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A geometric analysis of dynamical systems with singular Lagrangians

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Abstract. We study dynamics of singular Lagrangian systems described by implicit differential equations from a geometric point of view using the exterior differential systems approach. We analyze a concrete Lagrangian previously studied by other authors by methods of Dirac's constraint theory, and find its complete dynamics.[1]

1 Introduction

Singular (or degenerate) Lagrangian systems were first systematically considered by Dirac [2]. He was probably the first who noticed that the classical Hamilton equations make sense only for Lagrangians $L(t, q^\sigma, \dot{q}^\sigma)$ satisfying the regularity condition

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}\right) \neq 0,$$

and proposed a generalization to describe and understand dynamics of singular Lagrangians. Unfortunately, his approach was more heuristic than rigorous from the mathematical point of view, with an unpleasant consequence: study of the dynamics of concrete Lagrangian systems provided by different authors using Dirac's procedure can lead to different results.

A mathematically correct approach has been achieved later, with help of differential geometry. The dynamics of degenerate Lagrangian systems can be investigated in two geometrically distinct ways:

Indirect (image) approach concerns the well-known Hamiltonian formalism in symplectic geometry mapping a Lagrangian system from the tangent to the cotangent bundle: Hamiltonian dynamics then appears as image dynamics via Legendre

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map which is degenerate. An explicit study of the image (hamiltonian) dynamics is possible if the Legendre map has a constant rank (the image space is a submanifold in the cotangent space). Applying a procedure called "constraint algorithm" one can obtain, under some other assumptions on the Lagrangian, a *final constraint submanifold* where the image motion proceeds (among many references see e.g. [1], [5], [6], [7]).

Direct approach originally due to O. Krupková [8] concerns study of Hamiltonian exterior differential systems in jet bundles. This approach develops the idea of Goldschmidt and Sternberg [4] understanding Hamilton equations as equations for integral sections of an exterior differential system in the first jet bundle over a fibred manifold and is not restricted to some particular kind of Lagrangian systems (regarding rank, or order). Whatever is the Legendre map, in this approach there is a direct geometric relation to extremals (solutions of the Euler-Lagrange equations) as those integral sections of the Hamiltonian exterior differential system which are *holonomic* (i.e. take the form of prolongations). As proposed in [8], within the "direct" setting one can study in a unified way both the Hamiltonian (extended) and the Lagrangian (proper) dynamics of any Lagrangian system (including highly singular), and to obtain a *geometrically exact description of the dynamics*. From the point of view of mathematics, this is a *method of analysing the structure of solutions of implicit second (or higher) order differential equations*. It should be pointed out (also shown on examples in [8], [9]) that the resulting Hamiltonian (and Lagrangian) dynamics need *not* be bounded to a "final constraint submanifold", and may proceed rather in the whole phase space in a way which can be understood as a "controlled chaotic motion". Moreover, in cases when a "final constraint submanifold" exists, the motion typically is not described by a vector field along this submanifold but rather by a more complicated *family of vector fields* (vector distribution).

In this article we investigate a singular mechanical system given by the Lagrangian

$$L = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3.$$

This Lagrangian system has been studied by several authors (e.g. [3] and references therein) by means of the Dirac constraint algorithm, however, its dynamical properties were not clarified: the obtained results are incomplete and conclusions on the dynamical properties of this Lagrangian system made by different authors are not in agreement. It should be pointed out that the main result – to obtain Hamiltonian dynamics for this Lagrangian system, has not been achieved.

We show that problems of this kind can be rigorously solved by application of the above mentioned Hamiltonian exterior differential systems method. For the given Lagrangian, we obtain the corresponding dynamical distribution in the first jet bundle, and show that this distribution is not completely integrable and has a nonconstant rank. This means, however, that to obtain the dynamics one has to apply a general integration method developed in [8], called a "geometric constraint algorithm". With help of the geometric constraint algorithm we solve the problem completely: we compute the Euler-Lagrange equations and Hamilton equations in terms of the corresponding distributions and find the complete structure of solutions (so-called *proper dynamics* and *extended dynamics*, respectively). In particular, the Hamilton and Euler-Lagrange equations are not equivalent, and the dynamics are *not* representable by a *vector field*, they are even not representable by a *vector field* along a certain "final constraint submanifold" of the evolution space. It turns out that the dynamics *are* restricted to (the same) final constraint submanifold, however, along this submanifold the extended (Hamiltonian) and the proper (Lagrangian) motion are governed by distinct *nonintegrable* distributions of *rank greater than one*, the Lagrangian dynamics being characterized by the rank 2 semispray subdistribution of the distribution describing the Hamiltonian dynamics. Among others this means that neither the Hamiltonian nor the Lagrangian "solution pattern" follows within a foliation in the evolution space (or its submanifold). It can be shown, however, that the nonintegrable semispray subdistribution of rank 2 inherits an intrinsic structure such that, within the final constraint submanifold, (prolonged) extremals are constrained to a *family of submanifolds* parametrized by functions on the evolution space. Moreover, along each of these submanifolds the motion is regular, i.e. extremals are integral sections of a *semispray vector field*.

