overset{\star}{E}_j) + \overset{\star}{k}_i g(\zeta,\overset{\star}{E}_j)\big)\overset{\star}{E}_j\\ &\quad + \frac{1}{\phi^r}\overset{\star}{T}_r\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i + \eta\big(\overset{\star}{\widetilde{T}}_r\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{E}}_i - \frac{1}{\phi^r}\overset{\star}{T}_r\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i\big)\xi.\end{aligned}$$

The desired expression follows from direct substitution. The last claim is immediate by cancelling the screen term represented by the last summation in (20). □

Theorem 3. *Let (M^{n+1}, g, N) be a normalized null hypersurface of a Lorentzian manifold $(\bar{M}^{n+2}, \bar{g})$, and $r \in \{1, \ldots, n\}$ such that $\xi \cdot \overset{\star}{S}{}^i_r = 0$ for $i = 1, \ldots, n$. Then $\operatorname{div}^{\widetilde{\nabla}}(\overset{\star}{\widetilde{T}}_r) - \frac{1}{\phi^r} \operatorname{div}^{\nabla}(\overset{\star}{T}_r)$ is $TM^{\perp}$-valued for any change of normalization $\widetilde{N} = \phi N + \zeta$ with $\phi \in \mathbb{R}$, if and only if any point of M is r-umbilical or both maximal and $(r+1)$-maximal.*

Proof. Let $r \in \{1, \ldots, n\}$ and $\operatorname{div}^{\widetilde{\nabla}}(\overset{\star}{\widetilde{T}}_r) - \frac{1}{\phi^r} \operatorname{div}^{\nabla}(\overset{\star}{T}_r) \in \Gamma(\operatorname{Rad} TM)$ for any change of normalization $\widetilde{N} = \phi N + \zeta$. Then by (21),

$$\sum_{i=1}^{n} (\overset{\star}{S}{}^j_r - \overset{\star}{S}{}^i_r) \Big(g(\nabla_\xi \overset{\star}{E}_i, \overset{\star}{E}_j) g(\zeta, \overset{\star}{E}_i) + \overset{\star}{k}_i g(\zeta, \overset{\star}{E}_j) \Big) = 0, \quad \forall j = 1, \ldots, n.$$

Consider the particular changes $\widetilde{N} = N + \overset{\star}{E}_l$ for $l = 1, \ldots, n$. Then for each l,

$$(\overset{\star}{S}{}^j_r - \overset{\star}{S}{}^l_r) g(\nabla_\xi \overset{\star}{E}_l, \overset{\star}{E}_j) + \sum_{i=1}^{n} (\overset{\star}{S}{}^j_r - \overset{\star}{S}{}^i_r) \overset{\star}{k}_i \delta_{lj} = 0, \quad \forall j = 1, \ldots, n,$$

and setting $j = l$ yields

$$\overset{\star}{S}_1 \overset{\star}{S}{}^l_r - (r+1) \overset{\star}{S}_{r+1} = 0.$$

By Proposition 2 we deduce that any non-maximal point of M is r-umbilical. The converse is straightforward. □

3.3 Minkowski integral formulas

Using Newton transformations with respect to the shape operator $\overset{\star}{A}_\xi$ we introduce some Minkowski-type integral formulas on null hypersurfaces of Lorentzian manifolds carrying some conformal Killing vector field.

Recall that when a manifold M is provided with a linear connection D and X is a section of the tangent bundle of M, the map $DX : \Gamma(TM) \to \Gamma(TM)$ given by $T_pM \ni Y_p \mapsto D_{Y_p} X_p$ is an endomorphism at each point $p \in M$. The divergence of X (with respect to D) is defined as the trace of DX, that is

$$\operatorname{div}^D(X) = \operatorname{tr}(DX).$$

In particular on semi-Riemannian manifolds the default (natural) connection used in calculating the divergence is the Levi-Civita connection.

