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Generalized Birkhoffian realization of nonholonomic systems

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Abstract. Based on the Cauchy-Kowalevski theorem for a system of partial differential equations to be integrable, a kind of generalized Birkhoffian systems (GBSs) with local, analytic properties are put forward, whose manifold admits a presymplectic structure described by a closed 2-form which is equivalent to the self-adjointness of the GBSs. Their relations with Birkhoffian systems, generalized Hamiltonian systems are investigated in detail. Analytic, algebraic and geometric properties of GBSs are formulated, together with their integration methods induced from the Birkhoffian systems. As an important example, nonholonomic systems are reduced into GBSs, which gives a new approach to some open problems of nonholonomic mechanics.

1 Introduction

As it is well known, making use of the calculus of variations, any analytic, regular, holonomic, conservative mechanical systems can be formulated by Lagrange's equations or Hamilton's equations, which are basis of establishing, simplifying and integrating the equations of motion. Thus it is important to find out the solutions of inverse problems of the calculus of variations for different dynamical systems so as to make the most of the Lagrange's equations and Hamilton's equations. However, the Lagrangian or Hamiltonian formulation for a dynamical system, limited by the conditions of self-adjointness, such as the Helmholtz's conditions [10], [13], [15], [18], is not directly universal if the physical variables remain without use of Darboux transformations. Based on the Cauchy-Kowalevski theorem of the integrability conditions for partial differential equations and the converse of the Poincaré lemma, it can be proved that there exists a direct universality of Birkhoff's equations for local Newtonian systems by means of reduction of Newton's equations to a first-order form, which means all local, analytic, regular, finite-dimensional,

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unconstrained or holonomic, conservative or non-conservative, and self-adjoint or non-self-adjoint systems always admit, in a star-shaped neighborhood of a regular point of their variables, a representation in terms of first-order Birkhoff's equations in the coordinate and time variables of the experimenter [11], [14]. The systems whose equations of motion are represented by the first-order Birkhoff's equations on a symplectic or a contact manifold spanned by the physical variables are called Birkhoffian systems, which are self-adjoint. The self-adjointness of local, analytic, regular, holonomic mechanical systems means the existence of symplectic or contact structure of the manifold. The Lie algebraic structure only exists for the autonomous Birkhoffian systems.

The inverse problem of the calculus of variations for nonholonomic systems is very complicated [7], [12]. Only some special nonholonomic systems, such as some Chaplygin's systems, can admit a homogenous Lagrangian or Hamiltonian formulation. Since the Chaplygin's systems can be reduced into a kind of holonomic nonconservative systems, it is suitable to formulate such nonholonomic systems in Birkhoffian mechanics [9], [11]. For a general nonholonomic system, i.e. a n-dimensional mechanical system constrained by l nonlinear nonholonomic constraints which is a coupled dynamical system, whose equations of motion are $n+l$ fixed first- and second-order ordinary differential equations, their inverse problem of the calculus of variations can be geometrically analyzed in a singular Lagrangian [6] or represented in Birkhoffian framework on an $2n$-dimensional phase space [4], [11]. For the latter case, the nonholonomic systems are reduced into the conditional holonomic systems on a $2n$-dimensional phase space, whose initial conditions are not arbitrary but confined by the nonholonomic constraints. Because the conditional holonomic systems are of symmetry determined by the constraints, it is necessary to reduce the Birkhoff's equations on the $2n$-dimensional phase space to those on its constraint submanifold of minimal dimension $2n-l$. Such a symmetry reduction strongly relies on the dimension $2n-l$ of the constraint submanifold or the number l of the constraints acted upon the system. Therefore, in order to directly universally analyze the inverse problem of the calculus of variations for general nonholonomic systems, we need to generalize the Birkhoffian mechanics. This problem can arise in other coupled dynamical systems, such as control theory for systems, supermechanics, etc.

In section 2, we will review Birkhoffian formulation of Newtonian Systems, emphasizing its analytic, algebraic and geometric properties. Its relation with generalized Hamiltonian mechanics is pointed out. In section 3, generalized Birkhoff's equations for all analytic, regular first-order dynamical systems are constructed based on the Cauchy-Kowalevski theorem for existence theory of partial differential equations. The integration methods induced from Birkhoffian mechanics are listed in section 4. In section 5, general nonholonomic systems are reduced into generalized Birkhoffian systems(GBSs), whose equations of motion are represented by the generalized Birkhoff's equations.