2 Singular Lagrangian systems

We shall consider a fibred manifold $\pi\colon Y \to X$; $Y = R \times M$ where M is a smooth manifold of dimension m, and its first jet prolongation J^1Y. Local fibred coordinates are denoted by (t, q^σ), where $1 \le \sigma \le m$, and the corresponding coordinates on J^1Y are denoted by $(t, q^\sigma, \dot{q}^\sigma)$. The manifold J^1Y is called evolution space.

We shall use the following setting due to [8], [9]:

- A geometric description of the dynamics using a vector distribution on J^1Y.
- Formulation of Hamilton theory as a problem of finding all solutions of this distribution.
- Formulation of Lagrange theory as a problem of finding holonomic solutions of this distribution.

Equations of motion of a Lagrangian system defined by a Lagrangian $\lambda = Ldt$, $L = L(t, q^\sigma, \dot{q}^\sigma)$, are represented by the Euler-Lagrange form

$$E_\lambda = E_\sigma \omega^\sigma \wedge dt, \quad E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma},$$

$1 \le \sigma \le m$, where $\omega^\sigma = dq^\sigma - \dot{q}^\sigma dt$. In what follows we assume that the Euler-Lagrange equations are nontrivially of order two, and denote

$$E_\sigma = A_\sigma + B_{\sigma\nu}\ddot{q}^\nu.$$

It is known that there exists a unique 2-contact form F on J^2Y such that the 2-form $\alpha = E_\lambda + F$ is closed and projectable onto J^1Y [9].

The form α gives rise to the following two distributions on J^1Y which are in general distinct but their holonomic sections are the same and coincide with prolongations of extremals:

- The *characteristic distribution* of α

$$D = \mathrm{span}\{i_\xi \alpha\}, \quad \text{where } \xi \text{ runs over all vector fields on } J^1Y$$

- The *Euler-Lagrange distribution* of α

$$\Delta = \mathrm{span}\{i_\xi \alpha\}, \quad \text{where } \xi \text{ runs over all vertical vector fields on } J^1Y$$

Note that $D \subset \Delta$.

In terms of a Lagrangian

$$\alpha = d\theta_\lambda,$$

where

$$\theta_\lambda = Ldt + \frac{\partial L}{\partial \dot{q}^\sigma}\omega^\sigma$$

is the Cartan form.

Definition 1. Equations for integral sections of Δ are called *Hamilton equations*, solutions of the Hamilton equations are called *Hamilton extremals*.

Definition 2. λ is called *regular* if $\mathrm{rank}\,\Delta = 1$. λ is called *semiregular* if Δ is weakly horizontal (i.e. at each point $p \in J^1Y$ the vector space $\Delta(x)$ is not vertical) and $\mathrm{rank}\,\Delta$ is constant.

Dynamical properties of a Lagrangian system are determined by properties of its related distributions:

Theorem 1. *Δ is weakly horizontal at $x \in J^1Y$ if and only if $D(x) = \Delta(x)$.*

Definition 3. The set $\tilde{P} = \{x \in J^1Y | D(x) = \Delta(x)\}$ is called *the primary constraint set*.

$\widetilde{P} \subset J^1Y$ need not be a submanifold. This set has the meaning of "possibly admissible" initial conditions for the Hamilton equations – more precisely, $J^1Y - \widetilde{P}$ is a primary obstruction set for the hamiltonian initial conditions (outside $\widetilde{P}$ there passes no solution of the Hamilton equations, and consequently, no solution of the Euler-Lagrange equations).

