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On a class of nonlocal problem involving a critical exponent

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Abstract. In this work, by using the Mountain Pass Theorem, we give a result on the existence of solutions concerning a class of nonlocal p-Laplacian Dirichlet problems with a critical nonlinearity and small perturbation.

1 Introduction

This paper deals with the following elliptic problem

$$\begin{aligned} -M\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)\Delta_p u &= \beta h(x)|u|^{q-2}u + |u|^{p^*-2}u + f(x) \qquad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent, $1 < p < N$, β is a positive parameter, and $h \in L^{\frac{p^*}{p^*-q}}(\Omega)$, $f \in L^{p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$.

Where the functional M verifies,

$$M\colon (0,+\infty) \to (0,+\infty) \text{ is continuous and } m_0 = \inf_{s>0} M(s) > 0, \tag{2}$$

The problem (1) is called nonlocal because of the presence of the term $M\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)$, so it is not any more a pointwise identity. This leads us to some mathematical difficulties which makes the study of such a class of problem particularly interesting.

It is well known that the critical exponent case is often difficult because of the lack of compactness, so standard arguments cannot be carried out to handle the problem (1). As far as we know, very few results have been obtained in elliptic

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problems involving critical exponent, for instance we just quote [1], [2], [4], [5], [6], [7], [9], [11] and references therein. However, inspired by these interesting works, especially by [4], within which we will borrow some ideas, our goal will be to generalize some corresponding results partially and extend them to the case $p \neq 2$ with an existence of a perturbation f. We have to mention that [5] could be considered as the first work dealing with multivalued elliptic problem and the presence of which involves critical growth in an Orlicz-Sobolev space, where the nonlinearity can be discontinuous.

From now on, we make the following assumption:

$$\widehat{M}(t) \geq M(t)t \text{ for } t > 0 \text{ , with } \widehat{M}(t) = \int_0^t M(s)\,\mathrm{d}s. \tag{3}$$

Accordingly, we can report our main result,

Theorem 1. *Under the hypotheses* (2), (3) *and* $q \in (p, p^*)$, *there exists* $\beta^* > 0$, *such that the problem* (1) *has at least a nontrivial solutions for all* $\beta \geq \beta^*$, *provided* f *is small enough in the norm* $\|\cdot\|_*$ *of* $(W_0^{1,p}(\Omega))^*$.

Throughout this paper, we consider the C^1-functional energy

$$\phi(u) = \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u|^p\,\mathrm{d}x\right) - \frac{\beta}{q}\int_\Omega h(x)|u|^q\,\mathrm{d}x - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,\mathrm{d}x - \int_\Omega f(x)u\,\mathrm{d}x.$$

Note that

$$\begin{aligned}\phi'(u)\cdot v = M(\|u\|^p)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,\mathrm{d}x - \beta\int_\Omega h(x)|u|^{q-2}uv\,\mathrm{d}x \\ - \int_\Omega |u|^{p^*-2}uv\,\mathrm{d}x - \int_\Omega f(x)v\,\mathrm{d}x,\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. Where,

$$W_0^{1,p}(\Omega) = \Big\{u \in L^p(\Omega) : \int_\Omega |\nabla u|^p\,\mathrm{d}x < \infty, \quad u/\partial\Omega = 0\Big\}.$$

By a version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [10], [12], without Palais-Smale condition, there exists a sequence $(u_n)_n \subset W_0^{1,p}(\Omega)$ such that

$$\phi(u_n) \to c_\beta \quad \text{and} \quad \phi'(u_n) \to 0,$$

where

$$c_\beta = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\phi(\gamma(t)) > 0$$

with

$$\Gamma = \big\{\gamma \in C\big([0,1], W_0^{1,p}(\Omega)\big) : \gamma(0) = 0, \phi\big(\gamma(1)\big) < 0\big\}.$$

We recall that $u \in W_0^{1,p}(\Omega)$ is a weak solution of the problem (1) if it verifies

$$M(\|u\|^p)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \,\mathrm{d}x - \int_\Omega \beta h(x)|u|^{q-2}uv\,\mathrm{d}x \\ - \int_\Omega |u|^{p^*-2}uv\,\mathrm{d}x - \int_\Omega f(x)v\,\mathrm{d}x = 0,$$

for all $v \in W_0^{1,p}(\Omega)$.