2 Review of Birkhoffian formulation of Newtonian systems

Consider a *holonomic* dynamical system on a contact manifold $R \times TQ$ with local coordinates $\{q^i, \dot{q}^i\}$ $(i = 1, 2, \ldots, n)$ where Q is a n-dimensional configuration manifold. Let a *regular* Lagrangian be denoted by $L(t,q,\dot{q})$. Suppose the system is subject to non-conservative forces $f_i(t,q,\dot{q})$ which are *analytic*. The equations of motion for the system can be represented by non-homogeneous Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = f_i \tag{1}$$

The regularity condition $L_{ij} = \det(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}) \neq 0$ guarantees that these n second-order differential equations for q^i on the contact manifold $R \times TQ$ can be reduced into $2n$ first-order non-homogeneous Hamilton's equations on the contact manifold $R \times T^*Q$ with local coordinates $\{t, a^\mu\}$ $(\mu = 1, 2, \ldots, 2n)$:

$$\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H(t,a)}{\partial a^\mu} = F_\mu(t,a) \tag{2}$$

where $\{a^\mu\} = \{q^i, p_i\}$, $p_i = \frac{\partial L}{\partial \dot{q}^i}$, $H = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L$ is the Hamiltonian for the system and simple symplectic matrix is

$$\omega = (\omega_{\mu\nu})_{2n\times 2n} = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ +1_{n\times n} & 0_{n\times n} \end{pmatrix}_{2n\times 2n} \tag{3}$$

In general, equations (2) are non-self-adjoint. By means of the Cauchy-Kowalevski theorem, it can be proved that there exist integrating factors $\{h^\mu_\lambda\}$ for the equations (2) to become self-adjoint equations

$$\Omega_{\mu\nu}\dot{a}^\nu - \left[\frac{\partial B\,(t,a)}{\partial a^\mu} + \frac{\partial R_\mu\,(t,a)}{\partial t}\right] = 0 \tag{4}$$

in a star-shaped region of a regular point (t,a), where B is a Birkhoffian usually taken as the energy function of the system, R_μ are a set of Birkhoffian functions usually related with the function F_μ and $\Omega_{\mu\nu}$ is the covariant Birkhoff's tensor defined by

$$\Omega_{\mu\nu}\,(t,a) = \frac{\partial R_\nu\,(t,a)}{\partial a^\mu} - \frac{\partial R_\mu\,(t,a)}{\partial a^\nu} \tag{5}$$

is symplectic. The regularity condition $\det(\Omega_{\mu\nu}) \neq 0$ means that the $2n$ equations (4) are independent and can be transformed into the contravariant form

$$\dot{a}^\mu = \Omega^{\mu\nu}\left[\frac{\partial B\,(t,a)}{\partial a^\nu} + \frac{\partial R_\nu\,(t,a)}{\partial t}\right] \tag{6}$$

where $\Omega^{\mu\nu} = \Omega^{-1}_{\mu\nu}$.

The Birkhoff's equations (4) are analytic in the sense that they are derivable from the most general possible linear first-order action functional, the Pfaffian action

$$\mathcal{A}(\tilde{E}) = \int_{t_1}^{t_2} dt\,[R_\nu\,(t,a)\,\dot{a}^\nu - B\,(t,a)]\,(\tilde{E}) \tag{7}$$

where $\tilde{E}$ is a possible pass in the contact manifold. The discussion of gauge freedom and some methods for integrating the Birkhoff's equations can be found in [11], [14].

The conditions of self-adjointness of Birkhoff's Equations (4)

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0 \tag{8a}$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial a^{\tau}} + \frac{\partial \Omega_{\nu\tau}}{\partial a^{\mu}} + \frac{\partial \Omega_{\tau\mu}}{\partial a^{\nu}} = 0 \tag{8b}$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial t} = \frac{\partial}{\partial a^{\nu}} \left[\frac{\partial B\left(t,a\right)}{\partial a^{\mu}} + \frac{\partial R_{\mu}\left(t,a\right)}{\partial t} \right] - \frac{\partial}{\partial a^{\mu}} \left[\frac{\partial B\left(t,a\right)}{\partial a^{\nu}} + \frac{\partial R_{\nu}\left(t,a\right)}{\partial t} \right] \tag{8c}$$

are equivalent to the the integrability conditions for the 2-form on the contact manifold $R \times T^{*}Q$

$$\hat{\Omega} = \frac{1}{2} \hat{\Omega}_{\mu\nu}\left(\hat{a}\right) d\hat{a}^{\mu} \wedge d\hat{a}^{\nu}; \qquad \mu = 0, 1, 2, \ldots, 2n \tag{9}$$

to be closed, i.e.,

$$d\hat{\Omega} = 0 \tag{10}$$

where $\hat{\Omega}_{\mu\nu} = \Omega_{\mu\nu}$, $\hat{\Omega}_{0\nu} = -\hat{\Omega}_{\nu 0} = \frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t}$ $(\mu, \nu = 1, 2, \ldots, 2n)$. The above nonautonomous Birkhoff's equations (4) can be globally represented by a *general, local, Birkhoffian vector field* $\tilde{X}$ on $R \times T^{*}Q$ verifying the properties

$$i_{\tilde{X}} \hat{\Omega} = 0, \quad dt(\tilde{X}) = 1 \tag{11}$$

Locally the vector field admits

$$\tilde{X} = \frac{\partial}{\partial t} + \Omega^{\mu\nu} \left(\frac{\partial B}{\partial a^{\nu}} + \frac{\partial R_{\nu}}{\partial t} \right) \frac{\partial}{\partial a^{\mu}} \tag{12}$$

Evidently the universality of the Birkhoff's equations is not only direct but also global.

