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Supplementary balance laws for Cattaneo heat propagation

Serge Preston

Abstract. In this work we determine for the Cattaneo heat propagation system all the supplementary balance laws (shortly SBL) of the same order (zero) as the system itself and extract the constitutive relations (expression for the internal energy) dictated by the Entropy Principle. The space of all supplementary balance laws (having the functional dimension 8) contains four original balance laws and their deformations depending on 4 functions of temperature $(\lambda^0(\vartheta), K^A(\vartheta), A = 1, 2, 3)$. The requirements of the II law of thermodynamics leads to the exclusion of three functional degrees $(K^A = 0, A = 1, 2, 3)$ and to further restriction to the form of internal energy. In its final formulation, entropy balance represents the deformation of the energy balance law by the functional parameter $\lambda^0(\vartheta)$.

1 Introduction

Systems of balance equations form the cornerstone of the Continuum Thermodynamics, [1], [2], [4], [5]. With each system of this type, there is associated the space of "supplementary balance laws" (see next Section) playing, for the systems of balance equations, the role similar to the role the conservation laws play for general systems of differential equations. In this work we determine explicitly all supplementary balance laws for the Cattaneo heat propagation system (CHP-system) (1) of the same order (zero) that the Cattaneo system itself. We will solve directly the Lagrange-Liu system of differential equations associated with the CHP model [7], [8], and, on our way, specify the constitutive relation – the form of internal energy as the function of temperature θ and heat flux q. If this condition is fulfilled, the total space of SBL (modulo trivial balance laws) is 8-dimensional. If this condition does not hold, there are no new SBL except trivial (see [6]). Then we show that the positivity condition for the production in the new balance laws place additional restriction to the form of internal energy and determine the unique SBL having nonnegative production – the entropy balance law.

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2 Supplementary balance laws of a balance system

Let one have a system of balance equations for the fields $y^i(t, x^A, A = 1, 2, 3)$

$$\partial_t F_i^0 + \partial_{x^A} F_i^A = \Pi_i\,, \quad i = 1, \ldots, m, \tag{1}$$

with the densities F_i^0, fluxes $F_i^A, A = 1, 2, 3$ and sources Π_i being functions of space-time point $(t, x^A, A = 1, 2, 3)$, fields y^i and their derivatives (by t, x^A) up to the order $k \geqq 0$. Number k is called the order of balance system (1). In continuum thermodynamics people mostly work with the balance system of order 0 (case of Rational Extended Thermodynamics) and 1.

Definition 1. A balance law (2) of order r (in the same sense as the system (1) is of order k)

$$\partial_t K^0 + \sum_{A=1}^{3} \partial_{x^A} K^A = Q \tag{2}$$

is called a *supplementary balance law (SBL) for the system* (1) if *every solution of the system* (1) *is, at the same time, solution of the balance equation* (2).

Examples of supplementary balance laws are: entropy balance, provided the Entropy Principle is admitted for system (1) (see [5], [10], [11]), Noether balance laws corresponding to the Lie groups of symmetry (see [7], [8]) and some linear combinations of the balance equations of original balance system with variable coefficients satisfying some condition ("gauge symmetries" of system (1), see [7], [8]).

As a rule, in classical physics one looks for entropy balance laws of the same order as the original balance systems. Higher order SBL are also of an interest for studying the balance system (1) – for example, a study of integrable systems leads to the hierarchy of conservation laws (often having form of conservation laws themselves) of higher order.

For a balance system (1) of order 0 (case of Rational extended Thermodynamics, see [5]), density and flux of a SBL (2) satisfy the system of equations

$$\lambda^i F^\mu_{i,y^j} = K^\mu_{,y^j}\,, \tag{3}$$

where summation over repeated indices is considered. Functions $\lambda^i(y^j)$ (main fields in terminology of [5]) are to be found from the conditions of solvability of this system. We call this system the LL-system referring to the Liu method of using Lagrange method for formulating dissipative inequality for a system (1), see [3], [5]. Source/production in the system (2) is then found as $Q = \sum_i \lambda^i \Pi_i$.

3 Cattaneo Heat propagation balance system

Consider the heat propagation model containing the temperature ϑ and heat flux q as the independent dynamical fields $y^0 = \vartheta$, $y^A = q^A$, $A = 1, 2, 3$.

Balance equations of this model have the form

$$\begin{cases} \partial_t(\rho\epsilon) + \mathrm{div}(q) = 0\,, \\ \partial_t(\tau q) + \nabla\Lambda(\vartheta) = -q\,. \end{cases} \tag{4}$$

The second equation in (4) can be rewritten in the conventional form

$$\partial_t(\tau q) + \lambda \cdot \nabla\vartheta = -q\,,$$

where $\lambda = \frac{\partial \Lambda}{\partial \vartheta}$. If the coefficient λ may depend on the density ρ, the equation is more complex.

Constitutive relations specify dependence of the internal energy ϵ on ϑ, q and possible dependence of coefficients τ, Λ on the temperature (including the requirement $\Lambda_{,\vartheta} \neq 0$). The simplest case is the linear relation $\epsilon = k\vartheta$, but for our purposes it is too restrictive, see [2, Sec.2.1].