So the critical points of ϕ are solutions of the problem (1).

2 Auxiliary results

Let $L^s(\Omega)$ be the Lebesgue space equipped with the norm $|u|_s = \left(\int_\Omega |u|^s\,\mathrm{d}x\right)^{\frac{1}{s}}$, $1 \le s < \infty$ and let $W_0^{1,p}(\Omega)$ be the usual Sobolev space with respect to the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

Now we can define the best Sobolev constant

$$S = \inf_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\left(\int_\Omega |u|^{p^*}dx\right)^{\frac{p}{p^*}}}.$$

In the sequel, we are to compare the minimax level c_β with a suitable number which involves the constant S.

Lemma 1. *There exist $\sigma > 0, \rho > 0$ and $e \in W_0^{1,p}(\Omega)$ with $\|e\| > \rho$ such that*

(i) $\inf_{\|u\|=\rho}\phi(u) \ge \sigma > 0$;

(ii) $\phi(e) < 0$.

Proof. (i) From the Hölder's inequality and the compact embedding theorem, we have

$$\begin{aligned}\phi(u) &\ge \frac{m_0}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \frac{\beta}{q}|h|_\theta\int_\Omega |u|^q\,\mathrm{d}x - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,\mathrm{d}x - \int_\Omega f(x)u\,\mathrm{d}x \\ &\ge C_0\|u\|^p - \frac{C_1\beta}{q}|h|_\theta\|u\|^q - \frac{1}{p^*S^{\frac{p^*}{p}}}\|u\|^{p^*} - |f|_{p'}|u|_p \\ &\ge C_0\|u\|^p - \frac{C_1\beta}{q}|h|_\theta\|u\|^q - C_2\|u\|^{p^*} - C_3\|f\|_*\|u\|, \end{aligned} \tag{4}$$

with $\theta = \frac{p^*}{[p^*-q]}$ and $C_0, C_1, C_2, C_3 > 0$. Since $q \in (p, p^*)$ then for $\|u\| = \rho > 0$ small enough, we may find $\sigma > 0$ such that

$$\inf_{\|u\|=\rho}\phi(u) \ge \sigma > 0$$

where $\|f\|_*$ be small.

(ii) Fix $v \in C_0^\infty(\Omega \setminus \{0\})$ with $v \geq 0$ in Ω and $\|v\| = 1$.

$$\phi(tv) \leq A|t|^p - \beta|t|^\theta \int_\Omega h(x)v^\theta \,\mathrm{d}x + C - \frac{|t|^{p^*}}{p^*} \int_\Omega h(x)v^{p^*} \,\mathrm{d}x - |t| \int_\Omega f(x)v \,\mathrm{d}x,$$

with A and C are two positive constants, it follows that

$$\phi(tv) \to -\infty \quad \text{as} \quad |t| \to \infty. \qquad \square$$

Lemma 2. $\lim\limits_{\beta \to +\infty} c_\beta = 0.$

Proof. Let v the function given by the previous lemma 1, then there is $t_\beta > 0$ such that $\phi(t_\beta v) = \max\limits_{t \geq 0} \phi(tv)$, thereafter,

$$M(\|t_\beta v\|^p) t_\beta^p \|v\|^p = \beta t_\beta^q \int_\Omega h(x)|v|^q \,\mathrm{d}x + t_\beta^{p^*} \int_\Omega |v|^{p^*} \,\mathrm{d}x + t_\beta^2 \int_\Omega f(x)v^2 \,\mathrm{d}x, \tag{5}$$