For the autonomous Birkhoffian systems where[1] $\frac{\partial R_{\mu}}{\partial t} = 0$, the Birkhoff's equations are equivalent to the generalized Hamilton's equations [8] on an even-dimensional Poisson manifold

$$\dot{a}^{\mu} = \Omega^{\mu\nu} \frac{\partial B\left(t,a\right)}{\partial a^{\nu}} \tag{13}$$

In these cases the Poisson brackets can be defined by time evolution

$$\dot{A}\left(a\right) = \frac{\partial A}{\partial a^{\mu}} \dot{a}^{\mu} = \frac{\partial A}{\partial a^{\mu}} \Omega^{\mu\nu} \frac{\partial B}{\partial a^{\nu}} \overset{\text{def}}{=} [A, B] \tag{14}$$

verifying the Lie algebra axioms

$$[A, B] + [B, A] = 0 \tag{15a}$$

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 \tag{15b}$$

It should be pointed out that the Lie algebraic structure does not exist for a non-autonomous Birkhoffian system for the dependence of $R(t, a)$ on time t if we take Birkhoffian B to be total energy of the system.

[1] As will be seen in section 3, the so-called semi-autonomous Birkhoffian systems mentioned in [11], [14] are really autonomous ones. So the Birkhoffian systems can be classified into autonomous ones and non-autonomous ones.

3 Generalized Birkhoff's equations

In order to generalize the theory of Birkhoffian systems, it is necessary to analyze the fundamental conditions for the second-order dynamical systems to be capable of reduction to first-order self-adjoint Birkhoffian formulation, especially their role in the course of reduction to first-order systems.

(1) **Locality.** By locality we mean that the systems considered can be formulated by ordinary or partial differential equations, in which the interactions are independent of integro-differential type. This condition is necessary for the Lie algebra and symplectic geometry to be suitable to analyze the theories of some dynamical systems.

(2) **Analyticity.** Analyticity of a function means that it admits a convergent, multiple, power series expansion in a neighborhood of a point. The analyticity can be remained for a second-order dynamical system to be reduced into a first-order system. Evidently analyticity is based on locality. Analyticity does not depend on the self-adjointness of a dynamical system.

(3) **Regularity.** A system is called regular when it is of full rank or maximal rank, i.e., its functional Jacobi determinant is everywhere non-null in the region of a point, with the possible exception of finite number of isolated points. Regularity means maximal independency and invertibility. Thus a regular system can be recovered from its non-degenerately transformed form. Regularity is not an invariant character with respect to symmetry reduction, e.g., the canonical and Eulerian representations for the rotation of a rigid body with respect to a fixed point.

(4) **Holonomicity.** By a holonomic system we mean that constraints the system is subject to are integrable in the sense of Frobenius theorem. A holonomically constrained system can be reduced into a constraint-free system of lower dimensions. Therefore, holonomicity ensures the phase space for the first-order systems reduced from the second-order regular dynamical systems are even dimensional.

The universality of analytic/Lie/symplectic formulation in the most general possible form, i.e., Birkhoffian realization, does not depend on whether or not the original systems are conservative or self-adjoint. It should be pointed out that this symbiosis among analytic, Lie and symplectic techniques is comparatively flimsy because any one of the four legs under the symbiosis may be possibly broken. For example, the nonlocal type of interactions often occurs in several branches of physics, whose dynamical equations are integro-differential equations. Moreover, nonholonomic systems largely exist in the fields of physics, mechanics and engineering. Therefore it should be encouraged to generalize the Birkhoffian formulation to a new symbiosis among analytic, algebraic, geometric form in order to keep up with the process of mathematical and physical advances.

In this section, we consider a kind of GBSs from which the nonholonomic systems may be recovered if the second-order dynamical systems are reconstructed. Consider a dynamical system described by first-order differential equations

$$\dot{a}^I = \Xi^I(t, a^J), \qquad I, J = 1, 2, \dots, m \tag{16}$$

on an m-dimensional manifold M with local coordinates $\{t, a\}$ where $\Xi^I(t, a)$ are analytic on the regular points. It can be proved that the equations (16) admit an analytic and presymplectic structure whether they are self-adjoint or not. Two methods can be utilized to realize this goal. The first one is to find out an integrating factor matrix with the help of Cauchy-Kowalevski theorem for the partial differential equations to be integrable, so as to obtain a self-adjoint genotopic transformed covariant form of the equations (16) or a presymplectic form on the manifold M. Then the converse of Poincaré lemma is used to get the final result. The second method is a direct use of the following Cauchy-Kowalevski theorem.

Theorem 1. *Consider an initial problem consisting of $n+1$ first-order partial differential equations of the Cauchy-Kowalevski form*

$$\frac{\partial R_\alpha(t,a)}{\partial t} = \sum_{\beta=0}^{n}\sum_{I=1}^{m} \Xi_\alpha^{I\beta}(t,a)\frac{\partial R_\beta(t,a)}{\partial a^I} + \sum_{\beta=0}^{n}\Pi_\alpha^\beta(t,a)R_\beta(t,a) + \Theta(t,a) \tag{17}$$

in $n+1$ unknown functions R_α ($\alpha = 0,1,2,\dots,n$) and in $m+1$ independent variables $\{t,a^I\}$ ($I = 1,2,\dots,m$), and the $n+1$ initial conditions

$$R_\alpha(0,a^1,a^2,\dots,a^m) = \mathcal{R}_\alpha(a^1,a^2,\dots,a^m) \tag{18}$$

