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On some issues concerning polynomial cycles

Tadeusz Pezda

Abstract. We consider two issues concerning polynomial cycles. Namely, for a discrete valuation domain R of positive characteristic (for $N \geq 1$) or for any Dedekind domain R of positive characteristic (but only for $N \geq 2$), we give a closed formula for a set $\mathcal{CYCL}(R, N)$ of all possible cycle-lengths for polynomial mappings in R^N . Then we give a new property of sets $\mathcal{CYCL}(R, 1)$, which refutes a kind of conjecture posed by W. Narkiewicz.

1 Introduction

For a commutative ring R with unity and $\Phi = (\Phi_1, \dots, \Phi_N)$, where $\Phi_i \in R[X_1, \dots, X_N]$, we define a *cycle* for Φ as a k -tuple $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{k-1}$ of different elements of R^N such that

$$\Phi(\bar{x}_0) = \bar{x}_1, \Phi(\bar{x}_1) = \bar{x}_2, \dots, \Phi(\bar{x}_{k-1}) = \bar{x}_0.$$

The number k is called the *length* of this cycle.

Let $\mathcal{CYCL}(R, N)$ be the set of all possible cycle-lengths for polynomial mappings in N variables with coefficients from R (we clearly assume that the elements of the considered cycles lie in R^N). For a material on various aspects of polynomial mappings and arithmetic of dynamical systems, see [1] and [4].

In Section 2 we examine $\mathcal{CYCL}(R, N)$ for a discrete valuation domain R with maximal ideal P . We assume that the residue field R/P has p^f elements (if R/P is infinite, then $\mathcal{CYCL}(R, N) = \mathbf{N}$). It is known (see Fact 1 in Section 2) that any element $k \in \mathcal{CYCL}(R, N)$ is of the form $k = a \cdot p^\alpha$, where all possible a were completely determined by the author. Thus, in order to know $\mathcal{CYCL}(R, N)$ it suffices for a given ‘possible’ a (as explained before) to find all α such that $a \cdot p^\alpha \in \mathcal{CYCL}(R, N)$. It is known that for a finite ramification index e the numbers α are bounded from above by some explicit function depending on e, p, f, N . In Theorem 1 we give a bound from below (for a given ‘possible’ a) for the biggest α

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such that $a \cdot p^\alpha \in \mathcal{CYCL}(R, N)$. Namely, we receive

$$\alpha \geq \left\lfloor \log_p \left(\frac{\log_p e}{2fN} \right) \right\rfloor.$$

We see that for fixed f, p, N the right-hand side of the last inequality grows to ∞ (when $e \rightarrow \infty$). Note that for a discrete valuation domain R the set $\mathcal{CYCL}(R, N)$ does not depend solely on p, e, f, N . Sometimes some subtler properties of R should be taken into account.

As a consequence of Theorem 1, in Theorem 2 we determine the sets $\mathcal{CYCL}(S, N)$ for some Dedekind domains S of positive characteristic and some N .

In Section 3 we consider properties of $\mathcal{A} := \mathcal{CYCL}(R, 1)$ for a domain R . Any such \mathcal{A} satisfies the following three ‘obvious’ properties:

- (i) $1, 2 \in \mathcal{A}$;
- (ii) \mathcal{A} is closed under taking divisors;
- (iii) for any prime p from $p \in \mathcal{A}$ it follows that $[1, p] \subseteq \mathcal{A}$.

Since there were no other obvious properties of \mathcal{A} , in mid-nineties W. Narkiewicz conjectured that for $\mathcal{A} \subseteq \mathbf{N}$ satisfying (i), (ii), (iii) there exists a domain R such that $\mathcal{A} = \mathcal{CYCL}(R, 1)$. In Section 3 we give a negative answer to this question.

I think that it would be interesting to give a sensible conjecture concerning sets $\mathcal{CYCL}(R, N)$ for $N \geq 2$. In particular it is not clear whether the above property (iii) holds in this case.

2 Finding $\mathcal{CYCL}(R, N)$ for some rings of positive characteristic

Let R be a discrete valuation domain of any characteristic, and P is the unique maximal ideal of R . We assume that the field R/P is finite and has p^f elements (for prime p). Let π be a generator of the principal ideal P , and let v be the norm of R , normalized so that $v(\pi) = 1/p$. We denote by w the corresponding exponent, defined by

$$w(x) = -\frac{\log v(x)}{\log p} \quad \text{for } x \neq 0, \quad \text{and} \quad w(0) = \infty.$$

We put $e := w(p)$. Thus e is the ramification index of R . We extend w to R^N by putting $w(x_1, \dots, x_N) = \min\{w(x_1), \dots, w(x_N)\}$.