Since ρ is not considered here as a dynamical variable, we merge it with the field ϵ and from now on and till the end it will be omitted. On the other hand, in this model the energy ϵ depends on temperature ϑ *and on the heat flux* q (see [2, Sec.2.1.2]) or, by change of variables, temperature $\vartheta = \vartheta(\epsilon, q)$ will be considered as the function of dynamical variables.

Cattaneo equation $q + \tau\partial_t(q) = -\lambda \cdot \nabla\vartheta$ has the form of the vectorial balance law and, as a result there is no need for the constitutive relations to depend on the derivatives of the basic fields. No derivatives appear in the constitutive relation, therefore, this is the RET model. In the second equation there is a nonzero production $\Pi^A = -q^A$. The model is homogeneous, there is no explicit dependence of any functions on t, x^A.

4 LL-system for supplementary balance laws of CHP-system

To study the LL-system for the supplementary balance laws we start with the $i \times \mu$ matrix of density/flux components

$$F_i^\mu = \begin{pmatrix} \epsilon & \tau q^1 & \tau q^2 & \tau q^3 \\ q^1 & \Lambda(\vartheta) & 0 & 0 \\ q^2 & 0 & \Lambda(\vartheta) & 0 \\ q^3 & 0 & 0 & \Lambda(\vartheta) \end{pmatrix}.$$

Assuming that coefficients τ and the function Λ are independent on the heat flux variables q^A we get the "vertical (i.e. by fields ϑ, q^A) differentials" of densities and flux components F_i^μ

$$d_v F_i^\mu = \begin{pmatrix} \epsilon_\vartheta\, d\vartheta + \epsilon_{q^A}\, dq^A & \tau_\vartheta q^1\, d\vartheta + \tau\, dq^1 & \tau_\vartheta q^2\, d\vartheta + \tau\, dq^2 & \tau_\vartheta q^2\, d\vartheta + \tau\, dq^3 \\ dq^1 & \Lambda_\vartheta\, d\vartheta & 0 & 0 \\ dq^2 & 0 & \Lambda_\vartheta\, d\vartheta & 0 \\ dq^3 & 0 & 0 & \Lambda_\vartheta\, d\vartheta \end{pmatrix}.$$

Let now

$$\partial_t K^0(x,y) + \partial_{x^A} K^A(x,y) = Q(x,y) \tag{5}$$

be a supplementary balance law for the Cattaneo balance system (4). It is easy to see that the LL-system has the form:
Subsystem of LL-system with $\mu = 0$ has the form:

$$\begin{cases} \lambda^0 \epsilon_\vartheta + \tau_\vartheta \lambda^A q^A = K^0_{,\vartheta} \\ \lambda^0 \epsilon_{q^A} + \lambda^A \tau = K^0_{,q^A} \end{cases},\ A = 1,2,3\,. \tag{6}$$

For $\mu = A = 1,2,3$, using cyclic notations, we have LL-equations:

$$\begin{cases} \Lambda_\vartheta \lambda^A = K^A_{,\vartheta} \\ \lambda^0 = K^A_{,q^A} \\ 0 = K^A_{q^{A+1}} \\ 0 = K^A_{q^{A+2}} \end{cases}, \ A = 1,2,3\,. \tag{7}$$

Looking at systems (6), (7) we see that if we make the change of variables $\tilde{\vartheta} = \Lambda(\vartheta)$ then the system of equations (6), (7) takes the form (wherever is the derivative by ϑ we multiply this equation by $\Lambda_{,\vartheta}$)

$$\begin{cases} \lambda^0 \epsilon_{\tilde{\vartheta}} + \tau_{\tilde{\vartheta}} \lambda^A q^A = K^0_{,\tilde{\vartheta}} \\ \lambda^0 \epsilon_{q^A} + \lambda^A \tau = K^0_{q^A} \end{cases}; \quad \begin{cases} K^A_{,\tilde{\vartheta}} = \lambda^A \\ K^A_{q^B} = \lambda^0 \delta^A_B \end{cases}, \ A, B = 1,2,3. \tag{8}$$

The second subsystem is equivalent to the relation

$$d_v K^A = \lambda^A d\tilde{\vartheta} + \lambda^0 dq^A.$$

These integrability conditions imply the expression $K^A = K^A(x^\mu, \tilde{\vartheta}, q^A)$ and

$$K^A_{q^A} = \lambda^0,\ A = 1,2,3 \ \Rightarrow \lambda^0 = \lambda^0(\tilde{\vartheta}).$$

Integrating equation $K^A_{q^A} = \lambda^0(\tilde{\vartheta})$ by q^A we get

$$K^A = \lambda^0(\tilde{\vartheta}) q^A + \tilde{K}^A(\tilde{\vartheta}) \tag{9}$$

with some functions $\tilde{K}^A(\tilde{\vartheta})$.