If the functions $\Xi_\alpha^{I\beta}(t,a)$, $\Pi_\alpha^\beta(t,a)$, $\Theta(t,a)$ and $\mathcal{R}_\alpha(a)$ are real analytic functions at the regular point $A(a)$, then a unique analytic solution $R_0,R_1,\dots,R_n$ of the initial problem (17) *and* (18) *exists in a neighborhood of the point $A(a)$.*

Considering the need of analytic and presymplectic structure, we set $n=m$, $\Pi_\alpha^\beta = 0$, $\Theta = 0$, $R_0 = -B$, $a^0 = t$, $\Xi_J^{I\beta} = 2\delta_J^{[I}\Xi^{\beta]} = \delta_J^I\Xi^\beta - \delta_J^\beta\Xi^I$ (where $\Xi^0 = 1$, so $\Xi_J^{I0} = \delta_J^I$). Then equations (17) become

$$\frac{\partial R_I(t,a)}{\partial t} = \left[\frac{\partial R_J(t,a)}{\partial a^I} - \frac{\partial R_I(t,a)}{\partial a^J}\right]\Xi^J(t,a) - \frac{\partial B(t,a)}{\partial a^I} \tag{19a}$$

$$\frac{\partial B(t,a)}{\partial t} = \Xi_0^{I0}\frac{\partial B(t,a)}{\partial a^I} - \Xi_0^{IJ}\frac{\partial R_J(t,a)}{\partial a^I} \tag{19b}$$

The solution for $\{R_I,-B\}$ is uniquely determined by known functions $\Xi^J, \Xi_0^{I0}, \Xi_0^{IJ}$ due to the Cauchy-Kowalevski theorem. However, for the definite functions Ξ^I, different choices of functions Ξ_0^{I0}, Ξ_0^{IJ} can produce different solutions $\{R_I,-B\}$, which is not of physical meaning in general. Evidently there exist infinite solutions for the equations (19a). If the quantity B in the equations (19a) is given, the equations (19a) are complete and the theorem ensures unique existence of the functions R_I. In this case the equation (19b) for the functions Ξ_0^{I0}, Ξ_0^{IJ}, which is in fact algebraic, is not complete.

This analysis may be useful to easily find out the solution $\{R_I,-B\}$ based on a suitable choice of functions Ξ_0^{I0}, Ξ_0^{IJ}. For example, we can suppose that $\Xi_0^{I0} = 0, \Xi_0^{IJ} = \delta^{IJ}$. Then the equation (19b) becomes

$$\frac{\partial B(t,a)}{\partial t} + \frac{\partial R_I(t,a)}{\partial a^I} = 0 \tag{20}$$

Now we will observe the analytic, algebraic and geometric characteristic of the following generalized Birkhoff's equations

$$\left[\frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}\right]\dot{a}^J - \left(\frac{\partial B}{\partial a^I} + \frac{\partial R_I}{\partial t}\right) = 0, \quad I,J = 1,2,\dots,m \tag{21}$$

from which it is easy to infer that

$$\frac{dB}{dt} = \frac{\partial B}{\partial t} - \frac{\partial R_I}{\partial t}\dot{a}^I \tag{22}$$

The equations (21) are analytic because they are derivable from the Pfaffian action

$$\mathcal{A}(\tilde{E}) = \int_{t_1}^{t_2} dt\,[R_\nu\,(t,a)\,\dot{a}^\nu - B\,(t,a)]\,(\tilde{E}) \tag{23}$$

with the end points condition,where $\tilde{E}$ is a possible pass in the contact manifold M.

We can define 1-form $\Theta(t,a) = R_I(t,a)da^I - B(t,a)dt$ on the manifold M subject to the condition that its exterior derivative

$$\Omega = d(R_I\,da^I - B\,dt) = \frac{1}{2}\left(\frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}\right) da^I \wedge da^J + \left(\frac{\partial B}{\partial a^I} + \frac{\partial R_I}{\partial t}\right) dt \wedge da^I \tag{24}$$

is of maximal rank. Making use of the notation $\Omega_{IJ} = \frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}$, $\Gamma_I = \frac{\partial B}{\partial a^I} + \frac{\partial R_I}{\partial t}$, it is easy to verify the equivalence relation between the closure of the 2-form Ω and self-adjointness of equations (21), i.e.,

$$d\Omega = 0 \Longleftrightarrow \begin{cases} \Omega_{IJ} + \Omega_{JI} = 0 \\ \dfrac{\partial \Omega_{IJ}}{\partial a^K} + \dfrac{\partial \Omega_{JK}}{\partial a^I} + \dfrac{\partial \Omega_{KI}}{\partial a^J} = 0 \\ \dfrac{\partial \Omega_{IJ}}{\partial t} = \dfrac{\partial \Gamma_I}{\partial a^I} - \dfrac{\partial \Gamma_J}{\partial a^I} \end{cases} \tag{25}$$

It inferred that the self-adjointness of the systems is independent of the non-degeneracy of 2-form Ω.