A polynomial cycle $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{k-1}$ is called a (polynomial) \star -cycle if

$$w(\bar{x}_i - \bar{x}_j) \geq 1 \quad \text{for all } i, j.$$

Let $\mathcal{CYCL} \star(R, N)$ be the set of all possible lengths of \star -cycles for polynomial mappings in N variables with coefficients from R .

In the fact below we collect some properties of $\mathcal{CYCL}(R, N)$ already proved by the author (see [2], [3]).

Fact 1. *Let R, p, f, \dots be as above. Then*

- (i) *a number k lies in $\mathcal{CYCL}(R, N)$ if and only if $k = ab$, where $a \leq p^{fN}$ and b is the length of a suitable \star -cycle in R^N .*

(ii) If \widehat{R} is the completion of R with respect to the norm v , then

$$\mathcal{CYCL}(R, N) = \mathcal{CYCL}(\widehat{R}, N) \quad \text{and} \quad \mathcal{CYCL} \star (R, N) = \mathcal{CYCL} \star (\widehat{R}, N)$$

(note that for \widehat{R} the numbers p, e, f are the same as for R).

(iii) Let m be a positive integer not divisible by p . Then there is a \star -cycle of length m in R^N if and only if there are $r > 0$ and positive integers a_1, \dots, a_r with $a_1 + \dots + a_r \leq N$ such that m divides $[p^{fa_1} - 1, \dots, p^{fa_r} - 1]$.

(iv) Let S be a Dedekind domain, and let $\mathcal{P}(S)$ denote the family of all nonzero prime ideals of S . If $N \geq 2$, then

$$\mathcal{CYCL}(S, N) = \bigcap_{\mathfrak{p} \in \mathcal{P}(S)} \mathcal{CYCL}(S_{\mathfrak{p}}, N) = \bigcap_{\mathfrak{p} \in \mathcal{P}(S)} \mathcal{CYCL}(\widehat{S}_{\mathfrak{p}}, N),$$

where $\widehat{S}_{\mathfrak{p}}$ is the completion of $S_{\mathfrak{p}}$ with respect to the obvious valuation.

In particular, to find $\mathcal{CYCL}(R, N)$ it suffices to know p^f and, for each m dividing $[p^{fa_1} - 1, \dots, p^{fa_r} - 1]$ (for some a_1, \dots, a_r satisfying $a_1 + \dots + a_r \leq N$), to know for which $n \geq 0$ the number $m \cdot p^n$ lies in $\mathcal{CYCL} \star (R, N)$.

In this section we will prove that for any ‘possible’ m , as explained in Fact 1(iii), and for any n if the ramification index is sufficiently large, then $m \cdot p^n \in \mathcal{CYCL} \star (R, N)$. This, in turn, gives a closed formula for $\mathcal{CYCL}(S, N)$ for a Dedekind domain S of positive characteristic and $N \geq 2$. The fact that for any prime p and any $n \geq 0$ in $F_p[[X]]$ there are cycles of length p^n was established in the thesis of Zieve [5], who quoted an example due to Poonen.

Theorem 1. *Let R be as in this section. Let m be a divisor of $[p^{fa_1} - 1, \dots, p^{fa_r} - 1]$ for some a_1, \dots, a_r satisfying $a_1 + \dots + a_r \leq N$. If $e \geq p^{2fNp^n}$, then $m \cdot p^n \in \mathcal{CYCL} \star (R, N)$.*

Proof. Owing to Fact 1, we may assume that $n \geq 1$ and R is complete. It suffices to take $m = [p^{fa_1} - 1, \dots, p^{fa_r} - 1]$. Suppose that for $e \geq p^{2fNp^n}$ we have a \star -cycle of length $(p^{fa_1} - 1) \cdot p^n$ for a map $\Phi_1: R^{a_1} \rightarrow R^{a_1}$. For $i \geq 2$, by Fact 1(iii), in R^{a_i} there is a \star -cycle of length $p^{fa_i} - 1$ for some mapping Φ_i of R^{a_i} . We see that $\Phi = (\Phi_1, \dots, \Phi_r): R^{a_1 + \dots + a_r} \rightarrow R^{a_1 + \dots + a_r}$ constructed in the natural way has a \star -cycle of length $[(p^{fa_1} - 1) \cdot p^n, p^{fa_2} - 1, \dots, p^{fa_r} - 1] = m \cdot p^n$.

So, it suffices to prove for any $M \leq N$ that $(p^{fM} - 1) \cdot p^n \in \mathcal{CYCL} \star (R, M)$.