The first equation of each system now takes the form

$$\lambda^A = K^A_{\tilde{\vartheta}} = \lambda^0_{\tilde{\vartheta}} q^A + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta}). \tag{10}$$

Substituting these expressions for λ^A into the 0-th system

$$\begin{cases} \lambda^0 \epsilon_{\tilde{\vartheta}} + \tau_{\tilde{\vartheta}} \lambda^A q^A = K^0_{,\tilde{\vartheta}} \\ \lambda^0 \epsilon_{q^A} + \lambda^A \tau = K^0_{q^A} \end{cases}, \ A = 1,2,3,$$

we get

$$\begin{cases} K^0_{,\tilde{\vartheta}} = \lambda^0 \epsilon_{\tilde{\vartheta}} + \tau_{\tilde{\vartheta}} \big(\lambda^0_{\tilde{\vartheta}} \|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta}) q^A \big) \\ K^0_{q^A} = \lambda^0 \epsilon_{q^A} + \tau \big(\lambda^0_{\tilde{\vartheta}} q^A + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta}) \big) \end{cases}, \ A = 1,2,3, \tag{11}$$

where $\|q\|^2 = \sum_A q^{A\ 2}$.

Integrating A-th equation by q^A and comparing results for different A we obtain the following representation

$$K^0 = \lambda^0 \epsilon + \tau(\tilde{\vartheta}) \Big[\frac{1}{2} \lambda^0_{\tilde{\vartheta}} \|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta}) q^A \Big] + f(\tilde{\vartheta}) \tag{12}$$

for some function $f(\tilde{\vartheta}, x^\mu)$.

Calculate derivative by $\tilde{\vartheta}$ in the last formula for K^0 and subtract the first formula of the previous system. We get

$$0 = \lambda^0_{,\tilde{\vartheta}}\epsilon + \tau(\tilde{\vartheta})\Big[\Big(\frac{1}{2}\lambda^0_{\tilde{\vartheta}}\|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta})q^A\Big)\Big]_{,\tilde{\vartheta}} - \frac{1}{2}\tau_{,\tilde{\vartheta}}\lambda^0_{,\tilde{\vartheta}}\|q\|^2 + f_{,\tilde{\vartheta}}(\tilde{\vartheta}). \tag{13}$$

This is the compatibility condition for the system (6) for K^0. As such, it is realization of the general compatibility system (8).

Take $q^A = 0$ in the last equation, i.e. consider the case where *there is no heat flux*. Then the internal energy reduces to its *equilibrium value* $\epsilon^{\mathrm{eq}}(\tilde{\vartheta})$ and we get $f_{,\tilde{\vartheta}}(\tilde{\vartheta}) = -\lambda^0_{,\tilde{\vartheta}}\epsilon^{\mathrm{eq}}$. Integrating here we find

$$f(\tilde{\vartheta}) = f_0(x^\mu) - \int^{\tilde{\vartheta}} \lambda^0_{,\tilde{\vartheta}}(s)\epsilon^{\mathrm{eq}}(s)\, ds\,. \tag{14}$$

Substituting this value for f into the previous formula we get expressions for K^μ:

$$\begin{cases} K^0 = \lambda^0\epsilon - \displaystyle\int^{\tilde{\vartheta}} \lambda^0_{,\tilde{\vartheta}}\epsilon^{\mathrm{eq}}\, ds + \tau(\tilde{\vartheta})\Big[\frac{1}{2}\lambda^0_{\tilde{\vartheta}}\|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta})q^A\Big] + f_0 \\ K^A = \lambda^0(\tilde{\vartheta})q^A + \tilde{K}^A(\tilde{\vartheta}) \end{cases}, \quad A = 1, 2, 3. \tag{15}$$

In addition to this, from (13) and obtained expression for $f(\tilde{\vartheta})$, we get the *expression for internal energy*

$$\epsilon = \epsilon^{\mathrm{eq}}(\tilde{\vartheta}) + \frac{1}{2}\tau_{,\tilde{\vartheta}}\|q\|^2 - \frac{\tau(\tilde{\vartheta})}{\lambda^0_{\tilde{\vartheta}}(\tilde{\vartheta})}\Big[\frac{1}{2}\lambda^0_{,\tilde{\vartheta}\tilde{\vartheta}}\|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}\tilde{\vartheta}}(\tilde{\vartheta})q^A\Big]. \tag{16}$$

This form for internal energy presents the restriction to the constitutive relations in Cattaneo model placed on it by the entropy principle.

The zeroth main field λ^0 is an arbitrary function of $\tilde{\vartheta}$ while λ^A are given by the relations (15):

$$\lambda^A = (\lambda^0_{\tilde{\vartheta}}q^A + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta})). \tag{17}$$

Using this we find the source/production term for the SBL (5)

$$Q = \lambda^A\Pi_A = -\lambda^A q^A = -\big(\lambda^0_{\tilde{\vartheta}}\|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta})q^A\big)\,.$$