Let (M^{n+1}, g, N) be a normalized null hypersurface of a Lorentzian manifold $(\bar{M}^{n+2}, \bar{g})$ with rigged vector field ξ and $\tau^N = 0$. Let ∇ denote the linear connection induced by the rigging N and assume $K \in \Gamma(TM)$ is a conformal Killing vector field with smooth conformal factor 2Φ. For each $r \in \{0, \ldots, n+1\}$ we have

$$\operatorname{div}^{\nabla}(\overset{\star}{T}_r K) = \operatorname{tr}(\nabla \overset{\star}{T}_r K) = \sum_{i=1}^{n} g(\nabla_{\overset{\star}{E}_i} \overset{\star}{T}_r K, \overset{\star}{E}_i) + \bar{g}(\nabla_\xi \overset{\star}{T}_r K, N).$$

But

$$
\begin{aligned}
g(\nabla_{\overset{\star}{E}_i}\overset{\star}{T}_r K,\overset{\star}{E}_i) &= \overset{\star}{E}_i\cdot g(\overset{\star}{T}_r K,\overset{\star}{E}_i) - g(K,\overset{\star}{T}_r\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i) - \eta(\overset{\star}{T}_r K)B^N(\overset{\star}{E}_i,\overset{\star}{E}_i)\\
&= \overset{\star}{E}_i\cdot g(K,\overset{\star}{T}_r\overset{\star}{E}_i) + g(K,(\nabla_{\overset{\star}{E}_i}\overset{\star}{T}_r)\overset{\star}{E}_i) - g(K,\nabla_{\overset{\star}{E}_i}\overset{\star}{T}_r\overset{\star}{E}_i)\\
&\quad - \eta(\overset{\star}{T}_r K)B^N(\overset{\star}{E}_i,\overset{\star}{E}_i)\\
&= g(K,(\nabla_{\overset{\star}{E}_i}\overset{\star}{T}_r)\overset{\star}{E}_i) + (-1)^r\overset{\star}{S}{}^i_r g(\nabla_{\overset{\star}{E}_i}K,\overset{\star}{E}_i)\\
&\quad + \eta(K)B^N(\overset{\star}{E}_i,\overset{\star}{T}_r\overset{\star}{E}_i) - \eta(\overset{\star}{T}_r K)B^N(\overset{\star}{E}_i,\overset{\star}{E}_i).
\end{aligned}
$$

As $L_K g = 2\varphi g$ we have $g(\nabla_{\overset{\star}{E}_i}K,\overset{\star}{E}_i) = \varphi g(\overset{\star}{E}_i,\overset{\star}{E}_i)$. Hence

$$
\begin{aligned}
\operatorname{div}^\nabla(\overset{\star}{T}_r K) &= g(\operatorname{div}^\nabla(\overset{\star}{T}_r),K) + \varphi\big((-1)^{r-1}\overset{\star}{S}_r + \operatorname{tr}(\overset{\star}{T}_r)\big)\\
&\quad + \eta(K)\operatorname{tr}\left(\overset{\star}{A}_\xi\circ\overset{\star}{T}_r - (-1)^r\overset{\star}{S}_r\overset{\star}{A}_\xi\right) + \eta(\nabla_\xi\overset{\star}{T}_r K)\\
&= g(\operatorname{div}^\nabla(\overset{\star}{T}_r),K) + (-1)^r(n-r)\overset{\star}{S}_r\varphi\\
&\quad + \eta(K)\operatorname{tr}\left(\overset{\star}{A}{}^2_\xi\circ\overset{\star}{T}_{r-1}\right) + \eta(\nabla_\xi\overset{\star}{T}_r K).
\end{aligned}
$$

Now using Proposition 1 leads to

$$
\begin{aligned}
\operatorname{div}^\nabla(\overset{\star}{T}_r K) &= g(\operatorname{div}^\nabla(\overset{\star}{T}_r),K) + \eta(\nabla_\xi\overset{\star}{T}_r K)\\
&\quad + (-1)^r\left(c_r\overset{\star}{H}_r\varphi + c'_r\overset{\star}{H}_{r+1}\eta(K) - c''_r\overset{\star}{H}_1\overset{\star}{H}_r\eta(K)\right).
\end{aligned}
\tag{22}
$$

where

$$
c_r = (n-r)\binom{n+1}{r}, \qquad c'_r = (n+1)\binom{n}{r}, \quad c''_r = (n+1)\binom{n+1}{r}.
$$

Also, a straightforward computation gives

$$
\eta(\nabla_\xi\overset{\star}{T}_r K) = (-1)^r\xi(\overset{\star}{S}_r\eta(K)) + (-1)^r\overset{\star}{S}_r\tau^N(\xi)\eta(K) - \bar g(\overset{\star}{T}_r K, A_N\xi).
$$

We deduce the following.

