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Fixed point theorems of G-fuzzy contractions in fuzzy metric spaces endowed with a graph

Satish Shukla

Abstract. Let $(X, M, *)$ be a fuzzy metric space endowed with a graph G such that the set $V(G)$ of vertices of G coincides with X. Then we define a G-fuzzy contraction on X and prove some results concerning the existence and uniqueness of fixed point for such mappings. As a consequence of the main results we derive some extensions of known results from metric into fuzzy metric spaces. Some examples are given which illustrate the results.

1 Introduction

The concept of fuzzy sets was introduced by Zadeh [12]. He considered the nature of uncertainty in the behaviour of systems possessing fuzzy nature by means of a fuzzy set. The concept of fuzzy metric space was introduced by Kramosil and Michálek [7]. George and Veeramani [1] modified the definition of fuzzy metric spaces due to Kramosil and Michálek. The fixed point theory in fuzzy metric spaces was started by Grabiec [13] which has become of interest for several authors. Gregori and Sapena [15] introduced the concept of fuzzy contractive mappings and proved some fixed point results for fuzzy contractive mappings.

On the other hand, Jachymski [11] introduced the fixed point theory in the spaces endowed with a graph. The fixed point results on the spaces endowed with a graph generalize and unify several known results in the literature, e.g., the fixed point results on the spaces endowed with a partial order [3], [8], [10] and the fixed point results for the cyclic mappings (see [6] and [11]).

In this paper, we introduce the G-fuzzy contractions as an extension of Banach G-contraction (see [11]) in fuzzy metric spaces and prove some fixed point results for such mappings in complete fuzzy metric spaces in the sense of Grabiec [13]. Our results are the extension of results of Jachymski [11] and a generalization of result of Gregori and Sapena [15] in fuzzy metric spaces.

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2 Preliminaries

Firstly, we recall some known definitions and the properties about the fuzzy metric spaces.

Definition 1 (Schweizer and Sklar [4]). A binary operation $T\colon [0,1]\times[0,1]\to[0,1]$ is called a t-norm if the following conditions are satisfied:

(T1) $T(a,b)=T(b,a)$;

(T2) $T(a,b)\leq T(c,d)$ for $a\leq c, b\leq d$;

(T3) $T(T(a,b),c)=T(a,T(b,c))$;

(T4) $T(a,0)=0, T(a,1)=1$;

for all $a,b,c,d\in[0,1]$.

For $a,b\in[0,1]$, instead of $T(a,b)$ we will use the infix notation $a*b$. For $a_1,a_2,\dots,a_n\in[0,1]$ and $n\in\mathbb{N}$, the product $a_1*a_2*\cdots*a_n$ will be denoted by $\prod_{i=1}^{n}a_i$. For the details concerning t-norms the reader is referred to [5], [14].

In the present paper we will use the following definition of a fuzzy metric space:

Definition 2 (George and Veeramani [1]). A triple $(X,M,*)$ is called a fuzzy metric space if X is a nonempty set, $*$ is a continuous t-norm and $M\colon X^2\times(0,\infty)\to[0,1]$ is a fuzzy set satisfying following conditions:

(GV1) $M(x,y,t)>0$;

(GV2) $M(x,y,t)=1$ if and only if $x=y$;

(GV3) $M(x,y,t)=M(y,x,t)$;

(GV4) $M(x,z,t+s)\geq M(x,y,t)*M(y,z,s)$;

(GV5) $M(x,y,\cdot)\colon(0,\infty)\to[0,1]$ is a continuous mapping;

for all $x,y,z\in X$ and $s,t>0$.

Example 1 (George and Veeramani [1]). Let (X,d) be a metric space, then the triple $(X,M_d,*)$ is a fuzzy metric space, where $a*b=ab$ for all $a,b\in[0,1]$ and

$$M_d(x,y,t)=\frac{t}{t+d(x,y)}\text{ for all }x,y\in X, t>0.$$

M_d is called the standard fuzzy metric induced by the metric d.