Put $q = p^{fM}$, and let ξ be a primitive root of unity of order $q - 1$. By the usual Hensel’s lemma (here we use the completeness of R) we have that the minimal over R polynomial of ξ is of degree M . Thus $R^M \sim R[\xi]$ as modules over R . Using this canonical isomorphism, we obtain that to any polynomial $F(X) \in R[\xi][X]$ there corresponds a polynomial mapping $\Phi: R^M \rightarrow R^M$ with coefficients from R . One can see that $R[\xi]$ is a complete discrete valuation domain with maximal ideal $\pi R[\xi]$, and the corresponding residue field has p^{fM} elements. We thus have a notion of \star -cycles in $R[\xi]$, and to a \star -cycle in $R[\xi]$ there corresponds a \star -cycle in R^M .

Thus it suffices to find a \star -cycle in $R[\xi]$ of length $(q - 1)p^n$.

Take $F(X) = \pi + \xi X + \gamma X^q + X^d$, where $d = q^2$ and $q = p^{fM}$. We remember that $\binom{0}{0} = 1$.

Lemma 1. For any $T \geq 0$ the T -th iteration of F satisfies

$$\begin{aligned} F^T(0) &\equiv \sum_{t=1}^T \sum_{r=0}^{T-t} \xi^r \binom{T-t}{r} \pi^{d^{T-t-r}} + \gamma \sum_{t=1}^{T-1} \sum_{r=0}^{T-t-1} \xi^r \binom{T-t}{r} \pi^{d^{T-t-r-1}q} \\ &\equiv \sum_{t=0}^{T-1} \xi^{-t} \pi^{d^t} A_T(t) + \gamma \sum_{t=0}^{T-2} \xi^{-(t+1)} \pi^{d^t q} A_T(t+1) \pmod{(p\pi, \gamma^{q+1})}, \end{aligned}$$

where $A_T(t) = \sum_{k=0}^{T-1} \binom{k}{t} \xi^k$. Moreover,

$$A_T(t) + A_T(t+1) = \xi^{-1} \left(A_T(t+1) + \xi^T \binom{T}{t+1} \right).$$

Proof. We use direct induction. One only has to remember that $\xi^q = \xi^d = \xi$ and $(x+y)^p \equiv x^p + y^p \pmod{(p)}$. \square

Lemma 2. (i) If $q > 2$ and $T = (q-1)p^r$, then for $j = 0, 1, \dots, p^r - 1$ we have $w(A_T(j)) \geq e$, and $A_T(p^r)$ is invertible.

(ii) If $q = 2$, then $A_T(t) = \binom{T}{t+1}$.

Proof. (i) Since $\xi \neq 1$, we have

$$A_T(0) = 1 + \xi + \dots + \xi^{T-1} = 0.$$

Using $w(\xi - 1) = 0$, simple properties of binomial coefficients and

$$A_T(t) + A_T(t+1) = \xi^{-1} \left(A_T(t+1) + \xi^T \binom{T}{t+1} \right)$$

we obtain the assertion.

(ii) In this case we have $\xi = 1$, and therefore the assertion follows from Lemma 1. \square

Assume that $q > 2$. Put $\gamma = \pi^{d^{p^n-1}(d-q)}z$. In view of $(q+1)d^{p^n-1}(d-q) > d^{p^n}$ and $e \geq p^{2fNp^n} \geq d^{p^n}$ we obtain by Lemma 1 that

$$\begin{aligned} F^T(0) &\equiv \sum_{t=0}^{T-1} \xi^{-t} \pi^{d^t} A_T(t) + \pi^{d^{p^n-1}(d-q)}z \sum_{t=0}^{T-2} \xi^{-(t+1)} \pi^{d^t q} A_T(t+1) \\ &\pmod{(\pi^{d^{p^n}+1}R[\xi, z])}. \end{aligned}$$

In particular, taking $T = (q-1)p^n$ we get, using Lemma 2,

$$F^{(q-1)p^n}(0) = \pi^{d^{p^n}} \xi^{-p^n} A_{(q-1)p^n}(p^n)(1 + z + \pi h(z)),$$

for some polynomial $h \in R[\xi][X]$. Thus $F^{(q-1)p^n}(0) = 0$ if and only if

$$1 + z + \pi h(z) = 0.$$

The existence of (a unique) $z \in R[\xi]$ satisfying $F^{(q-1)p^n}(0) = 0$ follows from the Hensel's lemma. Fix such z .

Now it is sufficient to show that the smallest $j > 0$ satisfying $F^j(0) = 0$ is $j = (q - 1)p^n$.

