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A determinant formula for the relative class number of an imaginary abelian number field

Mikihito Hirabayashi

Abstract. We give a new formula for the relative class number of an imaginary abelian number field K by means of determinant with elements being integers of a cyclotomic field generated by the values of an odd Dirichlet character associated to K . We prove it by a specialization of determinant formula of Hasse.

1 Introduction

There are lots of formulas for the relative class number of an imaginary abelian number field K by means of determinant (see [5] for bibliography). In this paper we give such a new formula. We prove it by a specialization of the determinant formula for generalized group matrix which appears in [2, §13]. The key idea is a transformation of generalized Bernoulli numbers and a transformation of their product over the odd characters to one over the even characters. In our formula, elements of the determinant are integers of a cyclotomic field generated by the values of an odd Dirichlet character associated to K , whereas elements of the determinants are rational numbers for known formulas. We may regard our formula as an imaginary version of Hasse's formula [2, §16, (3)], which expresses the class number of a real abelian number field by means of determinant with elements being logarithms of cyclotomic units of its cyclic subfields.

2 Results

Let K be an imaginary abelian number field of degree n and with conductor f , and let K_0 be the maximal real subfield of K . Let H_0 be the subgroup of the group $(\mathbf{Z}/f\mathbf{Z})^\times$ of reduced residue classes modulo f corresponding to K_0 . Let X_0 be the set of Dirichlet characters associated to K_0 .

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We assume that the Dirichlet characters χ associated to K , which we call characters of K for short, are primitive and that, as usual, $\chi(x) = 0$ for an integer x not relatively prime to the conductor $f(\chi)$ of χ .

We classify the group X_0 by the following equivalence \sim : for characters $\chi, \psi \in X_0$ let $\chi \sim \psi$ if and only if there exists an integer m such that m is relatively prime to n_χ and that $\psi = \chi^m$, where n_χ is the order of χ . We call the classes classified by this equivalence Frobenius classes. Let $\{\psi_0\}$ be a system of representatives of the Frobenius classes. For a representative ψ_0 let t_{ψ_0} be an integer such that the quotient group $(\mathbf{Z}/f\mathbf{Z})^\times/H_{\psi_0}$ is generated by a class represented by $t_{\psi_0} \bmod f$, where $H_{\psi_0} = \{x \bmod f \in (\mathbf{Z}/f\mathbf{Z})^\times; \psi_0(x) = 1\}$.

We fix an odd character χ_1^* of K . As we will see, the elements of the determinant of our formula are integers of the field generated by the values of the character χ_1^* .

For an even character χ_0 of K and for an element $a \bmod f$ of $(\mathbf{Z}/f\mathbf{Z})^\times$ let

$$u_{\chi_0}(a) = -\chi_1^*(a) \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^f \chi_1^*(x)R_f(ax),$$

where $R_f(a)$ is the least positive residue modulo f of a . Then we define a matrix U by

$$U = (u_{\psi_0}(st_{\psi_0}^{-k}))_{(s \bmod f)H_0; \psi_0, 0 \leq k \leq \varphi(n_{\psi_0})-1},$$

where $(s \bmod f)H_0$ runs in the rows over the quotient group $(\mathbf{Z}/f\mathbf{Z})^\times/H_0$, which is isomorphic to the Galois group G_0 of K_0 ; ψ_0 and k run in the columns: $\{\psi_0\}$ is a system defined above and φ is the Euler totient function. Here, $t_{\psi_0}^{-k} \bmod f$ is the inverse of $t_{\psi_0}^k \bmod f$, i.e., $t_{\psi_0}^{-k}$ is an integer satisfying $t_{\psi_0}^{-k}t_{\psi_0}^k \equiv 1 \pmod{f}$.

With the notation above we have the following

Theorem 1. *For an imaginary abelian number field K of degree n and with conductor f , we have*

$$\det U = \pm \frac{(2f)^{n/2} c g^*}{Q w} h^*$$

where h^* is the relative class number of K , Q is the Hasse unit index of K , w is the number of roots of unity in K , and g^* is defined by

$$g^* = \prod_{\chi_1} \prod_{p|f} (1 - \chi_1(p))$$

where the products \prod_{χ_1} and $\prod_{p|f}$ are taken over the odd characters χ_1 of K and the prime numbers p dividing f , respectively, and c is a natural number expressed by

$$c = \prod_{p|n_0} p^{\frac{1}{2} \sum_{p^k|n_0} (q(\frac{n_0}{p^k}) - \frac{n_0}{p^k})},$$

where the product $\prod_{p|n_0}$ and the sum $\sum_{p^\mu|n_0}$ are taken over prime numbers p dividing $n_0 = n/2$ and the powers of p dividing n_0 , respectively, and $q(m)$ is the number of solutions of $x^m = 1$ in G_0 .

