

Geometric mechanics on nonholonomic submanifolds

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Abstract. In this survey article, nonholonomic mechanics is presented as a part of geometric mechanics. We follow a geometric setting where the constraint manifold is a submanifold in a jet bundle, and a nonholonomic system is modelled as an exterior differential system on the constraint manifold. The approach admits to apply coordinate independent methods, and is not limited to Lagrangian systems under linear constraints. The new methods apply to general (possibly nonconservative) mechanical systems subject to general (possibly nonlinear) nonholonomic constraints, and admit a straightforward generalization to higher order mechanics and field theory. In particular, we are concerned with the following topics: the geometry of nonholonomic constraints, equations of motion of nonholonomic systems on constraint manifolds and their geometric meaning, a nonholonomic variational principle, symmetries, a nonholonomic Noether theorem, regularity, and nonholonomic Hamilton equations.

1 Introduction

Nonholonomic mechanics is concerned with study of systems the motion of which is subject to constraints on time, positions and velocities. The interest to investigate mechanical systems with holonomic and nonholonomic constraints goes back to the 19th century, when D'Alembert's principle of virtual work and Gauss' principle of least action in presence of constraints were considered. It was discovered that holonomically constrained dynamics can be understood as motions subject to reactive forces of a gradient form, given by the constraints. As conjectured by Chetaev in early 30's of the last century, nonholonomic equations of motion could have a similar form, but now the reactive forces should take the form of derivatives with respect to the velocities [9]. Since that time, Chetaev's equations have been tested in many situations and on many examples in mechanics and engineering,

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and it turned out that (contrary to the so-called vakonomic equations proposed as alternative equations of motion), they really do describe motions of nonholonomic mechanical systems (see e.g. [6], [12]).

Within the classical analysis approach, only Lagrangian systems subject to linear integrable (semi-holonomic) constraints have been well-understood. In case of non-integrable, or even non-linear constraints, a satisfactory, complete theory, similar to the analytical dynamics of unconstrained systems, has been missing. On the other hand, during the last 20 years, in connection with the developments of geometric mechanics and global calculus of variations, methods of differential geometry and global analysis have turned out to be well suited and helpful for understanding nonholonomic systems. There have been proposed several geometric models, appropriate in different situations, applicable to Lagrangian systems in tangent bundles or in jet bundles. It should be stressed, however, that almost all the work on nonholonomic systems is concerned with the case of constraints linear (affine) in the velocities. The bibliography is very extensive and it is not possible to list here all important contributions; we refer at least to [2], [7], [10], [11], [13], [14], [15], [20], [29], [30], [33], [43], [44], [46], [47], [49] and references therein.

In this article we present the nonholonomic mechanics as a part of geometric mechanics. However, we should emphasize that we follow the setting where a nonholonomic system is modeled as an exterior differential system *on a constraint manifold* (subbundle of a jet bundle) [29], [30], [33], [35], [39], [51] (i.e., motion equations appear in the “reduced form”, without Lagrange multipliers). This approach consistently reflects the geometric character of nonholonomic constraints. It naturally admits to apply coordinate independent methods and transfer standard concepts and techniques of differential geometry and the calculus of variations on manifolds to the situation when differential constraints are present. Moreover, this approach is *not limited to Lagrangian systems under linear constraints*. In fact, both Lagrangian and non-conservative systems are treated in a unique way, and similarly, a unique geometric model of differential constraints (whatever they are: linear integrable or non-integrable, or nonlinear) is presented. Within this setting, a generalization to higher-order systems and constraints, and extension of nonholonomic mechanics to field theory is straightforward [31], [32], [34], [35], [36], [41], [42]. Remarkably, the new way of treating and understanding nonholonomic systems brings new methods for investigating concrete examples of nonholonomic systems, either with linear constraints (see [19]), or with nonlinear constraints (see [50] for problems of mechanics and engineering and [38] for applications in the special relativity theory).

The aim of the present article is to survey, in a consistent way, some of the recent results on first order mechanical systems. After a brief introduction to the standard geometric theory of first order mechanical systems in jet bundles (to be found e.g. in [29] or [40]) we turn to include nonholonomic constraints into the picture. We are concerned with the geometry of nonholonomic constraints, equations of motion of nonholonomic systems on constraint manifolds and their geometric meaning, including also the case of “implicit equations”, a nonholonomic variational principle, symmetries of nonholonomic systems and a nonholonomic Noether theorem, and finally we discuss regularity of nonholonomic equations, and nonholonomic

Hamilton equations. We note that there are also other interesting topics studied within nonholonomic mechanics, not included in this article, as e.g. the inverse variational problem in the nonholonomic setting [3], [39], nonholonomic reduction in presence of symmetries [4], [5], [8], integrability of nonholonomic systems and Hamilton-Jacobi theory [1], nonholonomic mechanics on Lie algebroids [11], [17], [44], etc.

2 Mechanical systems in jet bundles

Compared to the classical approach, geometric methods bring a new quality into the study of mechanical systems. The geometric language leads to an elegant and transparent formulation of results. It is important that concepts and formulas can be introduced in an *intrinsic* (coordinate independent) form: this is not only convenient for computations, but clarifies the geometric content and enables to distinguish between local and global results.

In this section we introduce structures for mechanics on fibred manifolds. We shall deal with both Lagrangian and nonconservative, generally time-dependent systems, the dynamics of which is described by systems of second order ordinary differential equations. In our approach, geometric concepts related with differential equations on manifolds play a central role. For more detailed exposition we refer to [21], [23], [48], and especially to the book [28] devoted to higher-order mechanics.

2.1 Basic structures

Throughout the paper we consider smooth manifolds and mappings. In coordinate formulas summation over repeated indices applies.

A smooth mapping $Y \rightarrow X$ between differentiable manifolds is called *submersion* if its rank is equal to $\dim X$ at each point $y \in Y$. A *surjective* submersion $\pi : Y \rightarrow X$ is called a *fibred manifold*. The manifold X is called *base*, Y *total space*, and the map π itself *projection*. The submanifold $\pi^{-1}(x)$ of Y , where $x \in X$, is called *fibre* over x . In case that all the fibres are diffeomorphic to each other, we speak about a *bundle* over X .

We shall consider fibred manifolds where $\dim X = 1$. This means that if X is connected, it is diffeomorphic either to \mathbb{R} or S^1 . We denote $\dim Y = m + 1$, hence m denotes the dimension of the fibres. From the definition of submersion it follows that to every point $y \in Y$ there exists a chart (V, ψ) on Y and (U, φ) on X such that V is a neighbourhood of y , $U = \pi(V)$, and the coordinate functions are of the form $\varphi = (t)$, $\psi = (t, q^\sigma)$, $1 \leq \sigma \leq m$. Charts of this kind are called *fibred charts*. Mostly we shall assume that $X = \mathbb{R}$: in this case we choose t on \mathbb{R} to be a global coordinate.

When dealing with dynamics of mechanical systems, we are concerned with a special kind of mappings between the base and the total space, called *sections*. By a section of the fibred manifold $\pi : Y \rightarrow X$ one means a (smooth) mapping $\gamma : X \rightarrow Y$, defined possibly on an open subset W of X , such that $\pi \circ \gamma = \text{id}_W$. Also, it is necessary to work with quantities dependent on first or higher derivatives of the corresponding sections. A precise mathematical setting is based on the concept of a *jet manifold*. We say that sections γ_1 and γ_2 defined on an open set $W \subset X$ have *contact of order one* at a point $x \in W$ if $\gamma_1(x) = \gamma_2(x)$, and if there is a fibred chart

around $\gamma_1(x) = \gamma_2(x)$ such that the derivatives of the components $\gamma_1^\sigma = q^\sigma \gamma_1 \varphi^{-1}$ and $\gamma_2^\sigma = q^\sigma \gamma_2 \varphi^{-1}$ of the sections γ_1 and γ_2 at the point $\varphi(x)$ coincide. The latter condition does not depend on the choice of fibred coordinates. In this way there arises an equivalence relation: the equivalence class can be easily visualized as a family of sections passing through the same point $y \in Y$ and possessing the same tangent vector. The equivalence class containing a section γ is called the *1-jet* of γ at x and is denoted by $J_x^1 \gamma$. Collecting all the equivalence classes for all the points $x \in X$ one obtains a set naturally endowed with a structure of a smooth manifold of dimension $2m + 1$, denoted by $J^1 Y$, and called *the first jet prolongation* of the fibred manifold $\pi : Y \rightarrow X$. Moreover, the manifold $J^1 Y$ is fibred over X (the fibred projection is denoted by π_1) as well as over Y (with the projection denoted by $\pi_{1,0}$). Consequently, one has on $J^1 Y$ coordinates, associated with fibred coordinates on Y . They are denoted by $(t, q^\sigma, \dot{q}^\sigma)$. The construction can be easily generalized to obtain *higher-order jets*: For every $x \in X$ one considers equivalence classes of sections passing through the same point over x and having at x the same derivatives up to the order r . In this way one gets a manifold $J^r Y$, called the *manifold of r -jets of local sections of π* , or briefly the *r -jet prolongation of π* . Similarly as in the first-order case, one has on $J^r Y$ coordinates naturally associated with fibred coordinates on Y denoted by $(t, q^\sigma, q_1^\sigma, q_2^\sigma, \dots, q_r^\sigma)$. Instead of q_1^σ and q_2^σ one often writes \dot{q}^σ and \ddot{q}^σ . From the definition of $J_x^r \gamma$ (which is a point in $J^r Y$) one can see that the values of the coordinate functions at $J_x^r \gamma$ can be regarded as the coefficients of the r -th order Taylor polynomial of the mapping γ around x . The manifold $J^r Y$ is fibred over X , Y , and all $J^s Y$, $s = 1, \dots, r - 1$. The corresponding projections are denoted by $\pi_r : J^r Y \rightarrow X$, $\pi_{r,0} : J^r Y \rightarrow Y$, $\pi_{r,s} : J^r Y \rightarrow J^s Y$, where $s < r$. For simplicity of notations, we also write $J^0 Y = Y$. In this paper we mostly use the first and second jet prolongations, $J^1 Y$ and $J^2 Y$.