Let $(X,M,*)$ be a fuzzy metric space. An open ball $B(x,r,t)$ with center $x\in X$ and radius r, $0<r<1$ and $t>0$ is defined by

$$B(x,r,t)=\{y\in X\colon M(x,y,t)>1-r\}.$$

The collection $\{B(x,r,t)\colon x\in X, 0<r<1, t>0\}$ is a neighbourhood system for the topology τ on X induced by the fuzzy metric M.

For topological properties of a fuzzy metric space in the sense of George and Veeramani the reader is referred to [1].

Remark 1 (George and Veeramani [2]). Let $(X, M, *)$ be a fuzzy metric space, then the function $M(x, y, \cdot)$ is a nondecreasing function.

Theorem 1 (George and Veeramani [1]). *Let $(X, M, *)$ be a fuzzy metric space, and τ be the topology induced by the fuzzy metric. Then for a sequence $\{x_n\}$ in X, $x_n \to x$ if and only if*

$$\forall_{t>0} \quad \lim_{n\to\infty} M(x_n, x, t) = 1\,.$$

In this paper, we use the following definitions of Cauchy sequence and complete fuzzy metric space.

Definition 3 (Grabiec [13]). Let $(X, M, *)$ be a fuzzy metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Cauchy sequence if

$$\forall_{t>0} \ \forall_{p>0} \quad \lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1\,.$$

A complete fuzzy metric space is a fuzzy metric space in which every Cauchy sequence is convergent.

Definition 4 (Gregori and Sapena [15]). Let $(X, M, *)$ be a fuzzy metric space. A mapping $T\colon X \to X$ is called t-uniformly continuous if for all $r \in (0, 1)$ there exists $s \in (0, 1)$ such that

$$\forall_{x,y\in X} \ \forall_{t>0} \quad \big[M(x, y, t) \geq 1 - s \quad \Rightarrow \quad M(Tx, Ty, t) \geq 1 - r\big]\,.$$

Remark 2. If T is t-uniformly continuous then it is uniformly continuous for the uniformity generated by M, thus it is continuous for the topology deduced from M. For the details concerning a uniform structure in a fuzzy metric space, see [15].

Definition 5 (Gregori and Sapena [15]). Let $(X, M, *)$ be a fuzzy metric space. A mapping $T\colon X \to X$ is called a fuzzy contractive mapping if there exists $\lambda \in (0, 1)$ such that

$$\forall_{x,y\in X} \ \forall_{t>0} \quad \frac{1}{M(Tx, Ty, t)} - 1 \leq \lambda \left[\frac{1}{M(x, y, t)} - 1\right]. \tag{1}$$

It is obvious that if T is a fuzzy contractive mapping then it is t-uniformly continuous and so continuous.

Following concepts about the graphs are similar to those in [11].

Let $(X, M, *)$ be a fuzzy metric space. Let Δ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, i.e., $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the fuzzy distance between its vertices.

By G^{-1} we denote the conversion of a graph G, i.e., the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(x, y) \in X \times X\colon (y, x) \in E(G)\}\,.$$

The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{G}) = E(G) \cup E(G^{-1})\,. \tag{2}$$

If x and y are vertices in a graph G, then a path in G from x to y of length l is a sequence $(x_i)_{i=0}^{l}$ of $l+1$ vertices such that $x_0 = x, x_l = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, l$. A graph G is called connected if there is a path between any two vertices of G. A graph G is weakly connected if $\widetilde{G}$ is connected. For a graph G such that $E(G)$ is symmetric and x is a vertex in G, the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of a relation R defined on $V(G)$ by the rule: yRz if there is a path in G from y to z. Clearly, G_x is connected.

Now we can state our main results.

3 Main results

Throughout this section we assume that X is nonempty set, G is a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$.

First we define the Cauchy equivalent sequence and G-fuzzy contraction in fuzzy metric spaces.

Definition 6. Let $(X, M, *)$ be a fuzzy metric space and G be a graph. Two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in X are said to be Cauchy equivalent if each of them is a Cauchy sequence and $\lim\limits_{n\to\infty} M(x_n, y_n, t) = 1$ for all $t > 0$.