We remark here that the elements $u_{\chi_0}(a)$ and the matrix U depend on the character χ_1^* , as we see in the examples below, and that, in addition, U depends on the choice of integers t_{ψ_0} . In fact, we have different U 's for different t_{ψ_0} 's in the case of $K = \mathbf{Q}(\zeta_7)$, the 7th cyclotomic field. Moreover, we note that the matrix U never coincides with any matrix in known formulas, because U always contains a constant column corresponding to the principal character $\psi_0 = 1$.

As seen by definition, the number g^* may be zero and then remains a problem of how to construct such a formula in Theorem 1 in case of $g^* = 0$.

For the cyclotomic fields of prime power conductor we have the following corollaries.

Corollary 1. *For the cyclotomic field $K = \mathbf{Q}(\zeta_{p^\rho})$ of conductor p^ρ ($\rho \geq 1$), p an odd prime, we have*

$$\begin{aligned} \det U &= \det(u_{\psi_0}(g^i t_{\psi_0}^{-k}))_{0 \leq i \leq \frac{p^{\rho-1}(p-1)}{2} - 1; \psi_0, 0 \leq k \leq \varphi(n_{\psi_0}) - 1} \\ &= \pm (2p^\rho)^{\frac{p^{\rho-1}(p-1)}{2} - 1} h^*, \end{aligned}$$

where g is a primitive root modulo p^ρ .

For the field $K = \mathbf{Q}(\zeta_{p^\rho})$ we can take $t_{\psi_0} = g$ for every $\psi_0 \neq 1$ and $t_{\psi_0} = 1$ for $\psi_0 = 1$.

Corollary 2. *For the cyclotomic field $K = \mathbf{Q}(\zeta_{2^\rho})$ of conductor 2^ρ ($\rho \geq 2$) we have*

$$\begin{aligned} \det U &= (u_{\psi_0}(5^i t_{\psi_0}^{-k}))_{0 \leq i \leq 2^{\rho-2} - 1; \psi_0, 0 \leq k \leq \varphi(n_{\psi_0}) - 1} \\ &= \pm 2^{(\rho+1)2^{\rho-2} - \rho} h^*. \end{aligned}$$

For the field $K = \mathbf{Q}(\zeta_{2^\rho})$ we can take $t_{\psi_0} = 5$ for every $\psi_0 \neq 1$ and $t_{\psi_0} = 1$ for $\psi_0 = 1$.

Here we give examples. We adopt the basic characters which Hasse used in [2]. For an odd prime p let χ_p be an odd character modulo p of order $p - 1$ and ψ_{p^ρ} ($\rho \geq 2$) an even character modulo p^ρ of order $p^{\rho-1}$; in addition $\psi_{p^\rho}^p = \psi_{p^{\rho-1}}$. For the prime 2 let χ_4 be the odd character modulo 4 and ψ_{2^ρ} ($\rho \geq 3$) an even character modulo 2^ρ of order $2^{\rho-2}$; in addition $\psi_{2^\rho}^2 = \psi_{2^{\rho-1}}$. The subscript of a basic character denotes the conductor.

For the following calculation of the values of $u_{\chi_0}(a)$, we use the identity

$$\sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^f \chi_1^*(x) R_f(ax) = \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^{\lfloor f/2 \rfloor} \chi_1^*(x) (2R_f(ax) - f).$$

Example 1. Let $K = \mathbf{Q}(\zeta_5)$, i.e., $p = 5$, $\rho = 1$. Take $g = 2$ and $\chi_1^* = \chi_5$. Then $\{\psi_0\} = \{1, \chi_5^2\}$ and

$$\begin{aligned} u_1(a) &= -\chi_5(a)(2R_5(a) - 5 + i(2R_5(2a) - 5)), \\ u_{\chi_5^2}(a) &= -\chi_5(a)(2R_5(a) - 5). \end{aligned}$$

Consequently

$$U = \begin{pmatrix} u_1(1) & u_{\chi_5^2}(1) \\ u_1(2) & u_{\chi_5^2}(2) \end{pmatrix} = \begin{pmatrix} 3+i & 3 \\ 3+i & i \end{pmatrix}$$

and hence $\det U = -2 \cdot 5$. Otherwise, by Corollary 1 and [2, Tafel II], $\det U = \pm(2 \cdot 5)^{\frac{5-1}{2}-1} \cdot 1 = \pm 2 \cdot 5$.

