COMMUNICATIONS IN MATHEMATICS

VOLUME 22/2014
No. 2

ISSN 1804-1388 (Print)
ISSN 2336-1298 (Online)

Available online at
http://cm.osu.cz

UNIVERSITAS OSTRAVIENSIS

Communications in Mathematics 22 (2014) 107–132

Bounds for Convex Functions of Čebyšev Functional Via Sonin's Identity with Applications

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Abstract. Some new bounds for the Čebyšev functional in terms of the Lebesgue norms

$$\left\| f - \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}$$

and the Δ-seminorms

$$\|f\|_p^{\Delta} := \left(\int_a^b \int_a^b |f(t) - f(s)|^p \,\mathrm{d}t\,\mathrm{d}s \right)^{\frac{1}{p}}$$

are established. Applications for mid-point and trapezoid inequalities are provided as well.

1 Introduction

For two Lebesgue integrable functions $f, g\colon [a,b] \to \mathbb{R}$, consider the *Čebyšev functional*

$$C(f,g) := \frac{1}{b-a} \int_a^b f(t)g(t)\,\mathrm{d}t - \frac{1}{(b-a)^2} \int_a^b f(t)\,\mathrm{d}t \int_a^b g(t)\,\mathrm{d}t\,.$$

In 1935, Grüss [7] showed that

$$|C(f,g)| \le \frac{1}{4}(M-m)(N-n)\,, \tag{1}$$

provided that there exists the real numbers m, M, n, N such that

$$m \le f(t) \le M \quad \text{and} \quad n \le g(t) \le N \quad \text{for a.e. } t \in [a,b]\,. \tag{2}$$

2010 MSC: 26D15, 25D10
Key words: Absolutely continuous functions, Convex functions, Integral inequalities, Čebyšev functional, Jensen's inequality, Lebesgue norms, Mid-point inequalities, Trapezoid inequalities

The constant $\frac{1}{4}$ is best possible in (1) in the sense that it cannot be replaced by a smaller quantity.

Another, however less known result, even though it was obtained by Čebyšev in 1882 [5], states that

$$|C(f,g)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2, \tag{3}$$

provided that f', g' exist and are continuous on $[a,b]$ and $\|f'\|_\infty = \sup_{t\in[a,b]} |f'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case.

Čebyšev inequality (3) also holds if $f, g\colon [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f', g' \in L_\infty[a,b]$ while $\|f'\|_\infty = \operatorname{ess\,sup}_{t\in[a,b]} |f'(t)|$.

A mixture between Grüss' result (1) and Čebyšev's one (3) is the following inequality obtained by Ostrowski in 1970 [12]:

$$|C(f,g)| \leq \frac{1}{8}(b-a)(M-m)\|g'\|_\infty, \tag{4}$$

provided that f is *Lebesgue integrable* and satisfies (2) while g is absolutely continuous and $g' \in L_\infty[a,b]$. The constant $\frac{1}{8}$ is best possible in (4).

The case of *euclidean norms* of the derivative was considered by A. Lupaş in [9] in which he proved that

$$|C(f,g)| \leq \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a), \tag{5}$$

provided that f, g are absolutely continuous and $f', g' \in L_2[a,b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Recently, Cerone and Dragomir [2] have proved the following results:

$$|C(f,g)| \leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \cdot \frac{1}{b-a}\left(\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right|^p \mathrm{d}t\right)^{\frac{1}{p}}, \tag{6}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ or $p = 1$ and $q = \infty$, and

$$|C(f,g)| \leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_1 \cdot \frac{1}{b-a} \operatorname*{ess\,sup}_{t\in[a,b]} \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right|,$$

provided that $f \in L_p[a,b]$ and $g \in L_q[a,b]$ ($p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$; $p = 1$, $q = \infty$ or $p = \infty$, $q = 1$).

Notice that for $q = \infty$, $p = 1$ in (6) we obtain

$$\begin{aligned}|C(f,g)| &\leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right| \mathrm{d}t \\ &\leq \|g\|_\infty \cdot \frac{1}{b-a}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right| \mathrm{d}t\end{aligned}$$

and if g satisfies (2), then

$$\begin{aligned}|C(f,g)| &\leq \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right| \mathrm{d}t \\ &\leq \left\| g - \frac{n+N}{2}\right\|_\infty \cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right| \mathrm{d}t \\ &\leq \frac{1}{2}(N-n)\cdot \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\right| \mathrm{d}t. \qquad (7)\end{aligned}$$

The inequality between the first and the last term in (7) has been obtained by Cheng and Sun in [6]. However, the sharpness of the constant $\frac{1}{2}$, a generalization for the abstract Lebesgue integral and the discrete version of it have been obtained in [3].

For other recent results on the Grüss inequality, see [8], [10] and [13] and the references therein.

In this paper, some new bounds for the Čebyšev functional in terms of the Lebesgue norms $\left\|f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right\|_{[a,b],p}$ and the Δ-seminorms are established. Applications for mid-point and trapezoid inequalities are provided as well.

2 Some Results Via Sonin's Identity

The following result for convex functions of Čebyšev functional holds.

Theorem 1. *Let $f, g\colon [a,b] \to \mathbb{R}$ be Lebesgue integrable functions on $[a,b]$. If $\Phi\colon \mathbb{R}\to\mathbb{R}$ is convex on $\mathbb{R}$ then we have the inequality*

$$\begin{aligned}\Phi[C(f,g)] &\leq \frac{1}{b-a}\inf_{\lambda\in\mathbb{R}} \int_a^b \Phi\left[\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\big(g(x)-\lambda\big)\right]\mathrm{d}x \\ &\leq \frac{1}{(b-a)^2}\inf_{\lambda\in\mathbb{R}} \int_a^b\int_a^b \Phi\Big[\big(f(x)-f(t)\big)\big(g(x)-\lambda\big)\Big]\,\mathrm{d}t\,\mathrm{d}x. \qquad (8)\end{aligned}$$

Proof. Start with Sonin's identity [11, p. 246]

$$C(f,g) = \frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\big(g(x)-\lambda\big)\,\mathrm{d}x$$

that holds for any $\lambda\in\mathbb{R}$.

