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Locally variational invariant field equations and global currents: Chern-Simons theories

M. Francaviglia, M. Palese, E. Winterroth

Abstract. We introduce the concept of conserved current variationally associated with locally variational invariant field equations. The invariance of the variation of the corresponding local presentation is a sufficient condition for the current beeing variationally equivalent to a global one. The case of a Chern-Simons theory is worked out and a global current is variationally associated with a Chern-Simons local Lagrangian.

1 Introduction

We are interested in the study of the relation between symmetries (i.e. invariance properties) of field equations and corresponding conservation laws. More precisely, the topic of this paper is the investigation of some aspects concerning the interplay between symmetries, conservation laws and *variational principles*. We shall consider Noether conservation laws associated with the invariance of global Euler--Lagrange morphisms generated by local variational problems of a given type.

We shall characterize symmetries of field equations having 'variational' meaning. In order to understand the *structure* of a phenomenon described by field equations, one should be interested in conservation laws more precisely characterized than those directly associated with invariance properties of field equations. Thus, we will look for conservation laws coming from invariance properties of a (*possibly local*) variational problem in its whole (rather than a field equation solely) to find a way of associating global conservation laws with a local Lagrangian *field* theory generating global Euler-Lagrange equations.

From a physical point of view, field equations appear to be a fundamental object, since they describe the changing of the field in base space. Somehow, we are well disposed to give importance to symmetries of equations, because they are transformations of the space leaving invariant the description of such a change

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provided by means of field equations. On the other hand the possibility of formulating a variational principle (i.e. a principle of *stationary* action) – from which both changing of fields and associated conservation laws (i.e. quantities not changing in the base space) could be obtained – has been one of the most important achievements in the history of mathematical and physical sciences in modern age. It allows, in fact, to keep account of both what (*and how*) changes and what (*and how*) is conserved. In the variational calculus perspective, we could say that *Euler--Lagrange field equations are 'adjoint' to stationary principles up to conservation laws*.

In line with Lepage's cornerstone papers [23], which pointed out the fact that the Euler-Lagrange operator is a quotient morphism of the exterior differential, we shall consider a geometric formulation of the calculus of variations on fibered manifolds for which the Euler-Lagrange operator is a morphism of a finite order exact sequence of sheaves according to [20]. The module in degree $(n+1)$, contains so-called (variational) dynamical forms; a given equation is globally an Euler--Lagrange equation if its dynamical form is closed in the complex of global sections (Helmholtz conditions) and its cohomolgy class is trivial. Dynamical forms which are only *locally variational*, i.e. closed in the complex and defining a non trivial cohomology class, admit a system of local Lagrangians, one for each open set in a suitable covering, which satisfy certain relations among them.

In her celebrated paper *Invariante Variationsprobleme* [25], Emmy Noether clearly pointed out how, considering invariance of *variational* problems, a major refinement in the description of associated conserved quantities is achieved. A formulation in modern language of Noether's results would say that symmetry properties of the Euler-Lagrange expressions introduce *a cohomology class which adds up to Noether currents*; it is important to stress that they are related with invariance properties of the first variation. Global projectable vector fields on prolongations of fibered manifold which are symmetries of dynamical forms, in particular of locally variational dynamical forms, and corresponding formulations of Noether theorem II can be considered in order to determine obstructions to the globality of associated conserved quantities [16]. The concept of global (and local) variationally trivial Lagrangians and in general of variationally trivial currents (i.e. $(n-1)$-forms) will be taken in consideration and for simplicity, in the sequel, a *locally variational form* will be any closed p-form in the variational sequence; inverse problems at any degree of variational forms will be considered.

In the present paper, we introduce the concept of conserved current variationally associated with locally variational invariant field equations. The invariance of the variation of the corresponding local presentation is a sufficient condition for the current beeing variationally equivalent to a global one. The case of a Chern-Simons gauge theory is worked out and a global current is variationally associated with a Chern-Simons local Lagrangian.