Taking $g = 2$ and $\chi_1^* = \chi_5^3$, we have

$$U = \begin{pmatrix} 3-i & 3 \\ 3-i & -i \end{pmatrix}$$

and hence $\det U = -2 \cdot 5$.

Example 2. Let $K = \mathbf{Q}(\zeta_{2^3})$, i.e., $p = 2, \rho = 3$. Take $\chi_1^* = \chi_4$. Then $\{\psi_0\} = \{1, \psi_{2^3}\}$ and

$$\begin{aligned} u_1(a) &= -2\chi_4(a)(R_{2^3}(a) - R_{2^3}(3a)), \\ u_{\psi_{2^3}}(a) &= -2\chi_4(a)(R_{2^3}(a) - 4). \end{aligned}$$

Consequently

$$U = \begin{pmatrix} u_1(1) & u_{\psi_{2^3}}(1) \\ u_1(5) & u_{\psi_{2^3}}(5) \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 4 & -2 \end{pmatrix}$$

and hence $\det U = -2^5$. Otherwise, by Corollary 2 and [2, Tafel II], $\det U = \pm 2^{(3+1)2^3-2-3} \cdot 1 = \pm 2^5$.

Taking $\chi_1^* = \chi_4\psi_8$, we have

$$U = \begin{pmatrix} 8 & 6 \\ 8 & 2 \end{pmatrix}$$

and hence $\det U = -2^5$.

Example 3. Let $K = \mathbf{Q}(\sqrt{-3}, \sqrt{5})$. Take $\chi_1^* = \chi_3$. Then $\{\psi_0\} = \{1, \chi_5^2\}$ and

$$\begin{aligned} u_1(a) &= -2\chi_3(a)(R_{15}(a) - R_{15}(2a) + R_{15}(4a) + R_{15}(7a) - 15), \\ u_{\chi_5^2}(a) &= -2\chi_3(a)(R_{15}(a) + R_{15}(4a) - 15). \end{aligned}$$

Consequently

$$U = \begin{pmatrix} u_1(1) & u_{\chi_5^2}(1) \\ u_1(2) & u_{\chi_5^2}(2) \end{pmatrix} = \begin{pmatrix} 10 & 20 \\ 10 & -10 \end{pmatrix}$$

and hence $\det U = -2^2 \cdot 3 \cdot 5^2$. Otherwise, since $c = 1, g^* = 2, w = 2 \cdot 3$ and $Q = 1$, which is obtained by [2, Tafel II], we have by Theorem 1

$$\det U = \pm \frac{(2f)^{n/2} c g^*}{Qw} h^* = \pm \frac{(2 \cdot 15)^2 \cdot 1 \cdot 2}{1 \cdot 2 \cdot 3} \cdot 1 = \pm 2^2 \cdot 3 \cdot 5^2.$$

Taking $\chi_1^* = \chi_3\chi_5^2$, we have

$$U = \begin{pmatrix} 30 & 20 \\ 30 & 10 \end{pmatrix}$$

and hence $\det U = -2^2 \cdot 3 \cdot 5^2$.

3 The determinant of a generalized group matrix

In the second chapter of the book [2] Hasse gave two transformations of the class number formula for a real abelian number field; the first transformation is an application of summations $\sum_x \chi(x)A_f(x)$ to the group matrix, $A_f(x)$ an ordinary distribution (cf. [2, p.18] or [4, Lemma 12.15]), and the second transformation is one for summations $\sum_s \chi(s)u_\chi(s)$ and for the matrix $U_{\mathfrak{G}}$ (see Lemma 1).

By the first transformation, replacing the distribution $A_f(x)$ in [2, p.18] with

$$A_f(x) = - \left(\frac{R_f(x)}{f} - \frac{1}{2} \right),$$

we can obtain the formula of Girstmair [1] with Maillet determinant for the relative class number of an imaginary abelian number field with conductor f .

