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Symmetries of a dynamical system represented by singular Lagrangians

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Abstract. Dynamical properties of singular Lagrangian systems differ from those of classical Lagrangians of the form $L = T - V$. Even less is known about symmetries and conservation laws of such Lagrangians and of their corresponding actions. In this article we study symmetries and conservation laws of a concrete singular Lagrangian system interesting in physics. We solve the problem of determining all point symmetries of the Lagrangian and of its Euler-Lagrange form, i.e. of the action.

It is known that every point symmetry of a Lagrangian is a point symmetry of its Euler-Lagrange form, and this of course happens also in our case. We are also interested in the converse statement, namely if to every point symmetry ξ of the Euler-Lagrange form E there exists a Lagrangian λ for E such that ξ is a point symmetry of λ. In the case studied the answer is affirmative, moreover we have found that the corresponding Lagrangians are all of order one.

1 Introduction

The aim of this paper is to investigate symmetry properties of a singular (degenerate) Lagrangian system, when the Euler-Lagrange equations cannot be put a normal form and rather form a system of implicit second order ordinary differential equations. While dynamical and symmetry properties of classical Lagrangians of the form $L = T - V$ (kinetic minus potential energy) have been in the focus of analytical mechanics and the calculus of variations from the very beginning of the art, there is still not much known about singular Lagrangian systems.

The problem of investigating singular Lagrangian systems goes back to the pioneer work of Dirac [4]. Nowadays there are two approaches to this topic, the first one coming from a generalization of the symplectic geometry [1], [2], [3], [6],

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[7], [8], [9], [10] to mention only a few, and the second (more recent) one based on the Lepagian theory of Lagrangian systems in jet bundles [12], [13], [14].

The Dirac algorithm is rather heuristic and it is known that applied to a concrete Lagrangian system different authors sometimes obtain confusing or even contradictory results. Also the application of the symplectic constraint algorithm in principle cannot provide complete information on the dynamics of the system and its symmetries – due to the fact that the image of the constrained dynamics in the symplectic manifold is not in one-to-one correspondence with the Lagrangian dynamics.

In this paper we apply the second way to singular systems, based on a model of a Lagrangian dynamics and the corresponding Hamiltonian dynamics defined in the same jet bundle [13]. We study a concrete singular Lagrangian interesting in physics, namely

$$L = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1 q^3. \tag{1}$$

Hamilton equations and symmetries of this Lagrangian were studied e.g. in [15], [16], [17], with incomplete results. With help of a "direct method" by Krupková we described the dynamics of this Lagrangian system completely in [3]. Here we continue in investigating the symmetry properties. Contrary to the preceding authors looking for symmetries of this Lagrangian [5] our approach to the problem is to find a complete solution of the symmetry conditions which take the form of a system of partial differential equations for the components of the invariance vector field. In this way we get all point symmetries of the Lagrangian (1) by solving the Noether equation, and all its generalized symmetries (that is point symmetries of the corresponding action) by solving the Noether–Bessel-Hagen equation.

It is known that every point symmetry of a Lagrangian is a point symmetry of its Euler-Lagrange form, and this of course happens also in our case. We are also interested in the converse statement, namely if to every point symmetry ξ of the Euler-Lagrange form E there exists a Lagrangian λ for E such that ξ is a point symmetry of λ. In the case studied the answer is affirmative, moreover we have found that the corresponding Lagrangians are *all of order one*.

In this context it is worth mention that in the well-known case of the kinetic energy Lagrangian, surprisingly, the situation is not so simple. The invariance group of the action – the Galilei group – also has the property that to every symmetry there is an invariant Lagrangian, however, not in all the cases the Lagrangian is of the first order. Namely, for Galilei transformations, one has only a *second order Lagrangian providing the same equations of motion as the kinetic energy* [14].

2 Symmetries and conservation laws

We consider a fibred manifold $\pi\colon Y \to X$, $\dim X = 1$, $\dim Y = m + 1$ and its jet prolongations $\pi_1\colon J^1Y \to X$, $\pi_2\colon J^2Y \to X$. In what follows, we denote by ∂_ξ the Lie derivative with respect to a vector field ξ.