Chern-Simons theories exhibit in fact many interesting and important properties: they are based on secondary characteristic classes and can be associated with new topological invariants for knots and three-manifolds; they appeared in physics as natural mass terms for gauge theories and for gravity in dimension three, and after quantization they lead to a quantized coupling constant as well as a mass

[10]. In particular, Chern-Simons gauge theory is also an example of a topological field theory [32]. Furthermore, as it was remarked in [2], obstructions to the construction of natural Lagrangians are in a one-to-one correspondence with the conformally invariant characteristic forms discovered by Chern and Simons in [9]. Finally, the Chern-Simons term is related to the anomaly cancellation problem in 2-dimensional conformal field theories [7].

2 Locally variational invariant field equations and variationally equivalent problems

We shall consider the variational sequence [20] defined on a fibered manifold $\pi\colon \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n+m$. For $r \geq 0$ we have the r-jet space $J_r\boldsymbol{Y}$ of jet prolongations of sections of the fibered manifold π. We have also the natural fiberings $\pi^r_s\colon J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $r \geq s$, and $\pi^r\colon J_r\boldsymbol{Y} \to \boldsymbol{X}$; among these the fiberings π^r_{r-1} are *affine bundles* which induce the natural fibered splitting

$$J_r\boldsymbol{Y} \times_{J_{r-1}\boldsymbol{Y}} T^*J_{r-1}\boldsymbol{Y} \simeq J_r\boldsymbol{Y} \times_{J_{r-1}\boldsymbol{Y}} (T^*\boldsymbol{X} \oplus V^*J_{r-1}\boldsymbol{Y}),$$

which, in turn, induces also a decomposition of the exterior differential on $\boldsymbol{Y}$ in the *horizontal* and *vertical differential*, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$. By $(j_r\Xi, \xi)$ we denote the jet prolongation of a *projectable vector field* (Ξ, ξ) on $\boldsymbol{Y}$, and by $j_r\Xi_H$ and $j_r\Xi_V$ the horizontal and the vertical part of $j_r\Xi$, respectively.

We have the *sheaf splitting* $\mathcal{H}^p_{(s+1,s)} = \bigoplus_{t=0}^p \mathcal{C}^{p-t}_{(s+1,s)} \wedge \mathcal{H}^t_{s+1}$ where $\mathcal{H}^p_{(s,q)}$ and $\mathcal{H}^p_s$ $(q \leq s)$ are sheaves of *horizontal forms*, and $\mathcal{C}^p_{(s,q)} \subset \mathcal{H}^p_{(s,q)}$ are subsheaves of *contact forms* [20]. Let us denote by h the projection onto the nontrivial summand with the higest value of t and by $d\ker h$ the sheaf generated by the corresponding presheaf and set then $\Theta^*_r \equiv \ker h + d\ker h$; the quotient sequence

$$0 \to I\!R_{\boldsymbol{Y}} \to \dots \xrightarrow{\mathcal{E}_{n-1}} \Lambda^n_r/\Theta^n_r \xrightarrow{\mathcal{E}_n} \Lambda^{n+1}_r/\Theta^{n+1}_r \xrightarrow{\mathcal{E}_{n+1}} \Lambda^{n+2}_r/\Theta^{n+2}_r \xrightarrow{\mathcal{E}_{n+2}} \dots \xrightarrow{d} 0$$

defines the *r–th order variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$; here Λ^p_s is the standard sheaf of p–forms on $J_s\boldsymbol{Y}$. The quotient sheaves (the sections of which are classes of forms modulo contact forms) in the variational sequence can be represented as sheaves $\mathcal{V}^k_r$ of k-forms on jet spaces of higher order. In particular, currents are classes $\nu \in (\mathcal{V}^{n-1}_r)_{\boldsymbol{Y}}$; Lagrangians are classes $\lambda \in (\mathcal{V}^n_r)_{\boldsymbol{Y}}$, while $\mathcal{E}_n(\lambda)$ is called a Euler-Lagrange form (being $\mathcal{E}_n$ the Euler-Lagrange morphism); dynamical forms are classes $\eta \in (\mathcal{V}^{n+1}_r)_{\boldsymbol{Y}}$ and $\mathcal{E}_{n+1}(\eta)$ is a Helmohltz form (being $\mathcal{E}_{n+1}$ the corresponding Helmholtz morphism).

