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Some geometric aspects of the calculus of variations in several independent variables

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Abstract. This paper describes some recent research on parametric problems in the calculus of variations. It explains the relationship between these problems and the type of problem more usual in physics, where there is a given space of independent variables, and it gives an interpretation of the first variation formula in this context in terms of cohomology.

1 Introduction

In this paper we consider some geometrical aspects of those problems in the calculus of variations which are known as 'parametric': see, for example, the classical work [9] for the difference between parametric and non-parametric variational problems. To illustrate this difference in a simple way, consider the following, superficially similar, examples of the two types of problem. For the first problem, suppose we are asked to find the trajectory of a free unit-mass particle in three-dimensional space with coordinates (u^1, u^2, u^3). For the second, suppose we are asked to find the shortest curve between two points in three-dimensional space with differently-labelled coordinates (y^1, y^2, y^3). A solution to the former problem is a map $[0, T] \to \mathbb{R}^3$, $t \mapsto (a^i t + b^i)$, and a Lagrangian for the problem is $\frac{1}{2}\left((\dot{u}^1)^2 + (\dot{u}^2)^2 + (\dot{u}^3)^2\right)$. In contrast, a solution to the latter problem is a straight line segment $[(p^i), (q^i)] \subset \mathbb{R}^3$, and a Lagrangian is $\sqrt{(\dot{y}^1)^2 + (\dot{y}^2)^2 + (\dot{y}^3)^2}$: note that this latter function is 'positively homogeneous'.

More generally, variational problems in physics are commonly defined on fibred manifolds $\pi : E \to M$ (for the free particle, this would be $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$). Extremals are local sections of π, and the Lagrangian is a function (or, more properly, a differential m-form, where $m = \dim M$) defined on a jet bundle $J^1\pi$ (or $J^k\pi$) of jets of local sections of π. But in geometry, variational problems are commonly defined on manifolds E without a given fibration. Extremals are then submanifolds

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of E, defined 'parametrically'. To see where the Lagrangian might be defined, we need to consider different types of jet bundle and the relationships between them.

We can illustrate this by examining the relationships between a vector space, an affine space and a projective space. If V is a vector space with $\dim V = n+1$, a basis $(e_0, e_1, \ldots, e_n)$ and corresponding coordinate functions $(\dot{y}^0, \dot{y}^1, \ldots, \dot{y}^n)$, then the set

$$A = \{v \in V : \dot{y}^0(v) = 1\}$$

is an n-dimensional affine space, whereas the set

$$P = (V - \{0\})/(\mathbb{R} - \{0\})$$

is an n-dimensional projective space; there is a natural injection $A \to P$.

Now let $\pi : E \to \mathbb{R}$ be a fibred manifold, with $\dim E = n+1$ and coordinates $(y^0 = t, y^1, \ldots, y^n)$; we can apply the remark above to the fibres of the tangent bundle to E. We write $J^1\pi$ for the manifold of jets of local sections of π, and $J^1(E,1)$ for the manifold of jets of immersed 1-dimensional submanifolds in E. The bundle $J^1\pi \to E$ is an affine bundle, and there is a canonical injection $J^1\pi \to TE$ whose image is given by $\dot{y}^0 = 1$. On the other hand, the bundle $J^1(E,1) \to E$ is isomorphic to the projective tangent bundle $PTE \to E$, and we may identify $J^1\pi$ with an open submanifold of $J^1(E,1)$ by mapping the jet of a local section to the jet of its image. Writing $\mathring{T}E$ for the slit tangent manifold, excluding the zero section, we may see that the bundle $\mathring{T}E \to J^1(E,1)$ is a principal bundle with structure group $\mathbb{R} - \{0\}$.

As an application of this structure, we mention the study of Finsler geometry (see, for example, [2]), or of its special case, Riemannian geometry. Here, we take a manifold E with local coordinates y^a $(0 \le a \le n)$. The Lagrangian (that is, the Finsler function) L is defined on $\mathring{T}E$, and the condition of positive homogeneity is that $\dot{y}^a \partial L/\partial \dot{y}^a = L$. The variational problem is to find extremals γ of the integral

$$\int j^1\gamma^*(L)\, dt$$

subject to suitable boundary conditions. If γ is an extremal then so is $\gamma \circ \phi$ where

$$\phi : \mathbb{R} \to \mathbb{R} \text{ is a diffeomorphism}, \qquad \phi' > 0$$

The problem may also be formulated on the quotient manifold $PTE^+ = \mathring{T}E/\mathbb{R}^+$, which is a double cover of the projective tangent bundle PTE.

Our task in this paper will be to extend these structures to provide a framework for the study of multiple-integral parametric variational problems, of first or higher order. In Section 2 we shall describe a geometrical background which is appropriate for a study of these problems, and in Section 3 we shall introduce a particular class of vector forms which will turn out to useful tools for our investigation. Section 4 contains a brief reminder, for comparison, of an approach to non-parametric problems defined on spaces of jets of fibred manifolds, and finally in Section 5 we show how an analogous approach may be devised for parametric problems with positively homogeneous Lagrangians.

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2 Geometrical background

In this section we shall describe a geometrical background which may be used for the study of parametric variational problems. Convenient references here are [3], [4]; see also [7].

Finsler geometry, a single-integral problem, is defined on the slit tangent bundle $\mathring{T}E$; first-order multiple integral problems are defined on a sub-bundle of the Whitney sum $\bigoplus^m TE$. The *bundle of regular velocities on* E is

$$\mathring{T}_{(m)}E = \{(\xi_1, \ldots, \xi_m) \in \bigoplus\nolimits^m TE : (\xi_i) \text{ linearly independent}\} ;$$

equivalently, we may say that $\mathring{T}_{(m)}E$ is the bundle of 'non-degenerate velocities', 1-jets $j^1_0\sigma$ at $0 \in \mathbb{R}^m$ of non-singular maps $\sigma : \mathbb{R}^m \to E$. If (y^a) are local coordinates on $Y \subset E$, then (y^a, y^a_i) $(1 \le i \le m)$ are local coordinates on $Y^1 \subset \mathring{T}_{(m)}E$, where

$$y^a_i(j^1_0\sigma) = \left.\frac{\partial\sigma^a}{\partial t^i}\right|_0 , \qquad y^a_i(\xi_1, \ldots, \xi_m) = \dot{y}^a(\xi_i)$$

and where $Y^1 = \tau_m^{-1}(Y)$ with $\tau_m : \mathring{T}_{(m)}E \to E$ the natural projection.