Definition 1. Let ξ be a π-projectable vector field on Y. Let λ be Lagrangian on J^1Y. A vector field ξ on Y is called point symmetry of λ, if

$$\partial_{J^1\xi}\lambda = 0. \tag{2}$$

This equation is called Noether equation.

In fibred coordinates, the Noether equation reads:

$$L\frac{\mathrm{d}\xi^0}{\mathrm{d}t} + \frac{\partial L}{\partial t}\xi^0 + \frac{\partial L}{\partial q^\sigma}\xi^\sigma + \frac{\partial L}{\partial \dot{q}^\sigma}\widetilde{\xi}^\sigma = 0, \tag{3}$$

where

$$\widetilde{\xi^\sigma} = \frac{\mathrm{d}\xi^\sigma}{\mathrm{d}t} - \dot{q}^\sigma\frac{\mathrm{d}\xi^0}{\mathrm{d}t}.$$

In this paper we shall use the Noether equation to solve the following problem:

Given a Lagrangian, find all its point symmetries and the corresponding first integrals. In this case one has to solve the Noether equation with respect to the vector field. First integrals are then found on the basis of the Noether Theorem:

Theorem 1. *Let λ be a Lagrangian defined on an open subset $W \subset J^1Y$, let θ_λ be its Cartan form. Let a π-projectable vector field ξ on Y be a point symmetry of the Lagrangian λ. Let γ be an extremal of λ defined on $\pi_1(W) \subset X$. Then*

$$i_{J^1\xi}\theta_\lambda \circ J^1\gamma = const.$$

Definition 2. A vector field ξ is called point symmetry of the Euler-Lagrange form E_λ, if

$$\partial_{J^2\xi}E_\lambda = 0. \tag{4}$$

This equation is called the Noether–Bessel-Hagen equation.

We shall use the Noether–Bessel-Hagen equation to solve the following problem:

Find (all) infinitesimal transformations of Y which leave given Euler-Lagrange expressions (a given Euler-Lagrange form) invariant. In this case (4) is considered as a system of PDE's for symmetries ξ of the given Euler-Lagrange form. (Solving this problem one gets all point symmetries of the corresponding action).

2.1 Symmetries of a singular Lagrangian

In what follows we shall be interested in symmetry properties of the following Lagrangian

$$L = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3.$$

With help of the Noether equation we shall find point symmetries of the Lagrangian (1).

We get:

$$0 = (\dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3)\dot{\xi_0} + q^3\xi^1 - \dot{q}^3\xi^2 + q^1\xi^3 + \dot{q}^3\widetilde{\xi^1} + (\dot{q}^1 - q^2)\widetilde{\xi^3}$$

and then:

$$0 = (\dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3)\frac{\mathrm{d}\xi^0}{\mathrm{d}t} + q^3\xi^1 - \dot{q}^3\xi^2 + q^1\xi^3 + \dot{q}^3\cdot\left(\frac{\mathrm{d}\xi^1}{\mathrm{d}t} - \dot{q}^1\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right) + (\dot{q}^1 - q^2)\cdot\left(\frac{\mathrm{d}\xi^3}{\mathrm{d}t} - \dot{q}^3\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right)$$

where $\xi^0 = \xi^0(t)$ a $\xi^\sigma = \xi^\sigma(t, \xi^1, \xi^2, \xi^3)$.
More explicitly:

$$0 = \dot{q}^1 \dot{q}^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t} - q^2 \dot{q}^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t} + q^1 q^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t} + q^3 \xi^1 - \dot{q}^3 \xi^2 + q^1 \xi^3 + \dot{q}^3 \frac{\partial \xi^1}{\partial t} + \dot{q}^1 \dot{q}^3 \frac{\partial \xi^1}{\partial q^1} + \\ + \dot{q}^2 \dot{q}^3 \frac{\partial \xi^1}{\partial q^2} + (\dot{q}^3)^2 \frac{\partial \xi^1}{\partial q^3} - \dot{q}^1 \dot{q}^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t} + \dot{q}^1 \frac{\partial \xi^3}{\partial t} + (\dot{q}^1)^2 \frac{\partial \xi^3}{\partial q^1} + \dot{q}^1 \dot{q}^2 \frac{\partial \xi^3}{\partial q^2} + \dot{q}^1 \dot{q}^3 \frac{\partial \xi^3}{\partial q^3} \\ - \dot{q}^1 \dot{q}^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t} - q^2 \frac{\partial \xi^3}{\partial t} - q^2 \dot{q}^1 \frac{\partial \xi^3}{\partial q^1} - q^2 \dot{q}^2 \frac{\partial \xi^3}{\partial q^2} - q^2 \dot{q}^3 \frac{\partial \xi^3}{\partial q^3} + q^2 \dot{q}^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t}$$