If γ is a section of $\pi : Y \rightarrow X$ then the mapping $x \rightarrow J_x^r \gamma$ is a section of the fibred manifold $\pi_r : J^r Y \rightarrow X$; it is called the *r -jet prolongation* of γ and denoted by $J^r \gamma$. It is important to note that a section of π_r need not be of the form of an r -jet prolongation of a section of π . A section δ of π_r such that $\delta = J^r \gamma$ is called *holonomic*. For example, in fibred coordinates, a section of $J^1 Y$ is a mapping $\delta(t) = (t, f^\sigma(t), g^\sigma(t))$ while a holonomic section takes the form $J^1 \gamma(t) = (t, f^\sigma(t), df^\sigma/dt)$.

Remark 1. Classical mechanics is often modeled on fibred manifolds of the form $\pi : \mathbb{R} \times M \rightarrow \mathbb{R}$, where M is a manifold of dimension m (called the *configuration space*). In this case $J^1 Y = \mathbb{R} \times TM$, $J^2 Y = \mathbb{R} \times T^2 M$ and sections of π are *graphs of curves* $c : \mathbb{R} \rightarrow M$.

In fibred manifolds, there are distinguished vector fields and differential forms, adapted to the fibred and prolongation structure.

A vector field ξ on Y is called *π -projectable* if there exists a vector field ξ_0 on X such that $T\pi.\xi = \xi_0 \circ \pi$, and *π -vertical* if it projects onto a zero vector field on X , i.e., $T\pi.\xi = 0$. In fibred coordinates, projectable vector fields have their $\partial/\partial t$ component dependent on t only, and vertical vector fields have this component equal to zero.

Similarly one defines a $\pi_{r,s}$ -projectable or a $\pi_{r,s}$ -vertical vector field on $J^r Y$, where $r > s$.

Local flows of projectable vector fields transfer sections into sections. Consequently, π -projectable vector fields on Y can be naturally prolonged to vector fields on $J^r Y$. The procedure is as follows: Let ξ be a π -projectable vector field, ξ_0 its projection, and denote $\{\phi_u\}$ and $\{\phi_{0u}\}$ the corresponding local one-parameter groups. For every u , the mapping ϕ_u is an isomorphism of the fibred manifold π meaning that $\pi \circ \phi_u = \phi_{0u} \circ \pi$. Then for every section γ , the composition $\gamma_u = \phi_u \circ \gamma \circ \phi_{0u}^{-1}$ is again a section and we can define the r -jet prolongation of ϕ_u by $J^r \phi_u(J_x^r \gamma) = J_{\phi_{0u}(x)}^r(\phi_u \gamma \phi_{0u}^{-1})$. Then $J^r \phi$ is a local flow corresponding to a vector field on $J^r Y$, denoted by $J^r \xi$ and called the r -jet-prolongation of ξ . The vector field $J^r \xi$ is both π_r -projectable and $\pi_{r,s}$ -projectable for $0 \leq s < r$, and its π_r -projection, (resp. $\pi_{r,s}$ -projection) is ξ_0 (resp. ξ , resp. $J^s \xi$, $1 \leq s \leq r-1$). In fibred coordinates, where

$$\xi = \xi^0(t) \frac{\partial}{\partial t} + \xi^\sigma(t, q^\nu) \frac{\partial}{\partial q^\sigma}, \quad (1)$$

one has for $1 \leq k \leq r$

$$J^r \xi = \xi^0 \frac{\partial}{\partial t} + \xi^\sigma \frac{\partial}{\partial q^\sigma} + \sum_{k=1}^r \xi_k^\sigma \frac{\partial}{\partial q_k^\sigma}, \quad \text{where} \quad \xi_k^\sigma = \frac{d\xi_{k-1}^\sigma}{dt} - q_k^\sigma \frac{d\xi^0}{dt}. \quad (2)$$

Above

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma} + \ddot{q}^\sigma \frac{\partial}{\partial \dot{q}^\sigma} + \ddot{\ddot{q}}^\sigma \frac{\partial}{\partial \ddot{q}^\sigma} + \dots \quad (3)$$

denotes the total derivative operator.

A differential k -form η on $J^r Y$ is called π_r -horizontal (resp. $\pi_{r,s}$ -horizontal) if it vanishes whenever at least one of its arguments is a π_r -vertical (resp. $\pi_{r,s}$ -vertical) vector field. A k -form η on $J^r Y$ is called *contact* if for every section γ of π

$$J^r \gamma^* \eta = 0. \quad (4)$$

Putting

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma dt, \quad \dot{\omega}^\sigma = d\dot{q}^\sigma - \ddot{q}^\sigma dt, \quad \dots, \quad \omega_{r-1}^\sigma = dq_{r-1}^\sigma - q_r^\sigma dt \quad (5)$$

$1 \leq \sigma \leq m$, we obtain a family of local contact 1-forms on $J^r Y$. Remarkably, the *contact ideal* on $J^r Y$ is locally generated by these one-forms and their exterior derivatives. We also note that one-forms (5) can be completed to a *basis of linear forms*

$$(dt, \omega^\sigma, \dots, \dot{\omega}_{r-1}^\sigma, dq_r^\sigma) \quad (6)$$

well adapted to the structure of $J^r Y$. Working in coordinates, it is much more convenient to use this basis instead of the canonical basis $(dt, dq^\sigma, \dots, dq_r^\sigma)$.

We have an important property of differential forms in jet bundles: Every k -form η on $J^r Y$, if lifted to $J^{r+1} Y$, admits a unique and invariant decomposition into two parts such that in the adapted basis the first and the second part contains

wedge products of exactly $k - 1$ and k basic contact forms (5), respectively. We write

$$\pi_{r+1,r}^* \eta = h\eta + p_1\eta, \quad \pi_{r+1,r}^* \eta = p_{k-1}\eta + p_k\eta \quad (7)$$

if $k = 1$ and $k \geq 2$, respectively. $h\eta$ is a horizontal form on $J^{r+1}Y$, called the *horizontal part* of η , $p_i\eta$ is then called the *i-contact part* of η (we also speak about a *i-contact form*). Note that for a function f we get $hdf = \frac{df}{dt} dt$.

2.2 Fibred mechanics

In what follows, let us consider a fibred manifold $\pi : Y \rightarrow \mathbb{R}$ with $\dim Y = m + 1$, and fibred coordinates denoted by (t, q^σ) , where t is a global coordinate on \mathbb{R} .

A *dynamical form* of order r is defined to be a 2-form E on J^rY which is 1-contact, and horizontal with respect to the projection onto Y . In fibred coordinates $E = E_\sigma \omega^\sigma \wedge dt$, where E_1, \dots, E_m are functions on an open subset of J^rY . Dynamical forms are appropriate objects to represent systems of *ordinary differential equations* on manifolds. In this paper we shall be interested in (at most) *second-order* ODE's. Then E is defined on J^2Y and its components E_σ depend upon $t, q^\nu, \dot{q}^\nu, \ddot{q}^\nu$ ($1 \leq \sigma, \nu \leq m$). Equation $E = 0$ determines a submanifold of J^2Y of codimension m . A section γ of π is called a *path* of E if it satisfies $E \circ J^2\gamma = 0$. In fibred coordinates this is a system of m (possibly *implicit*) second order ODE's

$$E_\sigma \left(t, \gamma^\nu(t), \frac{d\gamma^\nu}{dt}, \frac{d^2\gamma^\nu}{dt^2} \right) = 0, \quad 1 \leq \sigma \leq m \quad (8)$$

for components of γ .

In what follows it will be sufficient to restrict to the case of so-called J^1Y -pertinent dynamical forms that are distinguished by a significant property: the corresponding *dynamics proceeds in the manifold J^1Y* (sometimes called the *evolution space*).

Given a dynamical form E , we say that a 2-form α defined on an open subset $U \subset J^2Y$ is an *extension* of E on U if $E|_U = p_1\alpha$. E is called *pertinent with respect to J^1Y* if around every point in J^2Y it has a local extension α that is projectable onto an open subset of J^1Y . A second order dynamical form E is *pertinent with respect to J^1Y* if and only if

$$E_\sigma = A_\sigma(t, q^\rho, \dot{q}^\rho) + B_{\sigma\nu}(t, q^\rho, \dot{q}^\rho) \ddot{q}^\nu. \quad (9)$$

Then every local projectable extension of E takes the form

$$\alpha = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu} \omega^\sigma \wedge dq^\nu + F, \quad (10)$$

where F is a 2-contact 2-form on an open subset of J^1Y . The class $[\alpha]$ of 2-forms (10) is then called the (first-order) *Lepage class* of the dynamical form E . The corresponding ODE's are *affine in the second derivatives (accelerations)*,

$$A_\sigma \left(t, \gamma^\rho(t), \frac{d\gamma^\rho}{dt} \right) + B_{\sigma\nu} \left(t, \gamma^\rho(t), \frac{d\gamma^\rho}{dt} \right) \frac{d^2\gamma^\nu}{dt^2} = 0. \quad (11)$$

Equations (11) can be represented in a form of a *Pfaffian system*, or vector distribution Δ on J^1Y , called *dynamical distribution* of E [27], [32], as follows:

$$\begin{aligned} \Delta &= \text{span}\{i_\xi\alpha \mid \text{where } \xi \text{ runs over all vertical vector fields on } J^1Y\} \\ &= \text{span}\{A_\sigma dt + F_{\sigma\nu}\omega^\nu + B_{\sigma\nu}d\dot{q}^\nu, B_{\sigma\nu}\omega^\sigma\}, \end{aligned} \quad (12)$$

where $F_{\sigma\nu}$ are the components of F .