Definition 7. Let $(X, M, *)$ be a fuzzy metric space and G be a graph. The mapping $T\colon X \to X$ is said to be a G-fuzzy contraction if the following conditions hold:

(GF1) $\forall_{x,y\in X}\ \big((x,y) \in E(G) \Rightarrow (Tx, Ty) \in E(G)\big)$, i.e., T is edge-preserving;

(GF2) $$\exists_{\lambda\in(0,1)}\ \forall_{x,y\in X}\ \forall_{t>0}\ \left((x,y) \in E(G) \Rightarrow \frac{1}{M(Tx,Ty,t)} - 1 \le \lambda\left[\frac{1}{M(x,y,t)} - 1\right]\right),$$

where λ is called the contractive constant of T.

An obvious consequence of symmetry of $M(\cdot,\cdot,t)$ and (2) is the following remark.

Remark 3. If T is a G-fuzzy contraction then it is both a G^{-1}-fuzzy contraction and a $\widetilde{G}$-fuzzy contraction.

Example 2. Any constant function $T\colon X \to X$, that is $Tx = c$, $x \in X$, where $c \in X$ is fixed, is a G-fuzzy contraction with arbitrary value of $\lambda \in (0,1)$ since $E(G)$ contains all the loops.

Example 3. Any fuzzy contractive mapping is a G_0-fuzzy contraction with the same contractive constant, where the graph G_0 is defined by $E(G_0) = X \times X$.

Example 4. Let (X, d) be a metric space endowed with a partial order $\sqsubseteq$ and $T\colon X \to X$ be an ordered contraction, i.e.,

$$\exists_{\lambda \in (0,1)} \ \forall_{x,y \in X} \ \left(x \sqsubseteq y \ \Rightarrow \ d(Tx, Ty) \le \lambda d(x, y) \right).$$

Then T is a G_d-fuzzy contraction in the induced fuzzy metric space $(X, M_d, *)$ with contractive constant λ, where $G_d = \{(x, y) \in X \times X : x \sqsubseteq y\}$.

We see that every fuzzy contractive mapping is t-uniformly continuous. Following example shows that a G-fuzzy contraction need not be even continuous.

Example 5. Let $(\mathbb{R}^+, d)$ be the usual metric space of positive reals and $(\mathbb{R}^+, M_d, *)$ be the standard fuzzy metric space induced by d. Let G be the graph defined by $V(G) = X$ and

$$E(G) = \Delta \cup \{(x, y) \in X \times X : x, y \in \mathbb{Q} \cap \mathbb{R}^+ \text{ with } x \le y\}$$

Let the mapping $T\colon X \to X$ be defined by

$$Tx = \begin{cases} \dfrac{x}{2}, & \text{if } x \in \mathbb{Q} \cap \mathbb{R}^+; \\ 0, & \text{otherwise.} \end{cases}$$

Then it is clear that T is not continuous. Now one can see easily that T is a G-fuzzy contraction with $\lambda = \frac{1}{2}$.

Definition 8. Let $(X, M, *)$ be a fuzzy metric space and $T\colon X \to X$ be a mapping. We denote the nth iterate of T on $x \in X$ by $T^n x$ and $T^n x = TT^{n-1}x$ for all $n \in \mathbb{N}$ with $T^0 x = x$. T is called a Picard operator if T has a unique fixed point u and $\lim_{n\to\infty} M(T^n x, u, t) = 1$ for all $x \in X, t > 0$. T is called a weakly Picard operator if for all $x \in X$ there exists a fixed point $u_x \in X$ (which may depend on x) of T such that $\lim_{n\to\infty} M(T^n x, u_x, t) = 1$ for all $t > 0$.

Note that every Picard operator is a weakly Picard operator. Also, the fixed point of a weakly Picard operator may not be unique. In further discussion, we will denote the set of all fixed points of T by $\operatorname{Fix} T$. A subset $A \subset X$ is said to be T-invariant if $T(A) \subset A$.