Since the variational sequence is a soft resolution of the constant sheaf $I\!R_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$, the cohomology of the complex of global sections, denoted by $H^*_{VS}(\boldsymbol{Y})$, is naturally isomorphic to both the Čech cohomology of $\boldsymbol{Y}$ with coefficients in the constant sheaf $I\!R$ and the de Rham cohomology $H^k_{dR}\boldsymbol{Y}$ [20].

Let $\boldsymbol{K}^p_r \equiv Ker\ \mathcal{E}_p$. We have the short exact sequence of sheaves

$$0 \to \boldsymbol{K}^p_r \xrightarrow{i} \mathcal{V}^p_r \xrightarrow{\mathcal{E}_p} \mathcal{E}_p(\mathcal{V}^p_r) \to 0\,.$$

For any global section $\beta \in (\mathcal{V}^{p+1}_r)_{\boldsymbol{Y}}$ we have $\beta \in (\mathcal{E}_p(\mathcal{V}^p_r))_{\boldsymbol{Y}}$ if and only if $\mathcal{E}_{p+1}(\beta) = 0$, which are conditions of local variationality. A global inverse problem is to find necessary and sufficient conditions for such a locally variational β

to be globally variational. In particular $\mathcal{E}_n(\mathcal{V}^n_r)$ is the sheave of Euler-Lagrange morphisms and $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}}$ if and only if $\mathcal{E}_{n+1}(\eta) = 0$, which are Helmholtz conditions.

The above exact sequence gives rise to the long exact sequence in Čech cohomology

$$0 \to (\boldsymbol{K}^p_r)_{\boldsymbol{Y}} \to (\mathcal{V}^p_r)_{\boldsymbol{Y}} \to (\mathcal{E}_p(\mathcal{V}^p_r))_{\boldsymbol{Y}} \xrightarrow{\delta_p} H^1(\boldsymbol{Y}, \boldsymbol{K}^p_r) \to 0\,,$$

where the *connecting homomorphism* $\delta_p = i^{-1} \circ \mathfrak{d} \circ \mathcal{E}_p^{-1}$ is the mapping of cohomologies in the corresponding diagram of cochain complexes. In particular, every $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}}$ (i.e. locally variational) defines a cohomology class $\delta(\eta) \equiv \delta_n(\eta) \in H^1(\boldsymbol{Y}, \boldsymbol{K}^n_r)$. Furthermore, every $\mu \in (d_H(\mathcal{V}^{n-1}_r))_{\boldsymbol{Y}}$ (i.e. locally variationally trivial) defines a cohomology class $\delta'(\mu) \equiv \delta_{n-1}(\mu) \in H^1(\boldsymbol{Y}, \boldsymbol{K}^{n-1}_r)$.

Note that η is globally variational if and only if $\delta(\eta) = 0$. In the following we will be interested in the non trivial case $\delta(\eta) \neq 0$ whereby $\eta = \mathcal{E}_n(\lambda)$ can be solved only locally, i.e. for any countable good covering of $\boldsymbol{Y}$ there exists a local Lagrangian λ_i over each subset $U_i \subset \boldsymbol{Y}$ such that $\eta_i = \mathcal{E}_n(\lambda_i)$.

A *local variational problem* is a system of local sections λ_i of $(\mathcal{V}^n_r)_{U_i}$ such that $\mathcal{E}_n((\lambda_i - \lambda_j)|_{U_i \cap U_j}) = 0$. Note that $\mathfrak{d}\lambda = 0$ implies $\mathfrak{d}\eta_\lambda = 0$, while $\mathfrak{d}\eta_\lambda = 0$ only implies $\eta_{\mathfrak{d}\lambda} = 0$ i.e. $\mathfrak{d}\lambda \in C^1(\mathfrak{U}, \boldsymbol{K}^n_r)$ in Čech cohomology [5]. We call $(\{\boldsymbol{U}_i\}_{i\in\mathbb{Z}}, \lambda_i)$ a *presentation* of the local variational problem. Two local variational problems of degree p are *equivalent* if and only if they give rise to the same variational class of forms as the image of the corresponding morphism $\mathcal{E}_p$ in the variational sequence. This means that the coboundary is variationally trivial.