Remarkably, Δ need not have a constant rank, and need not be completely integrable. We say that a dynamical form E is *regular* if around every point in J^1Y it has a dynamical distribution which is everywhere of *rank one* [26], [27]. It can be shown that *every regular dynamical form E has a unique global rank one dynamical distribution*. It is locally annihilated by $2m$ one-forms ω^σ and $A_\sigma dt + B_{\sigma\nu}d\dot{q}^\nu$, or, equivalently, *spanned by one vector field*

$$\zeta = \frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma} - B^{\sigma\nu} A_\nu \frac{\partial}{\partial \dot{q}^\sigma}. \quad (13)$$

This geometrical model is used to study non-conservative time-dependent mechanical systems, to classify ODE's according to their dynamical properties, to study structure of solutions of both regular ODE's and ODE's "in implicit form" (non-representable by a vector field), to generalize Hamilton's equations to non-variational and non-regular equations, to study transformations of ODE's, and symmetries and first integrals, to develop exact integration methods based on symmetries and transformations (eg. generalized Liouville and Jacobi theorem of the calculus of variations), to study relations between variational and non-variational equations (the inverse problem of the calculus of variations, the problem of existence of variational multipliers), and much more (see eg. [24], [28], [32], [37], [40] and references therein).

Let us now turn to *variational equations*.

By a *Lagrangian* of order r , $r \geq 1$, we mean a *horizontal form* λ on J^rY . In fibred coordinates a Lagrangian reads $\lambda = L dt$, where L is a function on an open subset of J^rY . To every Lagrangian there exists a *unique* 1-form θ_λ on $J^{2r-1}Y$ such that $h\theta_\lambda = \lambda$ and $p_1d\theta_\lambda$ is a dynamical form [21]. The 1-form θ_λ is called the *Lepage equivalent* or the *Cartan form* of λ , and the related dynamical form $E_\lambda = p_1d\theta_\lambda$ is then called the *Euler-Lagrange form* of λ . We shall be mostly interested in first order Lagrangians. In this case $\lambda = L dt$ where L depends upon t, q^σ , and \dot{q}^σ , $1 \leq \sigma \leq m$, and

$$\theta_\lambda = L dt + \frac{\partial L}{\partial \dot{q}^\sigma} \omega^\sigma, \quad E_\lambda = \left(\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\sigma} \right) \omega^\sigma \wedge dt. \quad (14)$$

The components of E_λ are familiar as *Euler-Lagrange expressions*, and equations for paths of an Euler-Lagrange form are called *Euler-Lagrange equations*. We note that the same Euler-Lagrange form can arise from different Lagrangians, possibly even of different orders. Such Lagrangians are called *equivalent*; it is known that Lagrangians λ_1 of order r and λ_2 of order $k \geq r$ are equivalent iff around every point there is a function f of order $k-1$ such that $\lambda_2 = \lambda_1 + hdf$ (where, precisely, on the place of λ_1 one has to consider its lift by the projection $\pi_{k,r}$).

Let us recall some important properties of Euler-Lagrange dynamical forms [26], [28]. First, every second order Euler-Lagrange form is J^1Y -pertinent, since the Euler-Lagrange expressions are affine in the second derivatives. This means that E_λ on J^2Y is represented by a Lepage class $[\alpha]$ defined on J^1Y . Moreover, the Lepage class has a distinguished representative, independent upon the choice of a particular Lagrangian for $E = E_\lambda$, as follows:

Theorem 2. *Given an Euler-Lagrange form E on J^2Y , the associated Lepage class contains a unique global and closed representative α_E , defined on J^1Y . The 2-form α_E can be expressed by means of the Euler-Lagrange expressions as follows:*

$$\begin{aligned}\alpha_E &= E_\sigma \omega^\sigma \wedge dt + \frac{1}{4} \left(\frac{\partial E_\sigma}{\partial \dot{q}^\nu} - \frac{\partial E_\nu}{\partial \dot{q}^\sigma} \right) \omega^\sigma \wedge \omega^\nu + \frac{\partial E_\sigma}{\partial \dot{q}^\nu} \omega^\sigma \wedge \dot{\omega}^\nu \\ &= A_\sigma \omega^\sigma \wedge dt + \frac{1}{4} \left(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} - \frac{\partial A_\nu}{\partial \dot{q}^\sigma} \right) \omega^\sigma \wedge \omega^\nu + B_{\sigma\nu} \omega^\sigma \wedge d\dot{q}^\nu,\end{aligned}\tag{15}$$

where

$$A_\sigma = \frac{\partial L}{\partial \dot{q}^\sigma} - \frac{d'}{dt} \frac{\partial L}{\partial \dot{q}^\sigma}, \quad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}.\tag{16}$$

Moreover, for every (possibly local) Lagrangian λ of order $r \geq 1$ for E , the Cartan form θ_λ satisfies the following property: $d\theta_\lambda$ is projectable onto an open set in J^1Y , and on this set,

$$d\theta_\lambda = \alpha_E.\tag{17}$$

Above,

$$\frac{d'}{dt} = \frac{d}{dt} - \dot{q}^\nu \frac{\partial}{\partial \dot{q}^\nu} = \frac{\partial}{\partial t} + \frac{\partial}{\partial \dot{q}^\nu} \dot{q}^\nu\tag{18}$$

denotes so-called ‘‘cut total derivative’’ applied to functions on J^1Y .

Remarkably, also the converse holds true [26], [28], giving us a *one-to-one* relationship between *variational equations* and a class of *closed 2-forms*:

Theorem 3. *Let α be a 2-form on J^1Y such that $E = p_1\alpha$ is a dynamical form. If α is closed then E is locally variational, meaning that around every point in J^1Y there exists a Lagrangian λ such that over the domain of λ , $E = E_\lambda$.*

We note that the existence of a *global* Lagrangian for a locally variational dynamical form is related with topological properties of the manifold Y [23].

Euler-Lagrange equations can be obtained from the *variational principle*. Let us briefly recall the procedure. Denote by $\mathcal{S}_{[a,b]}(\pi)$ the set of sections of π with domains around an interval $[a, b] \subset \mathbb{R}$. Given a Lagrangian λ on J^1Y , consider the function

$$\mathcal{S}_{[a,b]}(\pi) \ni \gamma \rightarrow \int_a^b J^1\gamma^*\lambda = \int_a^b J^1\gamma^*\theta_\lambda \in \mathbb{R}\tag{19}$$

called the *action function* of λ over $[a, b]$. To get a correct concept of *variation* (one-parametric deformation) of a section γ , one has to restrict to consider π -*projectable* vector fields on Y : if ξ is a projectable vector field on Y with projection ξ_0 , and $\{\phi_u\}$, resp. $\{\phi_{0u}\}$ are the corresponding local one-parameter groups, we

get a one-parameter family $\{\gamma_u\}$ of *sections* where $\gamma_u = \phi_u \gamma \phi_{0u}^{-1}$ is defined in a neighbourhood of $\phi_{0u}([a, b]) \subset \mathbb{R}$, called *variation of the section γ induced by ξ* . The arising function

$$\mathcal{S}_{[a,b]}(\pi) \ni \gamma \rightarrow \left(\frac{d}{du} \int_{\phi_{0u}([a,b])} J^1 \gamma_u^* \lambda \right)_{u=0} = \int_a^b J^1 \gamma^* \mathcal{L}_{J^1 \xi} \lambda \in \mathbb{R} \quad (20)$$

is called the *first variation of the action function* of the Lagrangian λ over the interval $[a, b]$, *induced by ξ* . The *First Variation Formula* is a splitting of the above integral into a sum of two terms such that the first one does not depend upon “derivations of variations” (the Euler-Lagrange term) and the second one is a boundary term. With the Cartan form θ_λ the decomposition is available *directly* (without the integration by parts procedure), and in an *invariant* way [21]:

$$\begin{aligned} \int_a^b J^1 \gamma^* \mathcal{L}_{J^1 \xi} \lambda &= \int_a^b J^1 \gamma^* \mathcal{L}_{J^1 \xi} \theta_\lambda \\ &= \int_a^b J^1 \gamma^* i_{J^1 \xi} d\theta_\lambda + \int_a^b d(i_{J^1 \xi} \theta_\lambda \circ J^1 \gamma) \\ &= \int_a^b J^2 \gamma^* i_{J^2 \xi} E_\lambda + \text{the above boundary term.} \end{aligned} \quad (21)$$

A section γ of π is called an *extremal of λ on $[a, b]$* if the first variation of the action of λ on the interval $[a, b]$ vanishes for every vertical vector field ξ on Y with the support in $\pi^{-1}([a, b])$ (such a vector field is often called a *fixed-endpoints variation*). γ is called *extremal of λ* if it is an extremal on every interval $[a, b] \subset \mathbb{R}$.

With help of the First Variation Formula one obtains necessary and sufficient conditions for extremals as follows [21]:

Theorem 4. *Let λ be a Lagrangian on $J^1 Y$. A section γ of π is an extremal of λ if and only if γ satisfies one of the following equivalent conditions:*

- (1) $E_\lambda \circ J^2 \gamma = 0$, i.e. γ is a path of the Euler-Lagrange form of λ .
- (2) For every vertical vector field ξ on Y , $J^1 \gamma^* i_{J^1 \xi} d\theta_\lambda = 0$.
- (3) In every fibred chart γ satisfies the system of m second-order ordinary differential equations

$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\sigma} = 0, \quad 1 \leq \sigma \leq m. \quad (22)$$

Notice the meaning of condition (2) of the above theorem: it is a geometric interpretation of the Euler-Lagrange equations in terms of a dynamical distribution of the dynamical form $E = E_\lambda$. Namely, accounting Theorem 2 we can see that every Euler-Lagrange dynamical form possesses a distinguished *global* dynamical distribution related with the Lepage 2-form α_E ,

$$\Delta_E = \text{annih}\{i_\xi \alpha_E \mid \xi \text{ runs over all vertical vector fields on } J^1 Y\} \quad (23)$$

called the *Euler-Lagrange distribution* [25], [27]. By condition (2) of the above theorem, extremals (solutions of the Euler-Lagrange equations) are *holonomic* integral sections of the distribution Δ_E .