it follows from (3) that there is $c > 0$, such that

$$\widehat{M}(s) \leq c|s| \quad \text{for all } s > s_0 > 0\,.$$

Hence

$$ct_\beta^p \|v\|^p \geq \beta t_\beta^q \int_\Omega h(x)|v|^q \,\mathrm{d}x + t_\beta^{p^*} \int_\Omega |v|^{p^*} \,\mathrm{d}x + t_\beta^2 \int_\Omega f(x)v^2 \,\mathrm{d}x$$

and then t_β is bounded, so there exists a sequence $\beta_n \to +\infty$ and $t_* \geq 0$ with $t_{\beta_n} \to t_*$ as $n \to +\infty$ and thus

$$M(\|t_{\beta_n} v\|^p) t_{\beta_n}^p \|v\|^p < C, \quad \forall n \in \mathbb{N},$$

with C is a positive constant, which yields

$$\beta_n t_*^q \int_\Omega h(x)|v|^q \,\mathrm{d}x + t_*^{p^*} \int_\Omega |v|^{p^*} \,\mathrm{d}x \leq C, \quad \forall n \in \mathbb{N}.$$

Hence, we claim that $t_* = 0$, otherwise, $t_* > 0$ and then the last inequality becomes

$$\beta_n t_*^q \int_\Omega h(x)|v|^q \,\mathrm{d}x + t_*^{p^*} \int_\Omega |v|^{p^*} \,\mathrm{d}x \to +\infty$$

as $n \to +\infty$, which is absurd, so $t_* = 0$.

Taking $\gamma_0(t) = te$, with $\gamma_0 \in \Gamma$, then we get

$$0 < c_\beta \leq \max_{t \in [0,1]} \phi(\gamma_0(t)) \leq \frac{1}{p} \widehat{M}(t_\beta^p).$$

Since $\widehat{M}(t_\beta^p) \to 0$ then $\lim_{\beta \to \infty} c_\beta = 0$. $\square$

As consequence of the above lemma, there exists $\beta^* > 0$ such that for every $\beta \geq \beta^*$,

$$c_\beta < \left(1 - \frac{p}{p^*}\right)(m_0 S)^{\frac{N}{p}}.$$

Lemma 3. *Let $(u_n)_n \subset W_0^{1,p}(\Omega)$, with $\phi(u_n) \to c_\beta$, and $\phi'(u_n) \to 0$. Then $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Assume that $\phi(u_n) \to c_\beta$, and $\phi'(u_n) \to 0$, then we have

$$\begin{aligned} pc_\beta + o(1) + o(1)\|u_n\| &= p\phi(u_n) - (\phi'(u_n) \cdot u_n) \\ &\geq C_4\beta\left(1 - \frac{p}{q}\right)|h|_\theta\|u_n\|^q + C_5\left(1 - \frac{p}{p^*}\right)\|u_n\|^{p^*} \\ &\quad + (p-1)\int_\Omega f(x)u_n \,\mathrm{d}x, \end{aligned}$$

where $\theta = \frac{p^*}{p^*-q}, C_4, C_5 > 0$, we infer that $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$. □

3 Proof of the main result

Proof. (Theorem 1) As it was previously mentioned, we are to apply a version of the Mountain Pass theorem without Palais-Smale condition to obtain a sequence $(u_n)_n \subset W_0^{1,p}(\Omega)$ such that $\phi(u_n) \to c_\beta$ and $\phi'(u_n) \to 0$.

Because $(u_n)_n$ is a bounded sequence in $W_0^{1,p}(\Omega)$, passing to a subsequence, so we may find $\gamma > 0$ with

$$\|u_n\| \to \gamma,$$

it follows from the continuity of M that

$$M(\|u_n\|^p) \to M(\gamma^p).$$

On the other side, we know that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, then

$$u_n \to u \text{ in } L^r(\Omega), \quad \text{for } 1 < r < p^*$$

and

$$u_n(x) \to u(x) \quad \text{a.e. } x \in \Omega.$$

By the Lebesgue Dominated Theorem,

$$\int_\Omega h(x)|u_n|^q \,\mathrm{d}x \to \int_\Omega h(x)|u|^q \,\mathrm{d}x.$$