The concept of a *variational Lie derivative* operator $\mathcal{L}_{j_r\Xi}$ which is a local differential operator enables us to define symmetries of classes of forms of any degree in the variational sequence and the corresponding conservation theorems [19]. We notice that the variational Lie derivative acts on cohomology classes: closed variational forms defining nontrivial cohomology classes are trasformed in variational forms with trivial cohomology classes [29], [30]. Note, however, that an infinitesimal symmetry of a local presentation is not necessarily a symmetry of another local presentation [16].

In particular, if we have a 0-cocycle of currents ν_i $(\mathfrak{d}\nu_i \neq 0)$ such that $\mu = d_H\nu_i$ and $\mathfrak{d}\mu_\nu = 0$, then by using the representation of the Lie derivative of classes of variational forms of degree $p \leq n-1$ given in [19], we have $\mu_{\mathcal{L}_\Xi \nu_i} = d_H(\Xi_H \rfloor d_H\nu_i + \Xi_V \rfloor d_V\nu_i)$; Since we also have

$$\mathcal{L}_\Xi \mu_\nu = \mu_{\mathcal{L}_\Xi \nu_i} = d_H(\Xi_H \rfloor \mu_\nu + \Xi_V \rfloor p_{d_V\mu_\nu})\,,$$

from the definition of an equivalent variational problem, we can state that the local problem defined by $\mathcal{L}_\Xi\nu_i$ is variationally equivalent to the global problem defined by $\Xi_H \rfloor \mu_\nu + \Xi_V \rfloor p_{d_V\mu_\nu}$.

Moreover, if we have a 0-cocycle of Lagrangians (case $p = n+1$) or of variational forms of higher degree (in case $p = n+2$ we have a 0-cocycle of dynamical forms) λ_i $(\mathfrak{d}\lambda_i \neq 0)$ such that $\eta = \mathcal{E}_p(\lambda_i)$, then by linearity $\eta_{\mathcal{L}_\Xi\lambda_i} = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda)$; again, as a consequence of the fact that $\eta_{\mathcal{L}_\Xi\lambda_i} = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda)$, we have that the local problem defined by the local presentation $\mathcal{L}_\Xi\lambda_i$ is variationally equivalent to the global problem defined by $\Xi_V \rfloor \eta_\lambda$.

Resorting to the naturality of the variational Lie derivative we stated the following important result for the calculus of variations [29], [30], [18].

Lemma 1. *Let $\mu \in \mathcal{V}^p_r$, with $p \leq n$, be a locally variationally trivial p-form, i.e. such that $\mathcal{E}_p(\mu) = 0$ and let $\delta_p(\mu_\nu) \neq 0$. We have $\delta_p(\mathcal{L}_\Xi \mu_\nu) = 0$. Analogously, let $\eta \in \mathcal{V}^p_r$, with $p \geq n+1$, be a locally variational p-form, i.e. such that $\mathcal{E}_p(\eta) = 0$ and let $\delta_p(\eta_\lambda) \neq 0$. We have $\delta_p(\mathcal{L}_\Xi \eta_\lambda) = 0$.*

Geometric definitions of conserved quantities in field theories have been proposed within formulations based on symmetries of Euler-Lagrange operator rather than of the Lagrangian, see e.g. [31], [15], and strictly related with such an approach are also papers proposing the concept of *relative conservation laws*; see e.g. [12]. Accordingly, let us now consider the case of invariance of field equations, i.e. the case in which we will assume Ξ to be a generalized symmetry, i.e. a symmetry of a class of $(n+1)$-forms η in the variational sequence.