The following lemma will be useful in sequel.

Lemma 1. *Let $T\colon X \to X$ be a G-fuzzy contraction, then given $x \in X$ and $y \in [x]_{\widetilde{G}}$, we have $\lim_{n\to\infty} M(T^n x, T^n y, t) = 1$ for all $t > 0$.*

Proof. Let $x \in X$ and $y \in [x]_{\widetilde{G}}$. Then by definition there exists a path $(x_i)_{i=0}^m$ in $\widetilde{G}$ from x to y, i.e., $x_0 = x$, $x_m = y$ and $(x_i, x_{i-1}) \in E(\widetilde{G})$ for $i = 1, 2, \ldots, m$. By Remark 3, T is a $\widetilde{G}$-fuzzy contraction. Therefore by (GF1) we have $(T^n x_i, T^n x_{i-1}) \in E(\widetilde{G})$ and by (GF2), for $i = 1, 2, \ldots, m$ and $t > 0$ we have

$$\frac{1}{M(T^n x_{i-1}, T^n x_i, t)} - 1 \leq \lambda^n \left[\frac{1}{M(x_{i-1}, x_i, t)} - 1\right]. \tag{3}$$

Now we can choose a strictly decreasing sequence $(a_n)_{n\in\mathbb{N}}$ of positive numbers such that $\sum_{i=1}^{\infty} a_i = 1$ and then using (3) we obtain

$$\begin{aligned} M(T^n x, T^n y, t) &= M\Big(T^n x_0, T^n x_m, \sum_{i=1}^{\infty} a_i t\Big) \\ &\geq M\Big(T^n x_0, T^n x_m, \sum_{i=1}^{m} a_i t\Big) \geq \prod_{i=1}^{m} M(T^n x_{i-1}, T^n x_i, a_i t) \\ &\geq \prod_{i=1}^{m} \left[\frac{1}{1 - \lambda^n + \frac{\lambda^n}{M(x_{i-1}, x_i, a_i t)}}\right]. \end{aligned}$$

As $\lambda \in (0, 1)$ we obtain $\lim_{n\to\infty} M(T^n x, T^n y, t) = 1$ for all $t > 0$. □

The following theorem shows the equivalency of connectedness of graph and the convergence of an iterative sequences in fuzzy metric spaces.

Theorem 2. *The following statements are equivalent:*

(i) *G is weakly connected;*

(ii) *for any G-fuzzy contraction $T : X \to X$, given $x, y \in X$ the sequences $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent;*

(iii) *for any G-fuzzy contraction $T : X \to X$, $\mathrm{card}(\mathrm{Fix}\, T) \leq 1$.*

Proof. (i)⇒(ii): Let T be a G-fuzzy contraction and $x, y \in X$ then by hypothesis G is weakly connected, therefore $[x]_{\widetilde{G}} = X$ and so $T^p x \in [x]_{\widetilde{G}}$ for all $p \in \mathbb{N}$. Now by Lemma 1, we have $(T^n x)_{n\in\mathbb{N}}$ is a Cauchy sequence. Similarly, $(T^n y)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $[x]_{\widetilde{G}} = X$ therefore by Lemma 1, we have $\lim_{n\to\infty} M(T^n x, T^n y, t) = 1$ for all $t > 0$. Hence the sequences $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent.

(ii)⇒(iii): Let $x, y \in \mathrm{Fix}\, T$, where T is a G-fuzzy contraction. Since $x, y \in \mathrm{Fix}\, T^n$ and we have $M(x, y, t) = M(T^n x, T^n y, t)$. So by assumption $x = y$.

