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A note on normal generation and generation of groups

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Abstract. In this note we study sets of normal generators of finitely presented residually p-finite groups. We show that if an infinite, finitely presented, residually p-finite group G is normally generated by $g_1, \dots, g_k$ with order $n_1, \dots, n_k \in \{1, 2, \dots\} \cup \{\infty\}$, then

$$\beta_1^{(2)}(G) \le k - 1 - \sum_{i=1}^{k} \frac{1}{n_i},$$

where $\beta_1^{(2)}(G)$ denotes the first ℓ^2-Betti number of G. We also show that any k-generated group with $\beta_1^{(2)}(G) \ge k - 1 - \varepsilon$ must have girth greater than or equal $1/\varepsilon$.

1 Introduction

In the first part of this note we want to prove estimates of the number of normal generators of a discrete group in terms of its first ℓ^2-Betti number. It is well-known that if a non-trivial discrete group is generated by k elements, then

$$\beta_1^{(2)}(G) \le k - 1. \tag{1}$$

The proof of this statement is essentially trivial using the obvious Morse inequality. The following conjecture was first formulated in [13].

Conjecture 1. *Let G be a torsionfree discrete group. If G is normally generated by elements $g_1, \dots, g_k$, then*

$$\beta_1^{(2)}(G) \le k - 1.$$

If G is finitely presented, residually p-finite for some prime p, then Conjecture 1, i.e., the inequality $\beta_1^{(2)}(G) \le k-1$ is known to be true, see Remark 2. In this note, we give a proof of a variation of this conjecture, which also applies to the non-torsionfree case. In Theorem 7 we show the following result

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Theorem 7. *Let G be an infinite, finitely presented, residually p-finite group for some prime p. If G is normally generated by a subgroup Λ, then*

$$\beta_1^{(2)}(G) \leq \beta_1^{(2)}(\Lambda).$$

In particular, if G is normally generated by elements $g_1, \ldots, g_k \in G$ of order $n_1, \ldots, n_k \in \{1, 2, \ldots\} \cup \{\infty\}$, then

$$\beta_1^{(2)}(G) \leq k - 1 - \sum_{i=1}^{k} \frac{1}{n_i}.$$

The proof is based on some elementary calculations with cocycles on G taking values in $\mathbb{C}[G/H]$, for $H \subset G$ a normal subgroup of finite index, and Lück's Approximation Theorem [10].

In Section 5 we prove in Theorem 8 that if a k-generated group G satisfies $\beta_1^{(2)}(G) \geq k - 1 - \varepsilon$, then the shortest relation in terms of the generators must have length at least $1/\varepsilon$. A theorem of Pichot [15] already implied that the girth of the Cayley graph of G with respect to the natural generating set becomes larger and larger if ε is getting smaller and smaller. Our main result is a quantitative estimate that implies this qualitative result. We prove in Theorem 8:

Theorem 8. *Let G be a finitely generated group with generating set S. Then,*

$$\operatorname{girth}(G, S) \geq \frac{1}{k - 1 - \beta_1^{(2)}(G)}.$$

The main tool is an explicit uncertainty principle for the von Neumann dimension.

2 Residually p-finite groups

In this section we want to recall some basic results on the class of residually p-finite groups and show that various natural classes of groups are contained in this class of groups. Let us first recall some definitions.

Definition 1. Let p be a prime number. A group G is said to be residually p-finite, if for every non-trivial element $g \in G$, there exists a normal subgroup $H \subset G$ of p-power index such that $g \notin H$. A group G is called virtually residually p-finite if it admits a residually p-finite subgroup of finite index.

The following result relates residually p-finiteness to residual nilpotence and gives a large class of examples of groups which are residually p-finite.

Theorem 2 (Gruenberg). *Let G be a finitely generated group. If G is torsionfree and residually nilpotent, then it is residually p-finite for any prime p.*

Another source of residually p-finite groups is a result by Platonov, see [16], which says that any finitely generated linear group is virtually residually p-finite for almost all primes p. In [1], Aschenbrenner-Friedl showed that the same is true for fundamental groups of 3-manifolds. Gilbert Baumslag showed [3] that any one-relator group where the relator is a p-power is residually p-finite. For any group, its image in the pro-p completion is residually nilpotent.

We denote the group ring of G with coefficients in a ring R by RG. Its elements are formal finite linear combinations of the form $\sum_g a_g g$ with $a_g \in R$. The natural multiplication on G extends to RG. The natural homomorphism $\varepsilon\colon RG \to R$ given by

$$\varepsilon\Big(\sum_{g\in G} a_g g\Big) := \sum_{g\in G} a_g$$

is called augmentation. We denote by ω_R the kernel of $\varepsilon\colon RG \to R$; the so-called augmentation ideal.

In the proof of our main result, we will use the following characterization of finite p-groups that was obtained by Karl Gruenberg, see [7] and also [8], [5], will play an important role.

Theorem 3 (Gruenberg). *Let G be a finite group and let $\omega_{\mathbb{Z}} \subset \mathbb{Z}G$ be the augmentation ideal. The group G is of prime-power order if and only if*

$$\bigcap_{n=1}^{\infty} \omega^n = \{0\}.$$

3 ℓ^2-invariants of groups

3.1 Some definitions

ℓ^2-invariants of fundamental groups of compact aspherical manifolds where introduced by Atiyah in [2]. A definition which works for all discrete groups was given by Cheeger-Gromov in [4]. Later, a more algebraic framework was presented by Lück in [10]. We want to stick to this more algebraic approach.

Let G be a group and denote by $\mathbb{C}G$ the complex group ring. Note that the ring $\mathbb{C}G$ comes with a natural involution $f \mapsto f^*$ which is given by the formula

$$\Big(\sum_{g\in G} a_g g\Big)^* = \sum_{g\in G} \bar{a}_g g^{-1}.$$

We denote by $\tau\colon \mathbb{C}G \to \mathbb{C}$ the natural trace on $\mathbb{C}G$, given by the formula

$$\tau\Big(\sum_{g\in G} a_g g\Big) = a_e.$$

It satisfies $\tau(f^*f) \geq 0$ for all $f \in \mathbb{C}G$ and the associated GNS-representation is just the Hilbert space ℓ^2G with orthonormal basis $\{\delta_g \mid g \in G\}$ on which G (and hence $\mathbb{C}G$) acts via the left-regular representation. More explicitly, there exists a unitary representation $\lambda\colon G \to U(\ell^2G)$ and $\lambda(g)\delta_h = \delta_{gh}$. Similar to the left-regular representation, there is a right-regular representation $\rho\colon G \to U(\ell^2G)$, given by the formula $\rho(g)\delta_h = \delta_{hg^{-1}}$.

The group von Neumann algebra of a group is defined as

$$LG := B(\ell^2 G)^{\rho(G)} = \{T \in B(\ell^2 G) \mid \rho(g)T = T\rho(g), \forall g \in G\}.$$

It is obvious that $\lambda(\mathbb{C}G) \subset LG$, in fact it is dense in the topology of pointwise convergence on $\ell^2 G$. Recall that the trace τ extends to a positive and faithful trace on LG via the formula

$$\tau(a) = \langle a\delta_e, \delta_e \rangle.$$

For each $\rho(G)$-invariant closed subspace $K \subset \ell^2 G$, we denote by p_K the orthogonal projection onto K. It is easily seen that $p_K \in LG$. We set $\dim_G K := \tau(p_K) \in [0,1]$. The quantity $\dim_G K$ is called Murray-von Neumann dimension of K. Lück proved that there is a natural dimension function

$$\dim_{LG} \colon LG\text{-modules } \to [0,\infty]$$

satisfying various natural properties, see [10], such that $\dim_{LG} K = \dim_G K$ for every $\rho(G)$-invariant subspace of $\ell^2 G$.

We can now set

$$\beta_1^{(2)}(\Gamma) := \dim_{L\Gamma} H^1(\Gamma, L\Gamma),$$

where the group on the right side is the algebraic group homology of Γ with coefficients in the left $\mathbb{Z}\Gamma$-module $L\Gamma$. Since the cohomology group inherits a right $L\Gamma$-module structure a dimension can be defined.

Remark 1. The usual definition of ℓ^2-Betti numbers uses the group homology rather than the cohomology. Also, usually $\ell^2 G$ is used instead of LG. That the various definitions coincide was shown in [14].

3.2 Lück's Approximation Theorem

A striking result, due to Lück, states that for a finitely presented and residually finite group, the first ℓ^2-Betti number is a normalized limit of ordinary Betti numbers for a chain of subgroups of finite index, see [10] for a proof. The result says more precisely:

Theorem 4 (Lück). *Let G be a residually finite and finitely presented group. Let*

$$\cdots \subset H_{n+1} \subset H_n \subset \cdots \subset G$$

be a chain of finite index normal subgroups such that $\bigcap_{n=1}^{\infty} H_n = \{e\}$. Then,

$$\beta_1^{(2)}(G) = \lim_{n\to\infty} \frac{\operatorname{rk}((H_n)_{\text{ab}})}{[G:H_n]} = \lim_{n\to\infty} \frac{\dim_{\mathbb{C}} H^1(G, \mathbb{Z}[G/H_n]) \otimes_{\mathbb{Z}} \mathbb{C}}{[G:H_n]}.$$

This result has numerous applications and extensions, we call it Lück's Approximation Theorem.

3.3 Lower bounds on the first ℓ^2-Betti number

It is well known that the first ℓ^2-Betti number of a finitely generated group G is bounded from above by the number of generators of the group minus one. A more careful count reveals that a generator of order n counts only $1 - \frac{1}{n}$. Similarly, lower bounds can be found in terms of the order of the imposed relations in some presentation. More precesily, we find:

Theorem 5. *Let G be an infinite countable discrete group. Assume that there exist subgroups $G_1, \ldots, G_n$, such that*

$$G = \langle G_1, \ldots, G_n \mid r_1^{w_1}, \ldots, r_k^{w_k}, \ldots \rangle,$$

for elements $r_1, \ldots, r_k \in G_1 * \cdots * G_n$ and positive integers $w_1, \ldots, w_k$. We assume that the presentation is irredundant in the sense that $r_i^l \neq e \in G$, for $1 < l < w_i$ and $1 \leq i < \infty$. Then, the following inequality holds:

$$\beta_1^{(2)}(G) \geq n - 1 + \sum_{i=1}^{n} \left(\beta_1^{(2)}(G_i) - \frac{1}{|G_i|} \right) - \sum_{j=1}^{\infty} \frac{1}{w_j}.$$

A proof of this result was given in [14]. It can be used in many cases already if the groups G_i are isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$, see for example [13].

Another result says that the set of marked groups with first ℓ^2-Betti number greater or equal so some constant is closed in Grigorchuk's space of marked groups, see [15] for definitions and further references. More precisely, we have:

Theorem 6 (Pichot, see [15]). *Let $((G_n, S))_{n\in\mathbb{N}}$ be a convergent sequence of marked groups in Grigorchuk's space of marked groups. Then,*

$$\beta_1^{(2)}(G) \geq \limsup_{n\to\infty} \beta_1^{(2)}(G_n).$$

This applies in particular to limits of free groups and shows that they all have a positive first ℓ^2-Betti number. In particular, there is an abundance of finitely presented groups with positive first ℓ^2-Betti number.

4 Normal generation by torsion elements

The first main result in this note extends the trivial upper bound from Equation (1) (under some additional hypothesis) to the case where the group is *normally* generated by a certain finite set of elements. The additional hypothesis is that the group G be finitely presented and residually p-finite for some prime p. More precisely:

Theorem 7. *Let G be an infinite, finitely presented, residually p-finite group for some prime p. If G is normally generated by a subgroup Λ, then*

$$\beta_1^{(2)}(G) \leq \beta_1^{(2)}(\Lambda).$$

In particular, if G is normally generated by elements $g_1, \dots, g_k \in G$ of order $n_1, \dots, n_k \in \{1, 2, \dots\} \cup \{\infty\}$, then

$$\beta_1^{(2)}(G) \leq k - 1 - \sum_{i=1}^{k} \frac{1}{n_i}. \tag{2}$$

Proof. Let $\{g_i \mid i \in \mathbb{N}\}$ be a generating set for Λ. Let H be a finite index normal subgroup of G, such that G/H is of p-power order. We consider $Z^1(G, \mathbb{Z}[G/H])$, the abelian group of 1-cocycles of the group G with values in the G-module $\mathbb{Z}[G/H]$. In a first step, we will show that the restriction map

$$\sigma \colon Z^1(G, \mathbb{Z}[G/H]) \to Z^1(\Lambda, \mathbb{Z}[G/H])$$

is injective.

Note that there is a natural injective evaluation map

$$\pi \colon Z^1(\Lambda, \mathbb{Z}[G/H]) \to \mathbb{Z}[G/H]^{\oplus\infty}$$

which sends a 1-cocycle c to the values on the g_i, i.e. $c \mapsto (c(g_i))_{i=1}^{\infty}$.

We claim that $\pi \circ \sigma$ is injective. Indeed, assume that $c \in \ker(\pi \circ \sigma)$ and assume that $c(g) \in \omega^m$ for all $g \in G$, where m is some integer greater than or equal zero. Since g is in the normal closure of $\{g_i \mid i \in \mathbb{N}\}$, there exists some natural number $l \in \mathbb{N}$ and $h_1, \dots h_l \in G$, such that

$$g = \prod_{i=1}^{l} h_i g_{q(i)}^{\pm 1} h_i^{-1},$$

for some function $q \colon \{1, \dots, l\} \to \mathbb{N}$. Computing $c(g)$ using the cocycle relation and $c(g_i^{\pm}) = 0$, for $1 \leq i \leq k$, we get

$$c(g) = \sum_{i=1}^{l} \left(\prod_{j=1}^{i-1} h_j g_{q(j)}^{\pm} h_j^{-1} \right) \left(1 - h_i g_{q(i)}^{\pm} h_i^{-1} \right) c(h_i).$$

By hypothesis $c(h_i) \in \omega^m$ for $1 \leq i \leq l$ and we conclude that $c(g) \in \omega^{m+1}$. This argument applies to all $g \in G$. Since the hypothesis $c(g) \in \omega^m$ is obviously satisfied for $m = 0$, we finally get by induction that $c(g) \in \omega^m$ for all $m \in \mathbb{N}$ and hence

$$c(g) \in \bigcap_{m=1}^{\infty} \omega^m, \quad \forall g \in G.$$

By Theorem 3 and since G/H is of prime power order, we know that $\bigcap_{m=1}^{\infty} \omega^m = \{0\}$. Hence, $c(g) = 0$ for all $g \in G$. This proves that the map $\pi \circ \sigma$ and hence σ is injective.

Note that

$$\dim_{\mathbb{C}} H^1(G, \mathbb{C}[G/H_n]) = \dim_{\mathbb{C}} Z^1(G, \mathbb{C}[G/H_n]) - [G : H] + 1$$

and also

$$\dim_{\mathbb{C}} H^1(\Lambda, \mathbb{C}[G/H_n]) = \dim_{\mathbb{C}} Z^1(\Lambda, \mathbb{C}[G/H_n]) - [G:H] + 1,$$

as Λ surjects onto G/H, using that a finite p-group cannot be normally generated by a proper subgroup.

Now, by assumption, there exists a chain

$$\cdots \subset H_{n+1} \subset H_n \subset \cdots \subset G$$

of finite index subgroups with p-power index such that

$$\bigcap_{n=1}^{\infty} H_n = \{e\}.$$

The claim is now implied by Lück's Approximation Theorem (see Theorem 4). Indeed, Lück's Approximation Theorem applied to the chain of finite index subgroups of p-power index gives:

$$\beta_1^{(2)}(G) = \lim_{n\to\infty} \frac{\dim_{\mathbb{C}} H^1(G, \mathbb{C}[G/H_n])}{[G:H_n]} \leq \lim_{n\to\infty} \frac{\dim_{\mathbb{C}} H^1(\Lambda, \mathbb{C}[G/H_n])}{[G:H_n]} \leq \beta_1^{(2)}(\Lambda).$$

Here, we used Kazhdan's inequality in the last step (see [12, Theorem 1.1] for a proof). This finishes the proof of the first inequality.

The second claim follows from a simple and well-known estimate for the first ℓ^2-Betti number (see for example Theorem 3.2 in [14]) that we apply to Λ. More directly, for each H, we can estimate the dimension of the image of $\pi \otimes_{\mathbb{Z}} \mathbb{C}$. Since g_i has order n_i, we compute

$$0 = c(g_i^{n_i}) = \left(\sum_{j=0}^{n_i - 1} \bar{g}_i^j \right) c(g_i),$$

where we denote by $\bar{g}_i$ the image of g_i in G/H. If the order of the image of $\bar{g}_i$ is m_i, then $n_i^{-1} \sum_{j=0}^{n_i-1} \bar{g}_i^j$ is a projection of normalized trace m_i^{-1} such that

$$\dim_{\mathbb{C}} (\operatorname{im}(\pi) \otimes_{\mathbb{Z}} \mathbb{C}) \leq [G:H] \cdot \sum_{i=1}^{k} \left(1 - \frac{1}{m_i}\right) \leq [G:H] \cdot \sum_{i=1}^{k} \left(1 - \frac{1}{n_i}\right).$$

This implies that

$$\dim_{\mathbb{C}} H^1(\Lambda, \mathbb{Z}[G/H]) \otimes_{\mathbb{Z}} \mathbb{C} \leq [G:H] \cdot \sum_{i=1}^{k} \left(1 - \frac{1}{n_i}\right) - [G:H] + 1.$$

We now apply the first result. This finishes the proof, again using Lück's Approximation Theorem. □

Remark 2. Let G be an infinite, residually p-finite group. It follows from Proposition 3.7 in [9] in combination with Lück's Approximation Theorem that

$$\beta_1^{(2)}(G) \leq \dim_{\mathbb{Z}/p\mathbb{Z}} H^1(G, \mathbb{Z}/p\mathbb{Z}) - 1.$$

This implies that $\beta_1^{(2)}(G) \leq k-1$ in the situation that G is normally generated by $g_1, \dots, g_k$. Our result improves this estimate in the case when some of the elements $g_1, \dots, g_k$ have finite order.