If we use Jensen's integral inequality we have for any $\lambda\in\mathbb{R}$

$$\begin{aligned}\Phi[C(f,g)] &= \Phi\left[\frac{1}{b-a}\int_a^b \left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\big(g(x)-\lambda\big)\,\mathrm{d}x\right] \\ &\leq \frac{1}{b-a}\int_a^b \Phi\left[\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\big(g(x)-\lambda\big)\right]\mathrm{d}x \\ &= \frac{1}{b-a}\int_a^b \Phi\left[\frac{1}{b-a}\int_a^b \Big[\big(f(x)-f(t)\big)\big(g(x)-\lambda\big)\Big]\,\mathrm{d}t\right]\mathrm{d}x \\ &\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b \Phi\Big[\big(f(x)-f(t)\big)\big(g(x)-\lambda\big)\Big]\,\mathrm{d}t\,\mathrm{d}x\,.\end{aligned}$$

Taking the infimum over $\lambda\in\mathbb{R}$ we deduce the desired inequalities (8). □

Remark 1. If we write inequality (8) for the convex function $\Phi(x) = |x|^p$, $p \geq 1$, then we get the inequality

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p}} \inf_{\lambda\in\mathbb{R}} \left\{ \int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^p \left| g(x) - \lambda \right|^p \mathrm{d}x \right\}^{1/p} \tag{9}$$
$$\leq \frac{1}{(b-a)^{2/p}} \inf_{\lambda\in\mathbb{R}} \left\{ \int_a^b \int_a^b \left| f(x) - f(t) \right|^p \left| g(x) - \lambda \right|^p \mathrm{d}t\,\mathrm{d}x \right\}^{1/p}.$$

Utilising Hölder's integral inequality we have

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^p \left| g(x) - \lambda \right|^p \mathrm{d}x$$
$$\leq \operatorname*{ess\,sup}_{x\in[a,b]} \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^p \int_a^b \left| g(x) - \lambda \right|^p \mathrm{d}x$$
$$= \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty}^p \| g - \lambda \|_{[a,b],p}^p ,$$

b) for $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^p \left| g(x) - \lambda \right|^p \mathrm{d}x$$
$$\leq \left(\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^{p\beta} \mathrm{d}x \right)^{1/\beta} \left(\int_a^b \left| g(x) - \lambda \right|^{p\alpha} \mathrm{d}x \right)^{1/\alpha}$$
$$= \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta}^p \| g - \lambda \|_{[a,b],p\alpha}^p ,$$

c) for $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$\int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^p \left| g(x) - \lambda \right|^p \mathrm{d}x$$
$$\leq \operatorname*{ess\,sup}_{x\in[a,b]} \left| g(x) - \lambda \right|^p \int_a^b \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right|^p \mathrm{d}x$$
$$= \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}^p \| g - \lambda \|_{[a,b],\infty}^p .$$

Utilising (9) we can state the following result.

Theorem 2. *Let $f, g\colon [a,b] \to \mathbb{R}$ be Lebesgue measurable functions on $[a,b]$. Then*

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p}} \inf_{\lambda\in\mathbb{R}} \| g - \lambda \|_{[a,b],p} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty} ,$$

b) *for* $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \inf_{\lambda\in\mathbb{R}} \|g-\lambda\|_{[a,b],p\alpha} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta},$$

c) *for* $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \inf_{\lambda\in\mathbb{R}} \|g-\lambda\|_{[a,b],\infty} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}.$$

We have the following particular cases of interest.

Corollary 1. *Let* $f,g\colon [a,b]\to\mathbb{R}$ *be Lebesgue measurable functions on* $[a,b]$. *Then*

a) *for* $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \left\| g - \frac{1}{b-a}\int_a^b g(t)\,\mathrm{d}t \right\|_{[a,b],p} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty},$$

b) *for* $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \left\| g - \frac{1}{b-a}\int_a^b g(t)\,\mathrm{d}t \right\|_{[a,b],p\alpha} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta},$$

c) *for* $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \left\| g - \frac{1}{b-a}\int_a^b g(t)\,\mathrm{d}t \right\|_{[a,b],\infty} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}.$$

If one function is bounded, then we can state the following result.

Corollary 2. *Assume that* $f,g\colon [a,b]\to\mathbb{R}$ *are Lebesgue measurable functions on* $[a,b]$. *If there exist constants* n, N *such that* $n \le g(t) \le N$ *for a.e.* $t\in[a,b]$, *then*

a) *for* $f \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{2}(N-n) \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty},$$

b) *for* $f \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$|C(f,g)| \le \frac{1}{2}(N-n) \frac{1}{(b-a)^{1/p\beta}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta},$$

c) *for* $f \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{2}(N-n)\frac{1}{(b-a)^{1/p}}\left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}.$$

Proof. We observe that

$$\begin{aligned}
\left\| g - \frac{n+N}{2} \right\|_{[a,b],p} &= \left(\int_a^b \left| g(t) - \frac{n+N}{2} \right|^p \mathrm{d}t \right)^{1/p} \\
&\le \left(\int_a^b \left(\frac{N-n}{2} \right)^p \mathrm{d}t \right)^{1/p} = \frac{N-n}{2}(b-a)^{1/p}, \\
\left\| g - \frac{n+N}{2} \right\|_{[a,b],p\alpha} &= \left(\int_a^b \left| g(t) - \frac{n+N}{2} \right|^{p\alpha} \mathrm{d}t \right)^{1/p\alpha} \\
&\le \frac{N-n}{2}(b-a)^{1/p\alpha}
\end{aligned}$$

and

$$\left\| g - \frac{n+N}{2} \right\|_{[a,b],\infty} \le \frac{N-n}{2}.$$

Utilising Theorem 2 we deduce the desired result of Corollary 2. □

When one function is of bounded variation, then we can state the following result.

Corollary 3. *If* $f\colon [a,b] \to \mathbb{R}$ *is Lebesgue integrable and* $g\colon [a,b] \to \mathbb{R}$ *is of bounded variation, then*

a) *for* $f \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{2}\bigvee_a^b(g) \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty},$$

b) *for* $f \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le \frac{1}{2}\bigvee_a^b(g) \frac{1}{(b-a)^{1/p\beta}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta},$$

c) *for* $f \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{2}\bigvee_a^b(g) \frac{1}{(b-a)^{1/p}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p},$$

where $\bigvee_a^b(g)$ *is the total variation of the function* g *on the interval* $[a,b]$.