(iii)⇒(i): Suppose (iii) holds but G is not weakly connected, i.e., $\widetilde{G}$ is disconnected. Let $u \in X$, then both the sets $[u]_{\widetilde{G}}$ and $X \setminus [u]_{\widetilde{G}}$ are nonempty. Let $v \in X \setminus [u]_{\widetilde{G}}$ and define a mapping $T : X \to X$ by

$$Tx = \begin{cases} u, & \text{if } x \in [u]_{\widetilde{G}}; \\ v, & \text{if } x \in X \setminus [u]_{\widetilde{G}}. \end{cases}$$

Now clearly $\mathrm{Fix}\,T = \{u, v\}$. We show that T is a G-fuzzy contraction. If $(x, y) \in E(G)$ then by the definition we have $[x]_{\widetilde{G}} = [y]_{\widetilde{G}}$, so either $x, y \in [u]_{\widetilde{G}}$ or $u, v \in X \setminus [u]_{\widetilde{G}}$. In both the cases we have $Tx = Ty$ and so $(Tx, Ty) \in E(G)$ (since $E(G) \supseteq \Delta$) and (GF1) is satisfied. Also, $M(Tx, Ty, t) = 1$ for all $t > 0$ so (GF2) is satisfied. Thus T is a G-fuzzy contraction and $\mathrm{card}(\mathrm{Fix}\,T) = 2 > 1$. This contradiction proves the result. □

The following corollary is an immediate consequence of the above theorem.

Corollary 1. *Let $(X, M, *)$ be a complete fuzzy metric space. Then the following statements are equivalent:*

(i) G is weakly connected;

(ii) for any G-fuzzy contraction $T\colon X \to X$, there is $x^* \in X$ such that $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.

The proof of following proposition is similar as for the metric case (see, e.g., [11]).

Proposition 1. Assume that $T\colon X \to X$ is a G-fuzzy contraction such that for some $x_0 \in X$ we have $Tx_0 \in [x_0]_{\widetilde{G}}$. Let $\widetilde{G}_{x_0}$ be the component of $\widetilde{G}$ containing x_0. Then $[x_0]_{\widetilde{G}}$ is T-invariant and $T|_{[x_0]_{\widetilde{G}}}$ is a $\widetilde{G}_{x_0}$-fuzzy contraction. Moreover, if $x, y \in [x_0]_{\widetilde{G}}$, then the sequences $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent.

Definition 9. Let $(X, M, *)$ be a fuzzy metric space and G be a directed graph, $T\colon X \to X$ be a mapping and $x, x^* \in X$. Then we say that the 4-tuple $(X, M, *, G)$ have the property $(\mathcal{P}_T)$ if for any sequence $(T^n x)_{n\in\mathbb{N}}$, which converges to x^* with $(T^n x, T^{n+1} x) \in E(G)$ for all $n \in \mathbb{N}$ there exists is a subsequence $(T^{k_n} x)_{n\in\mathbb{N}}$ with $(T^{k_n} x, x^*) \in E(G)$ for $n \in \mathbb{N}$.

Theorem 3. *Let $(X, M, *)$ be a complete fuzzy metric space and G be a directed graph and let the 4-tuple $(X, M, *, G)$ have the property $(\mathcal{P}_T)$. Let $T\colon X \to X$ be a G-fuzzy contraction and $X_T = \{x \in X\colon (x, Tx) \in E(G)\}$, then the following statements hold:*

(A) if $x \in X_T$, then $T|_{[x]_{\widetilde{G}}}$ is a Picard operator;

(B) if $X_T \neq \emptyset$ and G is weakly connected, then T is a Picard operator;

(C) $\mathrm{Fix}\,T \neq \emptyset$ if and only if $X_T \neq \emptyset$;

(D) if $T \subseteq E(G)$, then T is a weakly Picard operator.