From this equation we obtain a system of equations for components of ξ as follows:

$$\frac{\partial \xi^1}{\partial q^1} + \frac{\partial \xi^3}{\partial q^3} - \frac{\mathrm{d}\xi^0}{\mathrm{d}t} = 0$$
$$\frac{\partial \xi^1}{\partial t} - q^2 \frac{\partial \xi^3}{\partial q^3} - \xi^2 = 0$$
$$q^1 q^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t} + q^3 \xi^1 + q^1 \xi^3 = 0$$

where $\xi^0 = \xi^0(t),\ \xi^1 = \xi^1(t, q^1),\ \xi^2 = \xi^2(t, q^1, q^2, q^3),\ \xi^3 = \xi^3(q^3)$.

Solving these equations we get:

Theorem 2. *Point symmetries of Lagrangian (1) are generated by two vector fields:*

$$\frac{\partial}{\partial t}; \qquad q^1 \frac{\partial}{\partial q^1} + q^2 \frac{\partial}{\partial q^2} - q^3 \frac{\partial}{\partial q^3}.$$

To the time translation generated by $\frac{\partial}{\partial t}$ *there corresponds the first integral*

$$F_1 = (q^1 q^3 - \dot{q}^1 \dot{q}^3) = -H,$$

where H *is the Hamiltonian. To* $q^1 \frac{\partial}{\partial q^1} + q^2 \frac{\partial}{\partial q^2} - q^3 \frac{\partial}{\partial q^3}$ *there corresponds the conserved function*

$$F_2 = \dot{q}^3 q^1 - (\dot{q}^1 + q^2) \cdot q^3 = p_1 \cdot q^1 - p_3 \cdot q^3,$$

where

$$p_1 = \dot{q}^3, \quad p_2 = 0, \quad p_3 = \dot{q}^1 - q^2$$

are momenta.

2.2 Symmetries of the Euler-Lagrange form of L

Now let us determine point symmetries of the action of L, that is solutions of the Noether–Bessel-Hagen equation. Since

$$E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^\sigma},$$

where $\sigma = 1, 2, 3$ we get from L (1) the following Euler-Lagrange expressions:

$$E_1 = q^3 - \ddot{q}^3 \quad E_2 = -\dot{q}^3 \quad E_3 = q^1 + \dot{q}^2 - \ddot{q}^1. \tag{5}$$

Substituting the second jet prolongation of ξ:

$$J^2\xi = \xi^0 \frac{\partial}{\partial t} + \xi^\sigma \frac{\partial}{\partial q^\sigma} + \widetilde{\xi^i} \frac{\partial}{\partial \dot{q}^\sigma} + \widetilde{\widetilde{\xi^\sigma}} \frac{\partial}{\partial \ddot{q}^\sigma},$$

where

$$\widetilde{\xi^\sigma} = \dot{\xi^\sigma} - \dot{q}^\sigma \dot{\xi}^0, \quad \widetilde{\widetilde{\xi^\sigma}} = \ddot{\xi}^\sigma - 2\ddot{q}^\sigma \dot{\xi}^0 - \dot{q}^\sigma \ddot{\xi}^0,$$

we obtain the Noether–Bessel-Hagen equation in the following form:

$$E_\nu \frac{\partial \xi^\nu}{\partial q^\sigma} + E_\sigma \dot{\xi}^0 + \frac{\partial E_\sigma}{\partial t}\xi^0 + \frac{\partial E_\sigma}{\partial q^\nu}\xi^\nu + \frac{\partial E_\sigma}{\partial \dot{q}^\nu}(\dot{\xi}^\nu - \dot{q}^\nu \dot{\xi}^0) + \frac{\partial E_\sigma}{\partial \ddot{q}^\nu}(\ddot{\xi}^\nu - 2\ddot{q}^\nu \dot{\xi}^0 - \dot{q}^\nu \ddot{\xi}^0) = 0.$$