Theorem 2. *The following conditions are equivalent:*

1. *λ is regular*
2. $\Delta = \mathrm{span}\{\frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma} - B^{\sigma\nu} A_\nu \frac{\partial}{\partial \dot{q}^\sigma}\} = \mathrm{annih}\{A_\sigma dt + B_{\sigma\nu} d\dot{q}^\nu, \omega^\sigma\}$
3. $\det B_{\sigma\nu} = \det\big(\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}\big) \neq 0.$

2.1 The geometric constraint algorithm

The dynamics of a smooth singular Lagrangian system cannot be characterized by a vector field, or even by a system of continuous vector fields in the evolution space. In this section we recall a general procedure which enables one to solve the Euler-Lagrange distribution explicitly [8].

Since in general the extended dynamics and proper dynamics do not coincide, we have to distinguish two levels of the integration problem:

(1) to find the *extended dynamics*, i.e., all integral sections of the Euler-Lagrange distribution (Hamilton extremals)

(2) to find the *proper dynamics*, i.e., holonomic integral sections.

2.1.1 Extended dynamics

We shall describe an algorithm for finding the structure of solutions of Hamilton equations (the dynamics of a Hamiltonian system).

Let us denote

$$F = \begin{pmatrix} \frac{1}{2}(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} - \frac{\partial A_\nu}{\partial \dot{q}^\sigma}) & B_{\sigma\nu} \\ -B_{\sigma\nu} & 0 \end{pmatrix}$$

and $(F|A)$ the matrix F extended by the column $A_\sigma, 1 \leq \sigma \leq m$, where:

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma} \dot{q}^\nu \quad \text{and} \quad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}.$$

Step 1: Find the primary constraint set $\widetilde{P}$. As proved in [8],

$$\widetilde{P} = \{x \in J^1 Y \,|\, \operatorname{rank} F = \operatorname{rank}(F|A)\}.$$

If $\widetilde{P} = \emptyset$, there is no extended dynamics, hence no dynamics at all. If $\widetilde{P} \neq \emptyset$, choose a point $x \in \widetilde{P}$, and proceed to the next step.

Step 2: Denote $M_{(1)} \subset \widetilde{P}$ a submanifold of maximal dimension around x and calculate the Euler-Lagrange distribution $\Delta_{(1)}$ along $M_{(1)}$.

Step 3: Exclude from $M_{(1)}$ the points where the restriction of $\Delta_{(1)}$ to the tangent bundle of $M_{(1)}$ is not weakly horizontal and denote the resulting set by $\widetilde{P}'$. Repeat Step 2 with $\widetilde{P}'$ instead of $\widetilde{P}$.

Continue until the procedure is finalized. Then take another (distinct) submanifold $M_{(2)}$ in $\widetilde{P}$ around x, repeat the procedure.

After sufficiently many steps one obtains either a bunch of final constraint submanifolds at x, or finds that there is no final constraint submanifold passing through x.

Considering then the collection of final constraint submanifolds together with to them constrained Euler-Lagrange distributions, we get the dynamical picture corresponding to the solutions of the Hamilton equations.

2.1.2 Proper dynamics

We have to exclude solutions of Hamilton equations which are not holonomic. First we find the set

$$P = \{x \in J^1 Y \mid \operatorname{rank} B = \operatorname{rank}(B|A)\},$$

called *primary semispray constraint set*. Again, it need not be a submanifold in $J^1 Y$. Outside this set, there exist no prolonged extremals, hence there is no motion. If $P \neq \emptyset$, we choose a point $x \in P$ and proceed in a similar way as described above in searching for the extended dynamics: however, in this case we consider as admissible only those submanifolds and vector fields belonging to Δ which along the submanifold can be identified with a semispray.

3 A singular Lagrangian dynamics

Let us consider the following singular Lagrangian

$$L = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3. \tag{1}$$

Its Euler-Lagrange equations are implicit second order differential equations

$$q^3 - \ddot{q}^3 = 0, \quad \dot{q}^3 = 0, \quad q^1 + \dot{q}^2 - \ddot{q}^1 = 0.$$

Momenta take the form

$$p_1 = \dot{q}^3, \quad p_2 = 0, \quad p_3 = \dot{q}^1 - q^2$$

and the Hamiltonian reads

$$H = \dot{q}^1\dot{q}^3 - q^1q^3. \tag{2}$$

We can use momenta as a part of new coordinates on J^1Y by considering a local coordinate transformation as follows

$$(t, q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \dot{q}^3) \to (t, q^1, q^2, q^3, p_3, \dot{q}^2, p_1), \tag{3}$$

and get for the Hamiltonian the expression

$$H = (p_3 + q^2)p_1 - q^1q^3. \tag{4}$$

Computing the Hessian matrix of L we get

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

hence the Legendre map $R \times TM \to R \times T^*M$ has constant rank equal to 2 and defines a submanifold of dimension 5.