Remark 3. Consider $G = PSL(2, \mathbb{Z}) = \langle a, b \mid a^2 = b^3 = e \rangle$. Then, $\beta_1^{(2)}(G) = \frac{1}{6} \neq 0$ and G is normally generated by the element $ab \in G$. Hence, the assumption that G is residually a p-group cannot be omitted in Theorem 7.

5 An uncertainty principle and applications

In this section we want to prove a quantitative estimate on the girth of a marked group in terms of its first ℓ^2-Betti number. In [13], Osin and the author constructed for given $\varepsilon > 0$ a k-generated simple groups with first ℓ^2-Betti number greater than $k - 1 - \varepsilon$. The construction involved methods from small cancellation theory and in particular, those groups did not admit any short relations in terms of the natural generating set. This in fact follows already from the main result in [15]. If $(G_i, S_i)_{i \in \mathbb{N}}$ is a sequence of marked groups with $|S_i| = k$ and $\lim_{i \to \infty} \beta_1^{(2)}(G_i) = k - 1$, then necessarily

$$\lim_{i \to \infty} \operatorname{girth}(G_i, S_i) = \infty,$$

where $\operatorname{girth}(G, S)$ denotes the length of the shortest cycle in the Cayley graph of G with respect to the generating set S. Indeed, by [15], any limit point (G, S) of the sequence $(G_i, S_i)_{i \in \mathbb{N}}$ satisfies $\beta_1^{(2)}(G) = k - 1$ and hence is a free group on the basis S. (This last fact is well known and is also a consequence of our next theorem.) In this section, we want to prove a quantitative version of this result.

Theorem 8. *Let G be a finitely generated group with generating set $S = \{g_1, \dots, g_k\}$. Then,*

$$\operatorname{girth}(G, S) \geq \frac{1}{k - 1 - \beta_1^{(2)}(G)}.$$

In order to prove this theorem, we need some variant of the so-called uncertainty principle. We denote by $\|.\|$ the usual operator norm on $B(\ell^2 G)$ and use the same symbol to denote the induced norm on $\mathbb{C}G$, i.e., $\|f\| = \|\lambda(f)\|$ for all $f \in \mathbb{C}G$. The 1-norm is denoted by $\|\sum_g a_g g\|_1 = \sum_g |a_g|$. For $f = \sum_g a_g g$ we define its support as $\operatorname{supp} := \{g \in G \mid a_g \neq 0\}$.

Theorem 9. *Let G be a group and $f \in \mathbb{C}G$ be a non-zero element of the complex group ring. Then,*

$$\dim_{LG}(f \cdot LG) \cdot |\operatorname{supp}(f)| \geq 1.$$

Proof. First of all we have $\dim_{LG}(f \cdot LG) = \tau(p_K)$, where K is the closure of the image of $\lambda(f)\colon \ell^2 G \to \ell^2 G$.

$$\tau(f^*f) \leq \dim_{LG}(f \cdot LG) \cdot \|f\|^2 \tag{3}$$

since

$$\tau(f^*f) = \tau(ff^*) = \tau(p_K ff^*) \leq \tau(p_K) \cdot \|ff^*\| = \dim_{LG}(f \cdot LG) \cdot \|f\|^2.$$

Secondly, using the fact that $\|f\|_2^2 = \tau(f^*f)$, we see that

$$\|f\|_1^2 \leq |\mathrm{supp}(f)| \cdot \tau(f^*f) \tag{4}$$

by the Cauchy-Schwarz inequality applied to $f \cdot \chi_{\mathrm{supp}(f)}$, where the product here is the pointwise product of coefficients and $\|f\|_1$ denotes the usual 1-norm on $\mathbb{C}[G]$. Combining Equations (3) and (4) we conclude

$$\dim_{LG}(f \cdot LG) \cdot |\mathrm{supp}(f)| \geq \left(\frac{\|f\|_1}{\|f\|} \right)^2. \tag{5}$$

Now, since each group element acts as a unitary, and hence with operator norm 1 on $\ell^2 G$, we get $\|f\|_1 \geq \|f\|$. This proves the claim and finishes the proof. □

The preceding result and the following corollary were proved as result of a question by Efremenko on MathOverflow.

Corollary 1. *Let G be a finite group and $f \in \mathbb{C}[G]$ be an arbitrary non-zero element. Then,*

$$\dim_{\mathbb{C}}(f \cdot \mathbb{C}[G]) \cdot |\mathrm{supp}(f)| \geq |G|.$$

We are now ready to prove Theorem 8.

Proof. (Theorem 8) Again, we study the map $\pi\colon Z^1(G, LG) \to LG^{\oplus k}$, which is given by $c \mapsto (c(g_i))_{i=1}^k$. If $w \in \mathbb{F}_k$ is some word such that $w(g_1, \ldots, g_k) = e$ in G, then

$$0 = c\big(w(g_1, \ldots, g_k)\big) = \sum_{i=1}^k \partial_i(w)(g_1, \ldots, g_k) \cdot c(g_i),$$

where $\partial_i\colon \mathbb{Z}[\mathbb{F}_k] \to \mathbb{Z}[\mathbb{F}_k]$ denotes the i-th Fox derivative for $1 \leq i \leq k$. Thus, the image of π lies is annihilated by the LG-linear map

$$(\xi_1, \ldots, \xi_k) \mapsto \sum_{i=1}^k \partial_i(w)(g_1, \ldots, g_k)\xi_i\,.$$

In particular, the image of π does not intersect with $K := \ker(\partial_1(w)) \oplus 0 \oplus \cdots \oplus 0$. The number of summands in $\partial_1(w)$ is equal to the number of occurences of the letters $g_1^{\pm}$ in w. Thus, we have and $\mathrm{im}(\pi) \cap K = \{0\}$ and $\dim_{LG}(K) \geq |\mathrm{supp}(\partial_1(w))|^{-1}$. Thus

$$\dim_{LG} \mathrm{im}(\pi) \leq k - |\mathrm{supp}(\partial_1(w))|^{-1} \leq k - \frac{1}{\ell(w)}.$$

For any infinite group $\beta_1^{(2)}(G) = \dim_{LG} Z^1(G, LG) - 1$. This implies that $\beta_1^{(2)}(G) \leq k - 1 - 1/\ell(w)$ and finishes the proof. □

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A subclass of strongly clean rings

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Abstract. In this paper, we introduce a subclass of strongly clean rings. Let R be a ring with identity, J be the Jacobson radical of R, and let $J^{\#}$ denote the set of all elements of R which are nilpotent in R/J. An element $a \in R$ is called *very $J^{\#}$-clean* provided that there exists an idempotent $e \in R$ such that $ae = ea$ and $a - e$ or $a + e$ is an element of $J^{\#}$. A ring R is said to be *very $J^{\#}$-clean* in case every element in R is very $J^{\#}$-clean. We prove that every very $J^{\#}$-clean ring is strongly π-rad clean and has stable range one. It is shown that for a commutative local ring R, $A(x) \in M_2\big(R[[x]]\big)$ is very $J^{\#}$-clean if and only if $A(0) \in M_2(R)$ is very $J^{\#}$-clean. Various basic characterizations and properties of these rings are proved. We obtain a partial answer to the open question whether strongly clean rings have stable range one.

This paper is dedicated to Professor Abdullah Harmanci on his 70th birthday

1 Introduction

Throughout this paper, all rings are associative with identity unless otherwise stated. Nicholson in [16] defined clean elements and clean rings, also in [17] Nicholson and Zhou introduced strongly clean rings and Chen continued studying strongly clean rings and introduced strongly J-clean rings in [5]. Other generalizations of clean notion of rings are investigated by many authors ([4], [6], [10], [12]). Let U denote the set of all invertible elements and J be the Jacobson radical of R. In this paper, the set of all elements of R which are nilpotent in R/J will be denoted by $J^{\#}$. Clearly, $J \subseteq J^{\#}$. Let a be an element of R. The element a is called *clean* provided that there exist $e^2 = e \in R$ and $u \in U$ such that $a = e + u$. The element a is *strongly clean* if there exist $e^2 = e \in R$ and $u \in U$ such that $a = e + u$ and $eu = ue$. An element a is called *very clean* if there exists $e^2 = e \in R$ and $u \in U$ such that $a = e + u$ or $a = -e + u$ and $eu = ue$. In general, $a \in R$ is (strongly or very) $\mathcal{T}$-clean if and only if there exists an idempotent $e \in R$ such that ($ae = ea$ and) $a - e$ (or potentially $a + e$ for very cleanness) is in the set related to $\mathcal{T}$. Here,

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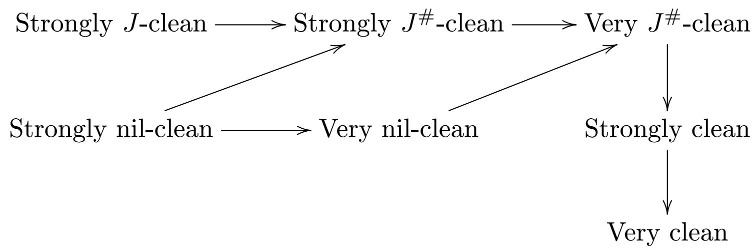
$\mathcal{T} \in \{\mathrm{Nil}, J, J^{\#}\}$, and the corresponding sets are $\mathrm{Nil}(R)$ (the set of all nilpotent elements of R), J and $J^{\#}$, respectively. A ring R is said to have *stable range one* if given $a, b \in R$ for which $aR + bR = R$, there exists $y \in R$ such that $a + by \in U$. One of the most important features of stable range one is the cancellation of related modules from direct sums. We know that stable range one in endomorphism rings implies cancellation in direct sums, that is, if A, B, C are modules such that $A \oplus B \cong A \oplus C$, and $\mathrm{End}(A)$ has stable range one, then $B \cong C$ [11, Theorem 2]. Further, if R is directly finite, i.e., any $x, y \in R$ satisfying $xy = 1$ also satisfy $yx = 1$, then so is $M_n(R)$ (for details one can see [6]). But so far it is unknown whether strongly clean rings have stable range one (see [17]). This motivates us to construct a natural subclass of strongly clean rings, namely, very $J^{\#}$-clean rings, which have stable range one.

Clearly, every commutative or Artinian strongly nil clean ring is strongly J-clean. But the converse is not true in general (see [5] or Example 2). Since $\mathrm{Nil}(R) \subseteq J^{\#}$ and $J \subseteq J^{\#}$, we know that strongly J-clean rings and strongly nil-clean rings are strongly $J^{\#}$-clean, and every very nil-clean ring is very $J^{\#}$-clean. Example 2 is a very $J^{\#}$-clean ring, which is not very nil-clean. Every strongly $J^{\#}$-clean ring is very $J^{\#}$-clean but Example 3 is very $J^{\#}$-clean, which is not strongly $J^{\#}$-clean. Any very $J^{\#}$-clean ring is strongly clean (see Theorem 1) but there exists a strongly clean ring which is not very $J^{\#}$-clean (e.g. $\mathbb{Z}_5$). Every strongly clean ring is very clean. Example 4 is a very clean ring, which is not strongly clean. Now we illustrate relations between these classes of rings in the following:



None of the implications in the diagram are reversible.

The paper is organized as follows: in Section 2, basic properties of very $J^{\#}$-clean rings are given. We give some examples concerning their relations with clean rings, strongly clean rings, strongly $J^{\#}$-clean rings. Further, we prove that if R is very $J^{\#}$-clean, then R has stable range one. In Section 3, we construct several examples of very $J^{\#}$-clean rings. For instance, if R is an abelian very $J^{\#}$-clean ring, then the ring $R[[x]]$ of power series over R is very $J^{\#}$-clean. In Section 4, we characterize the very $J^{\#}$-cleanness of matrices over commutative local rings. Further, we consider very $J^{\#}$-clean power series rings over such matrix rings.

In what follows, for a positive integer n, $\mathbb{Z}_n$ and $\mathbb{N}$ denote the ring of integers modulo n and the natural numbers, while for a prime integer p, $\mathbb{Z}_{(p)}$ denotes the ring of integers localized at the prime ideal (p), and we write $M_n(R)$ for the rings of all $n \times n$ matrices over a ring R. We write $R[[x]]$ and $\mathrm{Nil}(R)$ for the ring of power series over R and the set of all nilpotent elements of R, respectively. Let $\bar{R}$ denote the quotient ring R/J.

2 Elementary results

Recall that a ring R is called *local* if it has only one maximal left ideal (equivalently, maximal right ideal). It is well known that a ring R is local if and only if $a+b=1$ in R implies that either a or b is invertible if and only if $\bar{R}$ is a division ring. A ring R is said to be *reduced* if it has no non-zero nilpotent elements. Now we begin with the simple result.

Lemma 1. *For a ring R we have that $\bar{R}$ is reduced if and only if $J^{\#} = J$. In particular, $J^{\#} = J$ if R is commutative or local or $\bar{R}$ is the direct sum of division rings.*

It is clear from Lemma 1 that if R is a commutative or local ring, then $a \in R$ is strongly $J^{\#}$-clean if and only if $a \in R$ is strongly J-clean. Recall that a ring R is called *uniquely clean* if every element can be written uniquely as the sum of an idempotent and a unit (see [18]).

Lemma 2. *Let $\bar{R}$ be a direct sum of division rings. Then the following are equivalent.*

(1) R is strongly $J^{\#}$-clean.

(2) $\bar{R}$ is a direct sum of two-element fields.

Proof. Note that if $\bar{R}$ is a direct sum of division rings (every local ring or commutative Artinian ring has this property), then R is (strongly, very) $J^{\#}$-clean if and only if R is (strongly, very, respectively) J-clean, because, by Lemma 1, we have $J^{\#} = J$. Let $\mathbb{F}_n$ denote the field with n elements.

(1) $\Rightarrow$ (2) Since R is strongly J-clean, we have $\bar{R}$ is Boolean, and so $\bar{R} \cong \oplus\mathbb{F}_2$ because $\bar{R}$ is a direct sum of division rings.

(2) $\Rightarrow$ (1) Assume that $\bar{R}$ is a direct sum of two-element fields. Then R is uniquely clean by [18, Corollary 16]. This implies that R is abelian (that is, all idempotents in R are central) and for all $a \in R$ there exists a unique idempotent $e \in R$ such that $e - a \in J$ by [18, Theorem 20]. Thus R is strongly J-clean. □

One may suspect that if $\bar{R}$ is a direct sum of two- or three-element fields, then R is very $J^{\#}$-clean. The following example shows that this is not true in general.

Example 1. Let R denote the ring $\mathbb{Z}_9 \oplus \mathbb{Z}_9$. Then we have $\bar{R} = \mathbb{Z}_3 \oplus \mathbb{Z}_3$ and the only idempotents of the ring R are $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$. Further, note that $J^{\#} = J$. Hence $(2,4) \in R$ is not (strongly) very $J^{\#}$-clean.

(Strongly) Nil-clean elements (rings) are introduced by Diesl in [9], [10]. Clearly, every strongly nil-clean element (ring) is a strongly $J^{\#}$-clean element (ring). But there exists a strongly $J^{\#}$-clean element (ring) which is not strongly nil-clean element (ring) as the following example shows (see [5]).

Example 2. Let $R = \prod_{n=1}^{\infty} \mathbb{Z}_{2^n}$. For each $n \in \mathbb{N}$, $\mathbb{Z}_{2^n}$ is a local ring with the maximal ideal $2\mathbb{Z}_{2^n}$. Then $\mathbb{Z}_{2^n}/2\mathbb{Z}_{2^n} \cong \mathbb{Z}_2$. Hence R is strongly J-clean, and so R

is strongly $J^\#$-clean (and very $J^\#$-clean). Since the element $r = (0, 2, 2, \dots) \in R$ is not strongly nil-clean (and not very nil-clean), R is not strongly nil-clean (and not very nil-clean).

Every strongly $J^\#$-clean (strongly J-clean) ring is very $J^\#$-clean (very J-clean) but there exists a very $J^\#$-clean (very J-clean) ring which is not strongly $J^\#$-clean (strongly J-clean) as the following example shows.

Example 3. The ring $\mathbb{Z}_3$ is very $J^\#$-clean which is not strongly $J^\#$-clean.

Proof. Let $R = \mathbb{Z}_3$. Note that R is strongly (or very) $J^\#$-clean if and only if R is strongly (or very) J-clean because R is commutative, and we have $J = J^\# = 0$ by Lemma 1. Since $\bar{R}$ is not Boolean, R is not strongly $J^\#$-clean, but R is very $J^\#$-clean. □

Very clean elements (rings) are introduced by Chen et al. in [8]. Thus any very $J^\#$-clean ring is very clean. But the converse need not be true in general as shown below.

Example 4. $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ is a very clean ring which is not very $J^\#$-clean.

Proof. Set $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$. If R is very $J^\#$-clean, then, by Theorem 1, it is strongly clean, but it is not strongly clean by [8, Theorem 3.5] or by [2, Example 17]. □

The next result shows that for an element of a ring, being very $J^\#$-clean and strongly $J^\#$-clean coincide under some conditions.

Proposition 1. *Let R be a ring, $2 \in J$, and $a \in R$. Then a is very $J^\#$-clean if and only if it is strongly $J^\#$-clean.*

Proof. If $a \in R$ is strongly $J^\#$-clean, then it is very $J^\#$-clean. Conversely, assume that $a \in R$ is very $J^\#$-clean. Then there exist an idempotent $e \in R$ and $v \in J^\#$ such that $ae = ea$ and $a = e + v$ or $a = -e + v$. If $a = -e + v$, then $a = e + (v - 2e)$. As $2 \in J$, it easy to verify that $v - 2e \in J^\#$, hence $a \in R$ is strongly $J^\#$-clean. This completes the proof. □

Remark 1. If u is invertible, $v \in J^\#$ and $uv = vu$, then we have that $u + v$ and $u - v$ is invertible.