Substituting the Euler-Lagrange expressions (5), we get:

$$(q^3 - \ddot{q}^3)\frac{\partial \xi^1}{\partial q^1} - \dot{q}^3 \frac{\partial \xi^2}{\partial q^1} + (q^1 - \ddot{q}^1 + \dot{q}^2)\frac{\partial \xi^3}{\partial q^1} + (q^3 - \ddot{q}^3)\frac{\partial \xi^0}{\partial t} + \\ + \xi^3 - \left(\frac{\mathrm{d}^2 \xi^3}{\mathrm{d}t^2} - 2\ddot{q}^3 \frac{\partial \xi^0}{\partial t} - \dot{q}^3 \frac{\partial^2 \xi^0}{\partial t^2}\right) = 0$$

$$(q^3 - \ddot{q}^3)\frac{\partial \xi^1}{\partial q^2} - \dot{q}^3 \frac{\partial \xi^2}{\partial q^2} + (q^1 - \ddot{q}^1 + \dot{q}^2)\frac{\partial \xi^3}{\partial q^2} - \dot{q}^3 \frac{\partial \xi^0}{\partial t} - \left(\frac{\mathrm{d}\xi^3}{\mathrm{d}t} - \dot{q}^3 \frac{\partial \xi^0}{\partial t}\right) = 0$$

$$(q^3 - \ddot{q}^3)\frac{\partial \xi^3}{\partial q^1} - \dot{q}^3 \frac{\partial \xi^2}{\partial q^3} + (q^1 - \ddot{q}^1 + \dot{q}^2)\frac{\partial \xi^3}{\partial q^3} + (q^1 - \ddot{q}^1 + \dot{q}^2)\frac{\partial \xi^0}{\partial t} + \xi^1 + \\ + \left(\frac{\mathrm{d}\xi^2}{\mathrm{d}t} - \dot{q}^2 \frac{\partial \xi^0}{\partial t}\right) - \left(\frac{d^2 \xi^1}{\mathrm{d}t^2} - 2\ddot{q}^1 \frac{\partial \xi^0}{\partial t} - \dot{q}^1 \frac{\partial^2 \xi^0}{\partial t^2}\right) = 0$$

where $\xi^1 = \xi^1(t, q^1, q^2, q^3)$, $\xi^2 = \xi^2(t, q^1, q^2, q^3)$, $\xi^3 = \xi^3(t, q^1, q^2, q^3)$. More explicitly,

$$q^3 \frac{\partial \xi^1}{\partial q^1} - \ddot{q}^3 \frac{\partial \xi^1}{\partial q^1} - \dot{q}^3 \frac{\partial \xi^2}{\partial q^1} + q^1 \frac{\partial \xi^3}{\partial q^1} - \ddot{q}^1 \frac{\partial \xi^3}{\partial q^1} + \dot{q}^2 \frac{\partial \xi^3}{\partial q^1} + q^3 \frac{\partial \xi^0}{\partial t} - \ddot{q}^3 \frac{\partial \xi^0}{\partial t} - \\ - \frac{\partial^2 \xi^3}{\partial t^2} - 2\dot{q}^1 \frac{\partial^2 \xi^3}{\partial t \partial q^1} - 2\dot{q}^2 \frac{\partial^2 \xi^3}{\partial t \partial q^2} - 2\dot{q}^3 \frac{\partial^2 \xi^3}{\partial t \partial q^3} - \frac{\partial^2 \xi^3}{\partial (q^1)^2}(\dot{q}^1)^2 - \frac{\partial^2 \xi^3}{\partial (q^2)^2}(\dot{q}^2)^2 - \\ - \frac{\partial^2 \xi^3}{\partial (q^3)^2}(\dot{q}^3)^2 - 2\frac{\partial^2 \xi^3}{\partial q^1 \partial q^2}\dot{q}^1 \dot{q}^2 - 2\frac{\partial^2 \xi^3}{\partial q^1 \partial q^3}\dot{q}^1 \dot{q}^3 - 2\frac{\partial^2 \xi^3}{\partial q^2 \partial q^3}\dot{q}^2 \dot{q}^3 + 2\ddot{q}^3 \frac{\partial \xi^0}{\partial t} + \\ + \xi^3 + \dot{q}^3 \frac{\partial^2 \xi^0}{\partial t^2} - \frac{\partial \xi^3}{\partial q^1}\ddot{q}^1 - \frac{\partial \xi^3}{\partial q^2}\ddot{q}^2 - \frac{\partial \xi^3}{\partial q^3}\ddot{q}^3 = 0$$