Let then η_λ be the global Euler-Lagrange morphism of a local variational problem. It is a well known fact that Ξ being a generalized symmetry implies that $\mathcal{E}_n(\Xi_V \rfloor \eta) = 0$, thus locally $\Xi_V \rfloor \eta = d_H \nu_i$, then there exists a 0-cocycle ν_i, defined by $\mu_\nu = \Xi_V \rfloor \eta_\lambda \equiv d_H \nu_i$. Notice that $\mathfrak{d}\Xi_V \rfloor \eta_\lambda = 0$, but in general $\delta_n(\Xi_V \rfloor \eta_\lambda) \neq 0$ [16]. Along critical sections this implies the conservation law $d_H \nu_i = 0$ [1].

Noether's Theorem II implies that locally $\mathcal{L}_\Xi \lambda_i = d_H \beta_i$, thus we can write $\Xi_V \rfloor \eta_\lambda + d_H(\epsilon_i - \beta_i) = 0$, where ϵ_i is the usual *canonical* Noether current; the current $\epsilon_i - \beta_i$ is a *local* object and it is conserved along the solutions of Euler--Lagrange equations (critical sections). We stress that when Ξ is only a symmetry of a dynamical form and not a symmetry of the Lagrangian, the current $\nu_i + \epsilon_i$ *is not a conserved current* and it is such that $d_H(\nu_i + \epsilon_i)$ is locally equal to $d_H \beta_i$; see also [31]. We shall call $(\nu_i + \epsilon_i)$ a *strong Noether current*. Notice that if Ξ would be also a symmetry of the cochain of Lagrangians a strong Noether current would turn out to be a *conserved current* along any sections, not only along critical sections. Thus in this specific case we get the following.

Corollary 1. *Divergence expressions of the local problem defined by $\mathcal{L}_\Xi \nu_i$ coincide with divergence expressions for the global current $\Xi_H \rfloor \Xi_V \rfloor \eta_\lambda + \Xi_V \rfloor p_{d_V(\Xi_V \rfloor \eta_\lambda)}$.*

3 Currents variationally associated with locally variational field equations

We shall study variations of conserved currents in a quite general setting by determining the condition for the variation of a system of local strong Noether current to be equivalent to a system of *global conserved currents*. We now introduce the concept of conserved current variationally associated with locally variational invariant field equations and show that the invariance of the variation of the corresponding local presentation is a sufficient condition for the current beeing variationally equivalent to a global one.

Definition 1. We say a conserved current for an invariant field equation to be *variationally associated* if the symmetry of the field equation is also a symmetry for the variation of the local problem generating such a field equation.

[1] In this particular case ν_i is more precisely fixed, since $d_H \nu_i = \Xi_V \rfloor \eta$.

In other words, if λ_i is a local presentation we look for currents associated to a variation vector field Ξ satisfying $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$.

Suppose that, on the intersection of any two open sets, $\mathfrak{d}\lambda_i = d_H \gamma_{ij}$. By linearity, we have $\mathcal{L}_\Xi \mathfrak{d} \mathcal{L}_\Xi \lambda_i = \mathfrak{d} \mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i$; thus the condition $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$ implies $\mathcal{L}_\Xi \mathcal{L}_\Xi \mathfrak{d}\lambda_i = 0$. By Noether's Theorem II we must have $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = d_H \zeta_{ij}$, where ζ_{ij} is the sum of the Noether current associated with $\mathfrak{d}\lambda_i$ and a form locally given as $\mathfrak{d}\nu_i + d_H \rho_{ij}$. On the other hand $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = \mathfrak{d} d_H \epsilon_i$, where ϵ_i is the Noether current associated with λ_i. Of course, we have $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = \mathcal{L}_\Xi d_H \gamma_{ij}$ and again by Noether's Theorem II $\mathcal{L}_\Xi \lambda_i = d_H \beta_i$, hence by linearity we get, locally, $\mathfrak{d} d_H \beta_i = \mathcal{L}_\Xi d_H \gamma_{ij}$, where $\beta_i = \nu_i + \epsilon_i + d_H \omega_i$.