Proof. Since $v \in J^\#$ if and only if $-v \in J^\#$, we only need to prove one of $u + v \in U$ and $u - v \in U$. We prove that $u - v \in U$. Now, we have $v^n \in J$, thus $1 - u^{-n}v^n \in U$. Now

$$1 - u^{-n}v^n = 1 - (u^{-1}v)^n = (1 - u^{-1}v)\big(1 + u^{-1}v + \cdots + (u^{-1}v)^{n-1}\big)\,.$$

Hence $1 - u^{-1}v$ is invertible, and so $u - v = u(1 - u^{-1}v) \in U$, because $u \in U$. □

By the following result, we determine the set of all invertible elements of a very $J^\#$-clean ring.

Proposition 2. *If R is a very $J^\#$-clean ring, then*

$$U = \{u \in R \mid u - 1 \in J^\# \text{ or } u + 1 \in J^\#\}.$$

Proof. Let $u \in U$. Since R is very $J^\#$-clean, there exist an idempotent $e \in R$ and $v \in J^\#$ such that $ue = eu$ and $u = e + v$ or $u = -e + v$. Assume that $u = e + v$. Then $u - v = e \in U$ implies that $e = 1$ and so $u = v + 1$. Assume that $u = -e + v$. Then $v - u = e \in U$ implies that $e = 1$ and so $u = v - 1$.

On the other hand, suppose that $u = v - 1$ where $v \in J^\#$. Then we can find some $n \in \mathbb{N}$ such that $v^n \in J$. Hence $1 - av^n \in U$ for any $a \in R$. If $1 - v^n \in U$, then $1 - v \in U$ because

$$1 - v^n = (1 - v)(1 + v + \cdots + v^{n-1}),$$

and so $u \in U$. Suppose that $u = v + 1$ where $v \in J^\#$. Then we can find some $n \in \mathbb{N}$ such that $v^n \in J$. Therefore $1 - av^n \in U$ for any $a \in R$. If $1 + (-1)^{n-1}v^n \in U$, then $1 + v \in U$ because

$$1 + (-1)^{n-1}v^n = (1 + v)(1 - v + \cdots + (-1)^{n-1}v^{n-1}),$$

and so $u \in U$. Hence

$$U = \{u \in R \mid u - 1 \in J^\# \text{ or } u + 1 \in J^\#\},$$

as required. □

Every very nil-clean ring is very $J^\#$-clean, but there exists a very $J^\#$-clean ring which is not very nil-clean (see Example 2). Clearly, if J is nil, then $a \in R$ is very $J^\#$-clean if and only if $a \in R$ is very nil-clean.

Now we give the relations among strongly cleanness, very nil-cleanness and very $J^\#$-cleanness for the rings.

Theorem 1. *Let R be a ring. If R is very $J^\#$-clean, then R is strongly clean and $\bar{R}$ is very nil-clean. If R is strongly clean, $\bar{R}$ is very nil-clean and $2 \in J^\#$, then R is very $J^\#$-clean.*

Proof. Suppose that R is very $J^\#$-clean, and let $a \in R$. Then there exist an idempotent $e \in R$ and $v \in J^\#$ such that $ae = ea$ and $a = e + v$ or $a = -e + v$. This implies that $a = (1 - e) + (2e - 1 + v)$ or $a = 1 - e + v - 1$. As $ev = ve$ and $(2e - 1)^{-1} = 2e - 1$, we get $2e - 1 + v \in U$ or $v - 1 \in U$ by Remark 1 and Proposition 2. Hence $a \in R$ is strongly clean because $1 - e$ is an idempotent. Thus R is strongly clean. Further, $\bar{a} = \bar{e} + \bar{v}$ or $\bar{a} = -\bar{e} + \bar{v}$ where $\bar{v}^n = \bar{0}$ for some $n \in \mathbb{N}$. Therefore $\bar{R}$ is very nil-clean.

Assume that R is strongly clean, $\bar{R}$ is very nil-clean, $2 \in J^\#$ and let $a \in R$. Then there exists an idempotent $e \in R$ such that $a = e + u$ and $ea = ae$ where $u \in U$. As $\bar{R}$ is very nil-clean, we can find an idempotent $\bar{f} \in \bar{R}$ such that $\bar{u}\bar{f} = \bar{f}\bar{u}$ and $\bar{u} = \bar{f} + \bar{w}$ or $\bar{u} = -\bar{f} + \bar{w}$ where $\bar{w} \in \bar{R}$ is nilpotent. Further, $\bar{f} = \bar{u} - \bar{w} \in U(\bar{R})$

or $\bar{f} = \bar{w} - \bar{u} \in U(\bar{R})$, and then $\bar{f} = \bar{1}$. Hence $u = 1 + w + r$ or $u = -1 + w + r$ for some $r \in J$. Therefore

$$a = e + u = e + 1 + w + r = (1 - e) + (2e + w + r)$$

or

$$a = e + u = e - 1 + w + r = -(1 - e) + (w + r).$$

Obviously, $(w + r)^m \in J$ or $(2e + w + r)^m \in J$ for some $m \in \mathbb{N}$. Consequently, R is very $J^{\#}$-clean. □

Recall that an element $a \in R$ is called *strongly π-rad clean* provided that there exists an idempotent $e \in R$ such that $ae = ea$ and $a - e \in U$ and $(eae)^n = ea^n e \in J(eRe)$ for some integer $n \geq 1$. A ring R is said to be *strongly π-rad clean* in case every element in R is strongly π-rad clean (see [9]). For instance, if R is local, then it is strongly π-rad clean. It is well known that $eJe = J(eRe)$ for any $e^2 = e \in R$ (see [13, Theorem 1.3.3]).

Theorem 2. *If a ring R is very $J^{\#}$-clean, then it is strongly π-rad clean.*

Proof. Let R be a very $J^{\#}$-clean ring and $a \in R$. Then there exist an idempotent $e \in R$ and $v \in J^{\#}$ such that $ae = ea$ and $a = e + v$ or $a = -e + v$. Assume that $a = e + v$ where $v^n \in J$ for some $n \in \mathbb{N}$. This implies that $a = (1 - e) + (2e - 1 + v)$. As $ev = ve$ and $(2e - 1)^{-1} = 2e - 1$, it is easy to verify that $2e - 1 + v \in U$ by Remark 1. Hence $a(1 - e) = (1 - e)a$ and $a - (1 - e) \in U$ and

$$[(1 - e)a(1 - e)]^n = [(1 - e)v(1 - e)]^n = (1 - e)v^n(1 - e) \in (1 - e)J(1 - e)$$

for some $n \in \mathbb{N}$. Assume that $a = -e + v$ where $v^m \in J$ for some $m \in \mathbb{N}$. This implies that $a = (1 - e) + (v - 1)$. By Proposition 2, $v - 1 \in U$. Thus $a(1 - e) = (1 - e)a$ and $a - (1 - e) \in U$ and

$$[(1 - e)a(1 - e)]^m = [(1 - e)v(1 - e)]^m = (1 - e)v^m(1 - e) \in (1 - e)J(1 - e)$$

for some $m \in \mathbb{N}$. Therefore R is strongly π-rad clean, as asserted. □

The converse of Theorem 2 need not be true as the following example shows.

Example 5. Since $\mathbb{Z}_5$ is a local ring, it is strongly π-rad clean, but not very $J^{\#}$-clean. Because $\bar{2} \in \mathbb{Z}_5$ is not very $J^{\#}$-clean as $J^{\#}(\mathbb{Z}_5) = J(\mathbb{Z}_5) = 0$.

It is an open question that whether strongly clean rings have stable range one (see [17, Question 1]). In the next result, we obtain that very $J^{\#}$-clean rings have this property. So by Theorem 3, we can give a partial answer to the open question. We know from [19] that a ring R has stable range one if and only if $\bar{R}$ has stable range one. Recall that an element a of a ring R is called *strongly π-regular* if there exist a positive integer n and $x \in R$ such that $a^n = a^{n+1}x$. A ring R is said to be *strongly π-regular* if every element of R is strongly π-regular. Ara showed that if R is strongly π-regular, then R has stable range one (see [3, Theorem 4]).

Theorem 3. *Let R be a very $J^{\#}$-clean ring. Then $\bar{R}$ is strongly π-regular, hence R has stable range one.*

Proof. Let R be a very $J^{\#}$-clean ring and $a \in R$. Then there exist an idempotent $e \in R$ and $v \in J^{\#}$ such that $ae = ea$ and $a = e + v$ or $a = -e + v$. Assume that $a = e+v$ where $v^n \in J$ for some $n \in \mathbb{N}$. This implies that $a^n(1-e) = v^n(1-e) \in J$ and $a = (1-e) + (2e-1+v)$. As $ev = ve$ and $(2e-1)^{-1} = 2e-1$, we get $u := 2e-1+v \in U$ by Remark 1. Hence $\bar{a}^n = \bar{a}^n\bar{e} = \bar{u}^n\bar{e}$ and $\bar{a}^{n+1} = \bar{a}^{n+1}\bar{e} = \bar{u}^{n+1}\bar{e}$ in $\bar{R}$. This gives $\bar{a}^n = \bar{a}^{n+1}(\bar{u})^{-1} = (\bar{u})^{-1}\bar{a}^{n+1}$, that is, $\bar{a} \in \bar{R}$ is strongly π-regular. Suppose that $a = -e+v$ where $v^m \in J$ for some $m \in \mathbb{N}$. Write $a = (1-e)+(v-1)$. This implies that $a^m(1-e) = v^m(1-e) \in J$ and

$$a^m e = (ae)^m = \big((v-1)e\big)^m = (v-1)^m e.$$

Since $v^m \in J$, we have $v-1 \in U$. Hence $\bar{a}^m = \bar{a}^m\bar{e} = \overline{v-1}^m\bar{e}$ and

$$\bar{a}^{m+1} = \bar{a}^{m+1}\bar{e} = v \bar{-} 1^{n+1}\bar{e}$$

in $\bar{R}$. This gives

$$\bar{a}^m = \bar{a}^{m+1}(\overline{v-1})^{-1} = (\overline{v-1})^{-1}\bar{a}^{m+1},$$

that is, $\bar{a} \in \bar{R}$ is strongly π-regular, and so $\bar{R}$ is strongly π-regular. Thus $\bar{R}$ has stable range one from [3, Theorem 4]. By the remark above, R has stable range one. □

Let R be a ring and $a \in R$. Set

$$\mathrm{ann}_l(a) = \{r \in R \mid ra = 0\}$$

and

$$\mathrm{ann}_r(a) = \{r \in R \mid ar = 0\}.$$

Then we have the following lemma.

Lemma 3. *Let R be a ring and $a = e+v$ or $a = -e+v$ very $J^{\#}$-clean decomposition of a in R. Then $\mathrm{ann}_l(a) \subseteq \mathrm{ann}_l(e)$ and $\mathrm{ann}_r(a) \subseteq \mathrm{ann}_r(e)$.*

Proof. Let $r \in \mathrm{ann}_l(a)$. Then $ra = 0$. Since $ev = ve$, we have $re = rv$ or $re = -rv$, and so $re = rve = rev$ or $re = -rve = -rev$. It follows that $re(1-v) = 0$ or $re(1+v) = 0$, and so $re = 0$ because $1+v,\ 1-v \in U$. That is, $r \in \mathrm{ann}_l(e)$. Therefore $\mathrm{ann}_l(a) \subseteq \mathrm{ann}_l(e)$. Similarly, we can prove that $\mathrm{ann}_r(a) \subseteq \mathrm{ann}_r(e)$. □

Theorem 4. *Let R be a ring and $f \in R$ be an idempotent. Then $a \in fRf$ is very $J^{\#}$-clean in R if and only if a is very $J^{\#}$-clean in fRf.*

Proof. Suppose $a = e + v$, $e^2 = e \in fRf$, $v \in J^{\#}(fRf)$, and $ev = ve$. Obviously, $v \in J^{\#}$ because $v^n \in J(fRf) = fJf \subseteq J$ for some $n \in \mathbb{N}$. Hence $a \in fRf$ is very $J^{\#}$-clean in R. Similarly, one can show that if $a = -e + v$, $e^2 = e \in fRf$, $v \in J^{\#}(fRf)$, and $ev = ve$, then $a \in fRf$ is very $J^{\#}$-clean in R.

Conversely, suppose that $a = -e + v$, $e^2 = e \in R$, $v \in J^{\#}$, and $ev = ve$. As $a \in fRf$, we see that

$$1 - f \in \mathrm{ann}_l(a) \cap \mathrm{ann}_r(a) \subseteq \mathrm{ann}_l(e) \cap \mathrm{ann}_r(e).$$

Hence $(1-f)v = 0 = v(1-f)$ and $fv = vf = v$. We observe that $a = fef + fvf$, $(fef)^2 = fef$, and

$$(fvf)^m = v^m \in fJf = J(fRf) \subseteq J^{\#}(fRf)$$

for some $m \in \mathbb{N}$. Furthermore,

$$(fef)(fvf) = fevf = fvef = (fvf)(fef).$$

Similarly, one can prove that $a \in fRf$ is very $J^{\#}$-clean in fRf where $a = e + v$, $e^2 = e \in R$, $v \in J^{\#}$, and $ev = ve$. Therefore the proof is completed. □

Corollary 1. *A ring R is very $J^{\#}$-clean if and only if eRe is very $J^{\#}$-clean for any idempotent $e \in R$.*

Proof. Let $a \in eRe$. Since R is very $J^{\#}$-clean, we see that $a \in eRe$ is very $J^{\#}$-clean in R. According to Theorem 4, $a \in eRe$ is very $J^{\#}$-clean in eRe. The converse is clear by using $e = 1$. □

As is well known, every homomorphic image of a (strongly) clean ring is (strongly) clean (see [12], [16], [17]). Analogously, we can give the following result.

Proposition 3. *Every homomorphic image of very $J^{\#}$-clean rings is very $J^{\#}$-clean.*

Proof. Let R be a very $J^{\#}$-clean ring and $\varphi\colon R \to S$ a surjective ring homomorphism. Then for any $b \in S$, there exists $a \in R$ such that $\varphi(a) = b$. Since R is very $J^{\#}$-clean, we can find an idempotent $e \in R$ and $v \in J^{\#}$ such that $ae = ea$ and $a = e+v$ or $a = -e+v$. Assume that $a = -e+v$ and $v^n \in J$ for some $n \in \mathbb{N}$. Then $\varphi(a) = -\varphi(e) + \varphi(v)$ and $\varphi(a)\varphi(e) = \varphi(e)\varphi(a)$. Obviously, $(\varphi(e))^2 = \varphi(e) \in S$. Since $\varphi(J) \subseteq J(S)$, we have $\varphi(v^n) = \varphi(v)^n \in J(S)$ and so $\varphi(v) \in J^{\#}(S)$. Similarly, one can show that $\varphi(a) = \varphi(e) + \varphi(v) \in S$ is very $J^{\#}$-clean in S where $a = e+v$ and $v \in J^{\#}$. □

If I is a left ideal of a ring R, *idempotents lift modulo I* if, given $a \in R$ with $a^2 - a \in I$, there exists $e^2 = e \in R$ such that $a - e \in I$ (see [16]). Note that R is a clean ring if and only if R/J is a clean ring and idempotents lift modulo J (see [12, Proposition 6]). Recall that a ring R is called *abelian* if every idempotent is central.

Theorem 5. *Let I be an ideal of an abelian ring R with $I \subseteq J$. Then R is very $J^{\#}$-clean if and only if R/I is very $J^{\#}$-clean and idempotents lift modulo I.*

Proof. Assume that R is very $J^{\#}$-clean. Then R/I is very $J^{\#}$-clean by Proposition 3. Further, by Theorem 1, R is strongly clean, and so idempotents lift modulo I by [12, Proposition 6].

Conversely, suppose that R/I is very $J^{\#}$-clean and idempotents lift modulo I and let $a \in R$. By assumption, for $\bar{a} \in R/I$, there exists an idempotent $\bar{e} \in R/I$ such that $\bar{a}\bar{e} = \bar{e}\bar{a}$ and $\bar{a} - \bar{e}$ or $\bar{a} + \bar{e}$ is an element of $J^{\#}(R/I)$. Assume that $\bar{a} = -\bar{e} + \bar{v}$ where $\bar{v} \in J^{\#}(R/I)$. Then we can find some $t \in \mathbb{N}$ such that $\bar{v}^t \in J(R/I) = J/I$ and so $v \in J^{\#}$. Since idempotents lift modulo I, we may assume that $e^2 = e$. Hence $a + e - v \in I \subseteq J$ and so a is a very $J^{\#}$-clean element because e is central. Similarly, one can prove that if $\bar{a} = \bar{e} + \bar{v}$ and $\bar{v} \in J^{\#}(R/I)$, then a is a very $J^{\#}$-clean element. □

3 Examples

The purpose of this section is to construct several examples for very $J^{\#}$-clean rings.

Let R be a ring and σ be an endomorphism of R. Let $R[[x,\sigma]]$ be the set of all power series over the ring R. For any $\sum\limits_{i=0}^{\infty} a_i x^i, \sum\limits_{i=0}^{\infty} b_i x^i \in R[[x,\sigma]]$, we define

$$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i,$$

and

$$\Big(\sum_{i=0}^{\infty} a_i x^i\Big)\Big(\sum_{i=0}^{\infty} b_i x^i\Big) = \sum_{i=0}^{\infty} c_i x^i$$

where $c_i = \sum\limits_{k=0}^{i} a_k \sigma^k\big(b_{i-k}\big)$. Then $R[[x,\sigma]]$ is a ring under the preceding addition and multiplication. Clearly, $R[[x,\sigma]]$ is $R[[x]]$ only when σ is the identity morphism. Furthermore, $J\big(R[[x,\sigma]]\big) = J + xR[[x,\sigma]]$ (see [14, Ex. 5.6]).