If $F^j(0) \equiv 0 \pmod{\pi^2}$, then, by Lemma 1, $A_j(0) \equiv 0 \pmod{\pi}$ and $\xi^j \equiv 1 \pmod{\pi}$, $q - 1 \mid j$ follow. From the simple properties of cycles it follows that it suffices to show that $F^{(q-1)p^{n-1}}(0) \neq 0$. But, Lemma 1 gives

$$F^{(q-1)p^{n-1}}(0) \equiv \xi^{-p^{n-1}} A_{(q-1)p^{n-1}}(p^{n-1}) \pi^{d^{p^{n-1}}} \pmod{(\pi^{d^{p^{n-1}+1}})},$$

and, by Lemma 2, we are done.

Assume that $q = 2$. Put $\gamma = \pi^{d^{p^n-2}(d-q)}z$. In view of $(q + 1)d^{p^n-2}(d - q) > d^{p^n-1}$ and $e \geq p^{2fNp^n} \geq d^{p^n}$ we obtain by Lemma 1 that

$$F^T(0) = \sum_{t=0}^{T-1} \pi^{d^t} A_T(t) + \pi^{d^{p^n-2}(d-q)}z \sum_{t=0}^{T-2} \pi^{d^t q} A_T(t+1) \pmod{(\pi^{d^{p^n-1}+1}R[z])}.$$

In particular, taking $T = p^n$ we get, using Lemma 2,

$$F^{p^n}(0) = \pi^{d^{p^n-1}} A_{p^n}(p^n - 1)(1 + z + \pi h(z)),$$

for some polynomial $h \in R[X]$. Thus $F^{p^n}(0) = 0$ if and only if $1 + z + \pi h(z) = 0$. The existence of $z \in R$ satisfying $F^{p^n}(0) = 0$ follows from the Hensel's lemma. Fix such z .

Now it suffices to show that the smallest $j > 0$ satisfying $F^j(0) = 0$ is $j = p^n$.

From the simple properties of cycles it follows that it suffices to show that $F^{p^{n-1}}(0) \neq 0$. But, Lemma 1 gives

$$F^{p^{n-1}}(0) \equiv A_{p^{n-1}}(p^{n-1} - 1) \pi^{d^{p^{n-1}-1}} \pmod{(\pi^{d^{p^{n-1}-1}+1})},$$

and, by Lemma 2, we are done.

This finishes the proof of the theorem. □

Theorem 2. (i) Let S be a Dedekind domain of characteristic $p > 0$. Let $\mathcal{F}(S)$ be the set of all natural f such that there is a nonzero prime ideal \mathfrak{p} of S of norm p^f . Let $\mathcal{A}(f, N)$ consists of all numbers of the form $a \cdot b \cdot p^n$, where $a \leq p^{fN}$, $n \geq 0$ and $b \mid [p^{fa_1} - 1, \dots, p^{fa_r} - 1]$ for some a_1, \dots, a_r satisfying $a_1 + \dots + a_r \leq N$.

If $N \geq 2$, then

$$\mathcal{CYCL}(S, N) = \bigcap_{f \in \mathcal{F}(S)} \mathcal{A}(f, N).$$

(ii) Let S be a discrete valuation domain of characteristic $p > 0$ such that the residue field has p^f elements. Then

$$\mathcal{CYCL}(S, 1) = \{a \cdot b \cdot p^n : a \leq p^f, b \mid p^f - 1, n \geq 0\}.$$

Proof. Since $e = \infty$, the assertion follows from Theorem 1 and Fact 1. □

Remark 1. (i) If in Theorem 2(i) $\mathcal{F}(S)$ is empty, then $\mathcal{CYCL}(S, N) = \mathbf{N}$. The similar happens to $\mathcal{CYCL}(S, 1)$ in Theorem 2(ii) if $f = \infty$.

(ii) Note that $\mathcal{A}(f, N) \subseteq \mathcal{A}(fk, N)$ for any natural k . Hence, if all elements from $\mathcal{F}(S)$ are multiplicities of one element from $\mathcal{F}(S)$, then the formula in Theorem 2(i) may be significantly simplified.