More precisely, we can immediately see that the condition $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$ implies only $\mathfrak{d} d_H \nu_i = 0$, i.e. $d_H \nu_i$ is global [2]. In order to get $\mathfrak{d} d_H \beta_i = 0$, i.e. the divergence of the strong Noether current, $d_H \beta_i$, to be *global* we must require a stronger condition, which is $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = 0$. This condition, by linearity, means that the Lie derivative must drag the local problems in such a way that they coincide on the intersections of two open sets [3]. Under this condition we have the conservation law $d_H \mathcal{L}_\Xi(\nu_i + \epsilon_i) = 0$, where $\mathcal{L}_\Xi(\nu_i + \epsilon_i)$, the *variation of the strong Noether currents*, is a *local representative* of a global conserved current.

In fact, Ξ being a *generalized symmetry*, we have $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = d_H \mathcal{L}_\Xi(\nu_i + \epsilon_i)$. If the second variational derivative is vanishing, then we have the conservation law $d_H \mathcal{L}_\Xi(\nu_i + \epsilon_i) = 0$, where $\mathcal{L}_\Xi(\nu_i + \epsilon_i)$ is a local representative of the current given by

$$\Xi_H \rfloor \mu_{\nu+\epsilon} + \Xi_V \rfloor p_{d_V \mu_{\nu+\epsilon}} \equiv \Xi_H \rfloor d_H(\nu_i + \epsilon_i) + \Xi_V \rfloor p_{d_V (d_H(\nu_i+\epsilon_i))} .$$

This current is global if $d_H \mathfrak{d}(\nu_i + \epsilon_i) = 0$; a sufficient condition for this to hold true is $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = 0$.

The conserved current associated with a generalized symmetry, assumed to be also a symmetry of the variational derivative of the corresponding local inverse problem, is variationally equivalent to the variation of the strong Noether currents for the corresponding local system of Lagrangians. Moreover, if the variational Lie derivative of the local system of Lagrangians is a global object, such a variation is variationally equivalent to a global conserved current [18]. In this paper, we make explicit the latter result in the case of Chern-Simons equations.

3.1 Chern-Simons gauge theory

It is well known that Chern-Simons field theories [8], [9] constitute a model for classical and quantum gravitational fields and that gravity can be considered as a gauge theory: in all odd dimensions and particularly in dimension three, where the field equations reproduce exactly the Einstein field equations, a Chern-Simons Lagrangian can be considered (instead of the Hilbert–Einstein Lagrangian) in which the gauge potential is a linear combination of a frame and a spin connection; in particular, $2+1$ gravity with a negative cosmological constant can be formulated as

[2] Notice that the symmetry Ξ, besides beeing a generalized symmetry, is *also a symmetry of the variational problem* $\mathcal{L}_\Xi \lambda_i$.

[3] This is also equivalent to $\mathfrak{d}(\nu_i + \epsilon_i) = d_H(\psi_{ij} - \rho_{ij})$, i.e. the coboundary of the strong Noether currents is locally exact; for details see [18].

a Chern-Simons theory (see, e.g. [32], as well as [7] for higher dimensional Chern--Simons gravity). Developing a 3-dimensional Chern-Simons theory as a possible and simpler model to analyse $(2+1)$-dimensional gravity brought in particular results concerned with thermodynamics of higher dimensional black holes [3], which in turn produced a renewed interest in Chern-Simons theories and, consequently, in the problem of gauge symmetries and gauge charges for Chern-Simons theories.

Let us then take in consideration the 3-dimensional Chern-Simons Lagrangian

$$\lambda_{CS}(A) = \frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}(A_\mu d_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho)ds\,,$$