For the proof of our formula we need the following lemmas. Let \mathfrak{G} be an abelian group of order n and \mathfrak{X} the group of characters of \mathfrak{G} . For $\chi \in \mathfrak{X}$ let

$$\mathfrak{H}_\chi = \{x \in \mathfrak{G}; \chi(x) = 1\}.$$

For $s \in \mathfrak{G}$ and $\chi \in \mathfrak{X}$ let $u_\chi(s)$ be a complex-valued function satisfying the following conditions:

- (i) $u_\chi(s) = u_{\chi^\nu}(s)$ for $s \in \mathfrak{G}$ and $\nu \in \mathbf{Z}$ relatively prime to the order n_χ of χ .
- (ii) $u_\chi(s) = u_\chi(s')$ for $s, s' \in \mathfrak{G}$ with $\chi(s) = \chi(s')$.

We classify the group \mathfrak{X} by the Frobenius equivalence defined as in §2. Let $\{\psi\}$ be a system of representatives of the Frobenius classes of \mathfrak{X} . For a character ψ let t_ψ be a representative of a generator $t_\psi \mathfrak{H}_\psi$ of the cyclic group $\mathfrak{G}/\mathfrak{H}_\psi$. Then we define a matrix $U_{\mathfrak{G}}$ by

$$U_{\mathfrak{G}} = (u_\psi(st_\psi^{-k}))_{s \in \mathfrak{G}; \psi, 0 \leq k \leq \varphi(n_\psi)-1},$$

where s runs in the rows, and ψ and k run in the columns.

Lemma 1. [2, §14] For the matrix $U_{\mathfrak{G}}$ we have

$$\det U_{\mathfrak{G}} = \pm c_{\mathfrak{G}} \prod_{\chi \in \mathfrak{X}} \sum_{s \bmod \mathfrak{H}_\chi} \chi(s)u_\chi(s),$$

where $c_{\mathfrak{G}}$ is a positive number defined by

$$c_{\mathfrak{G}} = \pm \frac{1}{\det(\chi(s))_{s \in \mathfrak{G}, \chi \in \mathfrak{X}}} \prod_{\psi} \left(\left(\frac{n}{n_\psi} \right)^{\varphi(n_\psi)} \det(\psi(t_\psi)^{ik})_{\substack{1 \leq i \leq n_\psi \\ (i, n_\psi) = 1 \\ 0 \leq k \leq \varphi(n_\psi)-1}} \right)$$

and $s \bmod \mathfrak{H}_\chi$ in the sum $\sum_{s \bmod \mathfrak{H}_\chi}$ runs over the quotient group $\mathfrak{G}/\mathfrak{H}_\chi$.

Lemma 2. [2, §14 and §15] For an abelian group \mathfrak{G} of order n the number $c_{\mathfrak{G}}$ is a natural number and holds

$$c_{\mathfrak{G}} = \prod_{p|n} p^{\frac{1}{2} \sum_{p^k|n} (q(\frac{n}{p^k}) - \frac{n}{p^k})},$$

where the product and summation are taken over the prime numbers p dividing n and over the powers of p dividing n , and $q(m)$ is the number of solutions of $x^m = 1$ in \mathfrak{G} . Therefore $c_{\mathfrak{G}} = 1$ if and only if \mathfrak{G} is cyclic.

4 Proof of Theorem 1

Proof of Theorem 1. We start with the arithmetic class number formula for h^* ,

$$h^* = Qw \prod_{\chi_1} \left(-\frac{1}{2} B_{1, \chi_1} \right).$$

For any odd character χ_1 of K we have

$$B_{1, \chi_1} = \frac{1}{f(\chi_1)} \sum_{a=1}^{f(\chi_1)} \chi_1(a)a = \frac{1}{f} \sum_{a=1}^f \chi_1(a)a$$

and like as [4, Lemma 8.7] we have

$$\sum_{\substack{a=1 \\ (a, f)=1}}^f \chi_1(a)a = \prod_{p|f} (1 - \chi_1(p)) \cdot \sum_{a=1}^f \chi_1(a)a.$$

In fact, if $p \mid f$, we have $\chi(p) \sum_{a=1}^f \chi(a)a = \sum_{b=1}^{f/p} \chi(pb)(pb)$ and hence

$$\begin{aligned} \prod_{p|f} (1 - \chi_1(p)) \cdot \sum_{a=1}^f \chi_1(a)a &= \sum_{a=1}^f \chi(a)a + \sum_{\substack{d|f \\ d>1}} \left(\sum_{\substack{d'|d \\ d'>1}} \mu(d') \right) \chi(d)d \\ &= \sum_{a=1}^f \chi(a)a - \sum_{\substack{d|f \\ d>1}} \chi(d)d = \sum_{\substack{a=1 \\ (a, f)=1}}^f \chi_1(a)a, \end{aligned}$$

where $\mu(\cdot)$ is the Möbius function.