Proof. Since $g\colon [a,b] \to \mathbb{R}$ is of bounded variation, then for any $t \in [a,b]$ we have

$$\begin{aligned}\left|g(t) - \frac{g(a)+g(b)}{2}\right| &= \left|\frac{g(t)-g(a)+g(t)-g(b)}{2}\right| \\ &\le \frac{1}{2}\Big[\big|g(t)-g(a)\big| + \big|g(b)-g(t)\big|\Big] \le \frac{1}{2}\bigvee_a^b(g)\,.\end{aligned}$$

Then

$$\begin{aligned}\left\|g - \frac{g(a)+g(b)}{2}\right\|_{[a,b],p} &= \left(\int_a^b \left|g(t) - \frac{g(a)+g(b)}{2}\right|^p \mathrm{d}t\right)^{1/p} \\ &\le \left(\int_a^b \left(\frac{1}{2}\bigvee_a^b(g)\right)^p \mathrm{d}t\right)^{1/p} = \frac{1}{2}\bigvee_a^b(g)\,(b-a)^{1/p},\end{aligned}$$

$$\left\|g - \frac{g(a)+g(b)}{2}\right\|_{[a,b],p\alpha} \le \frac{1}{2}\bigvee_a^b(g)\,(b-a)^{1/p\alpha},$$

and

$$\left\|g - \frac{g(a)+g(b)}{2}\right\|_{[a,b],\infty} \le \frac{1}{2}\bigvee_a^b(g)\,.$$

Utilising Theorem 2 we deduce the desired result of Corollary 3. □

For functions h that are *Lipschitzian in the middle point* with the constant $L_{\frac{a+b}{2}}$ and the exponent $q > 0$, i.e. satisfying the condition

$$\left|h(t) - h\!\left(\frac{a+b}{2}\right)\right| \le L_{\frac{a+b}{2}}\left|t - \frac{a+b}{2}\right|^q$$

for any $t \in [a,b]$, we have the following result as well.

Corollary 4. *If $f\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable and $g\colon [a,b] \to \mathbb{R}$ is Lipschitzian in the middle point with the constant $L_{\frac{a+b}{2}}$ and the exponent $q > 0$, then*

a) for $f \in L_\infty[a,b]$

$$|C(f,g)| \le L_{\frac{a+b}{2}}\frac{(b-a)^q}{2^q(qp+1)^{1/p}}\left\|f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right\|_{[a,b],\infty}, \tag{10}$$

b) for $f \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le L_{\frac{a+b}{2}}\frac{(b-a)^{q-1/p\beta}}{2^q(qp\alpha+1)^{1/p\alpha}}\left\|f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right\|_{[a,b],p\beta}, \tag{11}$$

c) for $f \in L_p[a,b]$

$$|C(f,g)| \leq L_{\frac{a+b}{2}} \frac{(b-a)^{q-1/p}}{2^q} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}. \tag{12}$$

Proof. We have

$$\begin{aligned} \left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p} &= \left(\int_a^b \left| g(t) - g\left(\frac{a+b}{2}\right) \right|^p \mathrm{d}t \right)^{1/p} \\ &\leq \left(\int_a^b L^p_{\frac{a+b}{2}} \left| t - \frac{a+b}{2} \right|^{qp} \mathrm{d}t \right)^{1/p} \\ &= L_{\frac{a+b}{2}} \left(\int_a^b \left| t - \frac{a+b}{2} \right|^{qp} \mathrm{d}t \right)^{1/p}. \end{aligned} \tag{13}$$

Observe that

$$\begin{aligned} &\left(\int_a^b \left| t - \frac{a+b}{2} \right|^{qp} \mathrm{d}t \right)^{1/p} \\ &= \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right)^{qp} \mathrm{d}t + \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right)^{qp} \mathrm{d}t \right)^{1/p} \\ &= \left(2 \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right)^{qp} \mathrm{d}t \right)^{1/p} = \left(2 \frac{\left(t - \frac{a+b}{2}\right)^{qp+1}}{qp+1} \bigg|_{\frac{a+b}{2}}^b \right)^{1/p} \\ &= \left(2 \frac{\left(\frac{b-a}{2}\right)^{qp+1}}{qp+1} \right)^{1/p} = \left(\frac{(b-a)^{qp+1}}{2^{qp}(qp+1)} \right)^{1/p} = \frac{(b-a)^{q+1/p}}{2^q (qp+1)^{1/p}}. \end{aligned}$$

Then by (13) we have

$$\left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p} \leq L_{\frac{a+b}{2}} \frac{(b-a)^{q+1/p}}{2^q (qp+1)^{1/p}}.$$

Also

$$\left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p\alpha} \leq L_{\frac{a+b}{2}} \frac{(b-a)^{q+1/p\alpha}}{2^q (qp\alpha+1)^{1/p\alpha}}$$

and

$$\left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],\infty} \leq L_{\frac{a+b}{2}} \frac{(b-a)^q}{2^q}.$$

Utilising Theorem 2 we obtain

a) for $f \in L_\infty[a,b]$

$$\begin{aligned}
|C(f,g)| &\le \frac{1}{(b-a)^{1/p}} \left\| g - g\Big(\frac{a+b}{2}\Big) \right\|_{[a,b],p} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty} \\
&\le \frac{1}{(b-a)^{1/p}} L_{\frac{a+b}{2}} \frac{(b-a)^{q+1/p}}{2^q (qp+1)^{1/p}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty} \\
&= L_{\frac{a+b}{2}} \frac{(b-a)^q}{2^q (qp+1)^{1/p}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty},
\end{aligned}$$

b) for $f \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\begin{aligned}
|C(f,g)| &\le \frac{1}{(b-a)^{1/p}} \left\| g - g\Big(\frac{a+b}{2}\Big) \right\|_{[a,b],p\alpha} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta} \\
&\le \frac{1}{(b-a)^{1/p}} L_{\frac{a+b}{2}} \frac{(b-a)^{q+1/p\alpha}}{2^q (qp\alpha+1)^{1/p\alpha}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta} \\
&= L_{\frac{a+b}{2}} \frac{(b-a)^{q-1/p\beta}}{2^q (qp\alpha+1)^{1/p\alpha}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta},
\end{aligned}$$

c) and for $f \in L_p[a,b]$

$$\begin{aligned}
|C(f,g)| &\le \frac{1}{(b-a)^{1/p}} \left\| g - g\Big(\frac{a+b}{2}\Big) \right\|_{[a,b],\infty} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p} \\
&\le \frac{1}{(b-a)^{1/p}} L_{\frac{a+b}{2}} \frac{(b-a)^q}{2^q} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p} \\
&= L_{\frac{a+b}{2}} \frac{(b-a)^{q-1/p}}{2^q} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}.
\end{aligned}$$