Now we combine obtained expressions for components of a secondary balance law. We have to take into account that the LL-system defines K^μ only $\mathrm{mod}\, C^\infty(X)$. This means first of all that all the functions may depend explicitly on x^μ. For energy ϵ, field $\Lambda(\vartheta)$ and the coefficient τ this dependence is determined by constitutive relations and is, therefore, fixed. Looking at (16) we see that the coefficients of terms linear and quadratic by q^A are also defined by the constitutive relation, i.e. in the representation

$$\begin{aligned} \epsilon &= \epsilon^{\mathrm{eq}}(\tilde{\vartheta}) + \mu(\tilde{\vartheta})\|q\|^2 + M_A(\tilde{\vartheta})q^A \\ &= \epsilon^{\mathrm{eq}}(\tilde{\vartheta}) + \frac{1}{2}\tau_{,\tilde{\vartheta}}\|q\|^2 - \frac{\tau(\tilde{\vartheta})}{\lambda^0_{\tilde{\vartheta}}(\tilde{\vartheta})}\Big[\frac{1}{2}\lambda^0_{,\tilde{\vartheta}\tilde{\vartheta}}\|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}\tilde{\vartheta}}(\tilde{\vartheta})q^A\Big], \end{aligned} \tag{18}$$

coefficients

$$\mu(\tilde{\vartheta},x)=\frac{1}{2}\tau_{,\tilde{\vartheta}}-\frac{1}{2}\frac{\tau(\tilde{\vartheta})}{\lambda^0_{\tilde{\vartheta}}(\tilde{\vartheta})}\lambda^0_{\tilde{\vartheta}\tilde{\vartheta}}\,,\quad M_A=-\frac{\tau(\tilde{\vartheta})}{\lambda^0_{\tilde{\vartheta}}(\tilde{\vartheta})}\tilde{K}^A_{,\tilde{\vartheta}\tilde{\vartheta}}(\tilde{\vartheta}) \tag{19}$$

are defined by the CR – by expression of internal energy as the quadratic function of the heat flux.

More than this, quantities $\frac{\lambda^0_{\tilde{\vartheta}\tilde{\vartheta}}}{\lambda^0_{\tilde{\vartheta}}}$ and $\frac{\tilde{K}^A_{,\tilde{\vartheta}\tilde{\vartheta}}(\tilde{\vartheta})}{\lambda^0_{\tilde{\vartheta}}}$ are also defined by the constitutive relations.

Rewriting the first relation (18) we get

$$\begin{aligned}\left(\ln(\lambda^0_{\tilde{\vartheta}})\right)_{,\tilde{\vartheta}}=\ln(\tau)_{,\tilde{\vartheta}}-2\frac{\mu(\tilde{\vartheta})}{\tau(\tilde{\vartheta})}&\Rightarrow \ln(\lambda^0_{\tilde{\vartheta}})=\ln(\tau)+b^0-2\int^{\tilde{\vartheta}}\frac{\mu}{\tau}(s)\,ds\\ &\Rightarrow \lambda^0_{\tilde{\vartheta}}=\alpha\tau e^{-2\int^{\tilde{\vartheta}}\frac{\mu}{\tau}(s)ds},\ \alpha=e^{b^0}>0\,.\end{aligned}$$

From this relation we find

$$\lambda^0(\tilde{\vartheta},x)=a^0+\alpha\hat{\lambda}^0=a^0+\alpha\int^{\tilde{\vartheta}}\left[\tau e^{-2\int^u\frac{\mu(s)}{\tau(s)}\,ds}\right]du \tag{20}$$

Here a^0 and α are constants (or, maybe, functions of x^μ).

Using obtained expression for $\lambda^0(\tilde{\vartheta},x)$ in the second formula (19) we get the expression for coefficients $\tilde{K}^A$ and, integrating twice by $\tilde{\vartheta}$, for the functions $K^A(\tilde{\vartheta})$

$$\begin{aligned}\tilde{K}^A_{,\tilde{\vartheta}\tilde{\vartheta}}&=-M_A\cdot\frac{\lambda^0_{\tilde{\vartheta}}(\tilde{\vartheta})}{\tau(\tilde{\vartheta})}=-M_A\alpha e^{-2\int^{\tilde{\vartheta}}\frac{\mu}{\tau}(s)ds}\\ \Rightarrow \tilde{K}^A&=k^A\tilde{\vartheta}+m^A+\alpha\cdot\hat{K}^A(\tilde{\vartheta})\\ &=k^A\tilde{\vartheta}+m^A-\alpha\int^{\tilde{\vartheta}}dw\int^w[M_A(u)e^{-2\int^u\frac{\mu}{\tau}(s)ds}]\,du\,.\end{aligned} \tag{21}$$

Functions $\hat{K}^A(\tilde{\vartheta})$ are defined by the second formula in the second line.

Thus, functions $\lambda^0_{\vartheta},\tilde{K}^A_{,\vartheta\vartheta}$ are defined by the constitutive relations while coefficients $\alpha>0,a^0,k^A,m^A$ are arbitrary functions of x^μ.