Equations for (all) integral sections of the Euler-Lagrange distribution Δ_E , i.e. equations

$$\delta^* i_\xi \alpha_E = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \text{ on } J^1Y \quad (24)$$

for sections δ of the fibred manifold $\pi_1 : J^1Y \rightarrow \mathbb{R}$ are then called *Hamilton equations* [16], [27]. In case that $\text{rank } \Delta_E = 1$, i.e., E is *regular* (as a dynamical form), $\text{rank } \alpha_E$ is maximal (equal $2m$), and the Euler-Lagrange distribution is spanned by one vector field ζ . It is, up to a multiplier f , a unique solution of the equation $i_\zeta \alpha_E = 0$, and is called *Euler-Lagrange field* [16], or *Hamiltonian vector field*. The condition for regularity can be expressed by means of Lagrangians as follows:

(i) If Δ_E is defined on J^1Y (this means that the Euler-Lagrange equations are nontrivially second-order equations) the regularity condition takes the form

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}\right) \neq 0. \quad (25)$$

(ii) If α_E is projectable onto Y , i.e. Δ_E is defined on Y , then the regularity condition takes the form [26]

$$\det\left(\frac{\partial^2 L}{\partial q^\sigma \partial \dot{q}^\nu} - \frac{\partial^2 L}{\partial \dot{q}^\sigma \partial q^\nu}\right) \neq 0. \quad (26)$$

This is the case when the Euler-Lagrange equations are first-order equations, i.e. the corresponding Lagrangians are *affine* functions in the velocities.

For regular Lagrangians, i.e. satisfying either (25) or (26), the Cauchy problem has a unique solution, i.e., *through every point in the dynamical space (J^1Y , respectively Y) there passes a unique maximal solution of the Euler-Lagrange equations.*

The Cartan form θ_λ takes the coordinate form (14). Expressing the same form in the canonical basis $(dt, dq^\sigma, d\dot{q}^\sigma)$ one obtains

$$\theta_\lambda = -H dt + p_\sigma dq^\sigma, \quad \text{where } p_\sigma = \frac{\partial L}{\partial \dot{q}^\sigma}, \quad H = -L + p_\sigma \dot{q}^\sigma. \quad (27)$$

If L is not affine in velocities (meaning that Δ_E is defined on J^1Y) then the “momenta” p_σ are (local) functions on J^1Y . If, moreover, L is regular, we get on J^1Y local coordinates (t, q^σ, p_σ) , called *Legendre coordinates*. In these coordinates, Hamilton equations (24) take the “canonical form”

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q^\sigma}, \quad \frac{dq^\sigma}{dt} = \frac{\partial H}{\partial p_\sigma}. \quad (28)$$

Up to now we have been interested in the meaning of the first term in the decomposition of the first variation (21). The second term, however, is important as well, since it is connected with *conservation laws*.

We say that a π -projectable vector field ξ on Y is a *point symmetry* of a Lagrangian λ if

$$\mathcal{L}_{J^1\xi}\lambda = 0, \quad (29)$$

and a *generalized point symmetry* of λ if it is a point symmetry of its Euler-Lagrange form, i.e.,

$$\mathcal{L}_{J^2\xi}E_\lambda = 0. \quad (30)$$

Within the terminology of the classical calculus of variations, point symmetries of a Lagrangian correspond to infinitesimal transformations that leave invariant the action integral; similarly point symmetries of the Euler-Lagrange form correspond to transformations leaving the action integral invariant “up to a divergence”. Equation (29) and (30) is called *Noether equation* and *Noether–Bessel-Hagen equation*, respectively. It is known that every point symmetry of λ is a point symmetry of E_λ [22].

Substituting into the First Variation Formula (21) the symmetry condition and taking account of the extremal condition (2) in Theorem 4 we immediately obtain the following famous result [45], [22]:

Theorem 5. (Noether Theorem)

- (1) Assume that a π -projectable vector field ξ on Y is a point symmetry of a Lagrangian λ . Then, along every extremal of λ , the function $F = i_{J^1\xi}\theta_\lambda$ is constant.
- (2) Assume that a π -projectable vector field ξ on Y is a generalized point symmetry of a Lagrangian λ . Then (locally) $\mathcal{L}_{J^1\xi}\lambda = hdf$ for a function f , and along every extremal of λ , the function $F = i_{J^1\xi}\theta_\lambda - f$ is constant.

Within fibred mechanics one can easily consider also Lagrangian systems subject to external forces that need not be variational (so-called *nonconservative systems*) [32]. More precisely, by a *mechanical system* on a fibred manifold π we shall mean a pair (λ, Φ) where λ is a Lagrangian on J^1Y and Φ is a *first-order dynamical form*, called a *force*. It is generally assumed that λ is not affine in velocities (provides Euler-Lagrange equations that are nontrivially of order two). The corresponding dynamical form is then $E = E_\lambda - \pi_{2,1}^*\Phi$, and equations for paths of E take the form

$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\sigma} = \Phi_\sigma, \quad 1 \leq \sigma \leq m. \quad (31)$$

The corresponding Lepage class is represented by the Lepage 2-form

$$\begin{aligned} \alpha &= A_\sigma \omega^\sigma \wedge dt + \frac{1}{4} \left(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} - \frac{\partial A_\nu}{\partial \dot{q}^\sigma} \right) \omega^\sigma \wedge \omega^\nu + B_{\sigma\nu} \omega^\sigma \wedge d\dot{q}^\nu \\ &= d\theta_\lambda - \Phi - \frac{1}{4} \left(\frac{\partial \Phi_\sigma}{\partial \dot{q}^\nu} - \frac{\partial \Phi_\nu}{\partial \dot{q}^\sigma} \right) \omega^\sigma \wedge \omega^\nu, \end{aligned} \quad (32)$$

where A_σ and $B_{\sigma\nu}$ are defined as above by $E_\sigma = A_\sigma + B_{\sigma\nu} \ddot{q}^\nu$ and take the form

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma} \dot{q}^\nu - \Phi_\sigma, \quad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}. \quad (33)$$

The motion is described by the dynamical distribution $\Delta = \text{annih}\{i_\xi\alpha\}$ where ξ runs over all vertical vector fields on J^1Y .

Using theorems 1 and 2 we can see that the force Φ is conservative (potential) if and only if it is variational (as a first order dynamical form), i.e. if and only if the 2-form α is closed, hence

$$\Phi + \frac{1}{4} \left(\frac{\partial \Phi_\sigma}{\partial \dot{q}^\nu} - \frac{\partial \Phi_\nu}{\partial \dot{q}^\sigma} \right) \omega^\sigma \wedge \omega^\nu \quad (34)$$

is closed. It can be easily verified that Φ satisfies *Helmholtz conditions* [18]. Recall that in this case the Helmholtz conditions take the form

$$\begin{aligned} \frac{\partial \Phi_\sigma}{\partial \dot{q}^\nu} + \frac{\partial \Phi_\nu}{\partial \dot{q}^\sigma} &= 0, \\ \frac{\partial \Phi_\sigma}{\partial q^\nu} - \frac{\partial \Phi_\nu}{\partial q^\sigma} + \frac{d}{dt} \frac{\partial \Phi_\nu}{\partial \dot{q}^\sigma} &= 0. \end{aligned} \quad (35)$$

Since we assume the force Φ be of the first order, the latter condition gives

$$\frac{\partial^2 \Phi_\sigma}{\partial \dot{q}^\nu \partial \dot{q}^\rho} = 0, \quad (36)$$

i.e. that the force is affine in velocities,

$$\Phi_\sigma = a_{\sigma\rho} \dot{q}^\rho + b_\sigma, \quad (37)$$

and the first condition (35) then immediately means that the matrix of the coefficients $(a_{\sigma\rho})$ is skew-symmetric. Substituting now (37) to the second condition (35) we obtain the Helmholtz conditions for a force Φ in the familiar form

$$\begin{aligned} a_{\sigma\nu} &= -a_{\nu\sigma}, \\ \frac{\partial a_{\sigma\rho}}{\partial q^\nu} + \frac{\partial a_{\nu\sigma}}{\partial q^\rho} + \frac{\partial a_{\rho\nu}}{\partial q^\sigma} &= 0, \\ \frac{\partial b_\sigma}{\partial q^\nu} - \frac{\partial b_\nu}{\partial q^\sigma} + \frac{\partial a_{\nu\sigma}}{\partial t} &= 0. \end{aligned} \quad (38)$$

We note that conditions (38) mean that Φ is a *Lorentz-type force*. From the geometric point of view, (38) indeed are exactly the closedness conditions for the 2-form (34).

3 Holonomic constraints

Holonomic constraints on the motion appear in numerous applications in physics and engineering. Due to their importance they have been considered within analytical mechanics since the very beginning going back to Lagrange and Hamilton.

In classical mechanics holonomic constraints are constraints on positions of particles; they may be time-independent (not explicitly depending on time), or time dependent. In local coordinates (q^1, \dots, q^m) in \mathbb{R}^m , holonomic constraints are given by a system of (algebraic) equations

$$u^a(t, q^\sigma) = 0, \quad a = 1, 2, \dots, k < m, \quad (39)$$

satisfying the rank condition

$$\text{rank}\left(\frac{\partial u^a}{\partial q^\sigma}\right) = k. \quad (40)$$

The latter condition means that constraint conditions (39) can be expressed in a form

$$q^{m-k+a} = w^a(t, q^1, \dots, q^{m-k}). \quad (41)$$

There are two ways for considering constrained motions:

- *External – with Lagrange multipliers:*

The influence of the constraint on the motion is modeled via an external attractive force, called *constraint force*, proportional to $\text{grad } u$. *Equations of motion* of a Lagrangian system L subject to constraints (39) then take the form

$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\sigma} = -\mu_a \frac{\partial u^a}{\partial q^\sigma}, \quad 1 \leq \sigma \leq m, \quad (42)$$

where μ_a , $1 \leq a \leq k$, are Lagrange multipliers. Solutions to the problem are both curves in \mathbb{R}^m satisfying simultaneously the constraint conditions and the above motion equations, and Lagrange multipliers as functions of time.