$$q^3 \frac{\partial \xi^1}{\partial q^2} - \ddot{q}^3 \frac{\partial \xi^1}{\partial q^2} - \dot{q}^3 \frac{\partial \xi^2}{\partial q^2} + q^1 \frac{\partial \xi^3}{\partial q^2} - \ddot{q}^1 \frac{\partial \xi^3}{\partial q^2} + \dot{q}^2 \frac{\partial \xi^3}{\partial q^2} - \dot{q}^3 \frac{\partial \xi^0}{\partial t} - \frac{\partial \xi^3}{\partial t} - \\ - \frac{\partial \xi^3}{\partial q^1}\dot{q}^1 - \frac{\partial \xi^3}{\partial q^2}\dot{q}^2 - \frac{\partial \xi^3}{\partial q^3}\dot{q}^3 + \dot{q}^3 \frac{\partial \xi^0}{\partial t} = 0$$

$$
\begin{aligned}
q^3\frac{\partial\xi^1}{\partial q^3}-\ddot{q}^3\frac{\partial\xi^1}{\partial q^3}-\dot{q}^3\frac{\partial\xi^2}{\partial q^3}+q^1\frac{\partial\xi^3}{\partial q^3}-\ddot{q}^1\frac{\partial\xi^3}{\partial q^3}+\dot{q}^2\frac{\partial\xi^3}{\partial q^3}+q^1\frac{\partial\xi^0}{\partial t}-\ddot{q}^1\frac{\partial\xi^0}{\partial t}+\\
+\dot{q}^2\frac{\partial\xi^0}{\partial t}+\frac{\partial\xi^2}{\partial t}+\frac{\partial\xi^2}{\partial q^1}\dot{q}^1+\frac{\partial\xi^2}{\partial q^2}\dot{q}^2+\frac{\partial\xi^2}{\partial q^3}\dot{q}^3-\dot{q}^2\frac{\partial\xi^0}{\partial t}-\frac{\partial^2\xi^1}{\partial t^2}-2\dot{q}^1\frac{\partial^2\xi^1}{\partial t\partial q^1}-\\
-2\dot{q}^2\frac{\partial^2\xi^1}{\partial t\partial q^2}+\xi^1-2\dot{q}^3\frac{\partial^2\xi^1}{\partial t\partial q^3}-\frac{\partial^2\xi^1}{\partial(q^1)^2}(\dot{q}^1)^2-\frac{\partial^2\xi^1}{\partial(q^2)^2}(\dot{q}^2)^2-\\
-\frac{\partial^2\xi^1}{\partial(q^3)^2}(\dot{q}^3)^2-2\frac{\partial^2\xi^1}{\partial q^1\partial q^2}\dot{q}^1\dot{q}^2-2\frac{\partial^2\xi^1}{\partial q^1\partial q^3}\dot{q}^1\dot{q}^3-2\frac{\partial^2\xi^1}{\partial q^2\partial q^3}\dot{q}^2\dot{q}^3+2\ddot{q}^1\frac{\partial\xi^0}{\partial t}+\\
+\dot{q}^1\frac{\partial^2\xi^0}{\partial t^2}-\frac{\partial\xi^1}{\partial q^1}\ddot{q}^1-\frac{\partial\xi^1}{\partial q^2}\ddot{q}^2-\frac{\partial\xi^1}{\partial q^3}\ddot{q}^3=0
\end{aligned}
$$

From these equations we get conditions for components of ξ as follows:

$$
\begin{aligned}
\xi^1 &= k_1\cdot \mathrm{e}^t+k_2\cdot\mathrm{e}^{-t}-C_2q^1\\
\xi^2 &= C_3-C_2\cdot q^2+b(q^3)\\
\xi^3 &= C_2\cdot q^3\\
\xi^0 &= C_1
\end{aligned}
$$