5 Supplementary balance laws for CHP-system

Combining obtained results, returning to the variable ϑ (and using repeatedly the relation $f_{,\tilde{\vartheta}}=\vartheta_{,\tilde{\vartheta}}f_{,\vartheta}=(\tilde{\vartheta}_{,\vartheta})^{-1}f_{,\vartheta}=\Lambda^{-1}_{,\vartheta}f_{,\vartheta})$ we get the general expressions for admissible densities/fluxes of the supplementary balance laws

$$
\begin{cases}
K^0 = \lambda^0\epsilon - \displaystyle\int^{\tilde\vartheta} \lambda^0_{,\tilde\vartheta}\epsilon^{\mathrm{eq}}\,ds + \tau(\tilde\vartheta)\Big[\frac{1}{2}\lambda^0_{\tilde\vartheta}\|q\|^2 + \tilde K^A_{,\tilde\vartheta}(\tilde\vartheta)q^A\Big] + f_0 \\
\quad = (a^0+\alpha\hat\lambda^0)\epsilon - \alpha\displaystyle\int^{\vartheta}\hat\lambda^0_{,\vartheta}\epsilon^{\mathrm{eq}}\,ds \\
\qquad + \dfrac{\tau(\vartheta)}{\Lambda_{,\vartheta}}\Big[\frac{\alpha}{2}\hat\lambda^0_{\vartheta}\|q\|^2 + \big(\Lambda_{,\vartheta}k^A + \alpha\hat K^A_{,\vartheta}(\vartheta)\big)q^A\Big] + f_0, \\
K^A = \lambda^0(\tilde\vartheta)q^A + \tilde K^A(\tilde\vartheta) \\
\quad = (a^0+\alpha\hat\lambda^0(\vartheta))q^A + k^A\Lambda(\vartheta) + m^A + \alpha\hat K^A(\vartheta)\,, \quad A=1,2,3 \\
Q = -\lambda^A q^A = -\big(\lambda^0_{\tilde\vartheta}\|q\|^2 + \tilde K^A_{,\tilde\vartheta}(\tilde\vartheta)q^A\big) \\
\quad = -\Lambda^{-1}_{,\vartheta}\big(\lambda^0_{\vartheta}\|q\|^2 + \Lambda_{,\vartheta}k^Aq^A + \alpha\hat K^A_{,\vartheta}(\vartheta)q^A\big) \\
\quad = -\Lambda^{-1}_{,\vartheta}\big(\alpha\hat\lambda^0_{\vartheta}\|q\|^2 + \Lambda_{,\vartheta}k^Aq^A + \alpha\hat K^A_{,\vartheta}(\vartheta)q^A\big)\,.
\end{cases}
$$

Collecting previous results together we present obtained expressions for secondary balance laws first in short form and then in the form where original balance laws and the trivial balance laws are separated from the general form of SBL

$$
\begin{pmatrix} K^0 \\ K^1 \\ K^2 \\ K^3 \\ Q \end{pmatrix} = \begin{pmatrix} \lambda^0\epsilon - \int^{\vartheta}\lambda^0_{,\vartheta}\epsilon^{\mathrm{eq}}\,ds + \tau(\vartheta)\Lambda^{-1}_{\vartheta}[\frac{1}{2}\lambda^0_{\vartheta}\|q\|^2 + \alpha\tilde K^A_{,\vartheta}(\vartheta)q^A] + f_0 \\ \lambda^0(\vartheta)q^1 + \tilde K^1(\vartheta) \\ \lambda^0(\vartheta)q^2 + \tilde K^2(\vartheta) \\ \lambda^0(\vartheta)q^3 + \tilde K^3(\vartheta) \\ -\Lambda^{-1}_{,\vartheta}(\lambda^0_{,\vartheta}\|q\|^2 + \tilde K^A_{,\vartheta}(\vartheta)q^A) \end{pmatrix}
$$

$$
= a^0\begin{pmatrix} \epsilon \\ q^1 \\ q^2 \\ q^3 \\ 0 \end{pmatrix} + \sum_A k^A \begin{pmatrix} \tau(\vartheta)q^A \\ \delta^1_A\Lambda(\vartheta) \\ \delta^2_A\Lambda(\vartheta) \\ \delta^3_A\Lambda(\vartheta) \\ -q^A \end{pmatrix} + \begin{pmatrix} \alpha\tau\Lambda(\vartheta)^{-1}\hat K^A_{,\vartheta}(\vartheta)q^A \\ \hat K^1(\vartheta) \\ \hat K^2(\vartheta) \\ \hat K^3(\vartheta) \\ -\Lambda^{-1}_{,\vartheta}\hat K^A_{,\vartheta}(\vartheta)q^A \end{pmatrix} \tag{22}
$$

$$
+\,\alpha\begin{pmatrix} \hat\lambda^0\epsilon - \int^{\vartheta}\hat\lambda^0_{,\vartheta}\epsilon^{\mathrm{eq}}\,ds + \tau(\vartheta)\Lambda^{-1}_{\vartheta}[\frac{1}{2}\hat\lambda^0_{,\vartheta}\|q\|^2] \\ \hat\lambda^0(\vartheta)q^1 \\ \hat\lambda^0(\vartheta)q^2 \\ \hat\lambda^0(\vartheta)q^3 \\ -\Lambda^{-1}_{,\vartheta}\hat\lambda^0_{,\vartheta}\|q\|^2 \end{pmatrix} + \begin{pmatrix} f_0 \\ m^1 \\ m^2 \\ m^3 \\ 0 \end{pmatrix}.
$$

To get the second presentation of the SBL we use the decompositions (21) $\lambda^0 = \alpha\hat\lambda_0 + a^0$ and (20) $\tilde K^A(\tilde\vartheta) = k^A\tilde\vartheta + m^A - \hat K^A$.