Theorem 4. *Let (M^{n+1},g,N) be a normalized null hypersurface of a space-time $(\bar M^{n+2},\bar g)$ with rigged vector field ξ and $\tau^N=0$, carrying a compactly supported conformal Killing vector field K with smooth conformal factor 2Φ. Then, for each $r=1,\dots,n+1$, the following holds*

$$
\begin{aligned}
&\int_M\left(g(\operatorname{div}\overset{\star}{T}_{r-1},K) + \eta(\nabla_\xi\overset{\star}{T}_{r-1}K)\right)\mathrm{d}V\\
&\qquad = (-1)^r\int_M\left(c_{r-1}\overset{\star}{H}_{r-1}\varphi + \eta(K)(c'_{r-1}\overset{\star}{H}_r - c''_{r-1}\overset{\star}{H}_1\overset{\star}{H}_{r-1})\right)\mathrm{d}V,
\end{aligned}
\tag{23}
$$

where $\mathrm{d}V = i_N\mathrm{d}\bar V$ and $\mathrm{d}\bar V$ is the (fixed) volume element on $\bar M$ with respect to $\bar g$ and the given orientation.

In particular for horizontal conformal Killing vector fields K on M we have

$$\int_M \Big(g(\operatorname{div} \overset{\star}{T}_{r-1}, K)\Big)\, \mathrm{d}V = (-1)^r \int_M \Big(c_{r-1}\overset{\star}{H}_{r-1}\varphi\Big)\, \mathrm{d}V. \tag{24}$$

Proof. Since K is compactly supported, by Stoke's Theorem,

$$\int_M \operatorname{div}^{\nabla}(\overset{\star}{T}_r K)\, \mathrm{d}V = 0$$

and (23) is straightforward from (22). Now, assume K to be tangent to the screen structure $\mathscr{S}(N)$. Then $\eta(K) = 0$. Also, as $\tau^N = 0$ we have from Lemma 2 and (4) that $C^N(\xi, \overset{\star}{T}_{r-1}K) = 0$. Therefore

$$\nabla_\xi(\overset{\star}{T}_{r-1}K) = \overset{\star}{\nabla}_\xi(\overset{\star}{T}_{r-1}K) + C^N(\xi, \overset{\star}{T}_{r-1}K)\xi = \overset{\star}{\nabla}_\xi \overset{\star}{T}_{r-1}K \in \mathscr{S}(N).$$

Hence $\eta\big(\nabla_\xi \overset{\star}{T}_{r-1}K\big) = 0$ and the relation (24) follows (23). □

Remark 4. In Theorem 4 and below, the condition *compactly supported* may be removed and replaced by *compact null hypersurface without boundary.*

Corollary 1. *Let (M^{n+1}, g, N) be a normalized null hypersurface of a space-time $(\bar{M}^{n+2}, \bar{g})$ with rigged vector field ξ and $\tau^N = 0$, carrying a compactly supported conformal Killing vector field K with smooth conformal factor 2Φ. Suppose that for some $r = 1, \ldots, n+1$ the following condition holds*

$$\int_M g(\operatorname{div} \overset{\star}{T}_{r-1}, K)\, \mathrm{d}V = 0.$$

Then

$$\begin{aligned}\int_M \Big(c_{r-1}\overset{\star}{H}_{r-1}\Phi + c'_{r-1}\bar{g}(K,N)\overset{\star}{H}_r - c''_{r-1}\bar{g}(K,N)\overset{\star}{H}_1\overset{\star}{H}_{r-1}\Big)\, \mathrm{d}V& \\ = (-1)^r \int_M \eta(\nabla_\xi \overset{\star}{T}_{r-1}K)\, \mathrm{d}V.&\end{aligned} \tag{25}$$

In particular, (25) always holds when the ambient space-time $(\bar{M}^{n+2}, \bar{g})$ has constant sectional curvature and for the conformal factor 2Φ we have

$$\int_M \Phi\, \mathrm{d}V = -\frac{1}{n}\int_M \eta(\nabla_\xi K)\, \mathrm{d}V. \tag{26}$$

Moreover, if the conformal Killing vector field K is horizontal then $\int_M \Phi\, \mathrm{d}V = 0$.