Thus the inequalities (10)–(12) are proved. □

Remark 2. If the function g is Lipschitzian with the constant $L > 0$, then

a) for $f \in L_\infty[a,b]$

$$|C(f,g)| \le L \frac{b-a}{2(p+1)^{1/p}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],\infty}, \tag{14}$$

b) for $f \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le L \frac{(b-a)^{1-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}} \left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p\beta}, \tag{15}$$

c) for $f \in L_p[a,b]$

$$|C(f,g)| \le L\frac{(b-a)^{1-1/p}}{2}\left\| f - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right\|_{[a,b],p}. \tag{16}$$

3 Δ-Seminorms and Related Inequalities

For $f \in L_p[a,b]$, $p \in [1,\infty)$, we can define the functional (see [1] and [4])

$$\|f\|_p^{\Delta} := \left(\int_a^b \int_a^b \left| f(t) - f(s) \right|^p \mathrm{d}t\,\mathrm{d}s \right)^{\frac{1}{p}}$$

and for $f \in L_\infty[a,b]$, we can define

$$\|f\|_\infty^{\Delta} := \operatorname*{ess\,sup}_{(t,s)\in[a,b]^2} \left| f(t) - f(s) \right|.$$

If we consider $f_\Delta \colon [a,b]^2 \to \mathbb{R}$,

$$f_\Delta(t,s) = f(t) - f(s),$$

then obviously

$$\|f\|_p^{\Delta} = \|f_\Delta\|_p, \qquad p \in [1,\infty],$$

where $\|\cdot\|_p$ are the usual Lebesgue p-norms on $[a,b]^2$.

Using the properties of the Lebesgue p-norms, we may deduce the following seminorm properties for $\|\cdot\|_p^{\Delta}$:

(i) $\|f\|_p^{\Delta} \ge 0$ for $f \in L_p[a,b]$ and $\|f\|_p^{\Delta} = 0$ implies that $f = c$ (c is a constant) a.e. in $[a,b]$,

(ii) $\|f+g\|_p^{\Delta} \le \|f\|_p^{\Delta} + \|g\|_p^{\Delta}$ if $f,g \in L_p[a,b]$,

(iii) $\|\alpha f\|_p^{\Delta} = |\alpha|\,\|f\|_p^{\Delta}$.

We call $\|\cdot\|_p^{\Delta}$ as Δ-seminorms.

We note that if $p = 2$, then

$$\begin{aligned}\|f\|_2^{\Delta} &= \left(\int_a^b \int_a^b \big(f(t) - f(s)\big)^2 \mathrm{d}t\,\mathrm{d}s \right)^{\frac{1}{2}} \\ &= \sqrt{2}\left((b-a)\|f\|_2^2 - \left(\int_a^b f(t)\,\mathrm{d}t \right)^2 \right)^{\frac{1}{2}}.\end{aligned}$$

Using the inequalities (1), (3) and (5), we obtain the following estimate for $\|\cdot\|_2^{\Delta}$:

a) for $m \le f \le M$

$$\|f\|_2^{\Delta} \le \frac{\sqrt{2}}{2}(M-m)(b-a),$$

b) for $f' \in L_\infty[a,b]$

$$\|f\|_2^\Delta \le \frac{\sqrt{2}}{2\sqrt{3}}\|f'\|_\infty (b-a)^2 \,,$$

c) for $f' \in L_2[a,b]$

$$\|f\|_2^\Delta \le \frac{\sqrt{2}}{\pi}\|f'\|_2 (b-a)^{\frac{3}{2}} \,,$$

since

$$\|f\|_2^\Delta = \sqrt{2}(b-a)[C(f,f)]^{\frac{1}{2}}.$$

If $f\colon [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then we can point out the following bounds for $\|f\|_p^\Delta$ in terms of $\|f'\|_p$.

Theorem 3. *Assume that $f\colon [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$.*

(i) If $p \in [1,\infty)$, then we have the inequality

a) for $f' \in L_\infty[a,b]$

$$\|f\|_p^\Delta \le \frac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{\big[(p+1)(p+2)\big]^{\frac{1}{p}}}\|f'\|_\infty \,,$$

b) for $f' \in L_\alpha[a,b]$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$\|f\|_p^\Delta \le \frac{(2\beta^2)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{\big[(p+\beta)(p+2\beta)\big]^{\frac{1}{p}}}\|f'\|_\alpha \,,$$

c) for $f' \in L_1[a,b]$

$$\|f\|_p^\Delta \le (b-a)^{\frac{2}{p}}\|f'\|_1 \,.$$

(ii) If $p = \infty$, then we have the inequality

a) for $f' \in L_\infty[a,b]$

$$\|f\|_\infty^\Delta \le (b-a)\|f'\|_\infty \,,$$

b) for $f' \in L_\alpha[a,b]$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$\|f\|_\infty^\Delta \le (b-a)^{\frac{1}{\beta}}\|f'\|_\alpha \,,$$

c) for $f' \in L_1[a,b]$

$$\|f\|_\infty^\Delta \le \|f'\|_1 \,.$$

The following result of Grüss type holds, see [4].

Theorem 4. *Let $f, g\colon [a,b] \to \mathbb{R}$ be measurable on $[a,b]$. Then we have the inequality*

$$|C(f,g)| \le \frac{1}{2(b-a)^2} \|f\|_p^{\Delta} \|g\|_q^{\Delta},$$

where $p = 1$, $q = \infty$, or $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, or $q = 1$, $p = \infty$, provided all integrals involved exist.

The inequality is sharp in the sense that if we take $f(x) = g(x) = \operatorname{sgn}(x - \alpha)$ with $\alpha = \frac{a+b}{2}$, then the equality results.

Making use of the double integral inequality

$$|C(f,g)| \le \frac{1}{(b-a)^{2/p}} \inf_{\lambda \in \mathbb{R}} \left\{ \int_a^b \int_a^b \bigl|f(x) - f(t)\bigr|^p \bigl|g(x) - \lambda\bigr|^p \,\mathrm{d}t \,\mathrm{d}x \right\}^{1/p},$$

obtained in (9) we can state the following result as well.