Remarkably, equations of motion (42) can be obtained as standard Euler-Lagrange equations from the Lagrangian

$$\hat{L} = L + \mu_a u^a. \quad (43)$$

- *Internal (geometric) – without Lagrange multipliers:*

The point is that the constraints given by equations (39) with the accompanying rank condition have the *geometric meaning* of a *submanifold* of codimension k in $\mathbb{R} \times \mathbb{R}^m$. In terms of the fibred manifolds terminology, the constraint conditions define a fibred submanifold $\pi : \bar{Y} \rightarrow \mathbb{R}$ of the fibred manifold $pr_1 : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$. If we denote by ι the canonical embedding of the constraint submanifold \bar{Y} into $\mathbb{R} \times \mathbb{R}^m$, and by (t, q^s) , $1 \leq s \leq m-k = \dim \bar{Y} - 1$, adapted coordinates on \bar{Y} , then *equations of motion* of a Lagrangian system $L(t, q^\sigma, \dot{q}^\sigma)$ subject to constraints (39) take the form of “standard” Euler-Lagrange equations

$$\frac{\partial \bar{L}}{\partial q^s} - \frac{d}{dt} \frac{\partial \bar{L}}{\partial \dot{q}^s} = 0, \quad 1 \leq s \leq m-k, \quad (44)$$

for the Lagrangian $\bar{L} = L \circ J^1 \iota$ on the manifold $J^1 \bar{Y}$. It is essential that the latter equations and the equations with Lagrange multipliers above are, as equations for sections passing in the constraint submanifold (i.e. satisfying the constraint conditions), *equivalent*.

In analytical mechanics the manifold \bar{Y} is called “space of events”, (q^s) are “generalized coordinates”, $m-k$ is the “number of degrees of freedom”, $J^1 \bar{Y}$ is called “evolution space” or “phase space”, and \bar{L} is the “constrained Lagrangian”.

Due to the geometric nature of holonomic constraints, holonomic systems are very well understood within the fibred mechanics setting presented in the previous section. Indeed, they completely fit with the general scheme of the theory - and this concerns both Lagrangian and nonconservative systems. In a full generality, if $\pi : Y \rightarrow X$ is a fibred manifold, a *holonomic constraint* in Y is a fibred submanifold $\bar{\pi} : \bar{Y} \rightarrow X$ of π . Given a dynamical form E on J^2Y , on $J^2\bar{Y}$ there arises dynamical form $\bar{E} = J^2\iota^*E$. In particular, given a Lagrangian λ on J^1Y we obtain a Lagrangian $\bar{\lambda} = J^1\iota^*\lambda$ on $J^1\bar{Y}$, and the Euler-Lagrange equations of $\bar{\lambda}$ come from the restricted Euler-Lagrange form $E_{\bar{\lambda}} = J^2\iota^*E_{\lambda}$.

Note that if $\pi : \mathbb{R} \times M \rightarrow \mathbb{R}$, and \bar{Y} is of the form $\mathbb{R} \times N$ where N is a submanifold in M we can speak about a *time-independent holonomic constraint*, otherwise $\bar{\pi} : \bar{Y} \rightarrow \mathbb{R}$ is a *time-dependent holonomic constraint* in $\mathbb{R} \times M$.

4 Nonholonomic systems on constraint manifolds

In what follows we shall consider constraints on the motion that depend on time, positions and velocities, called *nonholonomic constraints*. In this case equations defining a constraint are *first order differential equations*. In terms of jet bundles constraints with this property are *submanifolds of the first jet manifold*.

As above, let us consider a fibred manifold $\pi : Y \rightarrow \mathbb{R}$, where $\dim Y = m + 1$. Precisely speaking, by a *nonholonomic constraint* in J^1Y we shall mean a submanifold $Q \subset J^1Y$, fibred over Y . When appropriate, we shall use notation $\iota : Q \rightarrow J^1Y$ for the canonical embedding. A constraint of codimension k ($1 \leq k < m$) in J^1Y is locally defined by a *system of k first order ordinary differential equations*

$$f^a(t, q^\sigma, \dot{q}^\sigma) = 0, \quad 1 \leq a \leq k, \quad (45)$$

where the functions f^a satisfy the rank condition

$$\text{rank}\left(\frac{\partial f^a}{\partial \dot{q}^\sigma}\right) = k. \quad (46)$$

Due to (46), equations of the constraint take a normal form

$$\dot{q}^{m-k+a} = g^a(t, q^\sigma, \dot{q}^1, \dots, \dot{q}^{m-k}), \quad 1 \leq a \leq k. \quad (47)$$

Similarly as in the case of holonomic constraints we can approach the nonholonomic dynamics in two ways:

- *External - with Lagrange multipliers:*

The influence of the constraint on the motion should be modeled via a *constraint force*. The problem now, however, is that it is not clear how such a force should look like. In [9] Chetaev proposed the following formula for the constraint force:

$$\mathcal{F}_\sigma = -\mu_a \frac{\partial f^a}{\partial \dot{q}^\sigma}, \quad (48)$$

where μ_a , $1 \leq a \leq k$, are Lagrange multipliers. *Equations of motion* of a Lagrangian system L subject to nonholonomic constraints (45) then read

$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\sigma} = -\mu_a \frac{\partial f^a}{\partial \dot{q}^\sigma}, \quad 1 \leq \sigma \leq m, \quad (49)$$

and are called *Chetaev equations*. In this case, the integration problem means to solve a system of $m + k$ mixed first and second order ordinary differential equations (45) and (49) for m components $q^\sigma(t)$ of the nonholonomic curves and k Lagrange multipliers $\mu_a(t)$.

It should be stressed, that in this case, rather surprisingly, Chetaev equations do not arise as Euler-Lagrange equations from a Lagrangian analogous to (43), i.e.

$$\hat{L} = L + \mu_a f^a, \quad (50)$$

since equations for extremals of (50), called *vakonomic equations*, take a different form

$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\sigma} = -\mu_a \left(\frac{\partial f^a}{\partial q^\sigma} - \frac{d}{dt} \frac{\partial f^a}{\partial \dot{q}^\sigma} \right) - \frac{d\mu_a}{dt} \frac{\partial f^a}{\partial \dot{q}^\sigma}, \quad 1 \leq \sigma \leq m. \quad (51)$$

Solutions of Chetaev and vakonomic equations are different unless the constraints are *semiholonomic* (linear, integrable), satisfying the integrability conditions

$$f^a = \frac{du^a}{dt}. \quad (52)$$

Investigations of different examples indicated that nonholonomic dynamics obey Chetaev equations. On the other hand, vakonomic equations seem to be valuable in control theory.

- *Internal (geometric) – without Lagrange multipliers:*

The second approach explores the geometric meaning of nonholonomic constraints as submanifolds in jet bundles. In what follows, we shall present namely this model and the arising geometric structures, first considered in our paper [29]. Remarkably, within this setting *nonholonomic systems and their dynamics are described by geometric structures on a corresponding constraint submanifold*, which has the physical meaning of a *constrained phase space*. The dynamics are governed by so-called *reduced equations* which represent a system of $m - k$ second order ordinary differential equations for sections of the constraint submanifold (as expected no Lagrange multipliers enter in these equations). For the study of the constrained systems concepts and techniques of fibred mechanics can be directly used or quite easily generalized. In this way, nonholonomic mechanics is a direct extension of fibred mechanics and admits a straightforward generalization to higher order and field theory.

Moreover, within the geometric model there arises a *new possibility* to understand and study constrained systems. Indeed, one can distinguish *two different situations* [39]:

- the constrained system arises from an unconstrained system defined in a neighbourhood of the constraint
- an internally defined constrained system on the constraint manifold is given, without reference to the ambient space J^1Y ; in this case *a corresponding unconstrained system need not exist*.

4.1 Constraint submanifolds in jet bundles

Consider a constraint submanifold $\iota : Q \rightarrow J^1Y$ of codimension $k < m$. This means that we have fibred manifolds $\bar{\pi}_{1,0} : Q \rightarrow Y$ where $\bar{\pi}_{1,0}$ is the restriction of the projection $\pi_{1,0} : J^1Y \rightarrow Y$ to Q , and $\bar{\pi}_1 : Q \rightarrow X$, where $\bar{\pi}_1 = \pi_1|_Q$.

We define the *first prolongation* \hat{Q} of the constraint Q to be a submanifold in J^2Y , consisting of all points $J_x^2\gamma$ such that $J_x^1\gamma \in Q$, $x \in X$. Locally \hat{Q} is defined by the equations of the constraint and their derivatives:

$$f^a = 0, \quad \frac{df^a}{dt} = 0, \quad 1 \leq a \leq k, \quad (53)$$

respectively, in normal form,

$$\dot{q}^{m-k+a} = g^a, \quad \ddot{q}^{m-k+a} = \frac{dg^a}{dt}. \quad (54)$$

We also use notation $\hat{\iota} : \hat{Q} \rightarrow J^2Y$ for the corresponding canonical embedding. The manifold \hat{Q} is fibred over Q , Y and X , the fibred projections are simply restrictions of the corresponding canonical projections of the underlying fibred manifolds. We write $\bar{\pi}_2 : \hat{Q} \rightarrow X$, $\bar{\pi}_{2,1} : \hat{Q} \rightarrow Q$ and $\bar{\pi}_{2,0} : \hat{Q} \rightarrow Y$.

Usually we shall use on Q adapted coordinates (t, q^σ, \dot{q}^s) , where $1 \leq s \leq m-k$, and on \hat{Q} associated coordinates $(t, q^\sigma, \dot{q}^s, \ddot{q}^s)$, $1 \leq \sigma \leq m$, $1 \leq s \leq m-k$.