Hence, we obtained:

$$
\xi=C_1\frac{\partial}{\partial t}+(k_1\cdot\mathrm{e}^t+k_2\cdot\mathrm{e}^{-t}-C_2q^1)\frac{\partial}{\partial q^1}+\big(C_3-C_2\cdot q^2+b(q^3)\big)\frac{\partial}{\partial q^2}+C_2\cdot q^3\frac{\partial}{\partial q^3}
$$

where C_1, C_2, C_3, k_1, k_2 are constants and $b(q^3)$ is a function depending only on q^3. This result can be formulated as follows:

Theorem 3. *Point symmetries of the Euler-Lagrange form of Lagrangian L (4) are generated by the following vector fields:*

$$
\frac{\partial}{\partial t};\quad q^1\frac{\partial}{\partial q^1}+q^2\frac{\partial}{\partial q^2}-q^3\frac{\partial}{\partial q^3};\quad \mathrm{e}^t\frac{\partial}{\partial q^1};\quad \mathrm{e}^{-t}\frac{\partial}{\partial q^1};\quad b(q^3)\frac{\partial}{\partial q^2},\tag{6}
$$

where $b(q^3)$ is an arbitrary function of the variable q^3.
The corresponding first integrals are

$$F_1=-H$$

for the transformation $\frac{\partial}{\partial t}$,

$$F_2=p_1q^1-p_3q^3$$

for the transformation $q^1\frac{\partial}{\partial q^1}+q^2\frac{\partial}{\partial q^2}-q^3\frac{\partial}{\partial q^3}$,

$$F_3=p_1\cdot\mathrm{e}^t$$

for the transformation $\mathrm{e}^t\frac{\partial}{\partial q^1}$,

$$F_4=p_1\cdot\mathrm{e}^{-t}$$

for the transformation $\mathrm{e}^{-t}\frac{\partial}{\partial q^1}$ and

$$F_5=0$$

for the transformation $b(q^3)\frac{\partial}{\partial q^2}$.

Theorem 2 and 3 demonstrate the known fact that every symmetry of Lagrangian λ is a symmetry of its Euler-Lagrange form (or the corresponding action). In the sequel we shall be interested in the converse problem, namely, if to every point symmetry of the action (that is of E) there exists a Lagrangian λ for E such that ξ is a point symmetry of λ.

To this end we shall represent our Lagrangian system in the form of the equivalence class of Lagrangians for E; the equivalence relation is given by

$$L \sim L' \qquad \text{iff} \qquad L' = L + \frac{\mathrm{d}f}{\mathrm{d}t},$$

where f is a function.

This means that

$$L' = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3 + \frac{\mathrm{d}f}{\mathrm{d}t},$$

where $f = f(t, q^1, q^2, q^3)$.

Substituting L' to the Noether equation we shall try to determine f for every symmetry (6)

(a) If $\xi = \mathrm{e}^t \frac{\partial}{\partial q^1} \Rightarrow \xi^1 = \mathrm{e}^t$
we get

$$q^3\xi^1 + \frac{\partial}{\partial q^1}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\xi^1 + \dot{q}^3\left(\dot{\xi}^1 - \dot{q}^1\dot{\xi}^0\right) + \frac{\partial}{\partial \dot{q}^1}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\left(\dot{\xi}^1 - \dot{q}^1\dot{\xi}^0\right) = 0.$$

From this equation we get equations:

$$\begin{aligned}
\frac{\partial^2 f}{(\partial q^1)^2} &= 0\\
\frac{\partial^2 f}{\partial q^1 \partial q^2} &= 0\\
\frac{\partial^2 f}{\partial q^1 \partial q^3} &= -1\\
\frac{\partial^2 f}{\partial q^1 \partial t} &= -q^3 - \frac{\partial f}{\partial q^1}
\end{aligned}$$

having the solution

$$f = a(t, q^2, q^3) - q^1q^3 + C_1e^{-t}q^1,$$

where $a(t, q^2, q^3)$ is arbitrary function of variables t, q^2 and q^3. C_1 is a constant.