As with any manifold of jets, we may define contact forms and other related structures on $\mathring{T}_{(m)}E$. We say that a differential form $\omega \in \Omega(\mathring{T}_{(m)}E)$ is a *contact form* if the pull-back $(j^1\sigma)^*\omega$ by the prolongation of any non-singular map $\sigma : \mathbb{R}^m \to E$ always vanishes. In coordinates, contact 1-forms are linear combinations of $(m+1) \times (m+1)$ determinants like

$$\begin{vmatrix} y^{a_1}_1 & y^{a_2}_1 & \cdots & y^{a_{m+1}}_1 \\ y^{a_1}_2 & y^{a_2}_2 & \cdots & y^{a_{m+1}}_2 \\ \vdots & \vdots & & \vdots \\ y^{a_1}_m & y^{a_2}_m & \cdots & y^{a_{m+1}}_m \\ dy^{a_1} & dy^{a_2} & \cdots & dy^{a_{m+1}} \end{vmatrix}$$

and so have a more complicated expression than the contact 1-forms $du^\alpha - u^\alpha_i dx^i$ on a jet bundle.

Next, for each function $f : E \to \mathbb{R}$, define the functions $d_i f : \mathring{T}_{(m)}E \to \mathbb{R}$ by

$$d_i f(j^1_0\sigma) = \frac{\partial(f \circ \sigma)}{\partial t^i} \qquad \text{where } \sigma : \mathbb{R}^m \to E ;$$

the operator d_i is a globally-defined vector field along $\tau_m : \mathring{T}_{(m)}E \to E$, called a *total derivative*. It is straightforward to check that a 1-form θ is a contact form exactly when $\langle d_i, \theta \rangle = 0$ for $1 \le i \le m$. In coordinates, we see that

$$d_i = y^a_i \frac{\partial}{\partial y^a} .$$

Finally, the Whitney sum $\bigoplus^m TE \to E$ is a vector bundle, and so supports a vertical lift operation, arising from the canonical isomorphism between a vector space and its tangent space at any point. Denote the vertical lift to (η_i) by

$$\bigoplus^m T_{\tau_m(\eta_i)}E \to T_{(\eta_i)}\left(\bigoplus^m TE\right), \qquad (\xi_i) \mapsto (\xi_i)^{\uparrow(\eta_i)}.$$

Then, for each vector $\zeta \in T_{(\eta_i)}\mathring{T}_{(m)}E$, define the vector $S^i\zeta \in T_{(\eta_i)}\mathring{T}_{(m)}E$ by

$$S^i\zeta = (0, \ldots, 0, T\tau_m(\zeta), 0, \ldots, 0)^{\uparrow(\eta_i)}.$$

With this definition S^i is a type $(1,1)$ tensor field on $\mathring{T}_{(m)}E$, called a *vertical endomorphism*; in coordinates we have

$$S^i = dy^a \otimes \frac{\partial}{\partial y^a_i}.$$

We may also relate the bundle of regular velocities $\mathring{T}_{(m)}E$ with the *Grassmannian bundle* $J^1(E,m)$: the former is a manifold of equivalence classes of non-degenerate maps $\mathbb{R}^m \to E$, whereas the latter is a manifold of equivalence classes of images of such maps, namely of m-dimensional subspaces of TE. We see that two regular velocities $j^1_0\sigma$, $j^1_0\hat{\sigma}$ represent the same subspace when

$$j^1_0\hat{\sigma} = j^1_0(\sigma \circ \phi)$$

for some diffeomorphism $\phi : \mathbb{R}^m \to \mathbb{R}^m$ with $\phi(0) = 0$. We may also consider the bundle $J^1(E,m)^+$ of oriented Grassmannians, where the diffeomorphism ϕ must preserve the orientation on $\mathbb{R}^m$. The natural projections give principal bundles

$$\begin{aligned} \rho &: \mathring{T}_{(m)}E \to J^1(E,m) && \text{(group } GL(m,\mathbb{R})\text{)} \\ \rho^+ &: \mathring{T}_{(m)}E \to J^1(E,m)^+ && \text{(group } GL(m,\mathbb{R})^+\text{)}, \end{aligned}$$

where a basis of fundamental vector fields is given by $\{\Delta^i_j = S^i(d_j)\}$. In coordinates, we therefore have

$$\Delta^i_j = y^a_j \frac{\partial}{\partial y^a_i}.$$

Note that any fibration $\pi : E \to M$ defines open submanifolds $J^1\pi \subset J^1(E,M)$ and $J^1\pi \subset J^1(E,M)^+$. If we take $m = 1$ we recover the special cases $J^1(E,1) = PTE$ and $J^1(E,1)^+ = PTE^+$.

We can finally, without too much conceptual difficulty although with increased computational complexity, extend these definitions to the case of higher-order regular velocities. We shall take the manifold of k-th order regular velocities $\mathring{T}^k_{(m)}E$ to be the set of all k-jets (at the origin) of non-singular maps $\mathbb{R}^m \to E$, with local coordinates y^a_I on $\mathring{T}^k_{(m)}E$, where $I \in \mathbb{N}^m$ is a symmetric multi-index with $0 \le |I| \le k$. The total derivatives d_i and vertical endomorphisms S^i have coordinate representations

$$d_i = \sum_{|I|=0}^{k-1} y^a_{I+1_i} \frac{\partial}{\partial y^a_I}, \qquad S^i = \sum_{|I|=0}^{k-1} \big(I(i)+1\big) dy^a_I \otimes \frac{\partial}{\partial y^a_{I+1_i}}$$

where different instances of each type of operator commute, so that we may use multi-index notation d_I, S^I where appropriate. We may again construct principal bundles

$$\rho^k : \mathring{T}^k_{(m)}E \to J^k(E,m)\,, \qquad \rho^{k+} : \mathring{T}^k_{(m)}E \to J^k(E,m)^+\,,$$

whose groups are the jet groups L^k_m, L^{k+}_m

$$\begin{aligned} L^k_m &= \{j^k_0\phi : \mathbb{R}^m \xrightarrow{\phi} \mathbb{R}^m \text{ is a diffeomorphism, } \phi(0) = 0\} \\ L^{k+}_m &= \{j^k_0\phi \in L^k_m : |\mathcal{J}(\phi)| > 0\} \end{aligned}$$

A basis for the space of fundamental vector fields of the principal bundles is given by

$$\{\Delta^I_j = S^I(d_j) : 1 \leq |I| \leq k\}\,;$$

we put i^I_j for the contraction with Δ^I_j, and d^I_j for the Lie derivative by Δ^I_j.