Remark 1. Notice the duality between the tensor structure of the basic fields of Cattaneo system – one scalar field (temperature ϑ) and one vector field (heat flux q^A, $A=1,2,3$) – and the structure of space $\mathcal{SBL}(C)$ of supplementary balance laws – elements of $\mathcal{SBL}(C)$ depend on one scalar function of temperature $\lambda^0(\vartheta)$ and one covector function of temperature $\hat{K}_A$.

Remark 2. It is easy to see that none of new SBL can be written as a linear combination of original balance equations with variable coefficients (Noether balance laws generated by vertical symmetries $v = v^k(y^i)\partial_{y^k}$, see [7], [8]). The easiest way to prove this is to compare the source terms of different balance equations.

Returning to the variable ϑ in the expression (16) and using the relation $\partial_{\tilde{\vartheta}} = \frac{1}{\Lambda(\vartheta)_{,\vartheta}}\partial_\vartheta$ we get the expression for the internal energy

$$\begin{aligned}\epsilon &= \epsilon^{\mathrm{eq}}(\vartheta) + \frac{\tau_{,\vartheta}}{2\Lambda_{,\vartheta}}\|q\|^2 - \frac{\tau(\vartheta)}{\lambda^0_{,\vartheta}}\left[\frac{1}{2}\left(\frac{\lambda^0_{,\vartheta}}{\Lambda_{,\vartheta}}\right)_{,\vartheta}\|q\|^2 + \left(\frac{\tilde{K}^A_{,\vartheta}}{\Lambda_{,\vartheta}}\right)_{,\vartheta} q^A\right]\\ &=^{\Lambda_{,\vartheta}=\kappa-\mathrm{const}} \epsilon^{\mathrm{eq}}(\vartheta) + \frac{\tau_{,\vartheta}}{2\kappa}\|q\|^2 - \frac{\tau(\vartheta)}{\kappa\lambda^0_{,\vartheta}}\left[\frac{1}{2}\lambda^0_{,\vartheta\vartheta}\|q\|^2 + \tilde{K}^A_{,\vartheta\vartheta}q^A\right].\end{aligned}$$

Notice that for $\lambda^0 = 0$, balance laws given by the 4th column in (22) (the one with coefficient α) vanish. The same is true for deformations of the Cattaneo equation (second column) defined by the third column when $\tilde{K}^A(\vartheta) = 0$.

The first and second balance laws in (22) are the balance laws of the original Cattaneo system. The last one is the *trivial balance law*. Third and fourth columns give the balance law

$$\begin{aligned}\partial_t\left[\hat{\lambda}^0\epsilon - \int^\vartheta \lambda^0_{,\vartheta}\epsilon^{\mathrm{eq}}\,ds + \tau(\vartheta)\Lambda^{-1}_{\vartheta}\left[\frac{1}{2}\lambda^0_{\vartheta}\|q\|^2 + \hat{K}^A_{,\vartheta}(\vartheta)q^A\right]\right]&\\ + \partial_{x^A}\left[\hat{\lambda}^0(\vartheta)q^A + \hat{K}^A(\vartheta)\right] = -\Lambda^{-1}_{,\vartheta}\left(\hat{\lambda}^0_{\vartheta}\|q\|^2 + \hat{K}^A_{,\vartheta}(\vartheta)q^A\right).& \quad (23)\end{aligned}$$

Source/production term in (23) equation has the form

$$\begin{aligned}-\Lambda^{-1}_{,\vartheta}\left(\hat{\lambda}^0_{\vartheta}\|q\|^2 + \hat{K}^A_{,\vartheta}(\vartheta)q^A\right) &= -\Lambda^{-1}_{,\vartheta}\hat{\lambda}^0_{\vartheta}\left(\|q\|^2 + \frac{\hat{K}^A_{,\vartheta}(\vartheta)}{\hat{\lambda}^0_{\vartheta}}q^A\right)\\ &= -\Lambda^{-1}_{,\vartheta}\hat{\lambda}^0_{\vartheta}\left[\sum_A\left(q^A + \frac{\hat{K}^A_{,\vartheta}(\vartheta)}{2\hat{\lambda}^0_{\vartheta}}\right)^2 - \sum_A\left(\frac{\hat{K}^A_{,\vartheta}(\vartheta)}{2\hat{\lambda}^0_{\vartheta}}\right)^2\right].\end{aligned}$$

By physical reasons, $\Lambda_{,\vartheta} > 0$. As (20) shows, $\lambda_{,\vartheta}$ may have any sign. We assume that this sign does not depend on ϑ.