Further,

$$\begin{aligned} |\nabla u_n|^p &\rightharpoonup |\nabla u|^p + \mu \quad \text{weak}^*\text{-sense of measure}, \\ |u_n|^{p^*} &\rightharpoonup |u|^{p^*} + \nu \quad \text{weak}^*\text{-sense of measure}. \end{aligned}$$

Afterwards, as a consequence of the concentration compactness principle due to Lion [8], there is an index set I, which is an at most countable set such that

$$\nu = \sum_{i\in I} \nu_i\delta_i, \quad \mu \geq \sum_{i\in I} \mu_i\delta_i$$

and

$$S\nu_i^{p/p^*} \leq \mu_i,$$

for any $i \in I$ with $(\mu_i)_i, (\nu_i)_i \subset [0, \infty)$, δ_i is the Dirac mass and $(\mu_i)_i, (\nu_i)_i$ are nonatomic positive measures. We claim that $I = \emptyset$, otherwise, we have $I \neq \emptyset$ and fix $i \in I$. Taking $\psi \in C_0^\infty(\Omega, [0,1])$ such that $\psi \equiv 1$ if $|x| < 1$ and $\psi \equiv 0$ when $|x| > 2$ with $|\nabla\psi|_\infty \leq 2$. Putting $\psi_\rho(x) = \psi\big(\frac{x-x_i}{\rho}\big)$ for $\rho > 0$, noting that $(\psi_\rho u_n)$ is bounded thus $\phi'(u_n) \cdot (\psi_\rho u_n) \to 0$, that is

$$\begin{aligned} M\left(\int_\Omega |\nabla u_n|^p\right) &\int_\Omega |\nabla u_n\ |^{p-2}\ \nabla u_n \cdot \nabla\psi_\rho u_n \,\mathrm{d}x \\ &= -M\left(\int_\Omega |\nabla u_n|^p\right) \int_\Omega |\nabla u_n\ |^p\ \psi_\rho \nabla u_n \,\mathrm{d}x + \int_\Omega |u_n|^{p^*-2} u_n \cdot \psi_\rho u_n \,\mathrm{d}x \\ &\qquad + \beta \int_\Omega h(x)|u_n|^{q-2} u_n \psi_\rho u_n \,\mathrm{d}x + \int_\Omega f(x)\psi_\rho u_n + O_n(1). \end{aligned}$$

As it is known that $B_{2\rho}(x_i)$ is the support of the functional ψ_ρ and by applying Hölder inequality then we get

$$\begin{aligned} \left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n.\nabla\psi_\rho u_n \,\mathrm{d}x\right| &\leq \int_{B_{2\rho}(x_i)} |\nabla u_n|^{p-1}\ |u_n \nabla\psi_\rho| \,\mathrm{d}x \\ &\leq \Big(\int_{B_{2\rho}(x_i)} |\nabla u_n|^p\Big)^{\frac{1}{p'}} \Big(\int_{B_{2\rho}(x_i)} |u_n\nabla\psi_\rho|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \\ &\leq C\Big(\int_{B_{2\rho}(x_i)} |u_n\nabla\psi_\rho|^p \,\mathrm{d}x\Big)^{\frac{1}{p}}. \end{aligned}$$

By the Dominated convergence Theorem we entail that

$$\int_{B_{2\rho}(x_i)} |u_n\nabla\psi_\rho|^p \,\mathrm{d}x \to 0$$

when $n \to \infty$ and $\rho \to 0$.