3 Vector forms

In the study of the parametric calculus of variations we use vectors of operators d_i, tensors S^i, and forms ϑ^i. These fit into a framework of *vector forms* [12], and the use of these forms will provide us with a convenient tool.

We consider forms on $\mathring{T}^k_{(m)}E$ taking values in the vector space $\mathbb{R}^{m*}$ and its exterior powers: put

$$\Omega^{r,s}_k = \left(\Omega^r \mathring{T}^k_{(m)}E\right) \otimes \left(\bigwedge{}^s \mathbb{R}^{m*}\right)\,.$$

Let the standard basis for $\mathbb{R}^{m*}$ be denoted by (dt^i); then a vector form Φ may be written canonically as

$$\Phi = \phi_{i_1\cdots i_s} \otimes dt^{i_1} \wedge \ldots \wedge dt^{i_s} \in \Omega^{r,s}_k$$

where the scalar forms $\phi_{i_1\cdots i_s}$ are skew-symmetric in their indices. Although this looks like a coordinate formula, in fact it is not: the indices $i_1, \ldots, i_s$ refer to a fixed basis of $\mathbb{R}^{m*}$, and so the formula is valid globally on $\mathring{T}^k_{(m)}E$.

We shall consider several operators on vector forms. First, obviously, we may use the de Rham differential $d : \Omega^{r,s}_k \to \Omega^{r+1,s}_k$, defined on decomposable forms by

$$d(\phi \otimes \eta) = d\phi \otimes \eta$$

and extended to arbitrary forms by linearity. But we may also use the total derivatives d_i to define two further operators, a contraction and a Lie derivative, by

$$i_{\mathrm{T}} : \Omega^{r,s}_k \to \Omega^{r-1,s+1}_{k+1}\,, \qquad i_{\mathrm{T}}(\phi \otimes \eta) = (d_i \lrcorner\, \phi) \otimes (dt^i \wedge \eta)$$

and

$$d_{\mathrm{T}} : \Omega^{r,s}_k \to \Omega^{r,s+1}_{k+1}\,, \qquad d_{\mathrm{T}}(\phi \otimes \eta) = d_i\phi \otimes (dt^i \wedge \eta)\,.$$

It is immediate, from the definitions and the properties of contraction and Lie derivative on scalar forms, that

$$dd_{\mathrm{T}} = d_{\mathrm{T}}d\,, \qquad d^2_{\mathrm{T}} = 0\,, \qquad d_{\mathrm{T}} = di_{\mathrm{T}} + i_{\mathrm{T}}d\,.$$

We may therefore construct a bicomplex, where in the first column it is convenient to write $\overline{\Omega}^{0,s}$ to denote the quotient $\Omega^{0,s} / \bigwedge^s \mathbb{R}^{m*}$ of vector-valued functions modulo constant functions. In the diagram we omit explicit mention of the order of the velocity manifolds on which the spaces are defined; if the order of the spaces for a given row is k then the order for the next row will be $k+1$. For small values of k of course only the lower part of the diagram will exist.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \overline{\Omega}^{0,0} \arrow[r, "d"] \arrow[d, "d_{\mathrm{T}}"'] & \Omega^{1,0} \arrow[r, "d"] \arrow[d, "d_{\mathrm{T}}"'] & \Omega^{2,0} \arrow[r, "d"] \arrow[d, "d_{\mathrm{T}}"'] & \Omega^{3,0} \arrow[r] \arrow[d, "d_{\mathrm{T}}"'] & {} \\
 & \cdots \arrow[d, "d_{\mathrm{T}}"'] & \cdots \arrow[d, "d_{\mathrm{T}}"'] & \cdots \arrow[d, "d_{\mathrm{T}}"'] & \cdots \arrow[d, "d_{\mathrm{T}}"'] & \\
0 \arrow[r] & \overline{\Omega}^{0,m-2} \arrow[r, "d"] \arrow[d, "d_{\mathrm{T}}"'] & \Omega^{1,m-2} \arrow[r, "d"] \arrow[d, "d_{\mathrm{T}}"'] & \Omega^{2,m-2} \arrow[r, "d"] \arrow[d, "d_{\mathrm{T}}"'] & \Omega^{3,m-2} \arrow[r] \arrow[d, "d_{\mathrm{T}}"'] & {} \\
0 \arrow[r] & \overline{\Omega}^{0,m-1} \arrow[r, "d"] \arrow[d, "d_{\mathrm{T}}"'] & \Omega^{1,m-1} \arrow[r, "d"] \arrow[d, "d_{\mathrm{T}}"'] & \Omega^{2,m-1} \arrow[r, "d"] \arrow[d, "d_{\mathrm{T}}"'] & \Omega^{3,m-1} \arrow[r] \arrow[d, "d_{\mathrm{T}}"'] & {} \\
0 \arrow[r] & \overline{\Omega}^{0,m} \arrow[r, "d"] & \Omega^{1,m} \arrow[r, "d"] & \Omega^{2,m} \arrow[r, "d"] & \Omega^{3,m} \arrow[r] & {}
\end{tikzcd}
$$

The bicomplex described above might appear to have some relation to the variational bicomplex for differential forms on the jet prolongations of fibred spaces, and the latter, when defined in the usual way on the infinite jet manifold, is locally exact: indeed, its interior columns are globally exact [1], [13], [14] (see [15] for a useful summary). The present bicomplex is, however, defined on (a family of) finite-order velocity manifolds, and the map $d_{\mathrm{T}} : \Omega^{r,s}_k \to \Omega^{r,s+1}_{k+1}$ is *not* exact, even locally. It is, however, globally exact modulo pull-backs (for $r \geq 1$).