The contact ideal on Q respectively \hat{Q} , is locally generated by one-forms

$$\bar{\omega}^s = dq^s - \dot{q}^s dt, \quad \bar{\omega}^{m-k+a} = dq^{m-k+a} - g^a dt, \quad (55)$$

respectively,

$$\hat{\omega}^s = dq^s - \dot{q}^s dt, \quad \hat{\omega}^{m-k+a} = dq^{m-k+a} - g^a dt, \quad \hat{\omega}^s = d\dot{q}^s - \ddot{q}^s dt, \quad (56)$$

and their exterior derivatives.

Due to the existence of the contact structure on constraint manifolds, it is possible to prolong projectable vector fields from the total space Y to the constraint and to its prolongations. The procedure was described in [35] and is as follows:

Let ξ be a projectable vector field on Y . A vector field ζ on Q (resp. \hat{Q}) is called the *first* (resp. *second*) *constrained prolongation* of ξ , and is denoted by $J_c^1\xi$ (resp. $J_c^2\xi$), if ζ is a symmetry of the contact ideal on Q (resp. \hat{Q}) and projects onto ξ . It should be stressed that not every projectable vector field on Y admits a constrained prolongation; conditions and formulas can be found in [35].

Similarly as in the unconstrained case, for every q -form η on Q one has a *unique decomposition* into a sum of a $\bar{\pi}_2$ -horizontal form and i -contact forms, $i = 1, 2, \dots, q$, on \hat{Q} [35]; we write

$$\bar{\pi}_{2,1}^*\eta = \bar{h}\eta + \bar{p}_1\eta + \dots + \bar{p}_q\eta. \quad (57)$$

Applying this decomposition to (locally) exact one-forms on Q we get an invariant *splitting of the exterior derivative* d to the horizontal and contact part, $\bar{\pi}_{2,1}^*d = \bar{h}d + \bar{p}_1d$. The operator $\bar{h}d$ has the component

$$\frac{d_c}{dt} = \frac{\partial}{\partial t} + \dot{q}^s \frac{\partial}{\partial q^s} + g^a \frac{\partial}{\partial q^{m-k+a}} + \ddot{q}^s \frac{\partial}{\partial \dot{q}^s}, \quad (58)$$

and is called the *constraint total derivative*.

For convenience of notations we also put

$$\frac{d'_c}{dt} = \frac{\partial}{\partial t} + \dot{q}^s \frac{\partial}{\partial q^s} + g^a \frac{\partial}{\partial q^{m-k+a}}. \quad (59)$$

4.2 The canonical distribution

The most important object in the constraint geometry is the *canonical distribution* (also called *Chetaev bundle*) [29] (see also [43]). Remarkably, it is an internal object – a bundle naturally arising over every nonholonomic constraint. The canonical distribution gives a geometric meaning to *virtual displacements* in the space of positions and velocities, and to the concept of *reactive (Chetaev) forces*; for more details and introduction of a *nonholonomic D'Alembert principle* we refer to [29] and [34].

The *canonical distribution* for a nonholonomic constraint $Q \subset J^1Y$ is a corank k distribution \mathcal{C} on the manifold Q , where $k = \text{codim } Q$, locally annihilated by the system of k linearly independent 1-forms

$$\varphi^a = \left(\frac{\partial f^a}{\partial \dot{q}^\sigma} \circ \iota \right) \bar{\omega}^\sigma = \bar{\omega}^{m-k+a} - \sum_{s=1}^{m-k} \frac{\partial g^a}{\partial \dot{q}^s} \bar{\omega}^s, \quad 1 \leq a \leq k, \quad (60)$$

or, equivalently, locally spanned by the system of $2(m-k) + 1$ independent vector fields

$$\begin{aligned} \frac{\partial_c}{\partial t} &\equiv \frac{\partial}{\partial t} + \sum_{a=1}^k \left(g^a - \sum_{l=1}^{m-k} \frac{\partial g^a}{\partial \dot{q}^l} \dot{q}^l \right) \frac{\partial}{\partial q^{m-k+a}} \\ \frac{\partial_c}{\partial q^s} &\equiv \frac{\partial}{\partial q^s} + \sum_{a=1}^k \frac{\partial g^a}{\partial \dot{q}^s} \frac{\partial}{\partial q^{m-k+a}} \\ &\frac{\partial}{\partial \dot{q}^s} \end{aligned} \quad (61)$$

where $1 \leq s \leq m-k$.

The annihilator of \mathcal{C} is denoted by \mathcal{C}^0 .

The ideal in the exterior algebra on Q locally generated by the 1-forms φ^a , $1 \leq a \leq k$, is called the *constraint ideal*, and denoted by $\mathcal{I}(\mathcal{C}^0)$. Differential forms belonging to the constraint ideal are called *constraint forms*.

Vector fields belonging to the canonical distribution are called *Chetaev vector fields*. Note that every Chetaev vector field takes a form

$$\begin{aligned} Z &= Z^0 \frac{\partial_c}{\partial t} + Z^s \frac{\partial_c}{\partial q^s} + \tilde{Z}^s \frac{\partial}{\partial \dot{q}^s} \\ &= Z^0 \frac{\partial}{\partial t} + Z^s \frac{\partial}{\partial q^s} + \sum_{a=1}^k \left(Z^0 \left(g^a - \sum_{l=1}^{m-k} \frac{\partial g^a}{\partial \dot{q}^l} \dot{q}^l \right) + Z^s \frac{\partial g^a}{\partial \dot{q}^s} \right) \frac{\partial}{\partial q^{m-k+a}} + \tilde{Z}^s \frac{\partial}{\partial \dot{q}^s}. \end{aligned} \quad (62)$$

We stress that the family of Chetaev vector fields *need not contain*

- vector fields *projectable* onto Y ,
- *prolongations* of vector fields defined on Y , even if the canonical distribution is projectable.

Remarkably, the following theorem holds [29]:

Theorem 6. *The constraint Q is given by equations affine in the first derivatives if and only if the canonical distribution \mathcal{C} on Q is $\bar{\pi}_{1,0}$ -projectable (i.e. the projection \mathcal{D} of \mathcal{C} is a distribution on Y).*

The distribution \mathcal{D} on Y is then locally spanned by vector fields

$$\frac{\partial}{\partial t} + \sum_{a=1}^k A^a \frac{\partial}{\partial q^{m-k+a}}, \quad \frac{\partial}{\partial q^s} + \sum_{a=1}^k B_s^a \frac{\partial}{\partial q^{m-k+a}}, \quad 1 \leq s \leq m-k, \quad (63)$$

or, annihilated by 1-forms $A^a dt + B_s^a dq^s - dq^{m-k+a}$, $1 \leq a \leq k$, where $g^a = A^a + B_s^a \dot{q}^s$.

The canonical distribution need not be completely integrable. We call a non-holonomic constraint Q *semiholonomic* if its canonical distribution \mathcal{C} is completely integrable. Properties of semiholonomic constraints can be summarized as follows [29], [33], [35]:

Theorem 7. *The following conditions are equivalent:*

- (1) Q is semiholonomic.
- (2) The canonical distribution \mathcal{C} on Q is projectable onto Y , and its projection is completely integrable.
- (3) The constraint ideal is closed.
- (4) Functions g^a defining locally the constraint satisfy

$$\frac{\partial_c g^a}{\partial q^s} - \frac{d_c}{dt} \frac{\partial g^a}{\partial \dot{q}^s} = 0, \quad 1 \leq s \leq m-k. \quad (64)$$

Theorem 8. *The canonical distribution \mathcal{C} of a semiholonomic constraint is spanned by vector fields $J_c^1 \xi$, where ξ belongs to the projection \mathcal{D} of \mathcal{C} , and $\bar{\pi}_{1,0}$ -vertical vector fields.*

We have seen that *constraints linear or affine in velocities can be alternatively modeled by means of a distribution \mathcal{D} on Y* , defined by (63) (that is completely integrable in case of semiholonomic constraints). The geometric description of non-holonomic constraints by a distribution on Y (on a “configuration space”, or “space of events”) is quite popular and frequently used. The reader should, however, keep in mind that using such a model means *restriction* to constraints *affine in velocities*.

The canonical distribution is naturally lifted to the distribution $\hat{\mathcal{C}}$ on \hat{Q} , defined with help of its annihilator by $\hat{\mathcal{C}}^0 = \bar{\pi}_{2,1}^* \mathcal{C}^0$.

4.3 Dynamics of nonholonomic systems: Reduced equations

Consider a nonholonomic constraint $\iota : Q \rightarrow J^1Y$ endowed with the canonical distribution \mathcal{C} as above. Let E be a J^1Y -pertinent dynamical form on J^2Y and $[\alpha]$ its Lepage class. Recall that $[\alpha]$ consists of local 2-forms on J^1Y , and contains a closed (global) 2-form if and only if the dynamical form E comes as the Euler-Lagrange form from (possibly local) Lagrangians. We keep notations used above, i.e. $E = E_\sigma \omega^\sigma \wedge dt$, $E_\sigma = A_\sigma + B_{\sigma\nu} \ddot{q}^\nu$, and

$$\alpha = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu} \omega^\sigma \wedge dq^\nu + F, \quad (65)$$

where F is a 2-contact 2-form on an open subset of J^1Y .