Corollary 1. *We have*

$$\mathcal{CYCL}(F_p[X], 2) = \{abp^n : a \leq p^2, b \mid p^2 - 1, n \geq 0\}$$

and

$$\mathcal{CYCL}(F_p[X], 3) = \{abp^n : a \leq p^3, n \geq 0 \text{ and } b \mid p^2 - 1 \text{ or } p^3 - 1\}.$$

On the other hand

$$\mathcal{CYCL}(F_p[X], 1) = \mathcal{CYCL}(F_p[X, Y], 1) = \mathcal{CYCL}(F_p, 1) = \{1, 2, \dots, p\}.$$

Proof. Taking into account Remark 1(ii) by Theorem 2 we obtain the first part. The second part follows from $\mathcal{CYCL}(A[X], 1) = \mathcal{CYCL}(A, 1)$ for any domain A . \square

3 A property of $\mathcal{CYCL}(R, 1)$

For a domain R with unity, the set $\mathcal{A} = \mathcal{CYCL}(R) := \mathcal{CYCL}(R, 1)$ satisfies

- (i) $1, 2 \in \mathcal{A}$;
- (ii) \mathcal{A} is closed under taking divisors;
- (iii) for a prime p , $p \in \mathcal{A}$ implies that $\{1, 2, \dots, p\} \subseteq \mathcal{A}$ (the last property follows from the Lagrange interpolation formula).

W. Narkiewicz asked in mid-nineties, whether for a subset \mathcal{A} of naturals, satisfying the above properties (i), (ii), (iii), there is a domain R with $\mathcal{CYCL}(R) = \mathcal{A}$.

In this section we emphasize another property of $\mathcal{CYCL}(R)$, and thus give a negative answer to the mentioned question.

Theorem 3. *For a domain R with unity, let $\mathcal{A} = \mathcal{CYCL}(R)$. Then for a prime number p we have that $p^2 \in \mathcal{A}$ implies $\{2r : r = 1, 2, \dots, p\} \subseteq \mathcal{A}$.*

Proof. Let a tuple $a_0, a_1, \dots, a_{p^2-1}$ be a cycle for $f(X) \in R[X]$. Then $0 = b_0$, $1 = b_1, b_2, \dots, b_{p^2-1}$, with $b_i = (a_i - a_0)/(a_1 - a_0) \in R$, is a cycle for

$$g(X) = (a_1 - a_0)^{-1} \left(f((a_1 - a_0)X + a_0) - a_0 \right) \in R[X].$$

So assume that $a_0 = 0$, $a_1 = 1$.

One proves that if $(j - i, p) = 1$, then $a_j - a_i$ is invertible. Put $d = a_p$. If $(j - i, p^2) = p$, then $a_j - a_i \sim d$. Fix $2 \leq r \leq p$. We are going to show that $a_0, a_1, \dots, a_{r-1}, a_p, a_{p+1}, \dots, a_{p+r-1}$ is a cycle (of length $2r$) for a suitable

polynomial $f(X)$ from $R[X]$. Namely let us take as $f(X)$ the unique polynomial of degree $\leq 2r - 1$ with coefficients from the field of fractions of R satisfying

$$\begin{aligned} f(a_0) = a_1, \quad f(a_1) = a_2, \quad \dots, \quad f(a_{r-1}) = a_p, \\ f(a_p) = a_{p+1}, \quad \dots, \quad f(a_{p+r-1}) = a_0. \end{aligned} \tag{1}$$

Put $f(X) = c_0 + c_1X + \dots + c_{2r-1}X^{2r-1}$. Then (1) is equivalent to a system of linear equations with c_0, \dots, c_{2r-1} to be found. From linear algebra we then get a formula for c_i .

Namely, putting $b_0 = a_0, \dots, b_{r-1} = a_{r-1}, b_r = a_p, b_{r+1} = a_{p+1}, \dots, b_{2r-1} = a_{p+r-1}$, we have $c_i = \Delta_i/\Delta$, where $\Delta = \prod_{0 \leq i < j \leq 2r-1} (b_j - b_i)$ and Δ_i is the determinant of the matrix

$$\begin{pmatrix} 1 & b_0 & \dots & b_0^{i-1} & b_1 & b_0^{i+1} & \dots & b_0^{2r-1} \\ 1 & b_1 & \dots & b_1^{i-1} & b_2 & b_1^{i+1} & \dots & b_1^{2r-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & b_{r-1} & \dots & b_{r-1}^{i-1} & b_r & b_{r-1}^{i+1} & \dots & b_{r-1}^{2r-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & b_{2r-1} & \dots & b_{2r-1}^{i-1} & b_0 & b_{2r-1}^{i+1} & \dots & b_{2r-1}^{2r-1} \end{pmatrix}.$$

We easily see that d divides all the terms in the differences of $r + 1$ -th and first rows, $r + 2$ -th and second rows, \dots , $2r$ -th and r -th rows. Thus $d^r \mid \Delta_i$. From the properties of the differences $a_j - a_i$ we get $\Delta \sim d^r$. Thus $c_i = \Delta_i/\Delta \in R$. \square

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