There are, perhaps surprisingly, *two* homotopy operators for d_{T} which are similar in formulation but subtly different in effect; the first was described in [12], and the second is a version for velocity manifolds of an operator described in [6]. The operators are $P, \widetilde{P} : \Omega^{r,s}_k \to \Omega^{r,s-1}_{(r+1)k-1}$, defined by

$$
\begin{aligned}
P(\Phi) &= P^j_{(s)}(\phi_{i_1 \cdots i_s}) \otimes \left\{ \frac{\partial}{\partial t^j} \,\lrcorner\, \left(dt^{i_1} \wedge \ldots \wedge dt^{i_s} \right) \right\} \\
\widetilde{P}(\Phi) &= \widetilde{P}^j_{(s)}(\phi_{i_1 \cdots i_s}) \otimes \left\{ \frac{\partial}{\partial t^j} \,\lrcorner\, \left(dt^{i_1} \wedge \ldots \wedge dt^{i_s} \right) \right\}
\end{aligned}
$$

where $P = \widetilde{P}$ when acting on vector 1-forms, or on first-order forms. The scalar operators $P^j_{(s)}$ and $\widetilde{P}^j_{(s)}$ are given by the formulæ

$$P^j_{(s)} = \sum_J \frac{(-1)^{|J|}(m-s)!|J|!}{r(m-s+|J|+1)!J!} d_J S^{J+1_j} ,$$
$$\widetilde{P}^j_{(s)} = \sum_J \frac{(-1)^{|J|}(m-s)!|J|!}{r^{|J|+1}(m-s+|J|+1)!J!} d_J \widetilde{S}^{J+1_j}$$

where, for a scalar form θ,

$$S^{1_{j_1}+1_{j_2}+\cdots+1_{j_r}}\theta = i_{S^{j_1}S^{j_2}\cdots S^{j_r}}\theta$$
$$\widetilde{S}^{1_{j_1}+1_{j_2}+\cdots+1_{j_r}}\theta = i_{S^{j_1}} i_{S^{j_2}} \cdots i_{S^{j_r}}\theta \, .$$

It is interesting to note that $\widetilde{P}^2 = 0$, but that $P^2 \neq 0$. Proofs that these operators really do act as homotopy operators modulo pullbacks may be found in the references cited (the proof for P is given in [6] for the related operator on jet manifolds, but the proof for velocity manifolds is essentially the same).

4 Variational problems on jet manifolds

For the purposes of comparison, we give a brief summary of the relevant part of variational theory on jet manifolds.

Let $\pi : E \to M$ be a fibred manifold, with $\dim M = m$ and $\dim E = m + n$, where the base manifold M is orientable; we take local coordinates x^i on M and (x^i, u^α) on E. We let $J^k\pi$ denote the manifold of k-th order jets of local sections of π [7], [10]. In this context a Lagrangian of order k is an m-form $\lambda = L\, d^m x$ on $J^k\pi$, horizontal over M. The fixed-boundary variational problem defined by λ is the search for submanifolds $\sigma(C) \subset E$ satisfying

$$\int_C ((j^k\sigma)^* X^k \lambda) = 0$$

for every variation field X on E satisfying $X|_{\sigma(\partial C)} = 0$, where X^k denotes the prolongation of X to $J^k\pi$.

Such a variational problem may be expressed in terms of certain other m-forms called Lepage forms [8]. The m-form θ on $J^l\pi$ (where $l \geq k$) is a *Lepage form* if $i_Y d\theta$ is a contact form whenever the vector field Y is vertical over E. It is a *Lepage equivalent* of λ if it is a Lepage form, and in addition $\pi^*_{l,k}\lambda - \theta$ is a contact form. Every Lagrangian m-form defined on $J^k\pi$ has a Lepage equivalent defined on $J^{2k-1}\pi$, although the question of whether there is a suitable geometric construction depends on the values of m and k.

The simplest cases, as might be expected, are for single-integral problems where $m = 1$. For a first-order Lagrangian $\lambda = L\, dx$ on $J^1\pi$ the 1-form

$$\theta = L\, dx + \frac{\partial L}{\partial \dot{u}^\alpha}(du^\alpha - \dot{u}^\alpha dx)$$

is the unique Lepage equivalent, the *Poincaré-Cartan form*; it is also defined on $J^1\pi$. For a higher-order Lagrangian $\lambda = L\,dx$ on $J^k\pi$ the 1-form

$$\theta = L\,dx + \sum_{p=0}^{k-1}\left(\sum_{q=0}^{k-p-1}(-1)^q\frac{d^q}{dx^q}\frac{\partial L}{\partial u^\alpha_{(p+q+1)}}\right)(du^\alpha_{(p)} - u^\alpha_{(p+1)}dx)$$

is the unique Lepage equivalent, and it is defined on $J^{2k-1}\pi$.

For a multiple integral variational problem where $m \geq 2$, a first-order Lagrangian $\lambda = L\,d^m x$ defined on $J^1\pi$ gives rise to three distinct globally-defined Lepage equivalents

$$\theta_1 = L\,d^m x + \frac{\partial L}{\partial u^\alpha_i}\omega^\alpha \wedge d^{m-1}x_i$$

$$\theta_2 = \frac{1}{L^{m-1}}\bigwedge_{i=1}^{m}\left(L\,dx^i + \frac{\partial L}{\partial u^\alpha_i}\omega^\alpha\right)$$

$$\theta_3 = \sum_{r=0}^{\min\{m,n\}}\frac{1}{(r!)^2}\frac{\partial^r L}{\partial u^{\alpha_1}_{i_1}\cdots\partial u^{\alpha_r}_{i_r}}\omega^{\alpha_1}\wedge\cdots\wedge\omega^{\alpha_r}\wedge d^{m-r}x_{i_1\cdots i_r}$$

where $\omega^\alpha = du^\alpha - u^\alpha_j dx^j$ (of course θ_2 is defined only where the Lagrangian does not vanish). For a second-order Lagrangian, Lepage equivalents similar to θ_1 and θ_2 may again be found; it is not known whether there is a Lepage equivalent similar to θ_3. If $m \geq 3$ then it is known that global Lepage equivalents cannot be constructed in a canonical way without the use of additional data such as a connection. A list of references for these various constructions may be found in [11].

