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Book Review

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Geometry of Nonholonomically Constrained Systems by R. Cushman, H. Duistermaat and J. Śniatycki. Advanced Series in Nonlinear Dynamics, Vol 26, World Scientific, 2010.

The admissible configurations and velocities of point masses or rigid bodies in a mechanical system are often observed to be restricted. In many cases these limitations can be handled by introducing constraint equations into the framework. One usually distinguishes between two types of such constraints. Holonomic constraints are restrictions on the position of the system only. A constraint is said to be nonholonomic if the restriction depends also on the velocities of the system, and if by no means it can be integrated to a holonomic constraint. Typical engineering problems that involve nonholonomic constraints arise for example in robotics, where the wheels of a mobile robot are often required to roll without slipping, or where one is interested in guiding the motion of a cutting tool or a skate. Some of the well-studied textbook examples of nonholonomic systems include the rolling disk, the rattleback, the rolling ball in a cylindrical tube, the problem of pursuit and the snakeboard. A classical reference for nonholonomic systems is the book by Neĭmark and Fufaev [3].

Since the second half of last century, tools and techniques from differential geometry (Riemann geometry, contact geometry, symplectic and Poisson geometry, Lie groups, fibre bundles, jet bundles, connections, distributions, etc.) have had an ever growing impact on the analysis of problems in mechanics. The discipline that emerged from the contact between geometry and mechanics is now commonly called 'Geometric Mechanics'. Two fairly recent books that deal specifically with geometric approaches to nonholonomic mechanical systems are e.g. the monographs by Bloch [1] and Cortés [2], and the book under review can be thought off as an new addition to this category. The book can be divided in two parts: while the first four chapters contain the theoretical material, the last three chapters concentrate on three specific examples.

Chapter 1 introduces the category of nonholonomically constrained mechanical systems. As in the bigger part of the literature, only non-holonomic constraints that

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are time-independent and that depend linearly on velocities are considered. The advantage is that they can geometrically be represented in the form of a distribution D on the configuration space Q. The authors only consider Lagrangians of so-called 'mechanical type', that is of the form $k - V$, where k is a kinetic energy function associated to a Riemannian metric on Q, and V is a potential function on Q. The authors work in what they call the 'distributional Hamiltonian formalism'. One should not think here of the adjective 'Hamiltonian' as meaning 'set in (a part of) phase space', but one should rather interpret it as 'set in a symplectic framework'. That is to say, one can pullback, by means of the Legendre transformation, the canonical symplectic two-form on T^*Q to a symplectic two-form on TQ, the so-called Poincaré-Cartan two-form. It can then be shown that the distribution D gives rise to a certain distribution H on the submanifold of TQ determined by D, and that the restriction ϖ of the Poincaré-Cartan form to H is non-degenerate. The object ϖ can therefore be thought off as playing the role of a symplectic form for nonholonomic systems. Analogously, if one assigns to the energy function $h = k+V$ on TQ the role of Hamiltonian, the symplectic-type equation $Y_h \lrcorner\, \varpi = \partial_H h$ defines a unique vector field Y_h on D whose base integral curves are solutions of the nonholonomic dynamics. After a few basic properties of Y_h, the rest of the first chapter deals mainly with the Dirac bracket and the almost Poisson structure that one can define in this context, with the projection principle (which states that the nonholonomic dynamical vector field Y_h is a certain projection of the free dynamical vector field) and with a nonholonomic version of Noether's Theorem (on the relation between infinitesimal symmetries and constants of motion).