Definition 1. A presymplectic structure on a manifold M can be defined by a closed 2-form Ω, which may be degenerate in the sense that for all vector fields $V \in \Gamma(M)$, $\exists X \neq 0$, $X \in \Gamma(M)$, such that $\Omega(X,V) = 0$. The pair (M,Ω) is called a presymplectic manifold.

Because a 2-form on the manifold M with odd dimension is degenerate,[2] such a closed 2-form can not define a one-to-one and onto map between the tangent space $T_{\{t,a\}}M$ and the cotangent space $T^*_{\{t,a\}}M$ unless the dimension of the manifold M is even, i.e., $m = 2n$ and $\det(\Omega_{IJ}) \neq 0$, which is the case of a Birkhoffian system. Therefore, if we denote the set of smooth real-valued functions on M by $\mathcal{F}(M)$, there does not, in general, exist the unique Hamiltonian vector field X_f on M such that $i_{X_f}\Omega = df$, $f \in \mathcal{F}(M)$. Although the dynamical vector field

[2]Denote the transpose of the matrix (Ω_{IJ}) by $(\Omega_{IJ})^T = (\Omega_{JI})$, then $\det(\Omega_{IJ})^T = \det(\Omega_{IJ})$. Since the matrix (Ω_{IJ}) is antisymmetric, i.e., $\Omega_{IJ} = -\Omega_{JI}$, we have $\det(-\Omega_{IJ}) = \det(\Omega_{IJ})$. It can be referred that $\det(-\Omega_{IJ}) = (-1)^m \det(\Omega_{IJ})$ from the relation $\det(\Omega_{IJ}) = \varepsilon^{IJ\ldots K}\Omega_{1I}\Omega_{2J}\cdots\Omega_{mK}$ where $\varepsilon^{IJ\ldots K}$ is the totally antisymmetric *Levi-Civita* symbol. Thus $\det(\Omega_{Ij}) = 0$ if m is odd.

$X = \frac{\partial}{\partial t} + \Xi^I(t,a)\frac{\partial}{\partial a^I}$ cannot locally represented by the Birkhoffian as equation (12), we fortunately still have a global formulation of the equations (21), i.e.,

$$i_X\Omega = 0, \quad i_X\, dt = 1. \tag{26}$$

which also include the time variation of the generalized Birkhoffian function B, i.e, the equation (22). It is interesting that the global form (26) is not only suitable to formulate Hamiltonian or Birkhoffian systems with contact (symplectic) structure but also enables to represent GBSs and other systems as nonholonomic systems [2], [3] of non-symplectic structure.

If the closed 2-form Ω is regular, the system is reduced to the Birkhoffian system. If the GBS is autonomous or semi-autonomous, the equation

$$\Omega_{IJ}\dot{a}^J - \frac{\partial B}{\partial a^I} = 0, \quad I, J = 1, 2, \ldots, m \tag{27}$$

can be globally formulated by

$$i_X\Omega = -dB \tag{28}$$

where $\Omega = \frac{1}{2}\Omega_{IJ}da^I \wedge da^J$. Combining the equation (22) with equation (28), yields that $\frac{\partial B}{\partial t} = 0$, i.e., the so-called semiautonomous Birkhoffian systems are really autonomous ones. Furthermore, if the closed 2-form is regular, the Lie algebra can be constructed by the Poisson bracket

$$\{f, g\} = \Omega^{-1}(df, dg) = \Omega(X_f, X_g) \tag{29}$$

in accordance with the time evolution law $\dot{a}^I = \Omega^{IJ}\frac{\partial B}{\partial a^J} = 0$. It should pointed out that Lie algebra structure does not generally exist for a GBS unless $\frac{\partial R_I}{\partial t} = 0$, $\det(\Omega_{IJ}) \neq 0$.

4 Integration of generalized Birkhoff's equations

As Birkhoffian mechanics, the most important tasks to study the generalized Birkhoffian mechanics mainly focus on both constructing the Birkhoffian, Birkhoffian functions and integrating the generalized Birkhoff's equations. The first procedure should be checking whether the methods utilized in Birkhoffian mechanics can be generalized to GBSs or not. Fortunately, all the existing methods to construct Birkhoffian functions can be used in GBSs because that the methods only rely upon the locality, analyticity of the integrand, independent of the regularity of the matrix Ω_{IJ} mentioned above.

A given first-order system verifying the conditions of locality, analyticity and regularity always admits infinite varieties of equivalent generalized Birkhoffian representations characterized by the gauge transformations

$$R_I(t,a) \to R'_I(t,a) = R_I(t,a) + \frac{\partial G(t,a)}{\partial a^I} \tag{30a}$$

$$B(t,a) \to B'(t,a) = B(t,a) - \frac{\partial G(t,a)}{\partial t} \tag{30b}$$

All the Birkhoffian representations are equivalent in the sense that Birkhoff's equations are the same for all possible functions (30), i.e.,

$$\left(\frac{\partial R'_\nu}{\partial a^\mu} - \frac{\partial R'_\mu}{\partial a^\nu}\right)\dot a^\nu - \left(\frac{\partial B'}{\partial a^\mu} + \frac{\partial R'_\mu}{\partial t}\right) = \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot a^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right) \quad (31)$$

The practical meaning is that we can choose a gauge function $G(t, a)$ to make the Birkhoffian be the physical energy of the system.