5 Homogeneous problems

We now consider m-dimensional variational problems on E, with fixed boundary conditions. For our purposes it is sufficient to consider submanifolds of the form $\sigma(C)$, where $\sigma : \mathbb{R}^m \to E$ and $C \subset \mathbb{R}^m$ is a compact m-dimensional submanifold: this is because variational problems are local, in the sense that an m-dimensional submanifold of E is extremal with fixed boundary conditions if, and only if, every small piece of it is extremal with fixed boundary conditions.

A vector function $\Lambda = L\,d^m t \in \Omega^{0,m}$ is called a *Lagrangian* for a parametric variational problem. It is called *positively homogeneous* if it is equivariant with respect to the action of the jet group L^{k+}_m, where k is the order of the Lagrangian. If Λ is positively homogeneous then the scalar function L satisfies

$$d^i_j L = \delta^i_j L\,, \qquad d^I_j L = 0 \text{ for } |I| \geq 2\,.$$

The fixed-boundary variational problem defined by Λ is the search for submanifolds $\sigma(C) \subset E$ satisfying

$$\int_C ((j\sigma)^* X^k L) d^m t = 0$$

for every variation field X on E satisfying $X|_{\sigma(\partial C)} = 0$, where X^k denotes the prolongation of X to $\mathring{T}^k_{(m)}E$. We may study this problem by looking for 'equivalents' of Lagrangians.

Definition 1. Let $\Lambda \in \Omega^{0,m}$ be a positively homogeneous Lagrangian. A scalar m-form $\Theta_m \in \Omega^{m,0}$ is called an *integral equivalent* of Λ if

$$\Lambda = \left(\frac{(-1)^{m(m-1)/2}}{m!} \right) i^m_{\mathrm{T}} \Theta_m \,.$$

A vector r-form $\Theta_r \in \Omega^{r,m-r}$ is called an *intermediate equivalent* if

$$\Lambda = \frac{(-1)^{r(r-1)/2}(m-r)!}{m!} \, i^r_{\mathrm{T}} \Theta_r \qquad 0 \leq r \leq m-1 \,.$$

It is clear that if Θ_{r+1} is an equivalent of Λ then

$$\Theta_r = \frac{(-1)^r}{m-r} \, i_{\mathrm{T}} \Theta_{r+1}$$

is also an equivalent. We use the terminology 'integral equivalent' because if $\sigma : \mathbb{R}^m \to E$ then $(j\sigma)^* \Lambda = (j\sigma)^* \Theta_m$, where by $j\sigma$ we mean the prolongation of σ to a map $\mathbb{R}^m \to \mathring{T}^l_{(m)}E$ for l sufficiently large, so that

$$\int_C (j\sigma)^* \Lambda = \int_C (j\sigma)^* \Theta_m \,,$$

from which we see that $\Lambda = \Theta_0$ and Θ_m have the same extremals.

We may also define some related forms which are used to obtain the Euler--Lagrange equations for the problem.

Definition 2. Let Θ_m be an integral equivalent of Λ; define the scalar $(m+1)$-form $\mathcal{E}_m \in \Omega^{m+1,0}$ by

$$\mathcal{E}_m = d\Theta_m \,.$$

Now let Θ_r be an intermediate equivalent of Λ for $0 \leq r \leq m-1$; define the vector form $\mathcal{E}_r \in \Omega^{r+1,m-r}$ by

$$\mathcal{E}_r = d\Theta_r - (-1)^r d_{\mathrm{T}} \Theta_{r+1} \,.$$

By a straightforward calculation we see that, corresponding to the relationships describing a family of intermediate equivalents, we have

$$\mathcal{E}_r = \frac{(-1)^{r+1}}{m-r} \, i_{\mathrm{T}} \mathcal{E}_{r+1} \qquad 0 \leq r \leq m-1 \,;$$

the form $\mathcal{E}_0$ is called the *Euler form* of Θ_m. The various forms we have defined inhabit two diagonals of our bicomplex.

We shall now impose an additional property on the equivalents of a Lagrangian.

$$\Theta_r = \frac{(-1)^r}{m-r}\, i_{\mathrm{T}}\Theta_{r+1}, \quad \Theta_0 = \Lambda$$

$$\begin{tikzcd}
 & & & & \Theta_m \in \Omega^{m,0} \arrow[r, "d"] \arrow[dl] \arrow[d, "d_{\mathrm{T}}"] & \Omega^{m+1,0} \ni \mathcal{E}_m \arrow[dl] \\
 & & & \cdots \arrow[dl] \arrow[d, "d_{\mathrm{T}}"] & {} \arrow[dl] & \\
 & & \Theta_2 \in \Omega^{2,m-2} \arrow[r, "d"] \arrow[dl] \arrow[d, "d_{\mathrm{T}}"] & \Omega^{3,m-2} \ni \mathcal{E}_2 \arrow[dl] & & \\
 & \Theta_1 \in \Omega^{1,m-1} \arrow[r, "d"] \arrow[dl] \arrow[d, "d_{\mathrm{T}}"] & \Omega^{2,m-1} \ni \mathcal{E}_1 \arrow[dl] & & & \\
 \Lambda \in \overline{\Omega}^{0,m} \arrow[r, "d"] & \Omega^{1,m} \ni \mathcal{E}_0 & & & &
\end{tikzcd}$$

$$\mathcal{E}_r = d\Theta_r - (-1)^r d_{\mathrm{T}}\Theta_{r+1}$$

Definition 3. Let Λ be a positively homogeneous Lagrangian, and let Θ_r be an equivalent of Λ $(1 \leq r \leq m)$. We say that Θ_r is *Lepagian* if the corresponding Euler form $\mathcal{E}_0 = \varepsilon_0 \otimes d^m t \in \Omega^{1,m}$ satisfies

$$S\mathcal{E}_0 = (S^i \varepsilon_0) \otimes d^{m-1} t_i = 0\,,$$

so that $\mathcal{E}_0$ is horizontal over E.

So far, we have described conditions which integral (or intermediate) equivalents and their Euler forms must satisfy, but we have not yet indicated whether such forms exist. We shall now remedy that deficiency.

Theorem 1. *The vector 1-form*

$$\Theta_1 = P d\Lambda$$

defined on $\overset{\circ}{T}{}^{2k-1}_{(m)} E$ *is an integral equivalent of* Λ $(m = 1)$ *or an intermediate equivalent* $(m \geq 2)$*, and is Lepagian. It is called the* Hilbert equivalent *of* Λ.