Theorem 5. *Let $f, g\colon [a,b] \to \mathbb{R}$ be Lebesgue measurable functions on $[a,b]$. Then*

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \inf_{\lambda \in \mathbb{R}} \|g - \lambda\|_{[a,b],p} \|f\|_\infty^{\Delta}, \tag{17}$$

b) for $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$ $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p + 1/p\beta}} \inf_{\lambda \in \mathbb{R}} \|g - \lambda\|_{[a,b],p\alpha} \|f\|_{p\beta}^{\Delta}, \tag{18}$$

c) for $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{2/p}} \inf_{\lambda \in \mathbb{R}} \|g - \lambda\|_{[a,b],\infty} \|f\|_p^{\Delta}. \tag{19}$$

Proof. Utilising Hölder's inequality for double integrals, we have

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$\begin{aligned}\int_a^b \int_a^b \bigl|f(x) - f(t)\bigr|^p \bigl|g(x) - \lambda\bigr|^p \,\mathrm{d}t \,\mathrm{d}x &\le \operatorname*{ess\,sup}_{(x,t) \in [a,b]^2} \bigl|f(x) - f(t)\bigr|^p \\ &\quad \times \int_a^b \int_a^b \bigl|g(x) - \lambda\bigr|^p \,\mathrm{d}t \,\mathrm{d}x \\ &= \bigl(\|f\|_\infty^{\Delta}\bigr)^p (b-a) \|g - \lambda\|_{[a,b],p}^p.\end{aligned}$$

Then

$$\begin{aligned}|C(f,g)|^p &\le \frac{1}{(b-a)^2} \bigl(\|f\|_\infty^{\Delta}\bigr)^p (b-a) \|g - \lambda\|_{[a,b],p}^p \\ &= \frac{1}{b-a} \bigl(\|f\|_\infty^{\Delta}\bigr)^p \|g - \lambda\|_{[a,b],p}^p.\end{aligned}$$

b) For $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$, we have

$$\begin{aligned}\int_a^b\int_a^b \big|f(x)-f(t)\big|^p\big|g(x)-\lambda\big|^p\,\mathrm{d}t\,\mathrm{d}x &\le \left(\int_a^b\int_a^b \big|f(x)-f(t)\big|^{p\beta}\,\mathrm{d}t\,\mathrm{d}x\right)^{1/\beta}\\ &\quad\times\left(\int_a^b\int_a^b \big|g(x)-\lambda\big|^{p\alpha}\,\mathrm{d}t\,\mathrm{d}x\right)^{1/\alpha}\\ &= \big(\|f\|_{p\beta}^{\Delta}\big)^p(b-a)^{1/\alpha}\|g-\lambda\|_{[a,b],p\alpha}^p\,.\end{aligned}$$

Then

$$\begin{aligned}|C(f,g)|^p &\le \frac{1}{(b-a)^2}\big(\|f\|_{p\beta}^{\Delta}\big)^p(b-a)^{1/\alpha}\|g-\lambda\|_{[a,b],p\alpha}^p\\ &= \frac{1}{(b-a)^{1+1/\beta}}\big(\|f\|_{p\beta}^{\Delta}\big)^p\|g-\lambda\|_{[a,b],p\alpha}^p\,.\end{aligned}$$

c) For $f \in L_p[a,b]$, $g \in L_\infty[a,b]$ we have

$$\begin{aligned}\int_a^b\int_a^b \big|f(x)-f(t)\big|^p\big|g(x)-\lambda\big|^p\,\mathrm{d}t\,\mathrm{d}x &\le \operatorname*{ess\,sup}_{x\in[a,b]}\big|g(x)-\lambda\big|^p\\ &\quad\times\int_a^b\int_a^b\big|f(x)-f(t)\big|^p\,\mathrm{d}t\,\mathrm{d}x\\ &= \|g-\lambda\|_{[a,b],\infty}^p\big(\|f\|_p^{\Delta}\big)^p\,.\end{aligned}$$

Then

$$|C(f,g)|^p \le \frac{1}{(b-a)^2}\|g-\lambda\|_{[a,b],\infty}^p\big(\|f\|_p^{\Delta}\big)^p\,.$$

Taking the power $\frac{1}{p}$ and then the infimum over $\lambda \in \mathbb{R}$, we get the desired results. □

Some particular cases of interest are as follows.

Corollary 5. *Let $f, g\colon [a,b] \to \mathbb{R}$ be Lebesgue measurable functions on $[a,b]$. Then*

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}}\left\|g-\frac{1}{b-a}\int_a^b g(t)\,\mathrm{d}t\right\|_{[a,b],p}\|f\|_\infty^{\Delta}\,,$$

b) for $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p+1/p\beta}}\left\|g-\frac{1}{b-a}\int_a^b g(t)\,\mathrm{d}t\right\|_{[a,b],p\alpha}\|f\|_{p\beta}^{\Delta}$$

c) for $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{2/p}} \left\| g - \frac{1}{b-a}\int_a^b g(t)\,\mathrm{d}t \right\|_{[a,b],\infty} \|f\|_p^\Delta .$$

The case when one function is bounded is as follows.

Corollary 6. *Assume that* $f, g\colon [a,b] \to \mathbb{R}$ *are Lebesgue integrable functions on* $[a,b]$*. If there exist constants* n, N *such that* $n \le g(t) \le N$ *for a.e.* $t \in [a,b]$*, then*

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{2}(N-n)\|f\|_\infty^\Delta , \tag{20}$$

b) for $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \le \frac{1}{2}(N-n)\frac{1}{(b-a)^{2/p\beta}}\|f\|_{p\beta}^\Delta \tag{21}$$

c) for $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \le \frac{1}{2}(N-n)\frac{1}{(b-a)^{2/p}}\|f\|_p^\Delta . \tag{22}$$

Proof. From (17)–(19) we have

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p}} \left\| g - \frac{N+n}{2} \right\|_{[a,b],p} \|f\|_\infty^\Delta . \tag{23}$$

Since

$$\left\| g - \frac{n+N}{2} \right\|_{[a,b],p} \le \frac{N-n}{2}(b-a)^{1/p}$$

then by (23) we get (20).

b) For $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have

$$|C(f,g)| \le \frac{1}{(b-a)^{1/p+1/p\beta}} \left\| g - \frac{N+n}{2} \right\|_{[a,b],p\alpha} \|f\|_{p\beta}^\Delta . \tag{24}$$

Since

$$\left\| g - \frac{n+N}{2} \right\|_{[a,b],p\alpha} \le \frac{N-n}{2}(b-a)^{1/p\alpha}$$

then by (24) we get (21).

c) For $f \in L_p[a,b]$, $g \in L_\infty[a,b]$ we have

$$|C(f,g)| \leq \frac{1}{(b-a)^{2/p}} \left\| g - \frac{N+n}{2} \right\|_{[a,b],\infty} \|f\|_p^{\Delta} . \tag{25}$$

Since

$$\left\| g - \frac{n+N}{2} \right\|_{[a,b],\infty} \leq \frac{N-n}{2},$$

then by (25) we get (22).