Proof. To prove (A) let $x \in X_T$. By definition of X_T, $(x, Tx) \in E(G)$ and so we have $Tx \in [x]_{\widetilde{G}}$. Now by Proposition 1, we have $T\colon [x]_{\widetilde{G}} \to [x]_{\widetilde{G}}$ and T is a $\widetilde{G}_x$-fuzzy contraction and if $y \in \widetilde{G}_x$ then $(T^n x)_{n\in\mathbb{N}}$ and $(T^n y)_{n\in\mathbb{N}}$ are Cauchy equivalent and so $(T^n x)_{n\in\mathbb{N}}$ is a Cauchy sequence. By completeness of X and Theorem 1 there exists $x^* \in X$ such that

$$\lim_{n\to\infty} M(T^n x, x^*, t) = 1 \quad \text{for all} \quad t > 0\,. \tag{4}$$

Since $(x, Tx) \in E(G)$ we have $(x, Tx) \in E(\widetilde{G})$ and so by (GF1) we have

$$(T^n x, T^{n+1}x) \in E(G) \quad \text{for all} \quad n \in \mathbb{N}\,. \tag{5}$$

Now by property $(\mathcal{P}_T)$ there exists a subsequence $(T^{k_n}x)_{n\in\mathbb{N}}$ such that $(T^{k_n}x, x^*) \in E(G)$ for all $n \in \mathbb{N}$. Hence, $(x, Tx, T^2x, \ldots, T^{k_n}x, x^*)$ is a path in G and so in $\widetilde{G}$. Therefore, $x^* \in [x]_{\widetilde{G}}$. Using (GF2) we have

$$\frac{1}{M(T^{k_n+1}x, Tx^*, t)} - 1 \leq \lambda \left[\frac{1}{M(T^{k_n}x, x^*, t)} - 1\right]$$

for all $t > 0$. Using the above inequality we obtain

$$\begin{aligned} M(x^*, Tx^*, t) &\geq M(x^*, T^{k_n+1}x, t/2) * M(T^{k_n+1}x, Tx^*, t/2) \\ &\geq M(x^*, T^{k_n+1}x, t/2) * \left[\frac{1}{1 - \lambda + \frac{\lambda}{M(T^{k_n}x, x^*, t/2)}}\right]. \end{aligned}$$

Letting $n \to \infty$ and using (4) in the above inequality we obtain $M(x^*, Tx^*, t) = 1$ for all $t > 0$. Thus $Tx^* = x^*$, i.e., $x^* \in [x]_{\widetilde{G}}$ is a fixed point of T and so by Theorem 2, $T|_{[x]_{\widetilde{G}}}$ is a Picard operator.

To prove (B) let $X_T \neq \emptyset$ and G is weakly connected then $[x]_{\widetilde{G}} = X$ for all $x \in X_T$ and so by (A) T is a Picard operator.

To prove (C), note that if $\operatorname{Fix} T \neq \emptyset$ then there is some $x \in \operatorname{Fix} T$ then $Tx = x$ and $E(G) \supseteq \Delta$ we have $(x, Tx) \in E(G)$. So $x \in X_T$ and $\operatorname{Fix} T \subseteq X_T \neq \emptyset$. If $X_T \neq \emptyset$, then by (A) for any $x \in X_T$, $T|_{[x]_{\widetilde{G}}}$ is a Picard operator and so $\operatorname{Fix} T \neq \emptyset$.

To prove (D) if $T \subseteq E(G)$, then $(x, Tx) \in E(G)$ for all $x \in X$ and so $X = X_T$. Now the result follows from (A). □

In the above theorem, if $x \in X_T$ then $T|_{[x]_{\widetilde{G}}}$ is a Picard operator, but if G is not weakly connected then T need not be a Picard operator on X, i.e., the fixed point of T need not be unique. The following example illustrates the above Theorem.

Example 6. Let $X = \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\} = X_o \cup X_e$, where $X_o = \left\{\frac{1}{2^n} : n \in \mathbb{N}_o\right\}$, $X_e = \left\{\frac{1}{2^n} : n \in \mathbb{N}_e\right\}$ and $\mathbb{N}_o$, $\mathbb{N}_e$ are the set of all odd and even natural numbers respectively. Let $*$ be the product norm, i.e., $a * b = ab$ for all $a, b \in [0, 1]$. Define the fuzzy set $M\colon X^2 \times (0, \infty) \to [0, 1]$ by

$$M(x, y, t) = \begin{cases} 1, & \text{if } x = y; \\ xy, & \text{otherwise} \end{cases} \qquad \forall t > 0\,.$$