Lemma 4. *If $R[[x,\sigma]]$ is abelian, then $\sigma(e) = e$ for every idempotent $e \in R$.*

Proof. Since $R[[x,\sigma]]$ is abelian, we have $xe = ex$ for every idempotent $e \in R$. Hence we get $xe = ex = \sigma(e)x$, and so $\sigma(e) = e$, as asserted. □

Proposition 4. *Let $R[[x,\sigma]]$ be an abelian ring. Then the following are equivalent.*

(1) R is very $J^{\#}$-clean.

(2) $R[[x,\sigma]]$ is very $J^{\#}$-clean.

Proof. (1) ⇒ (2) Let $a(x) \in R[[x,\sigma]]$. Then we can find an idempotent $e \in R$ and $v \in J^{\#}$ such that $a(0) = e + v$ or $a(0) = -e + v$. Assume that $a(0) = e + v$. Then $a(x) = e + v(x)$ where $v(x) = a(x) - e = v + a_1 x + a_2 x^2 + \cdots$. Since $\sigma(e) = e$ for any idempotent $e \in R$ by Lemma 4, we see that $ev(x) = v(x)e$. Further, we conclude that $v(x) \in J^{\#}\big(R[[x,\sigma]]\big)$ because $v \in J^{\#}$ and

$$J\big(R[[x,\sigma]]\big) = J + xR[[x,\sigma]].$$

This implies that $a(x) \in R[[x,\sigma]]$ is very $J^\#$-clean. Assume that $a(0) = -e + v$. Similarly, we can show that $a(x) \in R[[x,\sigma]]$ is very $J^\#$-clean. Thus $R[[x,\sigma]]$ is very $J^\#$-clean.

(2) $\Rightarrow$ (1) Let $a \in R$. Then we can find an idempotent $e(x) \in R[[x,\sigma]]$ and $v(x) \in J^\#\big(R[[x,\sigma]]\big)$ such that $ae(x) = e(x)a$ and $a = e(x) + v(x)$ or $a = -e(x) + v(x)$. Obviously, $e(0) \in R$ is an idempotent and $v(0) \in J^\#$. Since $a = e(0) + v(0)$ or $a = -e(0) + v(0)$ and $ae(0) = e(0)a$, we obtain that $a \in R$ is very $J^\#$-clean, and therefore R is very $J^\#$-clean. □

Remark 2. As in the proof of [1, Lemma 2.18], we can show that the idempotents of $R[[x,\sigma]]$ belong to R. Hence if R is abelian, then so is $R[[x,\sigma]]$.

The next result is a characterization of being very $J^\#$-clean for abelian rings.

Theorem 6. *Let R be an abelian ring. Then the following conditions are equivalent.*

(1) R is very $J^\#$-clean.

(2) $R[[x]]/\langle x^n\rangle$ is very $J^\#$-clean for all $n \geq 2$.

(3) $R[[x]]/\langle x^2\rangle$ is very $J^\#$-clean.

(4) $R[x]/\langle x^2\rangle$ is very $J^\#$-clean.

Proof. (1) $\Rightarrow$ (2) If R is very $J^\#$-clean, then $R[[x]]$ is very $J^\#$-clean by Proposition 4 and so $R[[x]]/\langle x^n\rangle$ is very $J^\#$-clean by Proposition 3 for all $n \geq 2$.

(2) $\Rightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) Since R is abelian, so is $R[[x]]$ by Remark 2. Note that $J(R[[x]]) = J + xR[[x]]$. Then $\langle x^2\rangle \subseteq J(R[[x]])$, and so R is very $J^\#$-clean by Theorem 5.

(3) $\Leftrightarrow$ (4) Since $R[x]/\langle x^2\rangle \cong R[[x]]/\langle x^2\rangle$, there is nothing to show. □

Let R be a ring and $\sigma\colon R \to R$ be an endomorphism. Set

$$D_2(R,\sigma) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a, b \in R \right\},$$

addition and multiplication are defined as follows:

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} + \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ 0 & a+c \end{pmatrix};$$
$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c \end{pmatrix} = \begin{pmatrix} ac & ad + b\sigma(c) \\ 0 & ac \end{pmatrix}.$$

Then $D_2(R,\sigma)$ is a ring with the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Denote $D_2(R, 1_R)$ by $D_2(R)$, where $1_R\colon R \to R$, $r \mapsto r$. Further, it can be verified that

$$J\big(D_2(R,\sigma)\big) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a \in J,\, b \in R \right\}.$$

Proposition 5. *Let R be an abelian ring and $\sigma\colon R \to R$ be an endomorphism. Then the following are equivalent.*

(1) R is very $J^{\#}$-clean.

(2) $D_2(R,\sigma)$ is very $J^{\#}$-clean.

Proof. Note that since R is abelian, $\sigma(e) = e$ for every idempotent $e \in R$ by Lemma 4 and Remark 2.

(1) $\Rightarrow$ (2) Let $A := \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in D_2(R,\sigma)$. Then there exists an idempotent $e \in R$ such that $ae = ea$ and $v := a - e \in J^{\#}$ or $v := a + e \in J^{\#}$. Assume that $v := a - e \in J^{\#}$ and $v^n \in J$ for some $n \in \mathbb{N}$. Since $V^n = \begin{pmatrix} v^n & * \\ 0 & v^n \end{pmatrix} \in J\big(D_2(R,\sigma)\big)$ where $V = \begin{pmatrix} v & b \\ 0 & v \end{pmatrix}$,

$$A - \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} = V \in J^{\#}\big(D_2(R,\sigma)\big).$$

As R is abelian and $\sigma(e) = e$, we see that $EA = AE$ where $E^2 = E = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$ (because $EA = AE$ if and only if $eb = b\sigma(e) = be$). Therefore $A \in D_2(R,\sigma)$ is very $J^{\#}$-clean. Assume that $v := a + e \in J^{\#}$. Similar to the preceding discussion, it can be shown that $A \in D_2(R,\sigma)$ is very $J^{\#}$-clean, as required.

(2) $\Rightarrow$ (1) Let $a \in R$. Then $A := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in D_2(R,\sigma)$. By hypothesis, there exists an idempotent $E := \begin{pmatrix} e & b \\ 0 & e \end{pmatrix} \in D_2(R,\sigma)$ such that $AE = EA$ and

$$A + E \in J^{\#}\big(D_2(R,\sigma)\big)$$

or $A - E \in J^{\#}\big(D_2(R,\sigma)\big)$. As E is an idempotent, we have $e = e^2$. Further, we get $ea = ae$, and that $a - e \in J^{\#}$ or $a + e \in J^{\#}$. Therefore R is very $J^{\#}$-clean. □

Let R be a ring and V an R-R-bimodule which is a general ring (possibly with no unity) in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in R$. Then *ideal-extension* (it is also called *Dorroh extension*) $I(R;V)$ of R by V is defined to be the additive abelian group $I(R;V) = R \oplus V$ with multiplication $(r,v)(s,w) = (rs, rw + vs + vw)$.

Proposition 6. *An ideal-extension $S = I(R;V)$ is very $J^{\#}$-clean if the following conditions are satisfied.*

(1) R is very $J^{\#}$-clean;

(2) If $e^2 = e \in R$, then $ev = ve$ for all $v \in V$;

(3) If $v \in V$, then $v + w + vw = 0$ for some $w \in V$.

Furthermore, if S is very $J^{\#}$-clean, then R is very $J^{\#}$-clean.

Proof. Suppose that (1), (2) and (3) are satisfied. Let $s = (r, w) \in S$ and (by (1)) write $r = e+v$ or $r = -e+v$, $e^2 = e$, $v \in J^{\#}$ and $re = er$. Assume that $r = -e+v$ and $v^n \in J$ for some $n \in \mathbb{N}$. Then $s = -(e, 0) + (v, w)$ and $(e, 0)^2 = (e, 0) \in S$. Note that $(0, V) \subseteq J(S)$ if and only if (3) holds (see [15]). Since $(v, w)^n = (v^n, *)$, it suffices to show that $(v^n, 0) \in J(S)$. For any $(p, q) \in S$,

$$(1, 0) - (v^n, 0)(p, q) = (1 - v^n p, -v^n q) \in U(S)$$

because

$$(1 - v^n p, -v^n q) = (1 - v^n p, 0)(1, (1 - v^n p)^{-1}(-v^n q))$$

and

$$(1, (1 - v^n p)^{-1}(-v^n q)) = (1, 0) + (0, (1 - v^n p)^{-1}(-v^n q)) \in U(S)$$

by $(0, V) \subseteq J(S)$. Thus $(v^n, 0) \in J(S)$ and so $(v, w) \in J^{\#}(S)$. By (2), $(r, w)(e, 0) = (e, 0)(r, w)$. The case where $r = e + v$ can be similarly handled.

On the other hand, suppose that S is very $J^{\#}$-clean and let $a \in R$. Then $(a, 0) = (e, t) + (v, w)$ or $(a, 0) = -(e, t) + (v, w)$, $(e, t)^2 = (e, t)$, $(v, w) \in J^{\#}(S)$ and $(a, 0)(e, t) = (e, t)(a, 0)$. Assume that $(a, 0) = (e, t)+(v, w)$ and $(v, w)^m \in J(S)$ for some $m \in \mathbb{N}$. Since $(v, w)^m \in J(S)$, $(e, t)^2 = (e, t)$ and $(a, 0)(e, t) = (e, t)(a, 0)$, we get $a = e + v$, $v^m \in J$, $e^2 = e \in R$, and $ae = ea$. Hence a is strongly $J^{\#}$-clean. Suppose $(a, 0) = -(e, t) + (v, w)$ and $(v, w)^n \in J(S)$ for some $n \in \mathbb{N}$. Similarly, it can be shown that $-a$ is strongly $J^{\#}$-clean and so R is very $J^{\#}$-clean. □

Example 6. Let R be an abelian very $J^{\#}$-clean ring, n a positive integer and

$$S = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \middle| \; a, a_{ij} \in R (i < j) \right\}.$$

Then S is very $J^{\#}$-clean and noncommutative if $n \geq 3$.

Proof. Let

$$V = \left\{ \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \middle| \; a_{ij} \in R (i < j) \right\}.$$

Then $S \cong I(R; V)$. By applying Proposition 6, (1) is clear; (2) holds because R is abelian and (3) follows because of $V \subseteq J(S)$. □

4 Very $J^{\#}$-clean 2×2 matrices

Let $f, g \in R[x]$ be polynomials over a commutative ring R and let (f, g) denote the ideal generated by f, g. A polynomial $f(x) \in R[x]$ is a monic polynomial of degree n if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_{n-1}, \ldots, a_1, a_0 \in R$. If $\varphi \in M_n(R)$, we use $\chi(\varphi)$ to stand for the characteristic polynomial $\det(xI_n - \varphi)$.

The aim of this section is to characterize a single very $J^{\#}$-clean 2×2 matrix over a commutative local ring by means of the factorization of its characteristic polynomial.

We begin with the following result from [7], and give the proof of it for the sake of completeness.

Lemma 5. *Let R be a commutative ring and $\varphi \in M_n(R)$. Then the following are equivalent.*

(1) $\varphi \in J^{\#}\big(M_n(R)\big)$.

(2) $\chi(\varphi) \equiv x^n \pmod J$.

(3) There exists a monic polynomial $h \in R[x]$ such that $h \equiv x^{\deg h} \pmod J$ for which $h(\varphi) = 0$.

Proof. Note that $J\big(M_n(R)\big) = M_n\big(J\big)$ and $M_n(R)/J\big(M_n(R)\big) = M_n(\bar{R})$. Furthermore, since R is commutative, we have that $\mathrm{Nil}(R) \subseteq J$.

(1) $\Rightarrow$ (2) If $\varphi \in J^{\#}\big(M_n(R)\big)$, then $\bar{\varphi}$ is nilpotent in $M_n(\bar{R})$. According to [4, Proposition 3.5.4], we get $\chi(\varphi) \equiv x^n \pmod{\mathrm{Nil}(R)}$. So $\chi(\varphi) \equiv x^n \pmod J$ because $\mathrm{Nil}(R) \subseteq J$.

(2) $\Rightarrow$ (3) Set $h = \chi(\varphi)$. Then $h \equiv x^{\deg h} \pmod J$. By Cayley-Hamilton Theorem, $h(\varphi) = 0$.

(3) $\Rightarrow$ (1) Assume that $h = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ where $a_i \in J$ for $0 \le i \le n-1$. Then $\bar{h} \equiv x^n \pmod{\mathrm{Nil}(\bar{R})}$ and $\bar{h}(\bar{\varphi}) = 0$. Again, by [4, Proposition 3.5.4], $\bar{\varphi}$ is nilpotent in $M_n(\bar{R})$. This gives $\varphi \in J^{\#}\big(M_n(R)\big)$. □

Definition 1. [7, Definition 2.4] For $r \in R$, define

$$\mathbb{J}_r = \{f \in R[x] \mid f \text{ is monic, and } f \equiv (x-r)^{\deg f} \pmod{J^{\#}}\}.$$

Remark 3. If R is commutative, then $J^{\#}$ is simply the Jacobson radical. So we get

$$\mathbb{J}_r = \{f \in R[x] \mid f \text{ is monic, and } f \equiv (x-r)^{\deg f} \pmod J\}.$$

By $f \equiv (x-r)^{\deg f} \pmod J$, we mean $f - (x-r)^{\deg f} \in J[x]$. Furthermore, it is well known that

$$\chi(\varphi) = x^2 - \mathrm{tr}(\varphi)x + \det(\varphi) \quad \text{and} \quad \chi(-\varphi) = x^2 + \mathrm{tr}(\varphi)x + \det(\varphi)$$

because $tr(-\varphi) = -\mathrm{tr}(\varphi)$ and $\det(\varphi) = \det(-\varphi)$ for $\varphi \in M_2(R)$. In general, note that

$$\chi(-\varphi)(x) = \det\big(xI_n - (-\varphi)\big) = (-1)^n \det\big((-x)I_n + \varphi\big) = (-1)^n\chi(\varphi)(-x)$$

and $\det(-\varphi) = (-1)^n \det(\varphi)$ for $\varphi \in M_n(R)$.

For an easy reference, we mention the following lemmas without proofs. Recall that a commutative ring R is called *projective-free* if every finitely generated projective R-module is free. Any commutative local ring is projective-free.

Lemma 6. *[7, Lemma 2.5] Let R be a projective-free ring and $h \in R[x]$ a monic polynomial of degree n, let $\varphi \in M_n(R)$. If $h(\varphi) = 0$ and there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$, then φ is strongly $J^\#$-clean.*

Lemma 7. *[7, Theorem 2.6] Let R be a projective-free ring and $h \in R[x]$ a monic polynomial of degree n. Then the following are equivalent.*

(1) Every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is strongly $J^\#$-clean.

(2) There exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$.

In the proof of Lemma 8 and Theorem 7, we refer to Lemma 6 and Lemma 7.

Lemma 8. *Let R be a commutative local ring and $h \in R[x]$ a monic polynomial of degree n, let $\varphi \in M_n(R)$. If $h(\varphi) = 0$ and there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1 \cup \mathbb{J}_{-1}$, then φ is very $J^\#$-clean.*

Proof. By hypothesis, there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1 \cup \mathbb{J}_{-1}$. If $h_1 \in \mathbb{J}_1$, then φ is strongly $J^\#$-clean by Lemma 6, and so φ is very $J^\#$-clean. Hence we assume that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_{-1}$. Then

$$h_0 \equiv x^{\deg(h_0)} \pmod J \quad \text{and} \quad h_1 \equiv (x-(-1))^{\deg(h_1)} \pmod J.$$

Set $t := -x$ and $g(t) := (-1)^{\deg(h)}h(-t)$. Then $g(t)$ factors as $g = g_0g_1$, where

$$g_0(t) = (-1)^{\deg(h_0)}h_0(-t) \quad \text{and} \quad g_1(t) = (-1)^{\deg(h_1)}h_1(-t).$$

Note that $\deg(g_0) = \deg(h_0)$ and $\deg(g_1) = \deg(h_1)$. Since $h_0 \equiv x^{\deg(h_0)} \pmod J$, we see that

$$g_0(t) = (-1)^{\deg(h_0)}h_0(-t) \equiv (-1)^{\deg(h_0)}x^{\deg(h_0)} \equiv t^{\deg(g_0)} \pmod J,$$

and so $g_0 \in \mathbb{J}_0$. Further, as $h_1 \equiv (x-(-1))^{\deg(h_1)} \pmod J$, we have

$$\begin{aligned} g_1(t) = (-1)^{\deg(h_1)}h_1(-t) &\equiv (-1)^{\deg(h_1)}(-t-(-1))^{\deg(h_1)} \\ &\equiv (t-1)^{\deg(g_1)} \pmod J, \end{aligned}$$

and so $g_1 \in \mathbb{J}_1$. We observe that $g(-\varphi) = 0$ because $h(\varphi) = 0$. In view of Lemma 6, $-\varphi \in M_n(R)$ is strongly $J^\#$-clean. That is, φ is very $J^\#$-clean. The proof is completed. □

Theorem 7. *Let R be a commutative local ring and $h \in R[x]$ a monic polynomial of degree n. Then the following are equivalent.*

(1) Every $\varphi \in M_n(R)$ with $\chi(\varphi) = h$ is very $J^\#$-clean.

(2) There exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1 \cup \mathbb{J}_{-1}$.