Let us turn to the analysis of the dynamics on $R \times TM$ with help of the corresponding distributions. To this end we need the Cartan form θ_λ and its exterior derivative $d\theta_\lambda$:

$$\theta_\lambda = (q^1q^3 - \dot{q}^1\dot{q}^3)dt + \dot{q}^3dq^1 + (\dot{q}^1 - q^2)dq^3,$$

$$d\theta_\lambda = (q^3dq^1 + q^1dq^3 - \dot{q}^3d\dot{q}^1 - \dot{q}^1d\dot{q}^3) \wedge dt + (d\dot{q}^1 - dq^2) \wedge dq^3 + d\dot{q}^3 \wedge dq^1.$$

Computing the distributions D and Δ we get:

$$D = \mathrm{annih}\{q^3dq^1 + q^1dq^3 - \dot{q}^3d\dot{q}^1 - \dot{q}^1d\dot{q}^3,\ q^3dt - d\dot{q}^3,\ dq^3,\ q^1dt + dq^2 - d\dot{q}^1,\ \omega^3,\ \omega^1\}$$

and

$$\Delta = \mathrm{annih}\{q^3dt - d\dot{q}^3,\ dq^3,\ q^1dt + dq^2 - d\dot{q}^1,\ \omega^3,\ \omega^1\}.$$

We can see that $D \subset \Delta$ and $D \neq \Delta$.

3.1 Extended dynamics

The Lagrangian system possesses primary dynamical constraints (primary obstructions to initial conditions for the Hamilton equations). The primary constraint set $\widetilde{P}$ is the set of points in the evolution space where $\operatorname{rank} F = \operatorname{rank}(F|A)$, hence

$$\widetilde{P} = \{x \in J^1Y |\ \dot{q}^3 = 0\},$$

and it is a closed submanifold in J^1Y of codimension 1.

Note that, indeed, outside the submanifold $\widetilde{P}$ the Euler-Lagrange distribution is spanned by two vector fields

$$\frac{\partial}{\partial q^2} + \frac{\partial}{\partial \dot{q}^1}, \quad \frac{\partial}{\partial \dot{q}^2}$$

which are vertical over the base R, hence among the integral curves there are no *sections* (the integral curves describe no evolution).

To get the Hamiltonian dynamics we have first to restrict our considerations to the admissible submanifold $\widetilde{P}$. Along this submanifold the Euler-Lagrange distribution Δ and the dynamical distribution D coincide and are spanned by the following three vector fields:

$$D = \Delta = \operatorname{span}\left\{ \frac{\partial}{\partial t} + \dot{q}^1 \frac{\partial}{\partial q^1} - q^1 \frac{\partial}{\partial q^2} + q^3 \frac{\partial}{\partial \dot{q}^3};\ \frac{\partial}{\partial q^2} + \frac{\partial}{\partial \dot{q}^1};\ \frac{\partial}{\partial \dot{q}^2} \right\}.$$

This distribution is weakly horizontal, but we have to exclude points where it is not tangent to $\widetilde{P}$, that is, the points where $q^3 \neq 0$. Indeed, at these points restriction of $D = \Delta$ to the tangent bundle of $\widetilde{P}$ is a vertical distribution. We obtain a submanifold

$$M = \{x \in J^1Y \mid q^3 = 0,\ \dot{q}^3 = 0\} \tag{5}$$

of $\widetilde{P}$, and along M the distribution

$$\begin{aligned} D_M = \Delta_M &= \operatorname{span}\left\{ \frac{\partial}{\partial t} + \dot{q}^1 \frac{\partial}{\partial q^1} - q^1 \frac{\partial}{\partial q^2};\ \frac{\partial}{\partial q^2} + \frac{\partial}{\partial \dot{q}^1};\ \frac{\partial}{\partial \dot{q}^2} \right\} \\ &= \operatorname{span}\left\{ f_1\left(\frac{\partial}{\partial t} + \dot{q}^1 \frac{\partial}{\partial q^1}\right) + (f_2 - f_1 q^1)\frac{\partial}{\partial q^2} + f_2 \frac{\partial}{\partial \dot{q}^1} + f_3 \frac{\partial}{\partial \dot{q}^2} \right\}, \end{aligned} \tag{6}$$

where f_1, f_2, f_3 are arbitrary functions on M. This distribution is tangent to M, and weakly horizontal at each point of M, as required. Note that its annihilator is spanned by the following two 1-forms: $q^1 dt + dq^2 - d\dot{q}^1$ and $\omega^1 = dq^1 - \dot{q}^1 dt$. We can see that $\operatorname{rank} \Delta_M$ is constant and equal to 3, however, Δ_M is not completely integrable. Summarizing, we have obtained the following structure of solutions of the Hamilton equations for our Lagrangian:

Theorem 3. *Hamilton equations of L are equations for integral sections of the not completely integrable rank 3 distribution Δ_M on the closed 5-dimensional manifold $M \subset J^1Y$ above.*

In fibred coordinates, the Hamilton equations are equations for sections $\delta(t) = (t, x^\sigma(t), y^\sigma(t))$ of J^1Y, where we have denoted $x^\sigma(t) = q^\sigma \circ \delta$ and $y(t) = \dot{q}^\sigma \circ \delta$, and take the following form:

$$\begin{aligned} &\frac{dx^1}{dt} = y^1, \quad &&\frac{dx^2}{dt} = g(t) - x^1, \quad &&\frac{dx^3}{dt} = 0, \\ &\frac{dy^1}{dt} = g(t), \quad &&\frac{dy^2}{dt} = h(t), \quad &&\frac{dy^3}{dt} = 0, \end{aligned}$$

where g, h are arbitrary functions on M, $g(t) = g \circ \delta$ and $h(t) = h \circ \delta$.

In a more conventional way, in "partial Legendre coordinates" defined by (3), and in terms of Hamiltonian (4) we can write

$$\begin{aligned} \frac{dq^1}{dt} &= \frac{\partial H}{\partial p_1}, & \frac{dq^2}{dt} &= g + \frac{\partial H}{\partial q^3}, & \frac{dq^3}{dt} &= 0, \\ \frac{dp_1}{dt} &= 0, & \frac{d\dot{q}^2}{dt} &= h, & \frac{dp_3}{dt} &= -\frac{\partial H}{\partial q^3}. \end{aligned}$$

For this Hamiltonian system M is the *final constraint submanifold* (having the meaning of a genuine evolution space, or phase space). Extended motion is constraint to this submanifold, is chaotic, not uniquely determined by initial conditions.

3.2 Proper dynamics

We are looking for *holonomic* Hamilton extremals = prolongations of extremals.

Computing the primary semispray-constraint set we get the following closed submanifold in the evolution space

$$P = \{x \in J^1Y \mid \dot{q}^3 = 0\}.$$

Outside this submanifold there are no (prolonged) extremals.

Since $P = \widetilde{P}$, the procedure of restricting the Euler-Lagrange distribution to P ends with the same submanifold M and the restricted distribution $D_M = \Delta_M$ as in the Hamiltonian case above. Now, however, this is not yet the end of the story, since we are interested in *holonomic* solutions, and the distribution Δ_M still has solutions which are not holonomic, hence do not correspond to extremals. We have to continue to another step in the geometric constraint algorithm in order to obtain a maximal submanifold of M with a distribution whose nonvertical vector fields are *semisprays*. It is easily seen that this is achieved by taking the manifold M itself and the rank 2 subdistribution of Δ_M which is obtained by choosing $f_1 = 1$ and $f_2 = \dot{q}^2 + q^1$, i.e. takes the form

$$\mathcal{D} = \operatorname{span}\left\{ \frac{\partial}{\partial t} + \dot{q}^1 \frac{\partial}{\partial q^1} + \dot{q}^2 \frac{\partial}{\partial q^2} + (\dot{q}^2 + q^1) \frac{\partial}{\partial \dot{q}^1} + f \frac{\partial}{\partial \dot{q}^2} \right\}, \tag{7}$$

where f is an arbitrary function on M. Hence we have obtained the following result:

Theorem 4. *The Euler-Lagrange equations of L are equations for integral sections of the not completely integrable rank 2 distribution $\mathcal{D}$ on the closed 5-dimensional manifold $M \subset J^1Y$ above.*

Theorem 4 gives a geometric solution to the extremal problem and a complete geometric "dynamical picture" for the proper dynamics of the given singular Lagrangian system (1). We can see that the motion is restricted to a final constraint submanifold of dimension 5, and is chaotic and indeterministic there (cannot be uniquely determined by initial conditions). Compared with the Hamiltonian dynamics obtained in the previous section, the final constraint submanifold is the same, and the Lagrangian dynamics is given by a rank 2 semispray subdistribution of the distribution describing the Hamiltonian dynamics.