Therefore, we can outline the three methods to construct Birkhoffian functions as follows.

Method 1. Let B be the total energy of the system and then solve the Cauchy--Kowalevski equations (19a) in the functions R_I.

Method 2. Via the method of the genotopic transformations starting from the equation (16), construct a self-adjoint covariant form

$$[\Omega_{IJ}(t,a)\,\dot a^J + \Gamma_I(t,a)]_{SA} = 0, \qquad I, J = 1, 2, \ldots, m \quad (32)$$

The functions R_I are then given by

$$R_I(t,a) = \left[\int_0^1 d\tau\,\tau\Omega_{IJ}(t,\tau a)\right] a^J \quad (33)$$

and the Birkhoffian is provided by the rule

$$B(t,a) = -\left[\int_0^1 d\tau \left(\Gamma_I + \frac{\partial R_I}{\partial t}\right)(t,\tau a)\right] a^I \quad (34)$$

This method is recommended when no physical condition is imposed on the meaning of the Birkhoffian and on the prescriptions for the construction of the first-order form. It is often preferable in practice, because of the greater freedom in the Birkhoffian functions.

Method 3. Suppose that the m obtained first integrals $\mathcal{I}^J$ of the first-order system (16) are independent in the sense that $\det(\partial\mathcal{I}^J/\partial a^I) \neq 0$. Then

$$R_I(t,a) = G_J \frac{\partial \mathcal{I}^J}{\partial a^I}, \quad B(t,a) = -G_J \frac{\partial \mathcal{I}^J}{\partial t} \quad (35)$$

where G_J are functions of the integrals $\mathcal{I}^I$, which are not constrained by the regularity condition $\det(\partial G_I/\partial\mathcal{I}^J - \partial G_J/\partial\mathcal{I}^I) \neq 0$.

5 Application of generalized Birkhoffian formulation to nonholonomic systems

As an important example of GBS, we consider a mechanical system on the contact manifold $R \times TQ$ with local coordinates $\{t, q^i, \dot q^i\}$ $(i = 1, 2, \ldots, n)$. Denote the Lagrangian of the system by $\mathcal{L}(t, q, \dot q)$ and suppose the system is subject to the nonholonomic constraints

$$\dot q^\alpha = \varphi^\alpha(t, q^i, \dot q^\mu), \quad \alpha = 1, 2, \ldots, l;\ \mu = 1, 2, \ldots, k = n - l \quad (36)$$

The dynamics of the nonholonomic system is uniquely determined by the four factors: (1) Lagrange-d'Alembert principle, (2) ideal constraints, (3) Chetaev's condition for the virtual displacement, and (4) the regularity of the Hessian matrix $\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j}\right)$. Of course, the locality and analyticity are understood. The equations of motion for the system form a set of mixed second- and first-order ordinary differential equations [6], [16], [17], [19]

$$\ddot{q}^\mu = f^\mu(t, q^\nu, \dot{q}^\nu, q^\alpha), \quad \nu = 1, 2, \ldots, k = n - l \tag{37a}$$

$$\dot{q}^\alpha = \varphi^\alpha(t, q^\nu, \dot{q}^\nu, q^\beta), \quad \beta = 1, 2, \ldots, l \tag{37b}$$

The Lagrangian and Hamiltonian inverse problem for such a coupled system is not universal [1], [7], [12]. The nonholonomicity of the system makes the Birkhoffian realization for the system to be not universal. However, the universality of self-adjointness for the nonholonomic system can be realized in the generalized Birkhoffian framework based on the conditions of locality, analyticity and regularity of the system.

Introduce l regular coordinates $\{x^\mu\}$

$$x^\mu = \xi^\mu(t, q^\nu, \dot{q}^\nu, q^\alpha), \quad \det(\partial \xi^\mu / \partial \dot{q}^\nu) \neq 0 \tag{38}$$

whose inverse transformation is

$$\dot{q}^\mu = \zeta^\mu(t, q^\nu, x^\nu, q^\alpha) \tag{39}$$

Substituting equation (39) into the equation (37) we get the following first-order system

$$\dot{q}^\mu = \zeta^\mu(t, q^\nu, x^\nu, q^\alpha) \tag{40a}$$

$$\dot{x}^\mu = \psi^\mu(t, q^\nu, x^\nu, q^\alpha) \tag{40b}$$

$$\dot{q}^\alpha = \varphi^\alpha(t, q^\nu, x^\nu, q^\beta) \tag{40c}$$

Sometimes we directly choose x^μ to be generalized velocity $\dot{q}^\mu$ or generalized momentum p_μ. Denote the $m = 2k + l$ local coordinates $\{q^\nu, x^\nu, q^\alpha\}$ on constraint manifold M by $\{a^I\}$ $(I = 1, 2, \ldots, m = 2k + l)$. Then the equations (37) can be reformulated by

$$\dot{a}^I = \Xi^I(t, a^J), \quad I, J = 1, 2, \ldots, m = 2k + l \tag{41}$$

The locality, analyticity and regularity of the functions $\Xi^I(t, a)$ make the equations (41) admit a generalized Birkhoffian formulation

$$\left[\frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}\right] \dot{a}^J - \left(\frac{\partial B}{\partial a^I} + \frac{\partial R_I}{\partial t}\right) = 0, \quad I, J = 1, 2, \ldots, m = 2k + l \tag{42}$$

where the total energy of the system can be taken as the Birkhoffian B and the functions R_I are related with the nonholonomic constraint forces. It should be remarked that the regularity of Hessian matrix for the original nonholonomic mechanical system does not assure the regularity of the matrix (Ω_{IJ}) which is determined by the integrability of the constraints or the nonholomicity of odd number of constraints. If l is even the system is a Birkhoffian system. For the case of odd l, no symplectic and Lie algebra structure exist on the constraint manifold M. However, the self-adjointness for the equation (41) is independent of parity of the number m of the nonholonomic constraints.