Proof. (1) ⇒ (2) Since φ is very $J^{\#}$-clean, φ or $-\varphi$ is strongly $J^{\#}$-clean. If φ is strongly $J^{\#}$-clean, then there exists a factorization $h = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1$ by Lemma 7. Suppose $-\varphi$ is strongly $J^{\#}$-clean. It follows by Lemma 7 that $g(t) := \chi(-\varphi)$ factors as $g = g_0g_1$ where $g_0 \in \mathbb{J}_0$ and $g_1 \in \mathbb{J}_1$. This implies

$$h(x) = \chi(\varphi) = (-1)^{\deg(h)}g(-x) = (-1)^{\deg(h)}g_0(-x)g_1(-x)\,.$$

Set $h_0(x) = (-1)^{\deg(g_0)}g_0(-x)$ and $h_1(x) = (-1)^{\deg(g_1)}g_1(-x)$. Then $h = h_0h_1$. Since $g_0(t) \equiv t^{\deg(g_0)} \pmod J$, we get

$$h_0(x) = (-1)^{\deg(g_0)}g_0(-x) \equiv x^{\deg(g_0)} \pmod J\,,$$

hence $h_0 \in \mathbb{J}_0$. In addition, as $g_1(t) \equiv (t-1)^{\deg(g_1)} \pmod J$, we see that

$$h_1(x) = (-1)^{\deg(g_1)}g_1(-x) \equiv (x+1)^{\deg(g_1)} \pmod J\,.$$

This gives $h_1 \in \mathbb{J}_{-1}$. That is, $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_1 \cup \mathbb{J}_{-1}$, as asserted.

(2) ⇒ (1) For any $\varphi \in M_2(R)$ with $\chi(\varphi) = h$, we have $h(\varphi) = 0$ by the Cayley-Hamilton Theorem. In light of Lemma 8, φ is very $J^{\#}$-clean. □

Corollary 2. *[7, Corollary 2.8] Let R be a commutative local ring and $\varphi \in M_2(R)$. Then φ is strongly $J^{\#}$-clean if and only if*

(1) $\chi(\varphi) \equiv x^2 \pmod J$*; or*

(2) $\chi(\varphi) \equiv (x-1)^2 \pmod J$*; or*

(3) $\chi(\varphi)$ *has a root in J and a root in $1 + J$.*

In analogy with Corollary 2, we have the following result.

Corollary 3. *Let R be a commutative local ring and $\varphi \in M_2(R)$. Then $-\varphi$ is strongly $J^{\#}$-clean if and only if*

(1) $\chi(\varphi) \equiv x^2 \pmod J$*; or*

(2) $\chi(\varphi) \equiv (x+1)^2 \pmod J$*; or*

(3) $\chi(\varphi)$ *has a root in J and a root in $-1 + J$.*

Proof. Suppose that $-\varphi$ is strongly $J^{\#}$-clean. As in the proof of Theorem 7, there exists a factorization $\chi(\varphi) = h_0h_1$ such that $h_0 \in \mathbb{J}_0$ and $h_1 \in \mathbb{J}_{-1}$. Consider the following cases:

Case I. $\deg(h_0) = 2$ and $\deg(h_1) = 0$. Then $h_0 = \chi(\varphi) = x^2 - \mathrm{tr}(\varphi)x + \det(\varphi)$ and $h_1 = 1$. As $h_0 \in \mathbb{J}_0$, it follows from Lemma 5 that $\varphi \in J^{\#}\big(M_2(R)\big)$ or equivalently, $\chi(\varphi) \equiv x^2 \pmod J$.

Case II. $\deg(h_0) = 0$ and $\deg(h_1) = 2$. Then $h_1(x) = \chi(\varphi) \equiv (x+1)^2 \pmod J$ because $h_1 \in \mathbb{J}_{-1}$.

Case III. $\deg(h_0) = 1$ and $\deg(h_1) = 1$. Then $h_0 = x - \alpha$ and $h_1 = x - \beta$. Since $h_0 \in \mathbb{J}_0$, we see that $h_0 \equiv x \pmod J$, and then $\alpha \in J$. As $h_1 \in \mathbb{J}_{-1}$, we have

$h_1 \equiv x+1 \pmod{J}$, and so $\beta \in -1+J$. Therefore $\chi(\varphi)$ has a root in J and a root in $-1+J$.

For the reverse implication, if (1) or (2) is valid, then $-\varphi \in J^{\#}\big(M_2(R)\big)$ or $I_2+\varphi \in J^{\#}\big(M_2(R)\big)$. This implies that $-\varphi$ is strongly $J^{\#}$-clean. Suppose that $\chi(\varphi)$ has a root in J and a root in $-1+J$ and $-\varphi$, $I_2+\varphi \notin J\big(M_2(R)\big)$. By Remark 3, we know that $\chi(\varphi)(-x)=\chi(-\varphi)(x)$. In this case, $\chi(\varphi)$ has a root in J and a root in $1+J$. According to [6, Theorem 16.4.31], φ is strongly J-clean, and therefore it is strongly $J^{\#}$-clean. □

For instance, choose $\varphi = \begin{pmatrix} 0 & 7 \\ 8 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_9)$. Note that $J(\mathbb{Z}_9)=3\mathbb{Z}_9$. Then $\chi(\varphi)=x^2+x+7=(x+1)^2+6x+6$. Hence $\chi(\varphi)\equiv (x+1)^2 \pmod{J(\mathbb{Z}_9)}$, and so $\varphi \in M_2(\mathbb{Z}_9)$ is very $J^{\#}$-clean by Corollary 3.

In the next, we investigate very $J^{\#}$-clean matrices over power series rings.

Theorem 8. *Let R be a commutative local ring. Then the following are equivalent.*

(1) $A(x) \in M_2\big(R[[x]]\big)$ is very $J^{\#}$-clean.

(2) $A(0) \in M_2(R)$ is very $J^{\#}$-clean.

Proof. (1) ⇒ (2) Since $A(x)$ is very $J^{\#}$-clean in $M_2\big(R[[x]]\big)$, there exist an

$$E(x)=E^2(x) \in M_2\big(R[[x]]\big) \quad \text{and} \quad V(x) \in J^{\#}\big(M_2(R[[x]])\big)$$

such that $E(x)V(x)=V(x)E(x)$, and

$$A(x)=E(x)+V(x) \quad \text{or} \quad A(x)=-E(x)+V(x)\,.$$

This implies that $E(0)V(0)=V(0)E(0)$ and

$$A(0)=E(0)+V(0) \quad \text{or} \quad A(0)=-E(0)+V(0)\,,$$

where $E(0)=E^2(0) \in M_2(R)$ and $V(0) \in J^{\#}\big(M_2(R)\big)$. Hence $A(0)$ is very $J^{\#}$-clean in $M_2(R)$.

(2) ⇒ (1) Since $R[[x]]/J\big(R[[x]]\big) \cong R/J$ and R is local, $R[[x]]$ is local. Assume that $-A(0)$ is strongly $J^{\#}$-clean. Then

- $-A(0) \in J^{\#}\big(M_2(R)\big)$;
- or $I_2+A(0) \in J^{\#}\big(M_2(R)\big)$;
- or the characteristic polynomial $\chi\big(A(0)\big)=y^2-\mu y+\lambda$ has a root $\alpha \in -1+J$ and a root $\beta \in J$.

If $-A(0) \in J^{\#}\big(M_2(R)\big)$, then

$$-A(x) \in J^{\#}\big(M_2(R[[x]])\big)\,.$$

If $I_2 + A(0) \in J^{\#}\big(M_2(R)\big)$, then

$$I_2 + A(x) \in J^{\#}\big(M_2(R[[x]])\big)\,.$$

Otherwise, we write $y = \sum\limits_{i=0}^{\infty} b_i x^i$ and

$$\chi(-A(x)) = y^2 - \mu(x)y + \lambda(x)\,.$$

Then $y^2 = \sum\limits_{i=0}^{\infty} c_i x^i$ where $c_i = \sum\limits_{k=0}^{i} b_k b_{i-k}$. Let

$$\mu(x) = \sum_{i=0}^{\infty} \mu_i x^i, \quad \lambda(x) = \sum_{i=0}^{\infty} \lambda_i x^i \in R[[x]]$$

where $\mu_0 = \mu$ and $\lambda_0 = \lambda$. Then

$$y^2 - \mu(x)y + \lambda(x) = 0$$

holds in $R[[x]]$ if the following equations are satisfied:

$$\begin{aligned} b_0^2 - b_0\mu_0 + \lambda_0 = 0;\\ (b_0b_1 + b_1b_0) - (b_0\mu_1 + b_1\mu_0) + \lambda_1 = 0;\\ (b_0b_2 + b_1^2 + b_2b_0) - (b_0\mu_2 + b_1\mu_1 + b_2\mu_0) + \lambda_2 = 0;\\ \vdots \end{aligned}$$

Obviously, $\mu_0 = \alpha + \beta \in U$ and $\alpha - \beta \in U$. Let $b_0 = \alpha$. Since R is commutative and $2b_0 - \mu_0 = 2\alpha - \mu = \alpha - \beta$, there exists some $b_1 \in R$ such that

$$b_1(2b_0 - \mu_0) = b_0\mu_1 - \lambda_1.$$

Further, there exists some $b_2 \in R$ such that

$$b_2(2b_0 - \mu_0) = b_0\mu_2 + b_1\mu_1 - b_1^2 - \lambda_2.$$

By iteration of this process, we get $b_3, b_4, \dots$. Then $y^2 - \mu(x)y + \lambda(x) = 0$ has a root $y_0(x) \in -1 + J\big(R[[x]]\big)$. If $b_0 = \beta \in J$, analogously, we can show that $y^2 - \mu(x)y + \lambda(x) = 0$ has a root $y_1(x) \in J\big(R[[x]]\big)$. In light of Corollary 3, $-A(x)$ is strongly $J^{\#}$-clean. Similarly, we can prove that if $A(0)$ is strongly $J^{\#}$-clean, then $A(x)$ is strongly $J^{\#}$-clean by Corollary 2. Therefore $A(x)$ is very $J^{\#}$-clean in $M_2\big(R[[x]]\big)$. □

Example 7. Let $R = \mathbb{Z}_9[[x]]$ and

$$A(x) = \begin{pmatrix} \bar{0} & \bar{2} - \sum\limits_{n=1}^{\infty} (\bar{1} + \bar{5}^n)x^n \\ \bar{1} & \bar{1} - \sum\limits_{n=1}^{\infty} (\bar{1} + \bar{7}^n)x^n \end{pmatrix} \in M_2(R).$$

Then

$$A(0) = \begin{pmatrix} \bar{0} & \bar{2} \\ \bar{1} & \bar{1} \end{pmatrix} = -\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix} + \begin{pmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{2} \end{pmatrix},$$

where $\begin{pmatrix} \bar{1} & \bar{0} \\ \bar{0} & \bar{1} \end{pmatrix}$ is an idempotent and

$$\begin{pmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{2} \end{pmatrix} \in J^{\#}\big(M_2(\mathbb{Z}_9)\big)$$

because

$$\begin{pmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{2} \end{pmatrix}^2 = \begin{pmatrix} \bar{3} & \bar{6} \\ \bar{3} & \bar{6} \end{pmatrix} \in J\big(M_2(\mathbb{Z}_9)\big).$$

Thus $A(x)$ is very $J^{\#}$-clean by Theorem 8. Note that $A(0)$ is not strongly J-clean.

Corollary 4. *Let R be a commutative local ring and $A(x) \in M_2\big(R[[x]]/(x^m)\big)$ $(m \geq 1)$. Then the following are equivalent.*

(1) $A(x) \in M_2\big(R[[x]]/(x^m)\big)$ is very $J^{\#}$-clean.

(2) $A(0) \in M_2(R)$ is very $J^{\#}$-clean.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Let $\psi\colon R[[x]] \to R[[x]]/(x^m)$ denote the natural homomorphism. Then ψ induces the surjective ring homomorphism

$$\psi^*\colon M_2\big(R[[x]]\big) \to M_2\big(R[[x]]/(x^m)\big).$$

Then there exists $B(x) \in M_2\big(R[[x]]\big)$ such that $\psi^*\big(B(x)\big) = A(x)$. Then Theorem 8 completes the proof. □

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Existence of solutions for Navier problems with degenerate nonlinear elliptic equations

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Abstract. In this paper we are interested in the existence and uniqueness of solutions for the Navier problem associated to the degenerate nonlinear elliptic equations

$$\Delta(v(x)\,|\Delta u|^{q-2}\Delta u) - \sum_{j=1}^{n} D_j\big[\omega(x)\mathcal{A}_j(x,u,\nabla u)\big] = f_0(x) - \sum_{j=1}^{n} D_j f_j(x), \text{ in } \Omega$$

in the setting of the weighted Sobolev spaces.

1 Introduction

In this paper we prove the existence and uniqueness of (weak) solutions in the weighted Sobolev space $X = W^{2,q}(\Omega, v) \cap W_0^{1,p}(\Omega, \omega)$ (see Definition 4) for the Navier problem

$$\begin{cases} Lu(x) = f_0(x) - \sum_{j=1}^{n} D_j f_j(x), & \text{in } \Omega \\ u(x) = 0, & \text{on } \partial\Omega \\ \Delta u(x) = 0, & \text{on } \partial\Omega \end{cases} \tag{1}$$

where L is the partial differential operator

$$Lu(x) = \Delta(v(x)\,|\Delta u|^{q-2}\Delta u) - \sum_{j=1}^{n} D_j\big[\omega(x)\mathcal{A}_j(x,u(x),\nabla u(x))\big]$$

where $D_j = \partial/\partial x_j$, Ω is a bounded open set in $\mathbb{R}^n$, ω and v are two weight functions, Δ is the usual Laplacian operator, $1 < p, q < \infty$ and the functions $\mathcal{A}_j : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ $(j = 1, \ldots, n)$ satisfy the following conditions:

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(H1) $x \mapsto \mathcal{A}_j(x,\eta,\xi)$ is measurable on Ω for all $(\eta,\xi) \in \mathbb{R} \times \mathbb{R}^n$
$(\eta,\xi) \mapsto \mathcal{A}_j(x,\eta,\xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^n$ for almost all $x \in \Omega$.

(H2) There exists a constant $\theta_1 > 0$ such that

$$[\mathcal{A}(x,\eta,\xi) - \mathcal{A}(x,\eta',\xi')] \cdot (\xi - \xi') \geq \theta_1 |\xi - \xi'|^p$$

whenever $\xi, \xi' \in \mathbb{R}^n$, $\xi \neq \xi'$, where $\mathcal{A}(x,\eta,\xi) = (\mathcal{A}_1(x,\eta,\xi), \ldots, \mathcal{A}_n(x,\eta,\xi))$ (where a dot denote here the Euclidean scalar product in $\mathbb{R}^n$).

(H3) $\mathcal{A}(x,\eta,\xi) \cdot \xi \geq \lambda_1 |\xi|^p$, where λ_1 is a positive constant.

(H4) $|\mathcal{A}(x,\eta,\xi)| \leq K_1(x) + h_1(x)|\eta|^{p/p'} + h_2(x)|\xi|^{p/p'}$, where K_1, h_1 and h_2 are non-negative functions, with h_1 and $h_2 \in L^\infty(\Omega)$, and $K_1 \in L^{p'}(\Omega,\omega)$ (with $1/p + 1/p' = 1$).

By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets on $\mathbb{R}^n$ through integration. This measure will be denoted by μ. Thus, $\mu(E) = \int_E \omega(x)\,\mathrm{d}x$ for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $\mathrm{W}^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1], [2] and [4]).

In various applications, we can meet boundary value problems for elliptic equations whose ellipticity is disturbed in the sense that some degeneration or singularity appears. This bad behaviour can be caused by the coefficients of the corresponding differential operator as well as by the solution itself. The so-called p-Laplacian is a prototype of such an operator and its character can be interpreted as a degeneration or as a singularity of the classical (linear) Laplace operator (with $p = 2$). There are several very concrete problems from practice which lead to such differential equations, e.g. from glaceology, non-Newtonian fluid mechanics, flows through porous media, differential geometry, celestial mechanics, climatology, petroleum extraction, reaction-diffusion problems, etc.

A class of weights, which is particulary well understood, is the class of A_p-weights (or Muckenhoupt class) that was introduced by B. Muckenhoupt (see [11]). These classes have found many useful applications in harmonic analysis (see [13]). Another reason for studying A_p-weights is the fact that powers of distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [10]). There are, in fact, many interesting examples of weights (see [9] for p-admissible weights).

In the non-degenerate case (i.e. with $v(x) \equiv 1$), for all $f \in L^p(\Omega)$, the Poisson equation associated with the Dirichlet problem

$$\begin{cases} -\Delta u = f(x), & \text{in } \Omega \\ u(x) = 0, & \text{on } \partial\Omega \end{cases}$$

is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [8]), and the nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p u = f(x), & \text{in } \Omega \\ u(x) = 0, & \text{on } \partial\Omega \end{cases}$$

is uniquely solvable in $W_0^{1,p}(\Omega)$ (see [3]), where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p- Laplacian operator. In the degenerate case, the weighted p-Biharmonic operator has been studied by many authors (see [12] and the references therein), and the degenerated p-Laplacian was studied in [4].

The following theorem will be proved in section 3.

Theorem 1. *Assume (H1) – (H4). If*

(i) $v \in A_q$, $\omega \in A_p$ *(with* $1 < p, q < \infty$*),*

(ii) $f_j/\omega \in L^{p'}(\Omega, \omega)$ $(j = 0, 1, \ldots, n)$,

then the problem (1) *has a unique solution* $u \in X = W^{2,q}(\Omega, v) \cap W_0^{1,p}(\Omega, \omega)$.

2 Definitions and basic results

Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega(x) < \infty$ almost everywhere. We say that ω belongs to the Muckenhoupt class A_p, $1 < p < \infty$, or that ω is an A_p-weight, if there is a positive constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)\,\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,\mathrm{d}x\right)^{p-1} \leq C$$

for all balls $B \subset \mathbb{R}^n$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$. If $1 < q \leq p$, then $A_q \subset A_p$ (see [7], [9] or [13] for more information about A_p-weights). The weight ω satisfies the doubling condition if there exists a positive constant C such that $\mu(B(x; 2\,r)) \leq C\mu(B(x; r))$, for every ball $B = B(x; r) \subset \mathbb{R}^n$, where $\mu(B) = \int_B \omega(x)\,\mathrm{d}x$. If $\omega \in A_p$, then μ is doubling (see Corollary 15.7 in [9]).