The distribution $\mathcal{D}$ is not completely integrable which means that the (prolonged) extremals do not proceed within leaves of a foliation of M. Nevertheless, as shown below, the geometric picture can be further refined to give us a more precise and fine description of the Lagrangian dynamics within the final constraint submanifold M.

Let us turn back to the distribution Δ_M (6) and note that it has the following rank 2 weakly horizontal subdistribution

$$\operatorname{span}\left\{\frac{\partial}{\partial t}+\dot{q}^1\frac{\partial}{\partial q^1}+(g-q^1)\frac{\partial}{\partial q^2}+g\frac{\partial}{\partial \dot{q}^1}+h\frac{\partial}{\partial \dot{q}^2}\right\}, \tag{8}$$

where g, h are arbitrary functions on M. Now, for every fixed $g(t, q^1, q^2, \dot{q}^1, \dot{q}^2)$ consider a manifold

$$M_g = \{x \in J^1Y \mid q^3 = \dot{q}^3 = 0,\ \dot{q}^2 = g - q^1\} \subset M. \tag{9}$$

If

$$\phi \equiv \frac{\partial g}{\partial \dot{q}^2} - 1 \neq 0, \tag{10}$$

then along M_g distribution (8) takes the form of a rank 2 *semispray* distribution spanned by the following vector fields:

$$\frac{\partial}{\partial t}+\dot{q}^1\frac{\partial}{\partial q^1}+\dot{q}^2\frac{\partial}{\partial q^2}+(\dot{q}^2+q^1)\frac{\partial}{\partial \dot{q}^1}+h\frac{\partial}{\partial \dot{q}^2}, \tag{11}$$

i.e., it is the distribution $\mathcal{D}$ restricted to the submanifold M_g. We have to find its subdistribution *tangent* to M_g. To this end let us consider local coordinates $\bar{t} = t$, $\bar{q}^1 = q^1$, $\bar{q}^2 = q^2$, $\dot{\bar{q}}^1 = \dot{q}^1$, $z = \dot{q}^2 - g + q^1$, adapted to the submanifold M_g. Note that regularity of the transformation means that at each point condition (10) holds true. Transforming (11) to the new coordinates we can see that there is a unique (up to a multiplier) vector field tangent to M_g, with

$$h = \frac{1}{\phi}(\dot{q}^1 - X(g)), \tag{12}$$

where we have denoted

$$X = \frac{\partial}{\partial t}+\dot{q}^1\frac{\partial}{\partial q^1}+\dot{q}^2\frac{\partial}{\partial q^2}+(\dot{q}^2+q^1)\frac{\partial}{\partial \dot{q}^1}.$$

Theorem 5. *Euler-Lagrange equations of L are equations for integral sections of the following family of rank one (hence completely integrable) constraint semispray distributions:*

$$\mathcal{S}_g = \operatorname{span}\left\{\frac{\partial}{\partial t}+\dot{q}^1\frac{\partial}{\partial q^1}+\dot{q}^2\frac{\partial}{\partial q^2}+(\dot{q}^2+q^1)\frac{\partial}{\partial \dot{q}^1}+\frac{1}{\phi}(\dot{q}^1 - X(g))\frac{\partial}{\partial \dot{q}^2}\right\}$$

each defined on the closed 4*-dimensional manifold*

$$M_g = \{x \in J^1Y \mid q^3 = \dot{q}^3 = 0,\ \dot{q}^2 - g + q^1 = 0\} \subset M \subset J^1Y,$$

where $g(t, q^1, q^2, \dot{q}^1, \dot{q}^2)$ *is an arbitrary function satisfying condition* (10).

Hence, the structure of extremals of the considered singular Lagrangian is completely described by a *family* of 4-dimensional submanifolds M_g of the 5-dimensional "final constraint submanifold" M, endowed with *semispray distributions of rank* 1. This means that every manifold M_g is foliated by one-dimensional foliation, and the family of these "constraint foliations" in M represents the structure of integral sections of the non-integrable rank 2 distribution $\mathcal{D}$ (7) on the final constraint submanifold in the evolution space.

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