Proof. From the definition of P,

$$P\Phi = P^j \phi \otimes d^{m-1} t_j\,, \quad \text{where} \quad P^j = \sum_J \frac{(-1)^{|J|}}{(|J|+1)J!} d_J S^{J+1_j}\,,$$

so that

$$\begin{aligned}
i_{\mathrm{T}} P d\Lambda &= i_{\mathrm{T}} P(dL \otimes d^m t) \\
&= i_{\mathrm{T}}(P^j dL \otimes d^{m-1} t_j) \\
&= i_k P^j dL \otimes dt^k \wedge d^{m-1} t_j \\
&= i_j P^j dL \otimes d^m t\,.
\end{aligned}$$

Then

$$\begin{aligned} i_j P^j dL &= i_j \bigg(\sum_J \frac{(-1)^{|J|}}{(|J|+1)J!} d_J S^{J+1_j} dL \bigg) \\ &= \sum_J \frac{(-1)^{|J|}}{(|J|+1)J!} d_J i_j S^{J+1_j} dL \end{aligned}$$

because $[i_k, d_j] = 0$; next

$$i_j P^j dL = \sum_J \frac{(-1)^{|J|}}{(|J|+1)J!} d_J (S^{J+1_j} i_j + S^{J+1_j-1_k} i^k_j) dL$$

because $[i_j, S^k] = i^k_j$ and $[i^J_j, S^k] = i^{J+1_k}_j$, but by homogeneity $i^J_j dL = d^J_j L = 0$ for $(|J| \geq 2)$; consequently

$$i_j P^j dL = i^j_j dL$$

because, when $|K| \geq 1$, S^K vanishes on functions and hence on $i_j dL$ and $i^k_j dL$; and so, finally,

$$i_j P^j dL = mL \,,$$

giving $i_{\mathrm{T}} P d\Lambda = m\Lambda$, so that Θ_1 is indeed an equivalent (integral or intermediate, as appropriate).

To show that Θ_1 is Lepagian, note that

$$\begin{aligned} S d_{\mathrm{T}} \Theta_1 &= S d_{\mathrm{T}} P d\Lambda \\ &= S d_{\mathrm{T}} (P^j dL \otimes d^{m-1} t_j) \\ &= S\big(d_i P^j dL \otimes (dt^i \wedge d^{m-1} t_j)\big) \\ &= S\big(d_j P^j dL \otimes d^m t\big) \\ &= S^i (d_j P^j dL) \otimes d^{m-1} t_i \\ &= S^i \bigg(\sum_{|J| \geq 0} \frac{(-1)^{|J|}}{(|J|+1)J!} d_{J+1_j} S^{J+1_j} dL \bigg) \otimes d^{m-1} t_i \\ &= S^i \bigg(\sum_{|K| \geq 1} \frac{(-1)^{|K|-1}}{K!} d_K S^K dL \bigg) \otimes d^{m-1} t_i \,; \end{aligned}$$

but $[S^i, d_k] = \delta^i_k$, giving $[S^i, d_K] = K(i) d_{K-1_i}$, so that

$$\begin{aligned} S d_{\mathrm{T}} \Theta_1 &= \sum_{|K| \geq 1} \frac{(-1)^{|K|-1}}{K!} (d_K S^{K+1_i} + K(i) d_{K-1_i} S^K) dL \otimes d^{m-1} t_i \\ &= S^i dL \otimes d^{m-1} t_i \\ &= S(dL \otimes d^m t) = S d\Lambda \end{aligned}$$

as the two parts of the sum over the multi-index K combine to give a collapsing sum. It it then immediate that $S\mathcal{E}_0 = 0$, as required. □

Theorem 2. *Let Λ be a homogeneous Lagrangian, with Hilbert equivalent Θ_1 and Euler form $\mathcal{E}_0$. If $\widetilde{\Theta}_1$ is any other Lepagian vector 1-form equivalent to Λ, with corresponding Euler form $\widetilde{\mathcal{E}}_0$, then*

$$\widetilde{\mathcal{E}}_0 = \mathcal{E}_0 \qquad \text{and} \qquad \widetilde{\Theta}_1 - \Theta_1 = d_{\mathrm{T}}\Phi$$

for some $\Phi \in \Omega^{r,m-2}$, so that if $m = 1$ then $\widetilde{\Theta}_1 = \Theta_1$.

Proof. It follows straightforwardly from the Lepagian condition $S\widetilde{\mathcal{E}}_0 = 0$ that $P\widetilde{\mathcal{E}}_0 = 0$, so that

$$\begin{aligned} 0 &= P\widetilde{\mathcal{E}}_0 \\ &= P(d\Lambda - d_{\mathrm{T}}\widetilde{\Theta}_1) \\ &= \Theta_1 - Pd_{\mathrm{T}}\widetilde{\Theta}_1 \\ &= \Theta_1 - (1 - d_{\mathrm{T}}P)\widetilde{\Theta}_1 \,, \end{aligned}$$

giving $\widetilde{\Theta}_1 - \Theta_1 = d_{\mathrm{T}}P\widetilde{\Theta}_1$ (or $\widetilde{\Theta}_1 = \Theta_1$ if $m = 1$). Thus

$$\begin{aligned} \widetilde{\mathcal{E}}_0 - \mathcal{E}_0 &= (d\Lambda - d_{\mathrm{T}}\widetilde{\Theta}_1) - (d\Lambda - d_{\mathrm{T}}\Theta_1) \\ &= -d_{\mathrm{T}}^2 P\widetilde{\Theta}_1 \\ &= 0 \,. \end{aligned}$$

□

In coordinates, if $\Lambda = L\, d^m t$ then the Hilbert equivalent and the Euler form are given by

$$\begin{aligned} \Theta_1 &= \sum_I \sum_J \frac{(-1)^{|I|}(I + J + 1_i)!|I|!|J|!}{(|I| + |J| + 1)!I!J!} d_I \left(\frac{\partial L}{\partial y^a_{I+J+1_i}} \right) dy^a_J \otimes d^{m-1}t_i \,, \\ \mathcal{E}_0 &= \sum_I (-1)^{|I|} d_I \left(\frac{\partial L}{\partial y^a_I} \right) dy^a \otimes d^m t \,. \end{aligned}$$