As an example of A_p-weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see Corollary 4.4, Chapter IX in [13]).

If $\omega \in A_p$, then

$$\left(\frac{|E|}{|B|}\right)^p \leq C\frac{\mu(E)}{\mu(B)}$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B (see 15.5 *strong doubling property* in [9]). Therefore, if $\mu(E) = 0$ then $|E| = 0$.

Definition 1. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $0 < p < \infty$ we define $L^p(\Omega, \omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\,\mathrm{d}x\right)^{1/p} < \infty\,.$$

If $\omega \in A_p$, $1 < p < \infty$, then $\omega^{-1/(p-1)}$ is locally integrable and we have $L^p(\Omega,\omega) \subset L^1_{\rm loc}(\Omega)$ for every open set Ω (see Remark 1.2.4 in [14]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$.

Definition 2. Let $\Omega \subset \mathbb{R}^n$ be open, $1 < p < \infty$ and $\omega \in A_p$. We define the weighted Sobolev space $W^{k,p}(\Omega,\omega)$ as the set of functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega,\omega)$, $1 \le |\alpha| \le k$. The norm of u in $W^{k,p}(\Omega,\omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p\omega(x)\,\mathrm{d}x + \sum_{1\le|\alpha|\le k}\int_\Omega |D^\alpha u(x)|^p\omega(x)\,\mathrm{d}x\right)^{1/p}. \quad (2)$$

We also define $W^{k,p}_0(\Omega,\omega)$ as the closure of $C^\infty_0(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}(\Omega,\omega)}$.

If $\omega \in A_p$, then $W^{k,p}(\Omega,\omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm (2.1) (see Theorem 2.1.4 in [14]). The spaces $W^{k,p}(\Omega,\omega)$ and $W^{k,p}_0(\Omega,\omega)$ are Banach spaces and the spaces $W^{k,2}(\Omega,\omega)$ and $W^{k,2}_0(\Omega,\omega)$ are Hilbert spaces.

It is evident that a weight function ω which satisfies $0 < c_1 \le \omega(x) \le c_2$ for $x \in \Omega$ (where c_1 and c_2 are constants), gives nothing new (the space $\mathrm{W}^{k,p}_0(\Omega,\omega)$ is then identical with the classical Sobolev space $\mathrm{W}^{k,p}_0(\Omega)$). Consequently, we shall be interested above in all such weight functions ω which either vanish somewhere in $\Omega \cup \partial\Omega$ or increase to infinity (or both).

In this paper we use the following results.

Theorem 2. *Let $\omega \in A_p$, $1 < p < \infty$, and let Ω be a bounded open set in $\mathbb{R}^n$. If $u_m \to u$ in $L^p(\Omega,\omega)$ then there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi \in L^p(\Omega,\omega)$ such that*

(i) $u_{m_k}(x) \to u(x)$, $m_k \to \infty$, μ-a.e. on Ω;

(ii) $|u_{m_k}(x)| \le \Phi(x)$, μ-a.e. on Ω; (where $\mu(E) = \int_E \omega(x)\,\mathrm{d}x$).

Proof. The proof of this theorem follows the lines of Theorem 2.8.1 in [6]. □

Theorem 3. *(The weighted Sobolev inequality) Let Ω be an open bounded set in $\mathbb{R}^n$ and $\omega \in A_p$ $(1 < p < \infty)$. There exist constants C_Ω and δ positive such that for all $u \in C^\infty_0(\Omega)$ and all k satisfying $1 \le k \le n/(n-1) + \delta$,*

$$\|u\|_{L^{kp}(\Omega,\omega)} \le C_\Omega \|\nabla u\|_{L^p(\Omega,\omega)}.$$

Proof. See Theorem 1.3 in [5]. □

Lemma 1. *Let $1 < p < \infty$.*

(a) There exists a constant α_p such that

$$\Big||x|^{p-2}x - |y|^{p-2}y\Big| \le \alpha_p|x-y|\big(|x|+|y|\big)^{p-2}y\,,$$

for all $x, y \in \mathbb{R}^n$;

(b) *There exist two positive constants* β_p, γ_p *such that for every* $x, y \in \mathbb{R}^n$

$$\beta_p(|x|+|y|)^{p-2}|x-y|^2 \leq \big(|x|^{p-2}x-|y|^{p-2}y\big)\cdot(x-y) \leq \gamma_p\big(|x|+|y|\big)^{p-2}|x-y|^2.$$

Proof. See [3], Proposition 17.2 and Proposition 17.3. □

Definition 3. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $v \in A_q$, $\omega \in A_p$, $1 < p, q < \infty$. We denote by $X = W^{2,q}(\Omega, v) \cap W_0^{1,p}(\Omega, \omega)$ with the norm

$$\|u\|_X = \|\nabla u\|_{L^p(\Omega,\omega)} + \|\Delta u\|_{L^q(\Omega\, v)}.$$

Definition 4. We say that an element $u \in X = W^{2,q}(\Omega, v) \cap W_0^{1,p}(\Omega, \omega)$ is a (weak) solution of problem (1) if for all $\varphi \in X$ we have

$$\int_\Omega |\Delta u|^{q-2}\,\Delta u\,\Delta\varphi\, v\,\mathrm{d}x + \sum_{j=1}^{n}\int_\Omega \omega\,\mathcal{A}_j(x, u(x), \nabla u(x)) D_j\varphi\,\mathrm{d}x$$
$$= \int_\Omega f_0\,\varphi\,\mathrm{d}x + \sum_{j=1}^{n}\int_\Omega f_j\, D_j\varphi\,\mathrm{d}x.$$

3 Proof of Theorem 1

The basic idea is to reduce the problem (1) to an operator equation $Au = T$ and apply the theorem below.

Theorem 4. *Let* $A : X \to X^*$ *be a monotone, coercive and hemicontinuous operator on the real, separable, reflexive Banach space* X. *Then for each* $T \in X^*$ *the equation* $Au = T$ *has a solution* $u \in X$.

Proof. See Theorem 26.A in [16]. □

To prove the existence of solutions, we define $B, B_1, B_2 \colon X \times X \to \mathbb{R}$ and $T \colon X \to \mathbb{R}$ by

$$B(u,\varphi) = B_1(u,\varphi) + B_2(u,\varphi)\,,$$
$$B_1(u,\varphi) = \sum_{j=1}^{n}\int_\Omega \omega\mathcal{A}_j(x,u,\nabla u)D_j\varphi\,\mathrm{d}x = \int_\Omega \omega\mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi\,\mathrm{d}x\,,$$
$$B_2(u,\varphi) = \int_\Omega |\Delta u|^{q-2}\Delta u\,\Delta\varphi v\,\mathrm{d}x\,,$$
$$T(\varphi) = \int_\Omega f_0\,\varphi\,\mathrm{d}x + \sum_{j=1}^{n}\int_\Omega f_j\, D_j\varphi\,\mathrm{d}x\,.$$

Then $u \in X$ is a (weak) solution to problem (1) if

$$B(u,\varphi) = B_1(u,\varphi) + B_2(u,\varphi) = T(\varphi), \quad \text{for all } \varphi \in X.$$

Step 1. For $j = 1, \ldots, n$ we define the operator $F_j \colon X \to L^{p'}(\Omega, \omega)$ by

$$(F_j u)(x) = \mathcal{A}_j(x, u(x), \nabla u(x)).$$

We now show that operator F_j is bounded and continuous.

(i) Using (H4) we obtain

$$\begin{aligned}\|F_j u\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |F_j u(x)|^{p'}\,\omega\,\mathrm{d}x = \int_\Omega |\mathcal{A}_j(x,u,\nabla u)|^{p'}\,\omega\,\mathrm{d}x \\ &\le \int_\Omega \Big(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\Big)^{p'}\,\omega\,\mathrm{d}x \\ &\le C_p \int_\Omega \Big[(K_1^{p'} + h_1^{p'}|u|^p + h_2^{p'}|\nabla u|^p)\omega\Big]\,\mathrm{d}x \\ &= C_p\Big[\int_\Omega K_1^{p'}\,\omega\,\mathrm{d}x + \int_\Omega h_1^{p'}|u|^p\omega\,\mathrm{d}x + \int_\Omega h_2^{p'}|\nabla u|^p\omega\,\mathrm{d}x\Big], \qquad (3)\end{aligned}$$

where the constant C_p depends only on p. We have, by Theorem 3,

$$\begin{aligned}\int_\Omega h_1^{p'}|u|^p\,\omega\,\mathrm{d}x &\le \|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega |u|^p\,\omega\,\mathrm{d}x \\ &\le C_\Omega^p\|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^p\,\omega\,\mathrm{d}x \\ &\le C_\Omega^p\|h_1\|^{p'}_{L^\infty(\Omega)}\|u\|^p_X,\end{aligned}$$

and

$$\int_\Omega h_2^{p'}|\nabla u|^p\omega\,\mathrm{d}x \le \|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega |\nabla u|^p\omega\,\mathrm{d}x \le \|h_2\|^{p'}_{L^\infty(\Omega)}\|u\|^p_X.$$

Therefore, in (3) we obtain

$$\|F_j u\|_{L^{p'}(\Omega,\omega)} \le C_p\Big(\|K\|_{L^{p'}(\Omega,\omega)} + (C_\Omega^{p/p'}\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|_X^{p/p'}\Big),$$

and hence the boundedness.

(ii) Let $u_m \to u$ in X as $m \to \infty$. We need to show that $F_j u_m \to F_j u$ in $L^{p'}(\Omega,\omega)$. We will apply the Lebesgue Dominated Convergence Theorem. If $u_m \to u$ in X, then $u_m \to u$ in $L^p(\Omega,\omega)$ and $|\nabla u_m| \to |\nabla u|$ in $L^p(\Omega,\omega)$. Using Theorem 2, there exist a subsequence $\{u_{m_k}\}$ and functions Φ_1 and Φ_2 in $L^p(\Omega,\omega)$ such that

$$\begin{aligned}u_{m_k}(x) &\to u(x),\ \mu_1\text{- a.e. in }\Omega,\\ |u_{m_k}(x)| &\le \Phi_1(x),\ \mu_1\text{ - a.e. in }\Omega,\\ |\nabla u_{m_k}(x)| &\to |\nabla u(x)|,\ \mu_1\text{ - a.e. in }\Omega,\\ |\nabla u_{m_k}(x)| &\le \Phi_2(x),\ \mu_1\text{ - a.e. in }\Omega.\end{aligned}$$

where $\mu_1(E) = \int_E \omega(x)\,\mathrm{d}x$. Hence, using (H4), we obtain

$$\begin{aligned}
\|F_j u_{m_k} - F_j u\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |F_j u_{m_k}(x) - F_j u(x)|^{p'}\omega\,\mathrm{d}x \\
&= \int_\Omega |\mathcal{A}_j(x,u_{m_k},\nabla u_{m_k}) - \mathcal{A}_j(x,u,\nabla u)|^{p'}\omega\,\mathrm{d}x \\
&\le C_p \int_\Omega \Big(|\mathcal{A}_j(x,u_{m_k},\nabla u_{m_k})|^{p'} + |\mathcal{A}_j(x,u,\nabla u)|^{p'}\Big)\omega\,\mathrm{d}x \\
&\le C_p \bigg[\int_\Omega \Big(K_1 + h_1|u_{m_k}|^{p/p'} + h_2|\nabla u_{m_k}|^{p/p'}\Big)^{p'}\omega\,\mathrm{d}x \\
&\quad + \int_\Omega \Big(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\Big)^{p'}\omega\,\mathrm{d}x\bigg] \\
&\le 2C_p \int_\Omega \Big(K_1 + h_1\Phi_1^{p/p'} + h_2\Phi_2^{p/p'}\Big)^{p'}\omega\,\mathrm{d}x \\
&\le 2C_p \bigg[\int_\Omega K_1^{p'}\omega\,\mathrm{d}x + \int_\Omega h_1^{p'}\Phi_1^{p}\omega\,\mathrm{d}x + \int_\Omega h_2^{p'}\Phi_2^{p}\omega\,\mathrm{d}x\bigg] \\
&\le 2C_p \bigg[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\int_\Omega \Phi_1^p\omega\,\mathrm{d}x \\
&\quad + \|h_2\|^{p'}_{L^\infty(\Omega)}\int_\Omega \Phi_2^p\omega\,\mathrm{d}x\bigg] \\
&\le 2C_p \bigg[\|K_1\|^{p'}_{L^{p'}(\Omega,\omega)} + \|h_1\|^{p'}_{L^\infty(\Omega)}\|\Phi_1\|^p_{L^p(\Omega,\omega)} \\
&\quad + \|h_2\|^{p'}_{L^\infty(\Omega)}\|\Phi_2\|^p_{L^p(\Omega,\omega)}\bigg].
\end{aligned}$$

By condition (H1), we have

$$F_j u_m(x) = \mathcal{A}_j(x,u_m(x),\nabla u_m(x)) \to \mathcal{A}_j(x,u(x),\nabla u(x)) = F_j u(x),$$

as $m \to +\infty$. Therefore, by the Dominated Convergence Theorem, we obtain $\|F_j u_{m_k} - F_j u\|_{L^{p'}(\Omega,\omega)} \to 0$, that is, $F_j u_{m_k} \to F_j u$ in $L^{p'}(\Omega,\omega)$. By the Convergence principle in Banach spaces (see Proposition 10.13 in [15]) we have

$$F_j u_m \to F_j u \text{ in } L^{p'}(\Omega,\omega). \tag{4}$$

Step 2. We define the operator $G\colon X \to L^{q'}(\Omega,v)$ by

$$(Gu)(x) = |\Delta u(x)|^{q-2}\,\Delta u(x).$$

We also have that the operator G is continuous and bounded. In fact,

(i) We have

$$\|Gu\|^{q'}_{L^{q'}(\Omega,v)} = \int_\Omega \big||\Delta u|^{q-2}\Delta u\big|^{q'} v\,\mathrm{d}x = \int_\Omega |\Delta u|^{(q-2)q'}|\Delta u|^{q'} v\,\mathrm{d}x = \int_\Omega |\Delta u|^q v\,\mathrm{d}x \le \|u\|^q_X.$$

Hence, $\|Gu\|_{L^{q'}(\Omega,v)} \le \|u\|_X^{q/q'}$.

(ii) If $u_m \to u$ in X then $\Delta u_m \to \Delta u$ in $L^q(\Omega,v)$. By Theorem 2, there exist a subsequence $\{u_{m_k}\}$ and a function $\Phi_3 \in L^q(\Omega,v)$ such that

$$\Delta u_{m_k}(x) \to \Delta u(x),\ \mu_2 \text{ - a.e. in } \Omega$$
$$|\Delta u_{m_k}(x)| \le \Phi_3(x),\ \mu_2 \text{ - a.e. in } \Omega.$$

where $\mu_2(E) = \int_E v(x)\,\mathrm{d}x$. Hence, using Lemma 1(a), we obtain, if $q \neq 2$

$$\begin{aligned}\|Gu_{m_k} - Gu\|^{q'}_{L^{q'}(\Omega,v)} &= \int_\Omega |Gu_{m_k} - Gu|^{q'} v\,\mathrm{d}x \\ &= \int_\Omega \Big||\Delta u_{m_k}|^{q-2}\Delta u_{m_k} - |\Delta u|^{q-2}\Delta u\Big|^{q'} v\,\mathrm{d}x \\ &\le \int_\Omega \Big[\alpha_q|\Delta u_{m_k} - \Delta u|(|\Delta u_{m_k}| + |\Delta u|)^{(q-2)}\Big]^{q'} v\,\mathrm{d}x \\ &\le \alpha_q^{q'} \int_\Omega |\Delta u_{m_k} - \Delta u|^{q'}(2\Phi_3)^{(q-2)q'} v\,\mathrm{d}x \\ &\le \alpha_q^{q'} 2^{(q-2)q'}\left(\int_\Omega |\Delta u_{m_k} - \Delta u|^q v\,\mathrm{d}x\right)^{q'/q} \\ &\quad \times \left(\int_\Omega \Phi_3^{(q-2)qq'/(q-q')} v\,\mathrm{d}x\right)^{(q-q')/q} \\ &\le \alpha_q^{q'} 2^{(q-2)q'}\|u_{m_k} - u\|_X^{q'}\|\Phi\|^{q-q'}_{L^q(\Omega,v)},\end{aligned}$$

since $(q-2)qq'/(q-q') = q$ if $q \neq 2$. If $q = 2$, we have

$$\|Gu_{m_k} - Gu\|^2_{L^2(\Omega,v)} = \int_\Omega |\Delta u_{m_k} - \Delta u|^2 v\,\mathrm{d}x \le \|u_{m_k} - u\|^2_X.$$

Therefore (for $1 < q < \infty$), by the Dominated Convergence Theorem, we obtain

$$\|Gu_{m_k} - Gu\|_{L^{q'}(\Omega,v)} \to 0\,,$$

that is, $Gu_{m_k} \to Gu$ in $L^{q'}(\Omega,v)$. By the Convergence principle in Banach spaces (see Proposition 10.13 in [15]), we have

$$Gu_m \to Gu \quad \text{in } L^{q'}(\Omega,v). \tag{5}$$

Step 3. We have, by Theorem 3,

$$
\begin{aligned}
|T(\varphi)| &\leq \int_\Omega |f_0||\varphi|\,\mathrm{d}x + \sum_{j=1}^n \int_\Omega |f_j||D_j\varphi|\,\mathrm{d}x \\
&= \int_\Omega \frac{|f_0|}{\omega}|\varphi|\omega\,\mathrm{d}x + \sum_{j=1}^n \int_\Omega \frac{|f_j|}{\omega}|D_j\varphi|\,\omega\,\mathrm{d}x \\
&\leq \|f_0/\omega\|_{L^{p'}(\Omega,\omega)}\|\varphi\|_{L^p(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\|D_j\varphi\|_{L^p(\Omega,\omega)} \\
&\leq \Big(C_\Omega\,\|f_0/\omega\|_{L^{p'}(\Omega,\omega)} + \sum_{j=1}^n \|f_j/\omega\|_{L^{p'}(\Omega,\omega)}\Big)\|\varphi\|_X.
\end{aligned}
$$