A large part of the book deals with the theory of Lie group symmetry reduction for nonholonomic systems. The benefits of exploiting symmetry are self-evident: if a dynamical system exhibits a symmetry, one may hope to reduce the system to one with fewer variables, possibly easier to solve. Throughout Chapters 2, 3 and 4 a special emphasis is put on the singular case, where the action is not necessarily free and proper, and this is precisely what sets this part of the book apart from the existing literature. In Chapter 2 the basic concepts related to Lie group actions on manifolds are reviewed. A free and proper action defines a principal fibre bundle structure on the orbit space, which in particular becomes a smooth manifold. The authors show that even in the singular case it is possible to define a kind of differential structure on the orbit space, by introducing the concept of a differential space. It is clear that when one works with spaces that do not necessarily possess a smooth manifold structure, one needs to rethink a lot of concepts, such as e.g. the definition of a tangent space and a vector field. Chapter 2 therefore mainly deals with re-inventing, in the more general set-up of differential spaces, familiar concepts known for manifolds. Chapter 3 contains the actual descriptions of the reduced distributional Hamiltonian systems, both for singular and regular Lie group symmetry reduction. Special attention is given to the subclass of Chaplygin nonholonomic systems (where the constraint distribution is the horizontal space of a principal connection, and where the reduced equations are of pure second-order type). Further, given that for nonholonomic systems the usual interplay between symmetries and conserved momenta is no longer valid, the authors situate the so-called momentum equations within the context of the reduced equations. Chapter 4 is about reconstruction equations (whose solutions enable one to reconstruct a complete solution from a solution of the reduced equations), about relative equilibria (i.e. equilibria of the reduced equations) and about relative periodic orbits. Again, we find in this chapter a careful analysis of the situation when the action is not necessarily free.

The following citation, taken from the introduction of Chapter 5, fully captures the spirit of the second part of the book: "In this chapter we will discuss the classical nonholonomically constrained system known as Carathéodory's sleigh. In order to illustrate the theory given in chapters 1,2 and 4, we will derive the equations of motion in five different ways, construct the reduced system in three different ways, and carry out reconstruction explicitly.". Indeed, the following three chapters contain a very comprehensive analysis of three famous examples of nonholonomic systems: Carathéodory's sleigh (which is a planar rigid body with a sharp edge in a vertical plane that makes contact with a horizontal plane in its lowest point), the example of a smooth, strongly convex rigid body rolling without slipping on a horizontal plane (under the influence of a constant vertical gravitational force), and the example of the rolling disk (which is not necessarily confined to roll vertically). These chapters do not only contain explicit formulae of reduced equations in many different forms and fashions, but, in particular in the chapter on the rolling disk, also contain a lot of information on the qualitative behaviour of particular solutions to the problems, such as e.g. a stability analysis for the relative equilibria and a study of the limiting behaviour of the disk when it nearly falls flat and then rises up again.

I believe that the book under review will become a standard reference work for people working in the field of geometric mechanics. I enjoyed the clear writing style of the main body of the text and I also appreciated the background information in the 'Notes' section at the end of each chapter. More importantly, in my opinion the book contains a lot of interesting research paths which makes it distinct from other books on this topic. One of them is the analysis of singular actions in the theoretical part, which is barely touched upon in e.g. [1] and [2]. An other major feat is that concrete simple examples are stripped down naked in an instructive manner and that they are shown to be the source of a very rich variety of interesting geometric problems. The new methods and techniques developed in the last three chapters for the specific examples may even become inspiration for future research on nonholonomic systems in general. A bit disappointingly, however, is that, as far as I could check, none of the Lie group actions involved in the last chapters were actually not-free, and that the singular tools from the first chapters were left unused in the last chapters. To my mind, this is a bit a missed opportunity.

References

[1] A. M. Bloch with the collaboration of J. Baillieul, P. Crouch and J. E. Marsden, *Nonholonomic Mechanics and Control*, Springer (2003).

[2] J. Cortés Monforte, *Geometric, Control and Numerical Aspects of Nonholonomic Systems*, Lecture Notes in Mathematics 1793, Springer (2002).

[3] J. I. Neĭmark and N. A. Fufaev, *Dynamics of Nonholonomic Systems*, Transl. of Math. Monographs 33, AMS (1972).

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