If the second-order equations are decoupled with the constraints, e.g., the Chaplygin's system

$$\ddot{q}^{\mu} = f^{\mu}(t, q^{\nu}, \dot{q}^{\nu}), \quad \nu = 1, 2, \ldots, k = n - l \tag{43a}$$

$$\dot{q}^{\alpha} = \varphi^{\alpha}(t, q^{\nu}, \dot{q}^{\nu}), \quad \alpha = 1, 2, \ldots, l \tag{43b}$$

the Birkhoffian formulation can be realized on a $2k$-dimensional subspace [9], [11].

Example 1. [9] Consider the motion of a simplified sleigh with unit mass and unit moment of inertia in $\mathbb{R}^2 \times T^1$ with coordinates (x, y, φ), subjected to the nonholonomic constraint $\dot{y} = \dot{x}\tan\varphi$. The Lagrangian is $L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2\right)$ and the Lagrangian embedded in the constraint is $\mathcal{L} = \frac{1}{2}\left(\dot{x}^2 \sec^2\varphi + \dot{\varphi}^2\right)$. Obviously the system is a Chaplygin's system and thus reduced to a holonomic nonconservative subsystem for x and φ on a submanifold $h_{\tau}^{s} \subset T\left(\mathbb{R}^2 \times T^1\right)$, decouped with the constraint. The Chaplygin's equations of motion are

$$\ddot{x} + \dot{x}\dot{\varphi}\tan\varphi = 0, \quad \ddot{\varphi} = 0, \quad \dot{y} - \dot{x}\tan\varphi = 0$$

Utilizing the Legendre transformation $\dot{x} = p_x \cos^2\varphi$, $\dot{\varphi} = p_{\varphi}$, the Hamiltonian embedded in the constraint is $\mathcal{H} = \frac{1}{2}\left(p_x^2 \cos^2\varphi + p_{\varphi}^2\right)$. The equations of motion are given by the matrix form

$$\begin{pmatrix} 0 & p_x \tan\varphi & -1 & 0 \\ -p_x \tan\varphi & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\varphi} \\ \dot{p}_x \\ \dot{p}_{\varphi} \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} p_x^2 \sin 2\varphi \\ p_x \cos^2\varphi \\ p_{\varphi} \end{pmatrix}$$

with four independent first integrals

$$I^1 = p_{\varphi}, \quad I^2 = \varphi - p_{\varphi} t, \quad I^3 = p_x \cos\varphi, \quad I^4 = \frac{1}{2}\left[\omega^2 x^2 + p_x^2 \cos^4\varphi\right]$$

where $\omega = \dot{\varphi}$ is constant.

Taking the conventional notations $a^J = \{x, \varphi, p_x, p_{\varphi}\}$ $(J = 1, 2, 3, 4)$, we find out a set of Birkhoffian functions by means of Hojman's method [5], [14]

$$\begin{aligned}
R_1 &= a^1 a^3 \left(a^4\right)^2 \cos a^2 \\
R_2 &= \frac{1}{2} a^4 + \frac{a^3 \cos a^2}{2a^4} - \left(a^3\right)^2 \cos^3 a^2 \sin 2a^2 \\
R_3 &= \left(a^3\right)^2 \cos^5 a^2 \\
R_4 &= -a^2 + \frac{1}{2} a^4 t - \frac{t a^3 \cos a^2}{2a^4} + \left(a^1\right)^2 a^3 a^4 \cos a^2 \\
B &= \frac{1}{2}\left[\left(a^4\right)^2 + a^3 \cos a^2\right]
\end{aligned}$$

Thus the symplectic tensor $\Omega_{IJ} = \frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}$ is given by the matrix elements

$$
\begin{aligned}
&\Omega_{11} = \Omega_{22} = \Omega_{33} = \Omega_{44} = 0 \\
&\Omega_{12} = -\Omega_{21} = a^1 a^3 \left(a^4\right)^2 \sin a^2 \\
&\Omega_{13} = -\Omega_{31} = -a^1 \left(a^4\right)^2 \cos a^2 \\
&\Omega_{14} = -\Omega_{41} = 0 \\
&\Omega_{23} = -\Omega_{32} = \left(a^3\right)^2 \cos^4 a^2 \sin a^2 - \frac{\cos a^2}{2a^4} \\
&\Omega_{24} = -\Omega_{42} = -\frac{3}{2} - \left(a^1\right)^2 a^3 a^4 \sin a^2 + \frac{a^3 \cos a^2 + t a^3 a^4 \sin a^2}{2(a^4)^2} \\
&\Omega_{34} = -\Omega_{43} = \left(a^1\right)^2 a^4 \cos a^2 - \frac{t \cos a^2}{2a^4}
\end{aligned}
$$

which satisfies the conditions of self-adjointness. It can be verified that the equations of motion can be represented by the nonautonomous Birkhoff's equations

$$\Omega_{IJ} \dot{a}^J - \frac{\partial B}{\partial a^I} - \frac{\partial R_I}{\partial t} = 0$$