If $m \geq 2$ then there can indeed be Lepagian vector 1-forms which are equivalent to a given Lagrangian but differ from its Hilbert equivalent. To see this, let $\Phi \in \Omega^{0,m-2}$, so that $d_{\mathrm{T}}d\Phi \in \Omega^{1,m-1}$. Then

$$i_{\mathrm{T}}(\Theta_1 + d_{\mathrm{T}}d\Phi) = i_{\mathrm{T}}\Theta_1 - d_{\mathrm{T}}i_{\mathrm{T}}d\Phi = \Lambda - d_{\mathrm{T}}^2\Phi = \Lambda$$

and

$$d\Lambda - d_{\mathrm{T}}(\Theta_1 + d_{\mathrm{T}}d\Phi) = d\Lambda - d_{\mathrm{T}}\Theta_1 - d_{\mathrm{T}}^2 d\Phi = \mathcal{E}_0 \,,$$

although there is no reason why we should have $d_{\mathrm{T}}d\Phi = 0$. For instance, when $m = 2$ we could take $\Phi = y^1 \in \Omega^{0,0}$ and then $d_{\mathrm{T}}d\Phi = dy^1_i \otimes dt^i \neq 0$.

We can now construct a version of the first variation formula for homogeneous variational problems, using the Hilbert equivalent Θ_1 of a Lagrangian Λ. Given a variation field X on E with $X|_{\sigma(\partial C)} = 0$, and its prolongation $\widehat{X}$ on $\mathring{T}^l_{(m)}E$

with l sufficiently large, we may use the standard formula $d_{\widehat{X}} = di_{\widehat{X}} + i_{\widehat{X}}d$, Stokes' Theorem, and the formula $d\Lambda = \mathcal{E}_0 + d_{\mathrm{T}}\Theta_1$ to obtain

$$\begin{aligned}\int_C (j\sigma)^* d_{\widehat{X}}\Lambda &= \int_C (j\sigma)^* di_{\widehat{X}}\Lambda + \int_C (j\sigma)^* i_{\widehat{X}} d\Lambda \\ &= \int_{\partial C} (j\sigma)^* i_{\widehat{X}}\Lambda + \int_C (j\sigma)^* i_{\widehat{X}} d\Lambda \\ &= \int_C (j\sigma)^* i_{\widehat{X}}(\mathcal{E}_0 + d_{\mathrm{T}}\Theta_1)\,.\end{aligned}$$

But

$$\int_C (j\sigma)^* i_{\widehat{X}} d_{\mathrm{T}}\Theta_1 = \int_C (j\sigma)^* d_{\mathrm{T}} i_{\widehat{X}}\Theta_1 = \int_C d\big((j\sigma)^* i_{\widehat{X}}\Theta_1\big) = \int_{\partial C} (j\sigma)^* i_{\widehat{X}}\Theta_1 = 0$$

because prolongations commute with total derivatives, and the pull-back of d_{T} to $\mathbb{R}^m$ is d; thus

$$\int_C (j\sigma)^* d_{\widehat{X}}\Lambda = \int_C (j\sigma)^* i_{\widehat{X}}\mathcal{E}_0 = \int_C (j\sigma)^* i_X\mathcal{E}_0$$

because $\mathcal{E}_0$ is horizontal over E.

Now if $m = 1$ then the Hilbert equivalent is an integral equivalent of Λ. But if $m \geq 2$ then this is no longer true, and we need some further work to find integral equivalents. Let $\Lambda = L\,d^m t$ be a positively homogeneous Lagrangian with $m \geq 2$, and write its Hilbert equivalent Θ_1 as

$$\Theta_1 = \vartheta^i \otimes d^{m-1}t_i\,;$$

the scalar 1-forms ϑ_i are called the *Hilbert forms* of Λ.

Definition 4. If Λ never vanishes, define the *Carathéodory equivalent* $\Theta_m \in \Omega^{m,0}$ by

$$\Theta_m = \frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \vartheta^i\,.$$

Theorem 3. *The Carathéodory equivalent* Θ_m *is an integral equivalent of* Λ.

Proof. We must show that $i^m_{\mathrm{T}}\Theta_m = (-1)^{m(m-1)/2}\, m!\Lambda$, so rewrite Θ_m as

$$\Theta_m = \frac{1}{m!L^{m-1}} \sum_{\sigma\in\mathfrak{S}_m} (-1)^\sigma \vartheta^{\sigma(1)} \wedge \cdots \wedge \vartheta^{\sigma(m)}\,,$$

where $\mathfrak{S}_m$ is the permutation group, and use induction. The calculation uses $d_j \lrcorner \vartheta^i = \delta^i_j L$, the proof of which is similar to that used to show that $i_{\mathrm{T}}\Theta_1 = m\Lambda$; we also define $\tau_{r,s} \in \mathfrak{S}_m$ by

$$\tau_{r,s}(i) = \begin{cases} m - s & (i = r) \\ i - 1 & (r + 1 \leq i \leq m - s) \\ i & \text{otherwise}\,. \end{cases}$$

Now

$$
\begin{aligned}
&i_{\mathrm{T}}\Big(\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\otimes dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\Big)\\
&=\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma d_j\lrcorner\big(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\big)\otimes dt^j\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\\
&=\sum_{r=1}^{m-s}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma(-1)^{r-1}\big(\vartheta^{\sigma(1)}\wedge\cdots\wedge(d_j\lrcorner\vartheta^{\sigma(r)})\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\big)\otimes\\
&\qquad\quad\otimes dt^j\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\\
&=L\sum_{r=1}^{m-s}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma(-1)^{r-1}\big(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(r-1)}\wedge\vartheta^{\sigma(r+1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\big)\otimes\\
&\qquad\quad\otimes dt^{\sigma(r)}\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\\
&=L\sum_{r=1}^{m-s}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma(-1)^{r-1}(-1)^{m-r-s}\Big\{\\
&\qquad\big(\vartheta^{\sigma\tau_{r,s}(1)}\wedge\cdots\wedge\vartheta^{\sigma\tau_{r,s}(r-1)}\wedge\vartheta^{\sigma\tau_{r,s}(r+1)}\wedge\cdots\wedge\vartheta^{\sigma\tau_{r,s}(m-s)}\big)\otimes\\
&\qquad\quad\otimes dt^{\sigma\tau_{r,s}(r)}\wedge dt^{\sigma\tau_{r,s}(m-s+1)}\wedge\cdots\wedge dt^{\sigma\tau_{r,s}(m)}\Big\}\\
&=(-1)^{m-s-1}L\sum_{r=1}^{m-s}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\big(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s-1)}\big)\otimes\\
&\qquad\quad\otimes dt^{\sigma(m-s)}\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\\
&=(-1)^{m-s-1}(m-s)L\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\big(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s-1)}\big)\otimes\\
&\qquad\quad\otimes dt^{\sigma(m-s)}\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\,,
\end{aligned}
$$