Moreover, using (H4) and the Hölder inequality, we also have

$$
\begin{aligned}
|B(u,\varphi)| &\leq |B_1(u,\varphi)| + |B_2(u,\varphi)| \\
&\leq \sum_{j=1}^n \int_\Omega |\mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\,\omega\,\mathrm{d}x + \int_\Omega |\,\Delta u|^{q-2}|\,\Delta u||\Delta\varphi|v\,\mathrm{d}x\,. \qquad (6)
\end{aligned}
$$

In (6) we have

$$
\begin{aligned}
&\int_\Omega |\mathcal{A}(x,u,\nabla u)|\,|\nabla\varphi|\,\omega\,\mathrm{d}x \leq \int_\Omega \Big(K_1 + h_1|u|^{p/p'} + h_2|\nabla u|^{p/p'}\Big)|\nabla\varphi|\,\omega\,\mathrm{d}x \\
&\leq \|K_1\|_{L^{p'}(\Omega,\omega)}\|\nabla\varphi\|_{L^p(\Omega,\omega)} + \|h_1\|_{L^\infty(\Omega)}\|u\|^{p/p'}_{L^p(\Omega,\omega)}\|\nabla\varphi\|_{L^p(\Omega,\omega)} \\
&\quad + \|h_2\|_{L^\infty(\Omega)}\|\nabla u\|^{p/p'}_{L^p(\Omega,\omega)}\|\nabla\varphi\|_{L^p(\Omega,\omega)} \\
&\leq \Big(\|K_1\|_{L^{p'}(\Omega,\omega)} + (C_\Omega^{p/p'}\,\|h_1\|_{L^\infty(\Omega)} + \|h_2\|_{L^\infty(\Omega)})\|u\|_X^{p/p'}\Big)\|\varphi\|_X,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_\Omega |\Delta u|^{q-2}|\Delta u||\Delta\varphi|v\,\mathrm{d}x &= \int_\Omega |\Delta u|^{q-1}|\Delta\varphi|v\,\mathrm{d}x \\
&\leq \Big(\int_\Omega |\Delta u|^q v\,\mathrm{d}x\Big)^{1/q'}\Big(\int_\Omega |\Delta\varphi|^q v\,\mathrm{d}x\Big)^{1/q} \\
&\leq \|u\|_X^{q/q'}\,\|\varphi\|_X.
\end{aligned}
$$

Hence, in (6) we obtain, for all $u, \varphi \in X$

$$
\begin{aligned}
|B(u,\varphi)| \leq \Big[&\|K_1\|_{L^{p'}(\Omega,\omega)} + C_\Omega^{p/p'}\,\|h_1\|_{L^\infty(\Omega)}\|u\|_X^{p/p'} \\
&+ \|h_2\|_{L^\infty(\Omega,\omega)}\|u\|_X^{p/p'} + \|u\|_X^{q/q'}\Big]\|\varphi\|_X.
\end{aligned}
$$

Since $B(u,\cdot)$ is linear, for each $u \in X$, there exists a linear and continuous operator $A\colon X \to X^*$ such that $\langle Au,\varphi\rangle = B(u,\varphi)$, for all $u,\varphi \in X$ (where $\langle f,x\rangle$

denotes the value of the linear functional f at the point x) and

$$\|Au\|_* \leq \|K_1\|_{L^{p'}(\Omega,\omega)} + C_\Omega^{p/p'}\,\|h_1\|_{L^\infty(\Omega)}\|u\|_X^{p/p'} + \|h_2\|_{L^\infty(\Omega,\omega)}\|u\|_X^{p/p'} + \|u\|_X^{q/q'}.$$

Consequently, problem (1) is equivalent to the operator equation

$$Au = T, \quad u \in X.$$

Step 4. Using condition (H2) and Lemma 1(b), we have

$$\begin{aligned}
&\langle Au_1 - Au_2, u_1 - u_2\rangle = B(u_1, u_1 - u_2) - B(u_2, u_1 - u_2)\\
&= \int_\Omega \omega\,\mathcal{A}(x, u_1, \nabla u_1).\nabla(u_1 - u_2)\,\mathrm{d}x + \int_\Omega |\,\Delta u_1|^{q-2}\Delta u_1\Delta(u_1 - u_2)v\,\mathrm{d}x\\
&\quad - \int_\Omega \omega\mathcal{A}(x, u_2, \nabla u_2)\cdot\nabla(u_1 - u_2)\,\mathrm{d}x - \int_\Omega |\,\Delta u_2|^{q-2}\,\Delta u_2\Delta(u_1 - u_2)\,v\,\mathrm{d}x\\
&= \int_\Omega \omega\Big(\mathcal{A}(x, u_1, \nabla u_1) - \mathcal{A}(x, u_2, \nabla u_2)\Big)\cdot\nabla(u_1 - u_2)\,\mathrm{d}x\\
&\quad + \int_\Omega (|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2)\Delta(u_1 - u_2)v\,\mathrm{d}x\\
&\geq \theta_1\int_\Omega \omega|\nabla(u_1 - u_2)|^p\,\mathrm{d}x + \beta_q\int_\Omega \big(|\,\Delta u_1| + |\,\Delta u_2|\big)^{q-2}|\Delta u_1 - \Delta u_2|^2 v\,\mathrm{d}x\\
&\geq \theta_1\int_\Omega \omega\,|\nabla(u_1 - u_2)|^p\,\mathrm{d}x + \beta_q\int_\Omega \big(|\Delta u_1 - \Delta u_2|\big)^{q-2}|\Delta u_1 - \Delta u_2|^2 v\,\mathrm{d}x\\
&= \theta_1\int_\Omega \omega|\nabla(u_1 - u_2)|^p\,\mathrm{d}x + \beta_q\int_\Omega |\Delta u_1 - \Delta u_2|^q v\,\mathrm{d}x\\
&\geq 0.
\end{aligned}$$

Therefore, the operator A is monotone. Moreover, using (H3), we obtain

$$\begin{aligned}
\langle Au, u\rangle &= B(u, u) = B_1(u, u) + B_2(u, u)\\
&= \int_\Omega \omega\,\mathcal{A}(x, u, \nabla u).\nabla u\,\mathrm{d}x + \int_\Omega |\,\Delta u|^{q-2}\,\Delta u\,\Delta u\,v\,\mathrm{d}x\\
&\geq \int_\Omega \lambda_1|\nabla u|^p\,\omega\,\mathrm{d}x + \int_\Omega |\,\Delta u|^q\,v\,\mathrm{d}x\\
&= \lambda_1\,\|\nabla u\|^p_{L^p(\Omega,\omega)} + \|\Delta u\|^q_{L^q(\Omega,v)}.
\end{aligned}$$

Hence, since $1 < p, q < \infty$, we have

$$\frac{\langle Au, u\rangle}{\|u\|_X} \to +\infty, \quad \text{as } \|u\|_X \to +\infty\,,$$

(using $\lim\limits_{t+s\to\infty}\dfrac{t^p + s^q}{t + s} = \infty$) that is, A is coercive.

Step 5. We need to show that the operator A is continuous.

Let $u_m \to u$ in X as $m \to \infty$. We have,

$$\begin{aligned}|B_1(u_m,\varphi)-B_1(u,\varphi)| &\le \sum_{j=1}^{n}\int_\Omega |\mathcal{A}_j(x,u_m,\nabla u_m)-\mathcal{A}_j(x,u,\nabla u)||D_j\varphi|\omega\,\mathrm{d}x\\ &= \sum_{j=1}^{n}\int_\Omega |F_ju_m-F_ju||D_j\varphi|\omega\,\mathrm{d}x\\ &\le \sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega)}\|D_j\varphi\|_{L^p(\Omega,\omega)}\\ &\le \sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega)}\|\varphi\|_X,\end{aligned}$$

and

$$\begin{aligned}|B_2(u_m,\varphi)-B_2(u,\varphi)| &= \left|\int_\Omega |\Delta u_m|^{q-2}\Delta u_m\Delta\varphi v\,\mathrm{d}x-\int_\Omega |\,\Delta u|^{q-2}\Delta u\,\Delta\varphi v\,\mathrm{d}x\right|\\ &\le \int_\Omega\Big||\Delta u_m|^{q-2}\Delta u_m-|\Delta u|^{q-2}\Delta u\Big||\Delta\varphi|v\,\mathrm{d}x\\ &= \int_\Omega |Gu_m-Gu||\Delta\varphi|v\,\mathrm{d}x\\ &\le \|Gu_m-Gu\|_{L^{q'}(\Omega,v)}\|\Delta\varphi\|_{L^q(\Omega,v)}\\ &\le \|Gu_m-Gu\|_{L^{q'}(\Omega,v)}\|\varphi\|_X,\end{aligned}$$

for all $\varphi\in X$. Hence,

$$\begin{aligned}|B(u_m,\varphi)-B(u,\varphi)| &\le |B_1(u_m,\varphi)-B_1(u,\varphi)|+|B_2(u_m,\varphi)-B_2(u,\varphi)|\\ &\le \Big[\sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega)}+\|Gu_m-Gu\|_{L^{q'}(\Omega,v)}\Big]\|\varphi\|_X.\end{aligned}$$

Then we obtain

$$\|Au_m-Au\|_* \le \sum_{j=1}^{n}\|F_ju_m-F_ju\|_{L^{p'}(\Omega,\omega)}+\|Gu_m-Gu\|_{L^{q'}(\Omega,v)}.$$

Therefore, using (4) and (5) we have $\|Au_m-Au\|_*\to 0$ as $m\to+\infty$, that is, A is continuous (and this implies that A is hemicontinuous).

Therefore, by Theorem 4, the operator equation $Au=T$ has a solution $u\in X$ and it is a solution for problem (1).

Step 6. Let us now prove the uniqueness of the solution. Suppose that $u_1,u_2\in X$ are two solutions of problem (1). Then,

$$\begin{aligned}&\int_\Omega |\Delta u_i|^{q-2}\,\Delta u_i\,\Delta\varphi\,v\,\mathrm{d}x+\sum_{j=1}^{n}\int_\Omega \omega\,\mathcal{A}_j(x,u_i(x),\nabla u_i(x))D_j\varphi\,\mathrm{d}x\\ &\qquad\qquad = \int_\Omega f_0\,\varphi\,\mathrm{d}x+\sum_{j=1}^{n}\int_\Omega f_j\,D_j\varphi\,\mathrm{d}x,\end{aligned}$$

for all $\varphi \in X$, and $i = 1, 2$. Hence, we obtain

$$\int_\Omega \Big(|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2 \Big) \Delta\varphi v \,\mathrm{d}x + \int_\Omega \omega \Big(\mathcal{A}(x, u_1(x), \nabla u_1(x)) - \mathcal{A}(x, u_2(x), \nabla u_2(x)) \Big) \cdot \nabla\varphi \,\mathrm{d}x = 0\,.$$

In particular, for $\varphi = u_1 - u_2 \in X$ we have, by (H2) and Lemma 1(b) (analogous to Step 4),

$$\begin{aligned} 0 &= \int_\Omega \omega \Big(\mathcal{A}(x, u_1, \nabla u_1) - \mathcal{A}(x, u_2, \nabla u_2) \Big) \cdot \nabla(u_1 - u_2) \,\mathrm{d}x \\ &\quad + \int_\Omega (|\Delta u_1|^{q-2}\Delta u_1 - |\Delta u_2|^{q-2}\Delta u_2)\Delta(u_1 - u_2) v \,\mathrm{d}x \\ &\geq \theta_1 \int_\Omega |\nabla(u_1 - u_2)|^p \omega \,\mathrm{d}x + \beta_q \int_\Omega |\Delta(u_1 - u_2)|^q v \,\mathrm{d}x\,. \end{aligned}$$

Hence, $\|\nabla(u_1 - u_2)\|_{L^p(\Omega,\omega)} = 0$ and $\|\Delta(u_1 - u_2)\|_{L^q(\Omega,v)} = 0$. Since $u_1, u_2 \in X$, then $u_1 = u_2$ μ_1-a.e. Therefore, since $\omega \in A_p$, we obtain that $u_1 = u_2$ a.e.

Example 1. Consider $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, the weight functions $\omega(x, y) = (x^2 + y^2)^{-1/2}$ and $v(x, y) = (x^2 + y^2)^{-2/3}$ ($\omega \in A_3$ and $v \in A_2$, $p = 3$, $q = 2$), and the function

$$\begin{gathered} \mathcal{A}\colon \Omega \times \mathbb{R}^2 \to \mathbb{R}^2 \\ \mathcal{A}((x, y), \eta, \xi) = h_2(x, y)|\xi|\xi, \end{gathered}$$

where $h(x, y) = 2\,\mathrm{e}^{(x^2+y^2)}$. Let us consider the partial differential operator

$$Lu(x, y) = \Delta\big((x^2 + y^2)^{-2/3}|\Delta u|\Delta u\big) - \operatorname{div}\big((x^2 + y^2)^{-1/2}\mathcal{A}((x, y), u, \nabla u)\big)\,.$$

Therefore, by Theorem 1, the problem (1)

$$\begin{cases} Lu(x) = \dfrac{\cos(xy)}{(x^2 + y^2)} - \dfrac{\partial}{\partial x}\left(\dfrac{\sin(xy)}{(x^2 + y^2)}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\sin(xy)}{(x^2 + y^2)}\right), & \text{in } \Omega \\ u(x) = 0, & \text{on } \partial\Omega \\ \Delta u(x) = 0, & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in X = W^{2,2}(\Omega, v) \cap W_0^{1,3}(\Omega, \omega)$.

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On a class of nonlocal problem involving a critical exponent

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Abstract. In this work, by using the Mountain Pass Theorem, we give a result on the existence of solutions concerning a class of nonlocal p-Laplacian Dirichlet problems with a critical nonlinearity and small perturbation.

1 Introduction

This paper deals with the following elliptic problem

$$\begin{aligned} -M\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)\Delta_p u &= \beta h(x)|u|^{q-2}u + |u|^{p^*-2}u + f(x) \qquad &&\text{in } \Omega,\\ u &= 0 &&\text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent, $1 < p < N$, β is a positive parameter, and $h \in L^{\frac{p^*}{p^*-q}}(\Omega)$, $f \in L^{p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$.

Where the functional M verifies,

$$M\colon (0,+\infty) \to (0,+\infty) \text{ is continuous and } m_0 = \inf_{s>0} M(s) > 0, \tag{2}$$

The problem (1) is called nonlocal because of the presence of the term $M\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)$, so it is not any more a pointwise identity. This leads us to some mathematical difficulties which makes the study of such a class of problem particularly interesting.

It is well known that the critical exponent case is often difficult because of the lack of compactness, so standard arguments cannot be carried out to handle the problem (1). As far as we know, very few results have been obtained in elliptic

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problems involving critical exponent, for instance we just quote [1], [2], [4], [5], [6], [7], [9], [11] and references therein. However, inspired by these interesting works, especially by [4], within which we will borrow some ideas, our goal will be to generalize some corresponding results partially and extend them to the case $p \neq 2$ with an existence of a perturbation f. We have to mention that [5] could be considered as the first work dealing with multivalued elliptic problem and the presence of which involves critical growth in an Orlicz-Sobolev space, where the nonlinearity can be discontinuous.

From now on, we make the following assumption:

$$\widehat{M}(t) \geq M(t)t \text{ for } t > 0 \text{ , with } \widehat{M}(t) = \int_0^t M(s)\,\mathrm{d}s. \tag{3}$$

Accordingly, we can report our main result,

Theorem 1. *Under the hypotheses* (2), (3) *and* $q \in (p, p^*)$, *there exists* $\beta^* > 0$, *such that the problem* (1) *has at least a nontrivial solutions for all* $\beta \geq \beta^*$, *provided* f *is small enough in the norm* $\|\cdot\|_*$ *of* $(W_0^{1,p}(\Omega))^*$.

Throughout this paper, we consider the C^1-functional energy

$$\phi(u) = \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u|^p\,\mathrm{d}x\right) - \frac{\beta}{q}\int_\Omega h(x)|u|^q\,\mathrm{d}x - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,\mathrm{d}x - \int_\Omega f(x)u\,\mathrm{d}x.$$

Note that

$$\begin{aligned}\phi'(u)\cdot v = M(\|u\|^p)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\,\mathrm{d}x - \beta\int_\Omega h(x)|u|^{q-2}uv\,\mathrm{d}x \\ - \int_\Omega |u|^{p^*-2}uv\,\mathrm{d}x - \int_\Omega f(x)v\,\mathrm{d}x,\end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. Where,

$$W_0^{1,p}(\Omega) = \Big\{u \in L^p(\Omega) : \int_\Omega |\nabla u|^p\,\mathrm{d}x < \infty, \quad u/\partial\Omega = 0\Big\}.$$

By a version of the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [10], [12], without Palais-Smale condition, there exists a sequence $(u_n)_n \subset W_0^{1,p}(\Omega)$ such that

$$\phi(u_n) \to c_\beta \quad \text{and} \quad \phi'(u_n) \to 0,$$

where

$$c_\beta = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\phi(\gamma(t)) > 0$$

with

$$\Gamma = \big\{\gamma \in C\big([0,1], W_0^{1,p}(\Omega)\big) : \gamma(0) = 0, \phi\big(\gamma(1)\big) < 0\big\}.$$

We recall that $u \in W_0^{1,p}(\Omega)$ is a weak solution of the problem (1) if it verifies

$$M(\|u\|^p)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v \,\mathrm{d}x - \int_\Omega \beta h(x)|u|^{q-2}uv\,\mathrm{d}x \\ - \int_\Omega |u|^{p^*-2}uv\,\mathrm{d}x - \int_\Omega f(x)v\,\mathrm{d}x = 0,$$

for all $v \in W_0^{1,p}(\Omega)$.