For a fixed ϑ expression for the production in the balance law (23) may have constant sign *for all values of q^A if and only if $\hat{K}^A_{,\vartheta}(\vartheta) = 0$, $A = 1,2,3$.* Therefore this is possible only if the internal energy has the form

$$\begin{aligned}\epsilon &= \epsilon^{\mathrm{eq}}(\vartheta) + \left[\frac{\tau_{,\vartheta}}{2\Lambda_{,\vartheta}} - \frac{\tau(\vartheta)}{2\hat{\lambda}^0_{,\vartheta}}\left(\frac{\hat{\lambda}^0_{,\vartheta}}{\Lambda_{,\vartheta}}\right)_{,\vartheta}\right]\|q\|^2\\ &=^{\tau-\mathrm{const},\ \Lambda_{,\vartheta}-\mathrm{const}} \epsilon^{\mathrm{eq}}(\vartheta) - \frac{\tau(\vartheta)}{2k\hat{\lambda}^0_{,\vartheta}}\hat{\lambda}^0_{,\vartheta\vartheta}\|q\|^2\end{aligned}$$

with some function $\hat{\lambda}^0(\vartheta)$. This being so, Cattaneo system has the entropy (supplementary balance) law

$$\partial_t\left[\hat{\lambda}^0\epsilon - \int^{\vartheta} \hat{\lambda}^0_{,\vartheta}\epsilon^{\mathrm{eq}}\, ds + \frac{1}{2}\tau(\vartheta)\Lambda_{\vartheta}^{-1}\hat{\lambda}^0_{,\vartheta}\|q\|^2\right] + \partial_{x^A}\left[\hat{\lambda}^0_{,\vartheta}q^A\right] = -\Lambda_{,\vartheta}^{-1}\hat{\lambda}^0_{,\vartheta}\|q\|^2$$

with the production term that may have constant sign – nonnegative, provided (we use the fact that $\hat{\lambda}^0_{,\vartheta} = \lambda^0_{,\vartheta}$)

$$\Lambda_{,\vartheta}^{-1}\lambda^0_{,\vartheta} \leqq 0\,. \tag{24}$$

This inequality (which is equivalent, if $\Lambda_{,\vartheta} \geqq 0$, to the inequality $\lambda^0_{,\vartheta} \leqq 0$) is the II law of thermodynamics for Cattaneo heat propagation model.

If we take $q = 0$ in obtained entropy balance we have to get the value of entropy at the equilibrium s^{eq}:

$$s^{\mathrm{eq}} = \hat{\lambda}^0\epsilon^{\mathrm{eq}} - \int^{\vartheta} \lambda^0_{,\vartheta}\epsilon^{\mathrm{eq}}\, ds = \int^{\vartheta} \hat{\lambda}^0\epsilon^{\mathrm{eq}}_{,\vartheta}\, d\vartheta.$$

From this it follows that *at a homogeneous state* $ds^{\mathrm{eq}} = \hat{\lambda}^0\, d\epsilon^{\mathrm{eq}}$. Comparing this with the Gibbs relation $d\epsilon^{\mathrm{eq}} = \vartheta ds^{\mathrm{eq}}$ we conclude that

$$\hat{\lambda}^0 = \frac{1}{\vartheta}. \tag{25}$$

Using (17) we also conclude that

$$\lambda^A = -\frac{q^A}{\vartheta^2}\,, \quad A = 1, 2, 3\,.$$

It follows from this that the condition (24) (II law) takes here the form well known from thermodynamics (see [2], [4], [5]):

$$\Lambda_{,\vartheta} \geqq 0\,.$$

Substituting (14) into (17) and calculating

$$\begin{aligned} -\frac{\tau(\vartheta)}{2\hat{\lambda}^0_{,\vartheta}}\left(\frac{\hat{\lambda}^0_{,\vartheta}}{\Lambda_{,\vartheta}}\right)_{,\vartheta} &= \frac{\tau(\vartheta)\vartheta^2}{2}\left(\frac{-1}{\vartheta^2\Lambda_{,\vartheta}}\right)_{,\vartheta} = -\frac{\tau(\vartheta)\vartheta^2}{2}\frac{-(2\vartheta\Lambda_{,\vartheta} + \vartheta^2\Lambda_{,\vartheta\vartheta})}{\vartheta^4\Lambda^2_{,\vartheta}} \\ &= \frac{\tau(\vartheta)}{\vartheta\Lambda_{,\vartheta}} + \frac{\tau(\vartheta)\Lambda_{,\vartheta\vartheta}}{2(\Lambda_{,\vartheta})^2} \end{aligned}$$

we get the expression for internal energy in the form

$$\begin{aligned} \epsilon &= \epsilon^{\mathrm{eq}}(\vartheta) + \left[\frac{\tau_{,\vartheta}}{2\Lambda_{,\vartheta}} + \frac{\tau}{\vartheta\Lambda_{\vartheta}} + \frac{\tau\Lambda_{,\vartheta\vartheta}}{2(\Lambda_{,\vartheta})^2}\right]\|q\|^2 \\ &=^{\tau-\mathrm{const},\ \Lambda_{,\vartheta}-\mathrm{const}} \epsilon^{\mathrm{eq}}(\vartheta) + \frac{\tau}{\vartheta\Lambda_{,\vartheta}}\|q\|^2. \end{aligned} \tag{26}$$