Hence,

$$\lim_{\rho\to 0}\left[\lim_n \int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n.\nabla\psi_\rho\right] = 0.$$

On the other hand, we recall that $M(\|u_n\|^p)$ converges to $M(\gamma^p)$, so we reach

$$\lim_{\rho\to 0}\left[\lim_n M(\|u_n\|^p) \int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n.\nabla\psi_\rho\right] = 0.$$

Similarly,

$$\lim_{\rho\to 0}\lim_n\left[\int_\Omega h(x)|u_n|^{q-2}u_n\psi_\rho u_n\right] = 0,$$

$$\lim_{\rho\to 0}\lim_n\left[\int_\Omega f(x)\psi_\rho u_n\right] = 0$$

and thus

$$\int_\Omega M(\gamma^p)\psi_\rho \,\mathrm{d}\mu + O_\rho(1) \leq \int_\Omega \psi_\rho \,\mathrm{d}\nu.$$

Tending ρ to zero we conclude that

$$\nu_i \geq M(\gamma^P)\mu_i \geq m_0\mu_i,$$

from the definition of ν and μ we have

$$\nu_i \geq (m_0 S)^{\frac{N}{p}}.$$

It does not make sense, indeed, let $i \in I$ such that

$$\nu_i \geq (m_0 S)^{\frac{N}{p}}.$$

Since $(u_n)_n$ is a $(PS)_{c_\beta}$ for the functional ϕ, then

$$\begin{aligned} pc_\beta &= p\phi(u_n) = p\phi(u_n) - \phi'(u_n)\cdot u_n + O_n(1) \\ &\geq \left(1 - \frac{p}{p^*}\right)\int_\Omega \psi_\rho |u_n|^{p^*}\,\mathrm{d}x + O_n(1), \end{aligned}$$

tending $n \to +\infty$, therefore

$$pc_\beta \geq \left(1-\frac{p}{p^*}\right)\sum_{i\in I}\psi_\rho(x_i)\nu_i = \left(1-\frac{p}{p^*}\right)\sum_{i\in I}\nu_i \geq \left(1-\frac{p}{p^*}\right)(m_0S)^{\frac{N}{p}},$$

which cannot occur (because $\lim_{\beta\to\infty} c_\beta = 0$), thereafter I is empty and thereby $u_n \to u$ in $L^{p^*}(\Omega)$.

On the other hand,

$$\begin{aligned} &M\left(\|u_n\|^p\right)\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\left(\nabla u_n - \nabla u\right)\,\mathrm{d}x \\ &\quad = \phi'(u_n).\,(u_n-u) + \beta\int_\Omega h(x)|u_n|^{q-2}u_n(u_n-u)\,\mathrm{d}x + \int_\Omega f(x)(u_n-u)\,\mathrm{d}x \\ &\qquad + \int_\Omega |u_n|^{p^*-2}u_n(u_n-u)\,\mathrm{d}x - M\left(\|u_n\|^p\right)\int_\Omega |\nabla u|^{p-2}\nabla u\left(\nabla u_n - \nabla u\right)\,\mathrm{d}x. \end{aligned}$$

In view of $u_n \rightharpoonup u$, a standard argument (similar to those found in [3]) shows that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{for a.e. } x \in \Omega,$$

and

$$u_n(x) \to u(x) \quad \text{for a.e. } x \in \Omega,$$

then

$$M\left(\|u_n\|^p\right)\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\left(\nabla u_n - \nabla u\right)\,\mathrm{d}x \to 0.$$

Using the following inequalities $\forall x, y \in \mathbb{R}^N$

$$|x-y|^\gamma \leq 2^\gamma(|x|^{\gamma-2}x - |y|^{\gamma-2}y)\cdot(x-y) \qquad \text{if } \gamma \geq 2,$$
$$|x-y|^2 \leq \frac{1}{\gamma-1}(|x|+|y|)^{2-\gamma}(|x|^{\gamma-2}x - |y|^{\gamma-2}y)\cdot(x-y) \qquad \text{if } 1<\gamma<2,$$

where $x \cdot y$ is the inner product in $\mathbb{R}^N$, we get

$$c\, m_0 \int_\Omega |\nabla u_n - \nabla u|^p \,\mathrm{d}x \leq M\left(\|u_n\|^p\right) \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\,\mathrm{d}x.$$

Consequently,

$$\|u_n - u\| \to 0,$$

which will imply that

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega).$$

Thus

$$\phi(u) = c_\beta, \quad \phi'(u) = 0$$

and we get the solution u_1, it is a mountain pass type. □

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