Formula (25) is the r-th Minkowski-type formula of the null hypersurface M, with respect to the shape operator $\overset{\star}{A}_\xi$.

Proof. Setting $\int_M g(\operatorname{div} \overset{\star}{T}_{r-1}, K)\,\mathrm{d}V$ to 0 in (23) leads to (25). From Remark 3 we know that when the ambient manifold has constant sectional curvature, $\operatorname{div}^{\nabla}(\overset{\star}{T}_{r-1})$ is $TM^{\perp}$-valued and the vanishing condition is fulfilled. Finally, set $r = 1$ in (25) to get (26), using the fact that $c_0 = n$, $c_0' = c_0''$ and $\overset{\star}{H}_0 = 1$. If in addition the conformal Killing vector field is tangent to the screen structure then by the screen Gauss formula and and $\tau^N = 0$ we have $C^N(\xi, \cdot) = 0$ and then $\nabla_\xi K \in \mathscr{S}(N)$, that is $\eta(\nabla_\xi K) = 0$ and the last claim follows. □

Corollary 2. *Let (M^{n+1}, g, N) be a normalized null hypersurface of a space-time $(\bar{M}^{n+2}, \bar{g})$ with rigged vector field ξ and $\tau^N = 0$, carrying a compactly supported conformal Killing vector field K with smooth conformal factor 2Φ. If M is r-totally umbilical for some $r = 1, \ldots, (n+1)$ and satisfies both $\xi \cdot \overset{\star}{S}{}_r^i = 0$ for $i = 1, \ldots, n$ and the r-th Minkowski-type formula* (25)*, then the same is true for all rigging of the form $\widetilde{N} = \psi N + \zeta$ with constant ψ.*

Proof. Consider from N a rigging $\widetilde{N} = \psi N + \zeta$. We pointed out in Theorem 3 that

$$\operatorname{div}^{\widetilde{\nabla}} \overset{\star}{\widetilde{T}}_{r-1} - \frac{1}{\psi^{r-1}} \operatorname{div}^{\nabla} \overset{\star}{T}_{r-1} \in \Gamma(\operatorname{Rad} TM).$$

Thus, $g(\operatorname{div}^{\widetilde{\nabla}} \overset{\star}{\widetilde{T}}_{r-1}, K) = \frac{1}{\psi^{r-1}} g(\operatorname{div}^{\nabla} \overset{\star}{T}_{r-1}, K)$. It follows that for constant ψ we have

$$\int_M g(\operatorname{div}^{\widetilde{\nabla}} \overset{\star}{\widetilde{T}}_{r-1}, K) = \frac{1}{\psi^{r-1}} \int_M g(\operatorname{div}^{\nabla} \overset{\star}{T}_{r-1}, K),$$

which shows that integrals from both sides vanish or not, simultaneously. □

4 Physical models

As usual, stationary and axisymmetric perfect fluid metrics are studied under the assumption of the existence of a conformal Killing vector field. Let $(\bar{M}^4, \bar{g})$ be the Einstein static fluid space-time with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (1 - \varrho^2)^{-1}\mathrm{d}\varrho^2 + \varrho^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big),$$

with the fluid four-velocity vector $u^a = \delta_0^a$ $(a = 0, 1, 2, 3)$. This space-time admits a conformal Killing vector field

$$K^a = (1 - \varrho^2)^{1/2} \cos t\, \delta_0^a - \varrho(1 - \varrho^2)^{1/2} \sin t\, \delta_1^a.$$