For the entropy density we have

$$\begin{aligned}
s &= s^{\mathrm{eq}} + \hat{\lambda}^0(\epsilon - \epsilon^{\mathrm{eq}}) + \frac{1}{2}\tau(\vartheta)\Lambda_{\vartheta}^{-1}\hat{\lambda}^0_{,\vartheta}\|q\|^2 = \\
&= s^{\mathrm{eq}} + \frac{1}{\vartheta}\left[\frac{\tau_{,\vartheta}}{2\Lambda_{,\vartheta}} + \frac{\tau}{\vartheta\Lambda_{\vartheta}} + \frac{\tau\Lambda_{,\vartheta\vartheta}}{2(\Lambda_{,\vartheta})^2}\right]\|q\|^2 - \frac{\tau(\vartheta)}{2\vartheta^2\Lambda_{\vartheta}}\|q\|^2 \\
&= s^{\mathrm{eq}} + \frac{1}{\vartheta}\left[\frac{\tau_{,\vartheta}}{2\Lambda_{,\vartheta}} + \frac{\tau}{2\vartheta\Lambda_{\vartheta}} + \frac{\tau\Lambda_{,\vartheta\vartheta}}{2(\Lambda_{,\vartheta})^2}\right]\|q\|^2 \\
&= s^{\mathrm{eq}} + \frac{\tau}{2\vartheta\Lambda_{,\vartheta}}\left[\frac{\tau_{,\vartheta}}{\tau} + \frac{1}{\vartheta} + \frac{\Lambda_{,\vartheta\vartheta}}{\Lambda_{,\vartheta}}\right]\|q\|^2 \\
&=^{\tau-\mathrm{const},\ \Lambda_{,\vartheta}-\mathrm{const}} s^{\mathrm{eq}} + \frac{\tau}{2\vartheta^2\Lambda_{,\vartheta}}\|q\|^2.
\end{aligned} \tag{27}$$

Correspondingly, the entropy balance law takes the form

$$\partial_t\left(s^{\mathrm{eq}} + \frac{\tau}{2\vartheta\Lambda_{,\vartheta}}\left[\frac{\tau_{,\vartheta}}{\tau} + \frac{1}{\vartheta} + \frac{\Lambda_{,\vartheta\vartheta}}{\Lambda_{,\vartheta}}\right]\|q\|^2\right) + \partial_{x^A}\left(\frac{q^A}{\vartheta}\right) = \frac{1}{\Lambda_{,\vartheta}}\left\|\frac{q}{\vartheta}\right\|^2.$$

Remark 3. If in the absence of the heat flow ($q = 0$) the "equilibrium state" is not homogeneous, more general constitutive relations with λ^0 different from (25) and more general form of energy and entropy balances satisfying the II law of Thermodynamics, are possible.

We collect obtained results in the following

Theorem 1. *1. For the Cattaneo heat propagation balance system* (1) *compatible with the entropy principle and having a nontrivial supplementary balance law that is not a linear combination of the original balance laws with constant coefficients, the internal energy has the form* (7). *If* (7) *holds, all supplementary balance laws for Cattaneo balance system (including original equations and the trivial ones) are listed in* (6). *New supplementary balance laws depend on the 4 functions of temperature –* $\hat{\lambda}^0(\vartheta)$, $\tilde{K}^A(\vartheta)$, $A = 1,2,3$. *Corresponding main fields* λ^μ, $\mu = 0,1,2,3$, *have the form* (17), (20).

2. The additional balance law (23) *given by the sum of third and fourth columns in* (22) has the nonnegative production term *if and only if the internal energy* ϵ *has the form* (26) *and, in addition, the condition* (24) *holds. Cattaneo systems satisfying these conditions depend on one arbitrary function of time* $\epsilon^{\mathrm{eq}}(\vartheta)$.

3. The supplementary balance law having nonnegative production term (entropy) is unique modulo linear combination of original balance laws and the trivial balance laws.

6 Conclusion

Description of the supplementary balance laws for Cattaneo heat propagation system given in this paper can probably be carried over for other systems of balance equations for the couples of fields: scalar + vector field.

One observes a kind of duality between the tensorial structure of dynamical fields (here ϑ, q) and the list of free functions of temperature $\lambda^0(\vartheta)$, $\tilde{K}^A(\vartheta)$, $A = 1, 2, 3$ entering the description of SBL.

It would be interesting to follow up if similar duality exists for the balance systems of more complex tensorial structure and for the systems of order 1 (recently the author completed the classification of SBL of order 0 and 1 for the Navier-Stokes fluid balance system, [9]).

In the case of Cattaneo heat propagation system, the II law of thermodynamics – existence of the SBL having the nonnegative production term – defines the entropy balance uniquely (modulo addition of trivial balance laws and the linear combination of the original balance laws). It would be interesting to look at other balance systems to determine the character of non-unicity of the SBL with the positive production – "abstract entropy balances" – to find the place of "physical entropy" in this list and to see if this "physical entropy balance" is "optimal" in some sense.

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