Example 2. Consider a nonholonomic system whose configuration is denoted by $\{q^1, q^2\}$. The Lagrangian of the system is $L = \frac{1}{2}\left((\dot{q}^1)^2 + (\dot{q}^2)^2\right)$. Suppose the system is constrained by a nonholonomic constraint

$$\dot{q}^1 + t\dot{q}^2 - q^2 + t = 0.$$

Then the differential equations of motion for the system are

$$
\begin{aligned}
(1 + t^2)\ddot{q}^2 + 2t\dot{q}^2 + 2\dot{q}^1 - 2q^2 + 3t = 0 \\
\dot{q}^1 + t\dot{q}^2 - q^2 + t = 0
\end{aligned}
$$

Let $a^1 = q^2$, $a^2 = \dot{q}^2$, $a^3 = q^1$, then the equations can be transformed into the first-order differential equations

$$
\begin{aligned}
\dot{a}^1 &= a^2, \\
\dot{a}^2 &= \frac{-t}{1 + t^2}, \\
\dot{a}^3 &= a^1 - ta^2 - t
\end{aligned}
$$

with three independent first integrals

$$
\begin{aligned}
I^1 &= a^3 - t\left(a^1 - ta^2 - t\right) - \frac{1}{2}\ln\left(1 + t^2\right), \\
I^2 &= a^1 - ta^2 - t + \arctan t, \\
I^3 &= a^2 + \frac{1}{2}\ln\left(1 + t^2\right)
\end{aligned}
$$

By using the Hojman's method, we can get the Birkhoffian functions

$$\begin{aligned}
R_1 &= G_1\frac{\partial I^1}{\partial a^1} + G_2\frac{\partial I^2}{\partial a^1} + G_3\frac{\partial I^3}{\partial a^1} = -G_1 t + G_2 \\
R_2 &= G_1\frac{\partial I^1}{\partial a^2} + G_2\frac{\partial I^2}{\partial a^2} + G_3\frac{\partial I^3}{\partial a^2} = G_1 t^2 - G_2 t + G_3 \\
R_3 &= G_1\frac{\partial I^1}{\partial a^3} + G_2\frac{\partial I^2}{\partial a^3} + G_3\frac{\partial I^3}{\partial a^3} = G_1 \\
B &= -\left[G_1\frac{\partial I^1}{\partial t} + G_2\frac{\partial I^2}{\partial t} + G_3\frac{\partial I^3}{\partial t}\right] \\
&= -G_1\left(2ta^2 + 2t - a^1 - \frac{t}{1+t^2}\right) - G_2\left(\frac{1}{1+t^2} - a^1 - 1\right) - G_3\frac{t}{1+t^2}
\end{aligned}$$

Set $G_1 = I^2, G_2 = 0, G_3 = I^3$. Then

$$\begin{aligned}
R_1 &= -t\left(a^1 - ta^2 - t + \arctan t\right) \\
R_2 &= t^2\left(a^1 - ta^2 - t + \arctan t\right) + a^2 + \frac{1}{2}\ln\left(1+t^2\right) \\
R_3 &= a^1 - ta^2 - t + \arctan t \\
B &= \left(2ta^2 + 2t - a^1 - \frac{t}{1+t^2}\right)\left(-a^1 + ta^2 + t - \arctan t\right) \\
&\quad - \frac{t}{1+t^2}\left[a^2 + \frac{1}{2}\ln\left(1+t^2\right)\right]
\end{aligned}$$

Thus the presymplectic tensor $\Omega_{IJ} = \frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}$ is given by the matrix

$$(\Omega_{IJ})_{3\times 3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -t \\ -1 & t & 0 \end{pmatrix}$$

It can be verified that the equations of motion can be represented by the generalized Birkhoff equations

$$\Omega_{IJ}\dot{a}^J - \frac{\partial B}{\partial a^I} - \frac{\partial R_I}{\partial t} = 0$$

Concluding remarks

As shown above the inverse problem of the calculus of variations for a dynamical system is characterized essentially by the self-adjointness conditions of the equations of motion in first-order form, which is equivalent to a closed 2-form on the manifold. Any local, analytic, regular, finite-dimensional, nonholonmic, self-adjoint or non-self-adjoint dynamical systems in first-order form always admit a generalized Birkhoffian formulation in a contractible region of regular point of variables. The sequence from self-adjointness to symplecticity and to Lie algebra of the formulation for the dynamics is a sequence for the conditions to become more and more strict. The symbiosis of self-adjoint/symplectic/Lie algebraic/physical formulation is only suitable to the local, analytic, regular, holonomic, autonomous dynamical systems.

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