Therefore, putting

$$S(\chi_1) = \sum_{\substack{a=1 \\ (a, f)=1}}^f \chi_1(a)a,$$

we have by the arithmetic class number formula for h^*

$$\frac{(-2f)^{n/2} g^* h^*}{Qw} = \prod_{\chi_1} S(\chi_1)$$

and hence our task is to show that the product of the right-hand side is $\pm c^{-1} \det U$.

Recall that χ_1^* is a fixed odd character of K . For an even character χ_0 of K let

$$H_{\chi_0} = \{x \bmod f \in (\mathbf{Z}/f\mathbf{Z})^\times : \chi_0(x) = 1\}.$$

Choose a system of representatives $s \bmod f$ of $(\mathbf{Z}/f\mathbf{Z})^\times / H_{\chi_0}$. Then, for an odd character $\chi_1 = \chi_0 \chi_1^*$ of K we have

$$S(\chi_1) = S(\chi_0 \chi_1^*) = \sum_{s \bmod H_{\chi_0}} \chi_0(s) u_{\chi_0}(s),$$

where

$$u_{\chi_0}(s) = \chi_1^*(s) \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^f \chi_1^*(x) R_f(sx).$$

Therefore we have

$$\prod_{\chi_1} S(\chi_1) = \prod_{\chi_0} \sum_{s \bmod H_{\chi_0}} \chi_0(s) u_{\chi_0}(s),$$

where the product \prod_{χ_0} is taken over the even characters χ_0 of K .

Here we use Lemmas 1 and 2 by letting \mathfrak{G} be the group $(\mathbf{Z}/f\mathbf{Z})^\times/H_0$ and by replacing n by $n/2$, χ by χ_0 , $U_{\mathfrak{G}}$ by U , $c_{\mathfrak{G}}$ by c , and $u_{\psi}(s)$ by $u_{\psi_0}(s)$.

To use Lemma 1, we need to check the $u_{\chi_0}(s)$ for meeting the conditions (i) and (ii) in §3. First let ν be an integer relatively prime to the order of χ_0 . Then $\chi_0^\nu(x) = 1$ if and only if $\chi_0(x) = 1$. Hence

$$\begin{aligned} u_{\chi_0^\nu}(s) &= \chi_1^*(s) \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0^\nu(x)=1}}^f \chi_1^*(x) R_f(sx) \\ &= u_{\chi_0}(s). \end{aligned}$$

Secondly let s, s' be integers relatively prime to f satisfying $\chi_0(s) = \chi_0(s')$. Hence

$$\begin{aligned} u_{\chi_0}(s') &= \chi_1^*(s') \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^f \chi_1^*(x) R_f(s'x) \\ &= \chi_1^*(s') \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^f \chi_1^*(s(s')^{-1}x) R_f(s' \cdot s(s')^{-1}x) \\ &= \chi_1^*(s') \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^f \chi_1^*(s) \chi_1^*(s')^{-1} \chi_1^*(x) R_f(sx) \\ &= \chi_1^*(s) \sum_{\substack{x=1 \\ (x,f)=1 \\ \chi_0(x)=1}}^f \chi_1^*(x) R_f(sx) \\ &= u_{\chi_0}(s). \end{aligned}$$

Here $(s')^{-1} \bmod f$ is the inverse of $s' \bmod f$. Therefore we have checked the conditions.

Consequently, by Lemma 1 we obtain

$$\frac{(-2f)^{n/2} g^* h^*}{Qw} = \prod_{\chi_1} S(\chi_1) = \frac{1}{\pm c} \det U,$$

that is,

$$\det U = \pm \frac{(2f)^{n/2} c g^* h^*}{Qw}$$

and by Lemma 2 we immediately obtain the expression of c . This completes the proof.

Corollaries 1 and 2 are directly obtained by Theorem 1, because for the cyclotomic fields K of prime power conductors we have $g^* = 1$ by definition, $c = 1$ by Lemma 2 and $Q = 1$ by [2, Satz 27].

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Author's address:

KANAZAWA INSTITUTE OF TECHNOLOGY, 7-1 OHGIGAOKA, NONOICHI, ISHIKAWA 921-8501
JAPAN

E-mail: hira@neptune.kanazawa-it.ac.jp

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