In fact in this space-time, the relation

$$\varrho = \cos t, \quad t \in \left]0, \frac{\pi}{2}\right[$$

defines a compact null hypersurface M for which the kernel of the degenerate induced metric g is spanned by the null conformal Killing vector field K. In other words, (M, g) is a compact totally umbilical null hypersurface. Indeed, consider the vector field

$$N^a = -\frac{1}{2\varrho^2}(1-\varrho^2)^{-1/2}\big[\cos t\,\delta_0^a + \varrho\sin t\,\delta_1^a\big].$$

This is a null rigging with associated screen structure $\mathscr{S}(N) = \operatorname{span}(\partial_\theta, \partial_\phi)$. It is easy to check that $\tau^N = 0$ and $\overset{\star}{A}_K = \big[(1-\varrho^2)^{1/2}\sin t\big]P$ where P denotes the projection morphism of TM onto $\mathscr{S}(N)$. The Newton transformations are given by

$$\overset{\star}{T}_0 = I, \quad \overset{\star}{T}_r = (1-\varrho^2)^{r/2}\sin^r t\left[\sum_{a=0}^{r}(-1)^a\binom{3}{a}\right]P, \quad r = 1, 2, 3.$$

The scale factor is given by $\Phi = -(1-\varrho^2)^{1/2}\sin t$. Also, for all $r \geq 1$,

$$\int_M g(\operatorname{div}\overset{\star}{T}_{r-1}, K)\,\mathrm{d}V = 0$$

and $\bar{g}(K, N) = 1$ and by direct calculation, we get $\int_M \Phi\,\mathrm{d}V = -2\pi$ which is non-zero. Observe that the conformal Killing vector field K is not compactly supported in M.

In general, when interested by perfect-fluid solutions of Einstein's field equations, it is well-known that there exist coordinates $\{t, x, y, z\}$ such that $U = \partial_y$ and $T = \partial_z$ are two Killing vector fields and in which the metric takes the form

$$\mathrm{d}s^2 = \frac{1}{S^2(t,x)}\Big[-\mathrm{d}t^2 + \mathrm{d}x^2 + F(t,x)\big(P^{-1}(t,x)\,\mathrm{d}y^2 + P(t,x)(\mathrm{d}z + W(t,x)\,\mathrm{d}y)^2\big)\Big].$$

Let us consider the 1-forms $\{\theta^a\}$ such that

$$\theta^0 = \frac{1}{S(t,x)}\,\mathrm{d}t \qquad \theta^1 = \frac{1}{S(t,x)}\,\mathrm{d}x$$

$$\theta^2 = \frac{1}{S(t,x)}\sqrt{\frac{F(t,x)}{P(t,x)}}\,\mathrm{d}y \qquad \theta^3 = \frac{1}{S(t,x)}\sqrt{F(t,x)P(t,x)}(\mathrm{d}z + W(t,x)\,\mathrm{d}y),$$

and let $S_{\alpha\beta}$ stand for the components of the Einstein tensor in the $\{\theta^a\}$ cobasis. Then the Einstein field equations can be written in terms of the $S_{\alpha\beta}$ and due to the symmetries inherent to this setting we are led to three inequivalent Lie algebras [16]. The Lie algebra A is given by

$$[U, T] = 0, \quad [U, K] = \frac{1}{2}(c+b)U, \quad [T, K] = \frac{1}{2}(c-b)T,$$

where b and c are arbitrary (possibly vanishing) constants and K is a conformal Killing vector field given by

$$K = \partial_t + \frac{1}{2}(c+b)yU + \frac{1}{2}(c-b)zT.$$

The line element in this case has the form

$$\begin{aligned}\mathrm{d}s^2 = \frac{1}{S^2(t,x)}\Big[&-\mathrm{d}t^2 + \mathrm{d}x^2 + F(x)P^{-1}(x)e^{-(b+c)t}\,\mathrm{d}y^2\\ &+ F(x)P(x)e^{(b-c)t}\big(\mathrm{d}z + W(x)e^{-bt}\,\mathrm{d}y\big)^2\Big].\end{aligned}$$