(b) If $\xi = \mathrm{e}^{-t} \frac{\partial}{\partial q^1} \Rightarrow \xi^1 = \mathrm{e}^{-t}$
we get

$$q^3\xi^1 + \frac{\partial}{\partial q^1}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\xi^1 + \dot{q}^3\left(\dot{\xi}^1 - \dot{q}^1\dot{\xi}^0\right) + \frac{\partial}{\partial \dot{q}^1}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\left(\dot{\xi}^1 - \dot{q}^1\dot{\xi}^0\right) = 0.$$

From this equation we get equations:

$$\frac{\partial^2 f}{(\partial q^1)^2} = 0$$
$$\frac{\partial^2 f}{\partial q^1 \partial q^2} = 0$$
$$\frac{\partial^2 f}{\partial q^1 \partial q^3} = 1$$
$$\frac{\partial^2 f}{\partial q^1 \partial t} = -q^3 + \frac{\partial f}{\partial q^1}$$

having the solution

$$f = a(t, q^2, q^3) + q^1 q^3 + C_2 \mathrm{e}^t q^1,$$

where $a(t, q^2, q^3)$ is arbitrary function of variables t, q^2 and q^3. C_2 is a constant.

(c) If $\xi = b(q^3)\frac{\partial}{\partial q^2} \Rightarrow \xi^2 = b(q^3)$
we get

$$-\dot{q}^3 \cdot b(q^3) + \frac{\partial}{\partial q^2}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) b(q^3) = 0$$

and from this equation we get a equations:

$$\frac{\partial^2 f}{\partial q^2 \partial q^1} = 0$$
$$\frac{\partial^2 f}{(\partial q^2)^2} = 0$$
$$\frac{\partial^2 f}{\partial q^2 \partial q^3} = 1$$
$$\frac{\partial^2 f}{\partial q^2 \partial t} = 0.$$

having the solution

$$f = a(t, q^1, q^3) + q^2 q^3,$$

where $a(t, q^1, q^3)$ is arbitrary function of variables t, q^1 and q^3.

Summarizing, we obtained the following result:

Theorem 4. *For every point symmetry ξ of the Euler-Lagrange form*

$$E = (q^3 - \ddot{q}^3)\mathrm{d}q^1 \wedge \mathrm{d}t - \dot{q}^3 \mathrm{d}q^2 \wedge \mathrm{d}t + (q^1 + \dot{q}^2 - \ddot{q}^1)\mathrm{d}q^3 \wedge \mathrm{d}t \tag{7}$$

there exists a first order Lagrangian λ' such that ξ is a point symmetry of λ'.

Explicitly:

- *For $\xi = \mathrm{e}^t \frac{\partial}{\partial q^1}$ it holds*

$$f = a(t, q^2, q^3) - q^1 q^3 + C_1 \mathrm{e}^{-t} q^1,$$

i.e. $L' = \dot{q}^1 \dot{q}^3 - q^2 \dot{q}^3 + q^1 q^3 + \frac{\mathrm{d}}{\mathrm{d}t}\left(a(t, q^2, q^3) - q^1 q^3 + C_1 \mathrm{e}^{-t} q^1\right).$

- *For* $\xi = \mathrm{e}^{-t}\frac{\partial}{\partial q^1}$ *it holds*

$$f = a(t, q^2, q^3) + q^1 q^3 + C_2 \mathrm{e}^t q^1,$$

i.e. $L' = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1 q^3 + \frac{\mathrm{d}}{\mathrm{d}t}\left(a(t, q^2, q^3) + q^1 q^3 + C_2 \mathrm{e}^t q^1\right).$

- *For* $\xi = b(q^3)\frac{\partial}{\partial q^2}$ *it holds*

$$f = a(t, q^1, q^3) + q^2 q^3,$$

i.e. $L' = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1 q^3 + \frac{\mathrm{d}}{\mathrm{d}t}\left(a(t, q^1, q^3) + q^2 q^3\right).$

By the above theorem, *the set of point symmetries of the class of equivalent first order Lagrangians for E locally coincides with the set of point symmetries of the Euler-Lagrange form* (7).

Remarkably, in this case, all the invariant Lagrangians are of *order one*. It is worth note that in the most often considered case of the Lagrangian $L = \frac{1}{2}mv^2$ (free particle of classical mechanics) this is *not* the case. Namely, one can prove that [14]:

– to every point symmetry ξ of the Euler-Lagrange form E ($E_i = m\ddot{x}^i$) there exists a Lagrangian L for E such that ξ is a point symmetry of L

– *such a Lagrangian need not be of order one:* in case of Galilei transformations L is a second order Lagrangian, equivalent with the kinetic energy.

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