where ds is a 3-dimensional volume density, $\kappa = \frac{l}{4G}$ (being G the Newton's constant and setting $c=1$), while $A_\mu = A^i_\mu\, J_i$ are the coefficients of the connection 1-form $A = A_\mu\, dx^\mu$ taking their values in any Lie algebra $\mathfrak{g}$ with generators J_i. By fixing $\mathfrak{g} = sl(2,\mathbb{R})$ and choosing the generators $J_k = \frac{1}{2}\sigma_k$, whith σ_k Pauli matrices, we have $[J_i, J_j] = \eta^{lk}\epsilon_{kij}\, J_l$ and $\mathrm{Tr}(J_i J_j) = \frac{1}{2}\eta_{ij}$, with $\eta = \mathrm{diag}(-1,1,1)$ and $\epsilon_{012} = 1$. Hence, we can explicitly write $\lambda_{CS}(A) = \frac{\kappa}{16\pi}\epsilon^{\mu\nu\rho}(\eta_{ij}F^i_{\mu\nu}A^j_\rho - \frac{1}{3}\epsilon_{ijk}A^i_\mu A^j_\nu A^k_\rho)ds$, where $F^i_{\mu\nu} = d_\mu A^i_\nu - d_\nu A^i_\mu + \epsilon^i_{jk}A^j_\mu A^k_\nu$ is the so-called field strength.

Note that, if we consider two independent $sl(2,\mathbb{R})$ connections A and $\bar{A}$, on the intersection of two open sets, the Lagrangian $\lambda_{CS}(A,\bar{A}) := \mathfrak{d}\lambda_{CS}(A) = \lambda_{CS}(A) - \lambda_{CS}(\bar{A})$ is a divergence. We recall for the sake of completeness that it is possible to perform a change of fiber coordinates, i.e. to define two new dynamical fields, e^i and ω^i, setting $A^i = \omega^i + \frac{e^i}{l}$ and $\bar{A}^i = \omega^i - \frac{e^i}{l}$, with l a constant, and in terms of these new variables we can write $\lambda_{CS}(A(\omega,e),\bar{A}(\omega,e)) = \frac{\kappa}{4\pi l}\sqrt{g}(g^{\mu\nu}R_{\mu\nu} + \frac{2}{l^2}) + d_\mu\{\frac{k}{4\pi l}\eta_{ij}\epsilon^{\mu\nu\rho}e^i_\nu\omega^j_\rho\}$, with $g_{\mu\nu} = \eta_{ij}\, e^i_\mu\, e^j_\nu$ and $R_{\mu\nu} = R^i_{j\rho\nu}\, e^j_\mu\, e^\rho_i$ the Ricci tensor of the metric g. In this expression, the non invariant term is under the total derivative and subtracting such a term from $\lambda_{CS}(A,\bar{A})$ one can get a global covariant Chern--Simons Lagrangian $\lambda_{CScov}(A(\omega,e),\bar{A}(\omega,e)) = \frac{\kappa}{4\pi\, l}\sqrt{g}(g^{\mu\nu}R_{\mu\nu} + \frac{2}{l^2})ds$, which can be recasted as $\lambda_{CScov}(A,\bar{A}) = \frac{k}{8\pi}\epsilon^{\mu\nu\rho}(\eta_{ij}\bar{F}^i_{\mu\nu}\, B^j_\rho + \eta_{ij}\bar{\nabla}_\mu B^i_\nu B^j_\rho + \frac{1}{3}\epsilon_{ijk}B^i_\mu B^j_\nu\, B^k_\rho)ds$, where $\bar{\nabla}_\mu$ is the covariant derivative with respect to the connection $\bar{A}$ and we set $B^i_\mu = A^i_\mu - \bar{A}^i_\mu$. As just explained, the procedure of writing the gauge potential A as a linear combination of the frame e and the spin connection ω enables one to split the non invariant divergence $\mathfrak{d}\lambda_{CS}(A)$ into a global piece plus a non covariant divergence and to generate a new Lagrangian. The latter is the difference of two local Lagrangians and it is covariant up to a divergence. It is well known that such a procedure can be applied to each Chern-Simons Lagrangian in dimension three, independently on the relevant gauge group of the theory, and it has been exploited in order to find Noether covariant charges (see e.g. [1], [4] and references therein). It should be noticed that such charges are associated with invariance properties of the thus obtained and above mentioned new Lagrangian; the interpretation of the relation with the conserved quantities associated with the original Euler-Lagrange equations for the Chern-Simons Lagrangian must be deeper investigated, see the discussion in [1].