Similarly, for the Lie Algebra B we have

$$[U,T] = 0, \quad [U;K] = \frac{1}{2}cU + aT, \quad [T,K] = \frac{1}{2}cT,$$

where a is a non-vanishing constant and K is a conformal Killing vector field given by

$$K = \partial_t + \frac{1}{2}cyU + \big(ay + \frac{1}{2}cz\big)T.$$

The corresponding line element has the form

$$\begin{aligned}\mathrm{d}s^2 = \frac{1}{S^2(t,x)}\Big[&-\mathrm{d}t^2 + \mathrm{d}x^2\\ &+ F(x)e^{-ct}\big(P^{-1}(x)\,\mathrm{d}y^2 + P(x)(\mathrm{d}z + [W(x) + at]\,\mathrm{d}y)^2\big)\Big].\end{aligned}$$

Finally for the Lie algebra VII (so named because it corresponds to the Bianchi type VII in Bianchi's classification of three-dimensional Lie algebra) the product is defined by

$$[U,T] = 0, \quad [U;K] = \frac{1}{2}cU - aT, \quad [T,K] = aU + \frac{1}{2}cT,$$

where $a \neq 0$ and c are constant and K is a conformal Killing vector field given by

$$K = \partial_t + \left(\frac{1}{2}cy + az\right)U + \left(-ay + \frac{1}{2}cz\right)T.$$

For each conformal Killing vector field K in above three (non equivalent) Lie algebras, the scale factor Φ is given by

$$\Phi = -\frac{S_{,t}}{S}.$$

Now, for each Lie algebra, consider the two distributions

$$\mathscr{D}_{U,K} = \mathrm{span}\{U,K\}, \quad \mathscr{D}_{T,K} = \mathrm{span}\{T,K\}$$

involving the conformal Killing vector field K. For the Lie algebra A the two distributions $\mathscr{D}_{U,K}$ and $\mathscr{D}_{T,K}$ are both integrable. For the Lie algebra B, only $\mathscr{D}_{T,K}$ is integrable and for the Lie algebra VII, none of them is integrable. Let M be any compact null hypersurface without boundary in the perfect-fluid space-time. Assume N is a rigging for M with screen structure $\mathscr{S}(N) = \mathscr{D}_{U,K}$ or $\mathscr{D}_{T,K}$ according to the Lie algebra A or $\mathscr{S}(N) = \mathscr{D}_{T,K}$ when dealing with the Lie algebra B. Then K is a horizontal conformal Killing vector field in the rigged null hypersurface M. If the $\mathrm{d}s^2$ have constant sectional curvature and τ^N is vanishing, we get thanks to Corollary 1 that

$$\int_M \frac{S_{,t}}{S}\,\mathrm{d}V = 0.$$

5 Newton transformation of the null hypersurface with respect to the shape operator A_N

Throughout this section the normalization is assumed to be closed. In this case Lemma 1 asserts that

$$\langle A_N X, Y\rangle - \langle A_N Y, X\rangle = \tau^N(X)\eta(Y) - \tau^N(Y)\eta(X) \tag{27}$$

for all $X, Y \in \Gamma(TM)$. It follows that the operator A_N is symmetric when restricted to the screen structure $\mathscr{S}(N)$. The ambient manifold will also considered to be Lorentzian which implies that the screen structure is Riemannian. Let $(E_0 = \xi, E_1, \ldots, E_n)$ be a quasi-orthonormal frame field of TM with $\mathscr{S}(N) = \operatorname{span}\{E_1, \ldots, E_n\}$. Then the matrix of A_N has the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \star & k_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & 0 & \cdots & k_n \end{pmatrix} \tag{28}$$

where $k_0, k_1, \ldots, k_n$ are the principal curvatures of the null hypersurface M with respect to the shape operator A_N. The scalar function $H_1 = \frac{1}{n+1}\operatorname{tr}(A_N)$ is the mean curvature of the null hypersurface with respect to A_N. For $0 \le r \le n+1$, the r-th mean curvature of the null hypersurface with respect to the shape operator A_N is defined by

$$H_r = \binom{n+1}{r}^{-1} \sigma_r(k_0, \ldots, k_n) \quad \text{and} \quad H_0 = 1,$$

and $S_r = \binom{n+1}{r} H_r$.