so if

$$
\begin{aligned}
i_{\mathrm{T}}^s\Theta_m=\frac{(-1)^{s(2m-s-1)/2}}{(m-s)!L^{m-s-1}}\Big\{&\\
&\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\otimes dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\Big\}
\end{aligned}
$$

then

$$
\begin{aligned}
i_{\mathrm{T}}^{s+1}\Theta_m=\frac{(-1)^{s(2m-s-1)/2}}{(m-s)!L^{m-s-1}}\Big\{&\\
&(-1)^{m-s-1}(m-s)L\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\big(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s-1)}\big)\otimes\\
&\qquad\otimes dt^{\sigma(m-s)}\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\Big\}
\end{aligned}
$$

$$= \frac{(-1)^{(s+1)(2m-s-2)/2}}{(m-s-1)!L^{m-s-2}} \sum_{\sigma\in\mathfrak{S}_m} (-1)^\sigma \big(\vartheta^{\sigma(1)} \wedge \cdots \wedge \vartheta^{\sigma(m-s-1)}\big) \otimes$$
$$\otimes\, dt^{\sigma(m-s)} \wedge dt^{\sigma(m-s+1)} \wedge \cdots \wedge dt^{\sigma(m)}$$

as required. Hence

$$\begin{aligned} i_{\mathrm{T}}^m \Theta_m &= \frac{(-1)^{m(m-1)/2}}{L^{-1}} \sum_{\sigma\in\mathfrak{S}_m} (-1)^\sigma dt^{\sigma(1)} \wedge \cdots \wedge dt^{\sigma(m)} \\ &= (-1)^{m(m-1)/2}\, m! L\, dt^1 \wedge \cdots \wedge dt^m \\ &= (-1)^{m(m-1)/2}\, m! \Lambda\,. \end{aligned}$$

□

We see also from the induction formula that

$$i_{\mathrm{T}}^{m-1}\Theta_m = (-1)^{m(m-1)/2}\,(m-1)!\Theta_1$$

where Θ_1 is the Hilbert equivalent; consequently Θ_m is Lepagian. The Carathéodory equivalent of a nonvanishing homogeneous Lagrangian is the 'parametric' version of the Lepage equivalent θ_2 for a variational problem on a jet manifold, with the difference that there is no longer a restriction to first or second order Lagrangians.

We can now create a 'variation formula' for Θ_m. For a variation field X on E with $X|_{\sigma(\partial C)} = 0$,

$$\begin{aligned} \int_C (j\sigma)^* d_{\widehat{X}}\Theta_m &= \int_C (j\sigma)^* i_{\widehat{X}} d\Theta_m + \int_C (j\sigma)^* d i_{\widehat{X}}\Theta_m \\ &= \int_C (j\sigma)^* i_{\widehat{X}}\mathcal{E}_m + \int_C d(j\sigma)^* i_{\widehat{X}}\Theta_m \\ &= \int_C (j\sigma)^* i_{\widehat{X}}\mathcal{E}_m + \int_{\partial C} (j\sigma)^* i_{\widehat{X}}\Theta_m \\ &= \int_C (j\sigma)^* i_{\widehat{X}}\mathcal{E}_m \\ &= \int_C (j\sigma)^* i_{\mathrm{T}}^m i_{\widehat{X}}\mathcal{E}_m \\ &= (-1)^m \int_C (j\sigma)^* i_{\widehat{X}} i_{\mathrm{T}}^m\mathcal{E}_m \\ &= (-1)^{m(m-1)/2} m! \int_C (j\sigma)^* i_{\widehat{X}}\mathcal{E}_0 \\ &= (-1)^{m(m-1)/2} m! \int_C (j\sigma)^* i_X\mathcal{E}_0 \end{aligned}$$

which is independent of the 'prolonged' part of $\widehat{X}$.

Finally, we consider the possibility of other integral equivalents of a Lagrangian Λ. We can, of course, obtain such an equivalent by adding any contact form to Θ_m; but it is of greater interest to see if we can obtain equivalents which have particular desirable properties. The analogy of Lepage equivalents for variational problems on jet manifolds suggests that there might be other possibilities in the parametric case, and there is indeed a version of θ_3 for first-order homogeneous Lagrangians. Let Λ be such a Lagrangian, and put

$$\widehat{\Theta}_{r+1} = (-1)^r P d\widehat{\Theta}_r \qquad (1 \leq r < m)\,,$$

so that each $\widehat{\Theta}_r$ is a first-order vector form. Using commutator relations as before, we obtain

$$\widehat{\Theta}_r = \frac{(-1)^r}{m-r}\, i_{\mathrm{T}}\widehat{\Theta}_{r+1}$$

so that $\widehat{\Theta}_m$ is a Lepagian integral equivalent of Λ, the *fundamental equivalent* of Λ. Thus $d\widehat{\Theta}_m = \widehat{\mathcal{E}}_m = 0$ if, and only if, $\widehat{\mathcal{E}}_0 = \mathcal{E}_0 = 0$, the same property satisfied by θ_3 in the jet manifold case [5].

6 Conclusions

Parametric variational problems are often studied on Grassmannian bundles. There is, however, some interest in considering the versions of the problem defined on velocity manifolds, subject to the homogeneity condition. The bicomplex of vector forms performs a similar rôle to the variational bicomplex in the jet bundle theory, but the intermediate and integral equivalents corresponding to the Lepage equivalents may be defined globally for forms of arbitrary order. It seems reasonable to expect that further study of the subject in this context would produce useful results concerning related concepts such as regularity, symmetry and the Helmholtz equations for the inverse problem of the calculus of variations.

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