□

The case when one function is of bounded variation, is as follows.

Corollary 7. *If $f\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable and $g\colon [a,b] \to \mathbb{R}$ is of bounded variation, then*

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \leq \frac{1}{2} \bigvee_a^b (g) \, \|f\|_\infty^{\Delta} ,$$

b) for $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \leq \frac{1}{2} \bigvee_a^b (g) \, \frac{1}{(b-a)^{2/p\beta}} \|f\|_{p\beta}^{\Delta} ,$$

c) for $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \leq \frac{1}{2} \bigvee_a^b (g) \, \frac{1}{(b-a)^{2/p}} \|f\|_p^{\Delta} .$$

Proof. From (17)–(19) we have

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p}} \left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],p} \|f\|_\infty^{\Delta} . \tag{26}$$

Since

$$\left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],p} \leq \frac{1}{2} \bigvee_a^b (g) \, (b-a)^{1/p} ,$$

then by (26) we get the desired result.

b) For $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p+1/p\beta}} \left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],p\alpha} \|f\|_{p\beta}^{\Delta} . \tag{27}$$

Since

$$\left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],p\alpha} \leq \frac{1}{2} \bigvee_a^b (g)\,(b-a)^{1/p\alpha} ,$$

then by (27) we get the desired result.

c) For $f \in L_p[a,b]$, $g \in L_\infty[a,b]$ we have

$$|C(f,g)| \leq \frac{1}{(b-a)^{2/p}} \left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],\infty} \|f\|_p^{\Delta} . \tag{28}$$

Since

$$\left\| g - \frac{g(a)+g(b)}{2} \right\|_{[a,b],\infty} \leq \frac{1}{2} \bigvee_a^b (g) ,$$

then by (28) we get the desired result.

□

Corollary 8. *If $f\colon [a,b] \to \mathbb{R}$ is Lebesgue integrable and $g\colon [a,b] \to \mathbb{R}$ is Lipschitzian in the middle point with the constant $L_{\frac{a+b}{2}}$ and the exponent $q > 0$, then*

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \leq \frac{1}{2^q} L_{\frac{a+b}{2}} \frac{(b-a)^q}{(qp+1)^{1/p}} \|f\|_\infty^{\Delta} ,$$

b) for $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \leq \frac{1}{2^q} L_{\frac{a+b}{2}} \frac{(b-a)^{q-2/p\beta}}{(qp\alpha+1)^{1/p\alpha}} \|f\|_{p\beta}^{\Delta} ,$$

c) for $f \in L_p[a,b]$, $g \in L_\infty[a,b]$

$$|C(f,g)| \leq \frac{1}{2^q} L_{\frac{a+b}{2}} (b-a)^{q-2/p} \|f\|_p^{\Delta} .$$

Proof. From (17)–(19) we have

a) for $f \in L_\infty[a,b]$, $g \in L_p[a,b]$

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p}} \left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p} \|f\|_\infty^{\Delta} . \tag{29}$$

Since

$$\left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p} \leq L_{\frac{a+b}{2}} \frac{(b-a)^{q+1/p}}{2^q (qp+1)^{1/p}} ,$$

then from (29) we deduce the desired result.

b) For $f \in L_{p\beta}[a,b]$, $g \in L_{p\alpha}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have

$$|C(f,g)| \leq \frac{1}{(b-a)^{1/p+1/p\beta}} \left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],p\alpha} \|f\|_{p\beta}^{\Delta}. \tag{30}$$

Since

$$\left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],\alpha} \leq L_{\frac{a+b}{2}} \frac{(b-a)^{q+1/p\alpha}}{2^q (qp\alpha+1)^{1/p\alpha}},$$

then from (30) we deduce the desired result.

c) For $f \in L_p[a,b]$, $g \in L_\infty[a,b]$ we have

$$|C(f,g)| \leq \frac{1}{(b-a)^{2/p}} \left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],\infty} \|f\|_{p}^{\Delta}. \tag{31}$$

Since

$$\left\| g - g\left(\frac{a+b}{2}\right) \right\|_{[a,b],\infty} \leq L_{\frac{a+b}{2}} \frac{(b-a)^{q}}{2^q},$$

then from (31) we deduce the desired result.

□

Remark 3. If the function g is Lipshitzian with the constant $L > 0$, then

a) for $f \in L_\infty[a,b]$

$$|C(f,g)| \leq \frac{1}{2} L \frac{b-a}{(p+1)^{1/p}} \|f\|_{\infty}^{\Delta},$$

b) for $f \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$|C(f,g)| \leq \frac{1}{2} L \frac{(b-a)^{1-2/p\beta}}{(p\alpha+1)^{1/p\alpha}} \|f\|_{p\beta}^{\Delta},$$

c) for $f \in L_p[a,b]$

$$|C(f,g)| \leq \frac{1}{2} L (b-a)^{1-2/p} \|f\|_{p}^{\Delta}.$$

4 Applications for Mid-point Inequalities

Consider absolutely continuous function $h\colon [a,b] \to \mathbb{R}$. We have the following well known representation

$$h\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b h(t)\,\mathrm{d}t = \frac{1}{b-a} \int_a^b K(t) h'(t)\,\mathrm{d}t,$$

where the kernel $K\colon [a,b] \to \mathbb{R}$ is defined by

$$K(t) := \begin{cases} t-a & \text{if } t \in \left[a, \frac{a+b}{2}\right], \\ t-b & \text{if } t \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Since $\int_a^b K(t)\,\mathrm{d}t = 0$, then

$$\frac{1}{b-a}\int_a^b K(t)h'(t)\,\mathrm{d}t = C(K,h').$$

Utilising Corollary 1 we have

a) for $h' \in L_\infty[a,b]$

$$\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{(b-a)^{1/p}}\|K\|_{[a,b],p}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty}, \quad (32)$$

b) for $h' \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{(b-a)^{1/p}}\|K\|_{[a,b],p\alpha}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p\beta}, \quad (33)$$

c) for $h' \in L_p[a,b]$

$$\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{(b-a)^{1/p}}\|K\|_{[a,b],\infty}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p}. \quad (34)$$