The characteristic polynomial of A_N is given by

$$P(t) = \det(A_N - tI) = \sum_{a=0}^{n+1} (-1)^a \binom{n+1}{a} H_a t^{n+1-a}.$$

In a similar way as for the operator $\overset{\star}{A}_\xi$ the Newton transformations T_r $(0 \le r \le n+1)$ of the null hypersurface M with respect to A_N are given by

$$T_r = \sum_{a=0}^{r} (-1)^a \binom{n+1}{a} H_a A_N^{r-a}.$$

Inductively,

$$T_0 = I \quad \text{and} \quad T_r = (-1)^r \binom{n+1}{r} H_r I + A_N \circ T_{r-1},$$

and the following items are straightforward.

Proposition 4. *(a) The transformations T_r $(0 \le r \le n+1)$ are self-adjoint on $\mathscr{S}(N)$ and commute with A_N.*

(b) $T_r E_i = (-1)^r S_r^i E_i$.

(c) $\operatorname{tr}(T_r) = (-1)^r (n+1-r) S_r$.

(d) $\operatorname{tr}(A_N \circ T_{r-1}) = (-1)^{r-1} r S_r$.

(e) $\operatorname{tr}(A_N^2 \circ T_{r-1}) = (-1)^r (S_1 S_r + (r+1) S_{r+1})$.

We prove the following.

Proposition 5. *For all $X \in \Gamma(TM)$,*

$$\begin{aligned} g(\operatorname{div}^\nabla(T_r), X) &= g\big(\operatorname{div}^\nabla(T_{r-1}), A_N X\big) - (-1)^r \big(g(X,N)\xi(S_r) - S_{r-1}^0 X(k_0)\big) \\ &\quad + g((\nabla_\xi A_N) T_{r-1}\xi, X) \\ &\quad + \sum_{i=1}^n \Big\{ \bar{g}(\bar{R}(E_i, X) T_{r-1} E_i, N) + \eta(X) B^N(E_i, A_N \circ T_{r-1} E_i) \\ &\quad + g(A_N X, E_i)\tau^N(E_i) - k_i \tau^N(\xi) \\ &\quad + E_i\big(\tau^N(T_{r-1}E_i)\eta(X) - \tau^N(X)\eta(T_{r-1}E_i)\big) \\ &\quad - \tau^N(T_{r-1}E_i)\eta(\nabla_{E_i} X) + \tau^N(\nabla_{E_i} X)\eta(T_{r-1}E_i) \\ &\quad - \tau^N(\nabla_{E_i} T_{r-1} E_i)\eta(X) + \tau^N(X)\eta(\nabla_{E_i} T_{r-1} E_i) \Big\}. \end{aligned} \tag{29}$$

Proof. Using iterative formula,

$$\begin{aligned} \operatorname{div}^\nabla(T_r) &= (-1)^r \operatorname{div}^\nabla(S_r I) + \operatorname{div}^\nabla(A_N \circ T_{r-1}) \\ &= \sum_{\alpha=0}^n \big((-1)^r (e_\alpha \cdot S_r) e_\alpha + (\nabla_{e_\alpha} A_N) T_{r-1} e_\alpha\big) + A_N(\operatorname{div}^\nabla(T_{r-1})). \end{aligned}$$

Hence, using (27),

$$\begin{aligned} g(\operatorname{div}^\nabla(T_r), X) &= (-1)^r PX(S_r) + g\big(\operatorname{div}^\nabla(T_{r-1}), A_N X\big) - \tau^N(X)\eta(\operatorname{div}^\nabla(T_{r-1})) \\ &\quad + \tau^N(\operatorname{div}^\nabla(T_{r-1}))\eta(X) + \sum_{\alpha=0}^n g\big((\nabla_{e_\alpha} A_N) T_{r-1} e_\alpha, X\big). \end{aligned} \tag{30}$$