Let $T\colon X \to X$ be a mapping defined by

$$T\left(\frac{1}{2^n}\right) = \begin{cases} \frac{1}{2}, & \text{if } x \in \mathbb{N}_o; \\ \frac{1}{4}, & \text{if } x \in \mathbb{N}_e. \end{cases}$$

Let G be the graph with $V(G) = X$ and

$$E(G) = (X_o \times X_o) \cup (X_e \times X_e).$$

Then it is easy to see that T is a G-fuzzy contraction with arbitrary $\lambda \in (0, 1)$ and by definition of T the condition $(\mathcal{P}_T)$ holds. Note that for all $k \in \mathbb{N}_o$ we have $\frac{1}{2^k} \in X_T$ and $\left[\frac{1}{2^k}\right]_{\widetilde{G}} = X_o$ and $T|_{X_o}$ is a Picard operator. Similarly, $\frac{1}{2^k} \in X_T$ and $\left[\frac{1}{2^k}\right]_{\widetilde{G}} = X_e$ for all $k \in \mathbb{N}_e$ and $T|_{X_e}$ is a Picard operator.

Now it is easy to see that G is not weakly connected and T is not a Picard operator on X since $\operatorname{Fix} T = \left\{\frac{1}{2}, \frac{1}{4}\right\}$. Also, $T \subseteq E(G)$ and T is a weakly Picard operator on X.

The next example shows that the results of this paper generalize the corresponding classical concepts in the classical metric space.

Example 7. Let $X = \left\{\frac{1}{2^{2^n}} : n \in \mathbb{N}_0\right\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Then the triple $(X, M_d, *)$ is a fuzzy metric space, where $a * b = ab$ for all $a, b \in [0, 1]$ and

$$M(x, y, t) = \begin{cases} 1, & \text{if } x = y\,; \\ xy, & \text{otherwise} \end{cases} \qquad \text{for all } t > 0\,.$$

Note that there exists no metric d on X satisfying $M(x, y, t) = \dfrac{t}{t + d(x, y)}$. Therefore, this fuzzy metric is not a standard fuzzy metric induced by a metric (in the sense of George and Veeramani [1]). Define a mapping $T \colon X \to X$ by

$$T\left(\frac{1}{2^{2^n}}\right) = \begin{cases} \dfrac{1}{2^{2^{n-1}}}, & \text{if } n \in \mathbb{N}\,; \\ \dfrac{1}{2}, & \text{if } n = 0\,. \end{cases}$$

Let G be the graph with $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X : x \leq y\}\,.$$

Then it is easy to see that T is a G-fuzzy contraction with $\lambda \in [1/2, 1)$. Also, the property $(\mathcal{P}_T)$ is satisfied trivially and $X_T \neq \emptyset$. By definition, the graph G is weakly connected and by (B) of Theorem 3, T is a Picard operator with $\operatorname{Fix} T = \left\{\frac{1}{2}\right\}$.

On the other hand, T is not a Banach contraction with respect to the usual metric d, and therefore it is not a fuzzy contractive mapping with respect to the standard fuzzy metric $M(x, y, t) = \frac{t}{t+d(x,y)}$ induced by d. To see this, take the points $x = \frac{1}{4}, y = \frac{1}{16} \in X$ and then T fails to be a Banach contraction with respect to d.

Now we give some consequences of Theorem 3. The following corollary is the fuzzy metric version and an improvement of the result of Nieto and Rodríguez-López [9].

Corollary 2. *Let $(X, M, *)$ be a complete fuzzy metric space and $\preceq$ be a partial order defined on X. Let $T\colon X \to X$ be a nondecreasing mapping (i.e., $x \preceq y \Rightarrow Tx \preceq Ty$) such that the following contractive condition is satisfied:*

$$\exists_{\lambda\in(0,1)} \forall_{x,y\in X} \forall_{t>0} \left(x \preceq y \Rightarrow \frac{1}{M(Tx,Ty,t)} - 1 \leq \lambda \left[\frac{1}{M(x,y,t)} - 1\right]\right).$$

Assume that the following condition holds:

if there is a nondecreasing sequence $(x_n)_{n\in\mathbb{N}}$ in X which converges to $x \in X$ and $x_{n+1} \preceq x_n$ for all $n \in \mathbb{N}$, then $x_n \preceq x$ or $x \preceq x_n$ for all $n \in \mathbb{N}$. $(\mathcal{P}')$

If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$ or $Tx_0 \preceq x_0$, then T has a fixed point in X.