Observe that for $q > 0$ we have

$$\begin{aligned}
\|K\|_{[a,b],q} &= \left[\int_a^b |K(t)|^q\,\mathrm{d}t\right]^{1/q} \\
&= \left[\int_a^{\frac{a+b}{2}} (t-a)^q\,\mathrm{d}t + \int_{\frac{a+b}{2}}^b (b-t)^q\,\mathrm{d}t\right]^{1/q} \\
&= \left[\frac{(t-a)^{q+1}}{q+1}\bigg|_a^{\frac{a+b}{2}} - \frac{(b-t)^{q+1}}{q+1}\bigg|_{\frac{a+b}{2}}^b\right]^{1/q} \\
&= \left[\frac{\left(\frac{b-a}{2}\right)^{q+1}}{q+1} + \frac{\left(\frac{b-a}{2}\right)^{q+1}}{q+1}\right]^{1/q} = \frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}}.
\end{aligned}$$

Then

$$\|K\|_{[a,b],p} = \frac{(b-a)^{1+1/p}}{2(p+1)^{1/p}}, \quad \|K\|_{[a,b],p\alpha} = \frac{(b-a)^{1+1/p\alpha}}{2(p\alpha+1)^{1/p\alpha}}.$$

We also have

$$\|K\|_{[a,b],\infty} = \frac{1}{2}(b-a).$$

Making use of (32)–(34) we get

a) for $h' \in L_\infty[a,b]$

$$\left|h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{b-a}{2(p+1)^{1/p}}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty},$$

b) for $h' \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{(b-a)^{1-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}} \left\| h' - \frac{h(b)-h(a)}{b-a} \right\|_{[a,b],p\beta},$$

c) for $h' \in L_p[a,b]$

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{2}(b-a)^{1-1/p} \left\| h' - \frac{h(b)-h(a)}{b-a} \right\|_{[a,b],p}.$$

For $p = 1$ we get the simpler inequalities

a) for $h' \in L_\infty[a,b]$

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{4}(b-a) \left\| h' - \frac{h(b)-h(a)}{b-a} \right\|_{[a,b],\infty},$$

b) for $h' \in L_1[a,b]$

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{2}(b-a) \left\| h' - \frac{h(b)-h(a)}{b-a} \right\|_{[a,b],1}.$$

Utilising Corollary 2 we have

a)

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{2}(\Gamma-\gamma)\|K\|_{[a,b],\infty},$$

b) for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{2}(\Gamma-\gamma)\frac{1}{(b-a)^{1/p\beta}}\|K\|_{[a,b],p\beta},$$

c)

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{2}(\Gamma-\gamma)\frac{1}{(b-a)^{1/p}}\|K\|_{[a,b],p},$$

provided that $\gamma \le h'(t) \le \Gamma$ for a.e. $t \in [a,b]$.

Utilising the above calculations we then have

a)

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le (\Gamma-\gamma)(b-a), \tag{35}$$

b) for $\alpha > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$

$$\left| h\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t \right| \le \frac{1}{2}(\Gamma-\gamma)\frac{(b-a)^{1+1/p\alpha-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}}, \tag{36}$$

c)

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{2}(\Gamma-\gamma)\frac{b-a}{2(p+1)^{1/p}}\,, \tag{37}$$

provided that $\gamma \le h'(t) \le \Gamma$ for a.e. $t \in [a,b]$.

In particular, for $p = 1$ in (37) we have

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{8}(\Gamma-\gamma)(b-a)\,,$$

which is the best inequality one can get from (35)–(37).

If we use Corollary 3 and assume that h' is of bounded variation on $[a,b]$, then

a)

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\bigvee_a^b(h')\,(b-a)\,,$$

b) for $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{2}\bigvee_a^b(h')\,\frac{(b-a)^{1+1/p\alpha-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}}\,,$$

c)

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{2}\bigvee_a^b(h')\,\frac{b-a}{2(p+1)^{1/p}}\,. \tag{38}$$

From (38) for $p = 1$ we get

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{8}(b-a)\bigvee_a^b(h').$$

If we use inequalities (14)–(16) and assume that h' is Lipschitzian with the constant $U > 0$, namely

$$\left|h'(t)-h'(s)\right|\le U|t-s| \text{ for } t,s\in(a,b),$$

then

a)

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le U\frac{1}{4}\frac{(b-a)^2}{(p+1)^{1/p}}\,,$$

b)

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le U\frac{1}{4}\frac{(b-a)^{2-1/p\beta+1/p\alpha}}{(p\alpha+1)^{2/p\alpha}}\,,$$

c)

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le U\frac{1}{4}\frac{(b-a)^2}{(p+1)^{1/p}}\,.$$

In particular, we get for $p=1$

$$\left|h\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le \frac{1}{8}(b-a)^2U.$$

5 Applications for Trapezoid Inequalities

Consider absolutely continuous function $h\colon[a,b]\to\mathbb{R}$. We have the following well known representation

$$\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t=\frac{1}{b-a}\int_a^b V(t)h'(t)\,\mathrm{d}t$$

where the kernel $V\colon[a,b]\to\mathbb{R}$ is defined by

$$V(t):=t-\frac{a+b}{2}.$$

Since $\int_a^b V(t)\,\mathrm{d}t=0$, then

$$\frac{1}{b-a}\int_a^b V(t)h'(t)\,\mathrm{d}t=C(V,h').$$

Utilising Corollary 1 we have

a) for $h'\in L_\infty[a,b]$

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{(b-a)^{1/p}}\|V\|_{[a,b],p}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty},\quad(39)$$

b) for $h'\in L_{p\beta}[a,b]$, $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{(b-a)^{1/p}}\|V\|_{[a,b],p\alpha}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p\beta},\quad(40)$$

c) for $h'\in L_p[a,b]$

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right|\le\frac{1}{(b-a)^{1/p}}\|V\|_{[a,b],\infty}\left\|h'-\frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p}.\quad(41)$$

Observe that, for $q>0$ we have

$$\|V\|_{[a,b],q}=\left[\int_a^b|V(t)|^q\,\mathrm{d}t\right]^{1/q}=\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)^q\mathrm{d}t+\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^q\mathrm{d}t\right]^{1/q}$$
$$=\left[2\int_{\frac{a+b}{2}}^b\left(t-\frac{a+b}{2}\right)^q\mathrm{d}t\right]^{1/q}=\left[\frac{2\left(\frac{b-a}{2}\right)^{q+1}}{q+1}\right]^{1/q}=\frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}}.$$