The concept of a conserved current variationally associated with locally variational invariant Chern-Simons field equations provides a global conserved current directly related with the Euler-Lagrange equations. Chern-Simons equations of motion are manifestly covariant with respect to spacetime diffeomorphism as well as with respect to gauge transformations, the Chern-Simons Lagrangian instead is not gauge invariant. Let $\nu_i + \epsilon_i$ be a 0-cocycle of strong Noether currents for the Chern-Simons Lagrangian and let $\beta_i = \nu_i + \epsilon_i + d_H\omega_i$ as above. We have the following important result.

Proposition 1. *Let Ξ be a symmetry of the Chern-Simons dynamical form, the global conserved current*

$$\Xi_H \rfloor \mathcal{L}_\Xi \lambda_{CS_i} + \Xi_V \rfloor p_{d_V \mathcal{L}_\Xi \lambda_{CS_i}} \,,$$

is associated with the invariance of the Chern-Simons equations and it is variationally equivalent *to the variation of the strong Noether currents $\nu_i + \epsilon_i$.*

Proof. Let $\mathcal{L}_\Xi \lambda_{CS_i}$ be a local Lagrangian presentation of the inverse problem associated with the Chern-Simons dynamical form, we have $\mathcal{L}_\Xi \lambda_{CS_i} = d_H \beta_i$ and it is easy to verify that for a Chern-Simons Lagrangian the relation $\mathfrak{d}\mathcal{L}_\Xi \lambda_{CS_i} = \mathcal{L}_\Xi d_H \gamma_{ij}$ holds. Comparing these equations, we have that $\mathfrak{d} d_H \beta_i = \mathcal{L}_\Xi d_H \gamma_{ij}$, thus, in particular, $\mathfrak{d} d_H(\nu_i + \epsilon_i) = \mathcal{L}_\Xi d_H \gamma_{ij}$. As stated in the section above it is clear that, if $\mathcal{L}_\Xi d_H \gamma_{ij} = 0$, then $d_H(\nu_i + \epsilon_i)$ is global. Generators of such a global current lie in the kernel of the second variational derivative and are symmetries of the variationally trivial Lagrangian $d_H\gamma_{ij}$. □

Example 1. From to the relation $\mathcal{L}_\Xi A^i_\mu = \nabla_\mu \Xi^i_v$, where Ξ^i_v is the component of the vertical part of Ξ with respect to a principal connection ω on the bundle of frames, we have $\mathcal{L}_\Xi \lambda_{CS}(A) = d_\mu(\frac{\kappa}{8\pi}\epsilon^{\mu\nu\rho}\eta_{ij}A^i_\nu d_\rho \Xi^j_v)$, therefore, writing $\Xi^i_v = \Xi^i_V + (A^i_\mu - \omega^i_\mu)\Xi^\mu$, for each $\Xi_V \in \mathfrak{k}$ the local expression of a global current associated with the gauge invariance of the Chern-Simons dynamical form is given by

$$\big[\Xi^\gamma d_\mu(\frac{\kappa}{8\pi}\epsilon^{\mu\nu\rho}\eta_{ij}A^i_\nu d_\rho(\Xi^j_V + (A^j_\tau - \omega^j_\tau)\Xi^\tau)) + \\ (\Xi^k - A^k_\lambda \Xi^\lambda) d_\mu\big(\frac{\kappa}{8\pi}\epsilon^{\mu\gamma\rho}\eta_{kj} d_\rho(\Xi^j_V + (A^j_\zeta - \omega^j_\zeta)\Xi^\zeta)\big)\big] ds_\gamma \,.$$

As a final remark, we mention that local conserved currents can be derived by using Lepagian equivalent of local systems of Lagrangians [6]. Therefore, a study of inverse problems within a sequence of Lepage equivalent forms following [21], [22], [24] is of great interest and will be the object of future investigations.

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Authors' address:
M. Francaviglia, M. Palese, E. Winterroth: Department of Mathematics, University of Torino, via C. Alberto 10, I-10123 Torino, Italy

E-mail: marcella.palese@unito.it

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