Also

$$\begin{aligned} g\big((\nabla_{E_i} A_N) T_{r-1} E_i, X\big) &= g\big(T_{r-1} E_i, (\nabla_{E_i} A_N) X\big) + (-1)^r \eta(X) k_i S_{r-1}^i B^N(E_i, E_i) \\ &\quad + \tau^N(\nabla_{E_i} X)\eta(T_{r-1}E_i) - \tau^N(T_{r-1}E_i)\eta(\nabla_{E_i} X) \\ &\quad + \tau^N(X)\eta(\nabla_{E_i} T_{r-1} E_i) - \tau^N(\nabla_{E_i} T_{r-1} E_i)\eta(X) \\ &\quad + E_i\big(\tau^N(T_{r-1}E_i)\eta(X) - \tau^N(X)\eta(T_{r-1}E_i)\big). \end{aligned} \tag{31}$$

Apply the Gauss-Codazzi equation (7) with the substitutions

$$X \to E_i, \quad Y \to X, \quad Z \to T_{r-1} E_i,$$

to get

$$g\big(T_{r-1}E_i,(\nabla_{E_i}A_N)X\big)=\bar{g}\big(\bar{R}(E_i,X)T_{r-1}E_i,N\big)-k_i\tau^N(X)\\ +g\big((\nabla_X A_N)E_i,T_{r-1}E_i\big)+g(A_NX,E_i). \tag{32}$$

Also, we have

$$\sum_{i=1}^{n}g\big(T_{r-1}E_i,(\nabla_X A_N)E_i\big)=(-1)^{r-1}\sum_{i=1}^{n}S^i_{r-1}X(k_i)\\ =(-1)^{r-1}\big(X(S_r-S^0_{r-1}X(k_0)\big). \tag{33}$$

Now, feeding back (33) into (32) and then the resulting expression into (31) we obtain by substitution in (30) the desired expression (29). □

For the rest of the section we assume $\tau^N=0$ which is equivalent to saying that the starred entries in the matrix of A_N (see (28)) are zero, that is $A_N\xi=0$. Then

$$g(\mathrm{div}^{\nabla}(T_r),X)=\sum_{a=0}^{r-1}\sum_{i=1}^{n}\bar{g}\big(\bar{R}(E_i,N)T_aE_i,A_N^{r-1-a}X\big)\\ -\eta(X)\Big(\mathrm{tr}(\overset{\star}{A}_\xi\circ A_N\circ T_{r-1})+(n+1-r)^{-1}\xi\big(\mathrm{tr}(T_r)\big)\Big).$$

In particular when the ambient manifold is Lorentzian with constant sectional curvature c we have

$$g(\mathrm{div}^{\nabla}(T_r),X)=\eta(X)\big(c\,\mathrm{tr}(T_{r-1})+(-1)^r cS_{r-1}-\mathrm{tr}(\overset{\star}{A}_\xi\circ A_N\circ T_{r-1})-(-1)^r\xi(S_r)\big). \tag{34}$$

Now we state the following

Theorem 5. *Let (M^{n+1},g,N) be a closed normalization of a null hypersurface of a Lorentzian space form $(\bar{M}(c)^{n+2},\bar{g})$ with rigged vector field ξ and $\tau^N=0$. Then, for all $r=0,\dots,n+1$, $\mathrm{div}^{\nabla}(T_r)$ is $TM^{\perp}$-valued and*

$$\xi(S_r)+c(n+1-r)S_{r-1}=(-1)^{r-1}\,\mathrm{tr}(\overset{\star}{A}_\xi\circ A_N\circ T_{r-1}). \tag{35}$$

Proof. Let $X\in\chi(M)$. We have

$$\begin{aligned}g(\mathrm{div}^{\nabla}(T_r),X)&=g(\mathrm{div}^{\nabla}(T_r),PX)\\ &\overset{(34)}{=}\eta(PX)\Big(c\,\mathrm{tr}(T_{r-1})+(-1)^r cS_{r-1}\\ &\qquad-\mathrm{tr}(\overset{\star}{A}_\xi\circ A_N\circ T_{r-1})-(-1)^r\xi(S_r)\Big)\\ &=0\end{aligned}$$

as $\eta(PX)=0$, which shows that $\mathrm{div}^{\nabla}(T_r)$ is $TM^{\perp}$-valued. It follows from the same equation (34) setting $X:=\xi$ and using the third item in Proposition 4 that (35) holds. □

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