According to [29], a *constrained mechanical system* associated with $[\alpha]$ is defined to be the class

$$[\bar{\alpha}] = \iota^* \alpha \text{ mod } \mathcal{I}(\mathcal{C}^0). \quad (66)$$

This means that $[\bar{\alpha}]$ is defined on the constraint Q and consists of all possibly local 2-forms on Q such that

$$\bar{\alpha} = \bar{A}_l \omega^l \wedge dt + \bar{B}_{ls} \omega^l \wedge dq^s + F + \varphi, \quad (67)$$

where F is a 2-contact and φ is a constraint 2-form on Q , and

$$\begin{aligned} \bar{A}_l &= \left(A_l + A_{m-k+j} \frac{\partial g^j}{\partial \dot{q}^l} + \left(B_{l,m-k+i} + B_{m-k+j,m-k+i} \frac{\partial g^j}{\partial \dot{q}^l} \right) \frac{d'g^i}{dt} \right) \circ \iota, \\ \bar{B}_{ls} &= \left(B_{ls} + B_{l,m-k+i} \frac{\partial g^i}{\partial \dot{q}^s} + B_{m-k+i,s} \frac{\partial g^i}{\partial \dot{q}^l} + B_{m-k+j,m-k+i} \frac{\partial g^j}{\partial \dot{q}^l} \frac{\partial g^i}{\partial \dot{q}^s} \right) \circ \iota. \end{aligned} \quad (68)$$

Note that if the matrix B is symmetric then so is \bar{B} , however, regularity of B does not imply regularity of \bar{B} . The latter has important consequences on dynamical properties of nonholonomically constrained systems making them much different from the holonomic ones. We shall discuss it in more detail below when dealing with the associated exterior differential systems.

If in particular α is related with a mechanical system (λ, Φ) , we have

$$\bar{A}_s = \frac{\partial_c \bar{L}}{\partial \dot{q}^s} - \frac{d'_c}{dt} \frac{\partial \bar{L}}{\partial \dot{q}^s} - \left(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota \right) \left(\frac{\partial_c g^a}{\partial \dot{q}^s} - \frac{d'_c}{dt} \frac{\partial g^a}{\partial \dot{q}^s} \right) - \bar{\Phi}_s - \bar{\Phi}_{m-k+a} \frac{\partial g^a}{\partial \dot{q}^s} \quad (69)$$

$$\bar{B}_{sr} = - \frac{\partial^2 \bar{L}}{\partial \dot{q}^r \partial \dot{q}^s} + \left(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota \right) \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s} \quad (70)$$

with the notation $\bar{L} = L \circ \iota$, $\bar{\Phi}_\sigma = \Phi_\sigma \circ \iota$.

In place of a single dynamical form $E = p_1 \alpha$ we have for the constrained system rather the class $[\bar{E}]$, on \hat{Q} , with

$$\bar{E} = \bar{p}_1 \bar{\alpha} = \hat{\iota}^* E + \varphi^a \wedge \nu_a \quad (71)$$

where φ^a are the canonical constraint 1-forms defined above and ν_a are horizontal forms.

Since $\mathcal{C} \rightarrow Q$ is a subbundle of the tangent bundle $TQ \rightarrow Q$, the class $[\bar{E}]$ gives rise to a *dynamical form along the canonical distribution*, called *constrained dynamical form*, $\bar{E}^c = (\bar{\iota}^* E)|_{\hat{\mathcal{C}}} \in \Lambda^2(\hat{\mathcal{C}})$ (see [29] for the definition and more details on forms along a distribution); we note that \bar{E}^c is the same for all $\bar{E} \in [\bar{E}]$.

Computations in adapted fibred coordinates yield the following formula:

$$\bar{E}^c = (\bar{A}_s + \bar{B}_{sr}\dot{q}^r)\bar{\omega}^s \wedge dt. \quad (72)$$

We shall be interested in *constrained sections* of π , that is in sections $\gamma : I \rightarrow Y$ such that $J^1\gamma(I) \subset Q$. Constrained sections satisfy the system of k first order ODE's of the constraint. In particular, every such a section is an integral section of the canonical distribution \mathcal{C} .

We have the following theorem (cf. [29]):

Theorem 9. *Equations of motion of a mechanical system α constrained to Q are equations for constrained sections of π , taking one of the following two equivalent intrinsic forms:*

$$\bar{E}^c \circ J^2\gamma = 0, \quad (73)$$

$$J^1\gamma^* i_Z \bar{\alpha} = 0 \quad \text{for every } \bar{\pi}_1\text{-vertical Chetaev vector field } Z \text{ on } Q \quad (74)$$

(where $\bar{\alpha}$ is (any) representative of the class $[\bar{\alpha}]$).

In coordinates,

$$\bar{A}_s + \bar{B}_{sr}\dot{q}^r = 0, \quad (75)$$

or, if α is given by means of a Lagrangian λ and a force Φ ,

$$\frac{\partial_c \bar{L}}{\partial q^s} - \frac{d_c \bar{L}}{dt} \frac{\partial \bar{L}}{\partial \dot{q}^s} - \left(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota \right) \left(\frac{\partial_c g^a}{\partial q^s} - \frac{d_c}{dt} \frac{\partial g^a}{\partial \dot{q}^s} \right) = \bar{\Phi}_s + \bar{\Phi}_{m-k+a} \frac{\partial g^a}{\partial \dot{q}^s}, \quad (76)$$

where $1 \leq s \leq m - k$.

It should be stressed that the above motion equations for nonholonomic systems are *differential equations on the constraint manifold Q* . They are called *reduced nonholonomic equations* (“without Lagrange multipliers”) [29].

Further it should be emphasized that the motion equations for nonholonomic systems are generally *equations in implicit form*. However, due to their interpretation as equations for an exterior differential system on the constraint manifold Q , apparent from (74), they are investigated with the same methods as motion equations in the unconstrained/holonomic case (see e.g. [29], [32], [40]).

4.4 The nonholonomic variational principle

Let us turn to the special case, when the force $\Phi(t, q^\nu, \dot{q}^\nu)$ is *conservative* (potential). Note that equations (31) are then variational being Euler-Lagrange equations of a Lagrangian $L' = L - V$, where V is a potential for Φ . Hence, without loss of generality, and for simplicity of notations, let us consider to have a Lagrangian system on J^1Y , given by a Lagrangian λ . The nonholonomic equations of motion then obviously take one of the equivalent forms:

$$\bar{E}_\lambda^c \circ J^2\gamma = 0, \quad (77)$$

$$J^1\gamma^*i_Z\iota^*d\theta_\lambda = 0 \quad \text{for every } \bar{\pi}_1\text{-vertical Chetaev vector field } Z \text{ on } Q, \quad (78)$$

$$\frac{\partial_c \bar{L}}{\partial q^s} - \frac{d_c}{dt} \frac{\partial \bar{L}}{\partial \dot{q}^s} - \left(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota \right) \left(\frac{\partial_c g^a}{\partial q^s} - \frac{d_c}{dt} \frac{\partial g^a}{\partial \dot{q}^s} \right) = 0, \quad 1 \leq s \leq m-k. \quad (79)$$

Reduced equations for constrained Lagrangian systems (79) were first considered in [46], and are equivalent with Chetaev equations.

In [35] a *variational principle for systems subject to nonholonomic constraints* was found, providing the above reduced equations as equations for “constrained extremals”. A generalization of the standard variational principle is in no case trivial or straightforward, and needs a careful review of basic variational concepts. Main points are as follows:

- The variational principle is formulated for the fibred manifold $\bar{\pi}_1 : Q \rightarrow \mathbb{R}$, endowed with the canonical distribution \mathcal{C} .
- “Admissible paths” are sections of the fibred manifold $\bar{\pi}_1 : Q \rightarrow \mathbb{R}$. (Note that they need not be holonomic, however, every admissible section δ has a counterpart in Y : it is a section γ of $\pi : Y \rightarrow \mathbb{R}$, given by $\gamma = \bar{\pi}_{1,0} \circ \delta$).
- “Admissible variations” are $\bar{\pi}_1$ -projectable vector fields belonging to the canonical distribution (Chetaev vector fields). (Note that the requirement of projectability onto the base is essential, since variations of this kind provide a one-parametric family of maps that all are *sections* of the constraint manifold. Also note that the family of admissible sections $\delta_u = \phi_u \delta \phi_{0u}^{-1}$, arising by deformation of a holonomic section $\delta = J^1\gamma$, may contain non-holonomic sections (which is a violation of the “classical” principle of virtual displacements); moreover, the projection of the family $\{\delta_u\}$, i.e. the family of sections of π of the form $\gamma_u = \bar{\pi}_{1,0} \phi_u J^1\gamma \phi_{0u}^{-1}$ is not induced by a vector field on Y unless the canonical distribution is projectable (meaning that the constraints are affine in velocities)—however, even in this case, $J^1\gamma_u = (J^1\gamma)_u$ need not be true).
- The integrand of the action function (taking the place of a “constrained Lagrangian”) is the 1-form $\iota^*\theta_\lambda$.

Definition 10. [35] Denote by $\mathcal{S}_{[a,b]}(\bar{\pi}_1)$ the set of sections of $\bar{\pi}_1$, defined around an interval $[a, b] \subset \mathbb{R}$, $a < b$. Given a Lagrangian λ on J^1Y , the function

$$\mathcal{S}_{[a,b]}(\bar{\pi}_1) \ni \delta \rightarrow \int_a^b \delta^* \iota^* \theta_\lambda \in \mathbb{R}, \quad (80)$$

is called *constrained (to Q) action function* of the Lagrangian λ over $[a, b]$.

Let $Z \in \mathcal{C}$ be a $\bar{\pi}_1$ -projectable vector field, and denote by ϕ and ϕ_0 the flows of Z and its projection Z_0 , respectively. The one-parameter family $\{\delta_u\}$ of sections of $\bar{\pi}_1$, where $\delta_u = \phi_u \delta \phi_{0u}^{-1}$, is called *constrained variation of δ* induced by Z . The function

$$\mathcal{S}_{[a,b]}(\bar{\pi}_1) \ni \delta \rightarrow \left(\frac{d}{du} \int_{\phi_{0u}([a,b])} \delta_u^* \iota^* \theta_\lambda \right)_{u=0} = \int_a^b \delta^* \mathcal{L}_Z \iota^* \theta_\lambda \in \mathbb{R} \quad (81)$$

is then called the *first constrained variation* of the action function of λ over $[a, b]$, induced by Z .