Proof. Let G be the graph defined by $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X : x \preceq y \vee y \preceq x\}.$$

Then since T is nondecreasing (GF1) holds and by the contractive condition (GF2) also holds. Therefore T is a G-fuzzy contraction. Also $(\mathcal{P}')$ implies $(\mathcal{P}_T)$ and by assumption $(x_0, Tx_0) \in E(G)$ so $x_0 \in X_T$. Therefore by (A) of Theorem 3, $T|_{[x_0]_{\tilde{G}}}$ is a Picard operator and so has a fixed point in $T|_{[x_0]_{\tilde{G}}}$. □

Recently, Kirk et al. [16] introduced the idea of cyclic contractions and established fixed point results for such mappings.

Let X be a nonempty set, m a positive integer, $A_i, i = 1, 2, \dots, m$ are nonempty subsets of X and $T\colon \bigcup_{i=1}^m A_i \to \bigcup_{i=1}^m A_i$ be a mapping, then $B = \bigcup_{i=1}^m A_i$ is said to be a cyclic representation of B with respect to T if

$$T(A_1) \subset A_2\,,\ T(A_2) \subset A_3\,,\ \dots\,,\ T(A_m) \subset T(A_1)$$

and then T is called a cyclic operator [16].

The following corollary is the fuzzy metric version of the result of Kirk et al. [16].

Corollary 3. *Let $(X, M, *)$ be a complete fuzzy metric space, m be a positive integer, $A_i, i = 1, 2, \dots, m$ be nonempty closed subsets of X and $B = \bigcup_{i=1}^m A_i$ be a cyclic representation of B with respect to T. Suppose $A_{m+i} = A_i$ for all $i \in \mathbb{N}$ and following condition holds:*

$$\exists_{\lambda\in(0,1)} \Bigg(x \in A_i\,,\ y \in A_{i+1}\,,\ i = 1, 2, \dots, m \Rightarrow \frac{1}{M(Tx,Ty,t)} - 1 \leq k \left[\frac{1}{M(x,y,t)} - 1\right]\Bigg).$$

Then T has a unique fixed point $x^* \in \bigcap_{i=1}^m A_i$.

Proof. Since $B = \bigcup_{i=1}^{m} A_i$ is closed so $(B, M, *)$ is complete. Let G be the graph defined by $V(G) = B$ and

$$E(G) = \Delta \cup \{(x, y) \in B \times B : x \in A_i \wedge y \in A_{i+1} : i = 1, 2, \dots, m\}.$$

Then since $B = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of B with respect to T, so (GF1) holds and by the given contractive condition (GF2) also hold. Now it is easy to see that the sequence $(T^n x)_{n\in\mathbb{N}}$ has infinitely many terms in each $A_i, i = 1, 2, \dots, m$ so if $(T^n x)_{n\in\mathbb{N}}$ converges to x^* then $x^* \in \bigcap_{i=1}^{m} A_i$. Therefore $(\mathcal{P}_T)$ holds good. Note that if $x \in B$ then $(x, Tx) \in E(G)$ therefore $T \subseteq E(G)$ and by (D) of Theorem 3, T has a fixed point. Uniqueness follows from the contractive condition and the fact that if $x \in \operatorname{Fix} T$ then $x \in \bigcap_{i=1}^{m} A_i$. □

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Author's address:
Department of Applied Mathematics, Shri Vaishnav Institute of Technology & Science, Sanwer Road, Indore, (M.P.) 453331, India

E-mail: `satishmathematics@yahoo.co.in`

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