Then

$$\|V\|_{[a,b],p} = \frac{(b-a)^{1+1/p}}{2(p+1)^{1/p}}, \qquad \|V\|_{[a,b],p\alpha} = \frac{(b-a)^{1+1/p\alpha}}{2(p\alpha+1)^{1/p\alpha}}.$$

We also have

$$\|V\|_{[a,b],\infty} = \frac{1}{2}(b-a).$$

Making use of (39)–(41) we get

a) for $h' \in L_\infty[a,b]$

$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{b-a}{2(p+1)^{1/p}}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty},$$

b) for $h' \in L_{p\beta}[a,b]$, $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{(b-a)^{1-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p\beta},$$

c) for $h' \in L_p[a,b]$

$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{2}(b-a)^{1-1/p}\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p}.$$

For $p=1$ we get the simpler inequalities

a) for $h' \in L_\infty[a,b]$

$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{4}(b-a)\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],\infty},$$

b) for $h' \in L_1[a,b]$

$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{2}(b-a)\left\|h' - \frac{h(b)-h(a)}{b-a}\right\|_{[a,b],p}.$$

Since the p-norms of the kernel V are the same as of K, then we can state the following results as well.

If $\gamma \le h'(t) \le \Gamma$ for a.e. $t \in [a,b]$, then we have

a)

$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le (\Gamma-\gamma)(b-a)\,, \tag{42}$$

b) for $\alpha > 1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$\left|\frac{h(b)+h(a)}{2} - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{2}(\Gamma-\gamma)\frac{(b-a)^{1+1/p\alpha-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}}\,, \tag{43}$$

c)

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{2}(\Gamma-\gamma)\frac{b-a}{2(p+1)^{1/p}}\,. \tag{44}$$

In particular, for $p=1$ in (44) we have

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{8}(\Gamma-\gamma)(b-a),$$

which is the best inequality one can get from (42)–(44).

If h' is of bounded variation on $[a,b]$, then

a)

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \bigvee_a^b (h')\,(b-a)\,,$$

b) for $\alpha>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{2}\bigvee_a^b (h')\,\frac{(b-a)^{1+1/p\alpha-1/p\beta}}{2(p\alpha+1)^{1/p\alpha}}\,,$$

c)

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{2}\bigvee_a^b (h')\,\frac{b-a}{2(p+1)^{1/p}}\,.$$

From (38) for $p=1$ we get

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{8}(b-a)\bigvee_a^b (h').$$

Assume that h' is Lipschitzian with the constant $U>0$ then

a)

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le U\frac{1}{4}\frac{(b-a)^2}{(p+1)^{1/p}}\,,$$

b)

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le U\frac{1}{4}\frac{(b-a)^{2-1/p\beta+1/p\alpha}}{(p\alpha+1)^{2/p\alpha}}\,,$$

c)

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le U\frac{1}{4}\frac{(b-a)^2}{(p+1)^{1/p}}\,.$$

In particular, we get for $p=1$

$$\left|\frac{h(b)+h(a)}{2}-\frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\right| \le \frac{1}{8}(b-a)^2U.$$

Some similar inequalities may be stated in terms of the Δ-seminorms. However the details are omitted.

6 Some Exponential Inequalities

We can state the following result.

Theorem 6. *Let $f, g\colon [a,b] \to \mathbb{R}$ be Lebesgue integrable functions on $[a,b]$. If $\Phi\colon \mathbb{R} \to \mathbb{R}$ is convex and monotonic nondecreasing on $\mathbb{R}$ then we have the inequality*

$$\Phi[C(f,g)] \le \frac{1}{b-a} \inf_{\mu\in\mathbb{R}} \int_a^b \Phi\left[\left(\frac{f(x)+g(x)}{2} - \mu\right)^2\right] \mathrm{d}x. \tag{45}$$

Proof. From Theorem 1 we have

$$\Phi[C(f,g)] \le \frac{1}{b-a}\int_a^b \Phi\left[\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\left(g(x) - 2\mu + \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\right]\mathrm{d}x$$

for any $\mu \in \mathbb{R}$.

Utilising the elementary inequality

$$\alpha\beta \le \left(\frac{\alpha+\beta}{2}\right)^2$$

that holds for any $\alpha, \beta \in \mathbb{R}$, we have

$$\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\left(g(x) - 2\mu + \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right) \le \left(\frac{f(x)+g(x)}{2} - \mu\right)^2$$

for any $x \in [a,b]$.

Since $\Phi\colon \mathbb{R} \to \mathbb{R}$ is monotonic nondecreasing on $\mathbb{R}$ then

$$\Phi\left[\left(f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\left(g(x) - 2\mu + \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)\right] \le \Phi\left[\left(\frac{f(x)+g(x)}{2} - \mu\right)^2\right] \tag{46}$$

for any $x \in [a,b]$.

Integrating (46) over x in $[a,b]$ and taking the infimum over $\mu \in \mathbb{R}$, we deduce the desired result (45). □

Remark 4. Writing the inequality (45) for $\Phi\colon \mathbb{R} \to \mathbb{R}$, $\Phi(x) = \exp x$ we have

$$\exp[C(f,g)] \le \frac{1}{b-a} \inf_{\mu\in\mathbb{R}} \int_a^b \exp\left[\left(\frac{f(x)+g(x)}{2} - \mu\right)^2\right] \mathrm{d}x. \tag{47}$$

This inequality can provide some exponential inequalities as follows.

Assume that $f\colon [a,b] \to \mathbb{R}$ is Lipschitzian with constant $L > 0$ and $g\colon [a,b] \to \mathbb{R}$ is Lipschitzian with constant $K > 0$. Then by taking

$$\mu = \frac{f\left(\frac{a+b}{2}\right) + g\left(\frac{a+b}{2}\right)}{2}$$

we have

$$\left(\frac{f(x)+g(x)}{2}-\frac{f\left(\frac{a+b}{2}\right)+g\left(\frac{a+b}{2}\right)}{2}\right)^2 \le \left(\frac{L+K}{2}\right)^2\left(x-\frac{a+b}{2}\right)^2$$

and by (47) we have

$$\exp[C(f,g)] \le \frac{1}{b-a}\int_a^b \exp\left[\left(\frac{L+K}{2}\right)^2\left(x-\frac{a+b}{2}\right)^2\right] \mathrm{d}x\,.$$

Acknowledgement

The author would like to thank the anonymous referees for their valuable comments that have been implemented in the final version of the paper.

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Received: 28 January, 2014
Accepted for publication: 12 April, 2014
Communicated by: Olga Rossi