To study *constrained sections of the fibred manifold* π , we have to *restrict the domain of definition* $\mathcal{S}_{[a,b]}(\bar{\pi}_1)$ of the function (81) to the subset $\mathcal{S}_{[a,b]}^h(\bar{\pi}_1)$ of *holonomic sections* of the projection $\bar{\pi}_1$, i.e. $\delta = J^1\gamma$ where $\gamma \in \mathcal{S}_{[a,b]}(\pi)$. Then the first constrained variation (81) can be regarded as a function

$$\mathcal{S}_{[a,b],Q}(\pi) \ni \gamma \rightarrow \int_a^b J^1\gamma^* \mathcal{L}_Z \iota^* \theta_\lambda \in \mathbb{R} \quad (82)$$

defined on a *subset of sections of the projection* $\pi : Y \rightarrow \mathbb{R}$.

We stress that (due to the properties of admissible variations mentioned above) the restricted first constrained variation *cannot* be obtained via a “variation procedure” from an action defined directly on the set $\mathcal{S}_{[a,b],Q}(\pi)$.

Applying to (82) Cartan’s formula for the decomposition of Lie derivative we obtain the *nonholonomic first variation formula*

$$\int_a^b J^1\gamma^* \mathcal{L}_Z \iota^* \theta_\lambda = \int_a^b J^1\gamma^* i_Z \iota^* d\theta_\lambda + \int_a^b J^1\gamma^* di_Z \iota^* \theta_\lambda, \quad (83)$$

where Z is a $\bar{\pi}_1$ -projectable Chetaev vector field.

Formula (83) gives the splitting of the first constrained variation to a “constrained Euler-Lagrange term” and a boundary term. One should notice that on the left-hand side of the nonholonomic first variation formula one cannot put the Lie derivative of the “constrained Lagrangian” $\bar{\lambda} = \iota^*\lambda$ instead of $\mathcal{L}_Z \iota^* \theta_\lambda$, since the difference $\mathcal{L}_Z \iota^* \theta_\lambda - \mathcal{L}_Z \bar{\lambda}$ need not be a contact form.

A section γ of $\pi : Y \rightarrow \mathbb{R}$ is called a *constrained extremal* of λ on $[a, b]$ if $\text{Im } J^1\gamma \subset Q$, and if the first constraint variation of the action on the interval $[a, b]$ vanishes for every “fixed endpoints” variation Z over $[a, b]$. γ is called a *constrained extremal of λ* if it is its constrained extremal on every interval $[a, b] \subset \text{Dom } \gamma$. With help of the nonholonomic first variation formula one proves that γ is a *constrained extremal of λ if and only if it satisfies equations of the constraint, and one of the (equivalent) equations (77)–(79)* [35]. Therefore we call any of these equations *nonholonomic Euler-Lagrange equations*.

Notice that for semiholonomic constraints equations (79) simplify to

$$\frac{\partial_c \bar{L}}{\partial q^s} - \frac{d_c}{dt} \frac{\partial \bar{L}}{\partial \dot{q}^s} = 0, \quad 1 \leq s \leq m - k, \quad (84)$$

completely determined by the “constrained Lagrangian” $\bar{\lambda} = \iota^*\lambda$.

Similarly as in the unconstrained/holonomic case, the second term on the right-hand side of the nonholonomic first variation formula (83) is connected with conservation laws. Let us recall a generalization of Noether theorem to nonholonomic systems, due to [36]:

A Chetaev vector field $Z \in \mathcal{C}$ is called a *constrained symmetry* of a Lagrangian λ if $\mathcal{L}_Z \iota^* \theta_\lambda$ is a constraint form.

Directly from (83) we obtain:

Theorem 11. (Nonholonomic Noether theorem)

Let λ be a Lagrangian on J^1Y , and Z be a constrained symmetry of λ . Then along every constrained extremal of λ , the function $F = i_Z \iota^* \theta_\lambda = i_Z \theta_\lambda$ is constant.

4.5 Regularity and Hamilton equations of nonholonomic systems

Consider a nonholonomic mechanical system (λ, Φ, Q) . Equations (74) represent an important form of the nonholonomic motion equations, since they provide a representation in form of an *exterior differential system* (particularly, a *distribution*) on the constraint Q . More precisely, solutions of equations (74) are *holonomic* integral sections of the distribution $\Delta_{\bar{\alpha}}^c$, locally annihilated by the system of 1-forms on Q ,

$$\varphi^a, \quad i_Z \bar{\alpha}, \quad (85)$$

where $1 \leq a \leq k$, and Z runs over all vertical vector fields in \mathcal{C} , called *constrained dynamical distribution* (note that $\Delta_{\bar{\alpha}}^c$ is a subdistribution of the canonical distribution \mathcal{C}). We shall call equations for (all) integral sections of the distribution $\Delta_{\bar{\alpha}}^c$ *nonholonomic Hamilton equations* (cf. [32], [42] for Lagrangian systems). Note that in this context, the constraint manifold Q has the meaning of a genuine *evolution space* for the constrained system.

The constrained dynamical distribution need not have a constant rank, and even if the rank is constant it need not be equal to one. We say that the nonholonomic system $[\bar{\alpha}]$ is *regular* if $\text{rank } \Delta_{\bar{\alpha}}^c = 1$ [29]. From (76) we conclude that for a constrained mechanical system (λ, Φ, Q) the *regularity condition* reads

$$\det \left(\frac{\partial^2 \bar{L}}{\partial \dot{q}^r \partial \dot{q}^s} - \left(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota \right) \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s} \right) \neq 0, \quad (86)$$

i.e. the matrix (\bar{B}_{sr}) (70) is regular, If the constrained system is regular then the distribution $\Delta_{\bar{\alpha}}^c$ is locally spanned by one *vector field* (constrained semispray)

$$\zeta = \frac{\partial}{\partial t} + \sum_{l=1}^{m-k} \dot{q}^l \frac{\partial}{\partial q^l} + \sum_{a=1}^k g^a \frac{\partial}{\partial q^{m-k+a}} - \sum_{l,s=1}^{m-k} \bar{B}^{ls} \bar{A}_s \frac{\partial}{\partial \dot{q}^l}, \quad (87)$$

where (\bar{B}^{ls}) is the inverse matrix to (\bar{B}_{ls}) and \bar{A}_s are given by (69), and the nonholonomic Hamilton equations are equivalent with the nonholonomic motion equations in Theorem 9.

Let us turn again to the case when the original mechanical system is Lagrangian. Then the nonholonomic Hamilton equations take the form

$$\begin{aligned} \delta^* i_Z \iota^* d\theta_\lambda &= 0 \quad \text{for every } \bar{\pi}_1\text{-vertical Chetaev vector field } Z \text{ on } Q, \\ \delta^* \varphi^a &= 0, \quad 1 \leq a \leq k. \end{aligned} \quad (88)$$

If the constrained system (λ, Q) is regular then the nonholonomic Hamilton equations are equivalent with the nonholonomic Euler-Lagrange equations ((77) or (78) or (79)). In this case we can introduce a *nonholonomic Legendre transformation* [51]:

Theorem 12. *Let $x \in Q$ be a point. Suppose that in a neighbourhood of x ,*

$$\frac{\partial \bar{B}_{ls}}{\partial \dot{q}^r} = \frac{\partial \bar{B}_{lr}}{\partial \dot{q}^s}, \quad 1 \leq l, r, s \leq m - k. \quad (89)$$

Then there exists a neighbourhood $U \subset Q$ of x , and, on U , functions P_l , $1 \leq l \leq m - k$, and a 1-form η , such that

$$\iota^* d\theta_\lambda = \eta \wedge dt + dP_l \wedge dq^l + F, \quad (90)$$

where F is a 2-contact form on Q . If, moreover, the constrained system (λ, Q) is regular, then $(t, q^\sigma, \dot{q}^l) \rightarrow (t, q^\sigma, P_l)$ is a coordinate transformation on U .

The integrability condition for the \bar{B}_{sl} 's (89) ensures that one can express functions P_l explicitly. To this purpose we consider a mapping $\chi : [0, 1] \times W \rightarrow W$ defined by $(u, t, q^\sigma, \dot{q}^l) \rightarrow (t, q^\sigma, u\dot{q}^l)$, where $W \subset Q$ is an appropriate open set. Then Poincaré Lemma gives us a solution

$$P_l = -\dot{q}^s \int_0^1 (\bar{B}_{ls} \circ \chi) du = \frac{\partial \bar{L}}{\partial \dot{q}^l} - \dot{q}^s \int_0^1 \left(\left(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota \right) \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s} \right) \circ \chi du. \quad (91)$$

We call the above functions P_l , $1 \leq l \leq k$, *nonholonomic momenta*, and the corresponding coordinate transformation *nonholonomic Legendre transformation* of λ . The 1-form η in (90) is called a *nonholonomic energy 1-form*.

The 1-form η is *determined up to a constraint 1-form*, and *need not be closed*. In constraint Legendre coordinates we can write

$$\eta = \eta_0 dt + \eta_l dq^l + \eta^l dP_l \text{ mod } \mathcal{I}(\mathcal{C}^0). \quad (92)$$

In nonholonomic Legendre coordinates the nonholonomic Hamilton equations take the following canonical form

$$\frac{d}{dt}(P_l \circ \delta) = \eta_l, \quad \frac{d}{dt}(q^l \circ \delta) = -\eta^l, \quad \frac{d}{dt}(q^{m-k+a} \circ \delta) = g^a, \quad (93)$$

where $1 \leq l \leq m - k$, $1 \leq a \leq k$.

For non-holonomic constraints *affine in velocities* the situation essentially simplifies: Indeed, then (89) is fulfilled identically and the nonholonomic momenta are defined by

$$P_l = \frac{\partial \bar{L}}{\partial \dot{q}^l}, \quad 1 \leq l \leq m - k. \quad (94)$$

The regularity condition takes the form

$$\det \left(\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} \right) \neq 0. \quad (95)$$

Moreover, if the constraint Q is semiholonomic then the family of energy 1-forms (92) contains a closed 1-form equal to $-d\bar{H}$, where

$$\bar{H} = -\bar{L} + P_l \dot{q}^l. \quad (96)$$

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