So the critical points of ϕ are solutions of the problem (1).

2 Auxiliary results

Let $L^s(\Omega)$ be the Lebesgue space equipped with the norm $|u|_s = \left(\int_\Omega |u|^s\,\mathrm{d}x\right)^{\frac{1}{s}}$, $1 \le s < \infty$ and let $W_0^{1,p}(\Omega)$ be the usual Sobolev space with respect to the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

Now we can define the best Sobolev constant

$$S = \inf_{u\in W^{1,p}(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\left(\int_\Omega |u|^{p^*}dx\right)^{\frac{p}{p^*}}}.$$

In the sequel, we are to compare the minimax level c_β with a suitable number which involves the constant S.

Lemma 1. *There exist $\sigma > 0, \rho > 0$ and $e \in W_0^{1,p}(\Omega)$ with $\|e\| > \rho$ such that*

(i) $\inf_{\|u\|=\rho}\phi(u) \ge \sigma > 0$;

(ii) $\phi(e) < 0$.

Proof. (i) From the Hölder's inequality and the compact embedding theorem, we have

$$\begin{aligned}\phi(u) &\ge \frac{m_0}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x - \frac{\beta}{q}|h|_\theta \int_\Omega |u|^q\,\mathrm{d}x - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,\mathrm{d}x - \int_\Omega f(x)u\,\mathrm{d}x \\ &\ge C_0\|u\|^p - \frac{C_1\beta}{q}|h|_\theta\|u\|^q - \frac{1}{p^* S^{\frac{p^*}{p}}}\|u\|^{p^*} - |f|_{p'}|u|_p \\ &\ge C_0\|u\|^p - \frac{C_1\beta}{q}|h|_\theta\|u\|^q - C_2\|u\|^{p^*} - C_3\|f\|_*\|u\|, \end{aligned} \tag{4}$$

with $\theta = \frac{p^*}{[p^*-q]}$ and $C_0, C_1, C_2, C_3 > 0$. Since $q \in (p, p^*)$ then for $\|u\| = \rho > 0$ small enough, we may find $\sigma > 0$ such that

$$\inf_{\|u\|=\rho}\phi(u) \ge \sigma > 0$$

where $\|f\|_*$ be small.

(ii) Fix $v \in C_0^\infty(\Omega \setminus \{0\})$ with $v \geq 0$ in Ω and $\|v\| = 1$.

$$\phi(tv) \leq A|t|^p - \beta|t|^\theta \int_\Omega h(x)v^\theta \,\mathrm{d}x + C - \frac{|t|^{p^*}}{p^*} \int_\Omega h(x)v^{p^*} \,\mathrm{d}x - |t| \int_\Omega f(x)v \,\mathrm{d}x,$$

with A and C are two positive constants, it follows that

$$\phi(tv) \to -\infty \quad \text{as} \quad |t| \to \infty.$$

□

Lemma 2. $\lim\limits_{\beta \to +\infty} c_\beta = 0.$

Proof. Let v the function given by the previous lemma 1, then there is $t_\beta > 0$ such that $\phi(t_\beta v) = \max\limits_{t \geq 0} \phi(tv)$, thereafter,

$$M(\|t_\beta v\|^p) t_\beta^p \|v\|^p = \beta t_\beta^q \int_\Omega h(x)|v|^q \,\mathrm{d}x + t_\beta^{p^*} \int_\Omega |v|^{p^*} \,\mathrm{d}x + t_\beta^2 \int_\Omega f(x)v^2 \,\mathrm{d}x, \quad (5)$$

it follows from (3) that there is $c > 0$, such that

$$\widehat{M}(s) \leq c|s| \quad \text{for all } s > s_0 > 0\,.$$

Hence

$$ct_\beta^p \|v\|^p \geq \beta t_\beta^q \int_\Omega h(x)|v|^q \,\mathrm{d}x + t_\beta^{p^*} \int_\Omega |v|^{p^*} \,\mathrm{d}x + t_\beta^2 \int_\Omega f(x)v^2 \,\mathrm{d}x$$

and then t_β is bounded, so there exists a sequence $\beta_n \to +\infty$ and $t_* \geq 0$ with $t_{\beta_n} \to t_*$ as $n \to +\infty$ and thus

$$M(\|t_{\beta_n} v\|^p) t_{\beta_n}^p \|v\|^p < C, \quad \forall n \in \mathbb{N},$$

with C is a positive constant, which yields

$$\beta_n t_*^q \int_\Omega h(x)|v|^q \,\mathrm{d}x + t_*^{p^*} \int_\Omega |v|^{p^*} \,\mathrm{d}x \leq C, \quad \forall n \in \mathbb{N}.$$

Hence, we claim that $t_* = 0$, otherwise, $t_* > 0$ and then the last inequality becomes

$$\beta_n t_*^q \int_\Omega h(x)|v|^q \,\mathrm{d}x + t_*^{p^*} \int_\Omega |v|^{p^*} \,\mathrm{d}x \to +\infty$$

as $n \to +\infty$, which is absurd, so $t_* = 0$.

Taking $\gamma_0(t) = te$, with $\gamma_0 \in \Gamma$, then we get

$$0 < c_\beta \leq \max_{t \in [0,1]} \phi(\gamma_0(t)) \leq \frac{1}{p} \widehat{M}(t_\beta^p).$$

Since $\widehat{M}(t_\beta^p) \to 0$ then $\lim_{\beta \to \infty} c_\beta = 0$.

□

As consequence of the above lemma, there exists $\beta^* > 0$ such that for every $\beta \geq \beta^*$,

$$c_\beta < \left(1 - \frac{p}{p^*}\right)(m_0 S)^{\frac{N}{p}}.$$

Lemma 3. *Let $(u_n)_n \subset W_0^{1,p}(\Omega)$, with $\phi(u_n) \to c_\beta$, and $\phi'(u_n) \to 0$. Then $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Assume that $\phi(u_n) \to c_\beta$, and $\phi'(u_n) \to 0$, then we have

$$\begin{aligned} pc_\beta + o(1) + o(1)\|u_n\| &= p\phi(u_n) - (\phi'(u_n) \cdot u_n) \\ &\geq C_4 \beta \left(1 - \frac{p}{q}\right)|h|_\theta \|u_n\|^q + C_5 \left(1 - \frac{p}{p^*}\right)\|u_n\|^{p^*} \\ &\quad + (p-1)\int_\Omega f(x) u_n \,\mathrm{d}x, \end{aligned}$$

where $\theta = \frac{p^*}{p^*-q}, C_4, C_5 > 0$, we infer that $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$. □

3 Proof of the main result

Proof. (Theorem 1) As it was previously mentioned, we are to apply a version of the Mountain Pass theorem without Palais-Smale condition to obtain a sequence $(u_n)_n \subset W_0^{1,p}(\Omega)$ such that $\phi(u_n) \to c_\beta$ and $\phi'(u_n) \to 0$.

Because $(u_n)_n$ is a bounded sequence in $W_0^{1,p}(\Omega)$, passing to a subsequence, so we may find $\gamma > 0$ with

$$\|u_n\| \to \gamma,$$

it follows from the continuity of M that

$$M(\|u_n\|^p) \to M(\gamma^p).$$

On the other side, we know that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, then

$$u_n \to u \text{ in } L^r(\Omega), \quad \text{for } 1 < r < p^*$$

and

$$u_n(x) \to u(x) \quad \text{a.e. } x \in \Omega.$$

By the Lebesgue Dominated Theorem,

$$\int_\Omega h(x)|u_n|^q \,\mathrm{d}x \to \int_\Omega h(x)|u|^q \,\mathrm{d}x.$$

Further,

$$\begin{aligned} |\nabla u_n|^p &\rightharpoonup |\nabla u|^p + \mu \quad \text{weak}^*\text{-sense of measure}, \\ |u_n|^{p^*} &\rightharpoonup |u|^{p^*} + \nu \quad \text{weak}^*\text{-sense of measure}. \end{aligned}$$

Afterwards, as a consequence of the concentration compactness principle due to Lion [8], there is an index set I, which is an at most countable set such that

$$\nu = \sum_{i \in I} \nu_i \delta_i, \quad \mu \geq \sum_{i \in I} \mu_i \delta_i$$

and

$$S\nu_i^{p/p^*} \leq \mu_i,$$

for any $i \in I$ with $(\mu_i)_i, (\nu_i)_i \subset [0, \infty)$, δ_i is the Dirac mass and $(\mu_i)_i, (\nu_i)_i$ are nonatomic positive measures. We claim that $I = \emptyset$, otherwise, we have $I \neq \emptyset$ and fix $i \in I$. Taking $\psi \in C_0^\infty(\Omega, [0,1])$ such that $\psi \equiv 1$ if $|x| < 1$ and $\psi \equiv 0$ when $|x| > 2$ with $|\nabla \psi|_\infty \leq 2$. Putting $\psi_\rho(x) = \psi\big(\frac{x - x_i}{\rho}\big)$ for $\rho > 0$, noting that $(\psi_\rho u_n)$ is bounded thus $\phi'(u_n) \cdot (\psi_\rho u_n) \to 0$, that is

$$\begin{aligned} M\left(\int_\Omega |\nabla u_n|^p\right) \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \psi_\rho u_n \, \mathrm{d}x \\ = -M\left(\int_\Omega |\nabla u_n|^p\right) \int_\Omega |\nabla u_n|^p \psi_\rho \nabla u_n \, \mathrm{d}x + \int_\Omega |u_n|^{p^*-2} u_n \cdot \psi_\rho u_n \, \mathrm{d}x \\ + \beta \int_\Omega h(x) |u_n|^{q-2} u_n \psi_\rho u_n \, \mathrm{d}x + \int_\Omega f(x) \psi_\rho u_n + O_n(1). \end{aligned}$$

As it is known that $B_{2\rho}(x_i)$ is the support of the functional ψ_ρ and by applying Hölder inequality then we get

$$\begin{aligned} \left| \int_\Omega |\nabla u_n|^{p-2} \nabla u_n . \nabla \psi_\rho u_n \, \mathrm{d}x \right| &\leq \int_{B_{2\rho}(x_i)} |\nabla u_n|^{p-1} \, |u_n \nabla \psi_\rho| \, \mathrm{d}x \\ &\leq \Big(\int_{B_{2\rho}(x_i)} |\nabla u_n|^p\Big)^{\frac{1}{p'}} \Big(\int_{B_{2\rho}(x_i)} |u_n \nabla \psi_\rho|^p \, \mathrm{d}x\Big)^{\frac{1}{p}} \\ &\leq C\Big(\int_{B_{2\rho}(x_i)} |u_n \nabla \psi_\rho|^p \, \mathrm{d}x\Big)^{\frac{1}{p}}. \end{aligned}$$

By the Dominated convergence Theorem we entail that

$$\int_{B_{2\rho}(x_i)} |u_n \nabla \psi_\rho|^p \, \mathrm{d}x \to 0$$

when $n \to \infty$ and $\rho \to 0$.

Hence,

$$\lim_{\rho \to 0} \Big[\lim_n \int_\Omega u_n |\nabla u_n|^{p-2} \nabla u_n . \nabla \psi_\rho \Big] = 0.$$

On the other hand, we recall that $M(\|u_n\|^p)$ converges to $M(\gamma^p)$, so we reach

$$\lim_{\rho \to 0} \Big[\lim_n M(\|u_n\|^p) \int_\Omega u_n |\nabla u_n|^{p-2} \nabla u_n . \nabla \psi_\rho \Big] = 0.$$

Similarly,

$$\lim_{\rho \to 0} \lim_n \Big[\int_\Omega h(x) |u_n|^{q-2} u_n \psi_\rho u_n \Big] = 0,$$

$$\lim_{\rho \to 0} \lim_n \Big[\int_\Omega f(x) \psi_\rho u_n \Big] = 0$$

and thus

$$\int_{\Omega} M(\gamma^p)\psi_\rho \, \mathrm{d}\mu + O_\rho(1) \leq \int_{\Omega} \psi_\rho \, \mathrm{d}\nu.$$

Tending ρ to zero we conclude that

$$\nu_i \geq M(\gamma^P)\mu_i \geq m_0\mu_i,$$

from the definition of ν and μ we have

$$\nu_i \geq (m_0 S)^{\frac{N}{p}}.$$

It does not make sense, indeed, let $i \in I$ such that

$$\nu_i \geq (m_0 S)^{\frac{N}{p}}.$$

Since $(u_n)_n$ is a $(PS)_{c_\beta}$ for the functional ϕ, then

$$\begin{aligned} pc_\beta &= p\phi(u_n) = p\phi(u_n) - \phi'(u_n)\cdot u_n + O_n(1) \\ &\geq \left(1 - \frac{p}{p^*}\right) \int_{\Omega} \psi_\rho |u_n|^{p^*} \, \mathrm{d}x + O_n(1), \end{aligned}$$

tending $n \to +\infty$, therefore

$$pc_\beta \geq \left(1 - \frac{p}{p^*}\right) \sum_{i\in I} \psi_\rho(x_i)\nu_i = \left(1 - \frac{p}{p^*}\right) \sum_{i\in I} \nu_i \geq \left(1 - \frac{p}{p^*}\right) (m_0 S)^{\frac{N}{p}},$$

which cannot occur (because $\lim_{\beta\to\infty} c_\beta = 0$), thereafter I is empty and thereby $u_n \to u$ in $L^{p^*}(\Omega)$.

On the other hand,

$$\begin{aligned} &M\left(\|u_n\|^p\right) \int_{\Omega} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u) \, \mathrm{d}x \\ &\quad = \phi'(u_n).\,(u_n - u) + \beta \int_{\Omega} h(x)|u_n|^{q-2}u_n(u_n - u)\,\mathrm{d}x + \int_{\Omega} f(x)(u_n - u)\,\mathrm{d}x \\ &\qquad + \int_{\Omega} |u_n|^{p^*-2}u_n(u_n - u)\,\mathrm{d}x - M\left(\|u_n\|^p\right) \int_{\Omega} |\nabla u|^{p-2}\nabla u\,(\nabla u_n - \nabla u)\,\mathrm{d}x. \end{aligned}$$

In view of $u_n \rightharpoonup u$, a standard argument (similar to those found in [3]) shows that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{for a.e. } x \in \Omega,$$

and

$$u_n(x) \to u(x) \quad \text{for a.e. } x \in \Omega,$$

then

$$M\left(\|u_n\|^p\right) \int_{\Omega} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\,\mathrm{d}x \to 0.$$

Using the following inequalities $\forall x, y \in \mathbb{R}^N$

$$|x-y|^\gamma \le 2^\gamma (|x|^{\gamma-2}x - |y|^{\gamma-2}y)\cdot(x-y) \qquad \text{if } \gamma \ge 2,$$
$$|x-y|^2 \le \frac{1}{\gamma-1}(|x|+|y|)^{2-\gamma}(|x|^{\gamma-2}x - |y|^{\gamma-2}y)\cdot(x-y) \qquad \text{if } 1<\gamma<2,$$

where $x\cdot y$ is the inner product in $\mathbb{R}^N$, we get

$$c\, m_0 \int_\Omega |\nabla u_n - \nabla u|^p \,\mathrm{d}x \le M\left(\|u_n\|^p\right) \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)(\nabla u_n - \nabla u)\,\mathrm{d}x.$$

Consequently,

$$\|u_n - u\| \to 0,$$

which will imply that

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega).$$

Thus

$$\phi(u) = c_\beta, \quad \phi'(u) = 0$$

and we get the solution u_1, it is a mountain pass type. □

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Newton transformations on null hypersurfaces

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Abstract. Any rigged null hypersurface is provided with two shape operators: with respect to the rigging and the rigged vector fields respectively. The present paper deals with the Newton transformations built on both of them and establishes related curvature properties. The laters are used to derive necessary and sufficient conditions for higher-order umbilicity and maximality we introduced in passing, and develop general Minkowski-type formulas for the null hypersurface, supported by some physical models in perfect-fluid space-times.

1 Introduction

It is a well-known fact that null hypersurfaces are exclusive objects of pseudo--Riemannian manifolds in the sense that they have no Riemannian counterpart and hence are interesting on their own from a (differential) geometric point of view. They also play an important role in general relativity namely in the study of black hole horizons (regions of space-time which contains a huge amount of mass compacted into an extremely small volume). From a more technical aspect, they are hypersurfaces having (induced) metrics with (pointwise) vanishing determinants and this degeneracy leads to several difficulties. In pseudo-Riemannian case, due to the causal character of three categories of vector fields (namely, spacelike, timelike and null), the induced metric on a hypersurface is a non-degenerate metric tensor field or a degenerate symmetric tensor field depending on whether the normal vector field is of the first two types or the third one. On non-degenerate hypersurfaces one can consider all the fundamental intrinsic and extrinsic geometric notions. In particular, a well defined (up to sign) notion of the unit orthogonal vector field is known to lead to a canonical splitting of the ambient tangent space into two factors: a tangent and an orthogonal one. Therefore, by respective projections, one has fundamental equations such as the Gauss, the Codazzi, the Weingarten equations, ... along with the second fundamental form, shape operator, induced connection, etc.

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The null hypersurface case is precisely when the normal vector field is null (also called lightlike) and since (contrary to the non-degenerate counterpart) the normal vector bundle intersects (non trivially) with the tangent bundle, one cannot find natural projector (and hence there is no preferred induced connection such as Levi-Civita) to define induced geometric objects as usual. This degeneracy of the induced metric makes it impossible to study them as part of standard submanifold theory, forcing to develop specific techniques and tools. For the most part, these tools are specific to a given problem, or sometimes with auxiliary non-canonical choices on which, unfortunately, depends the constructed null geometry. Indeed, Duggal and Bejancu in [12] introduced a non-degenerate screen distribution (or equivalently a null transversal line vector bundle as we may see below) so as to get a three factors splitting of the ambient tangent space and derive the main induced geometric objects such as second fundamental forms, shape operators, induced connections, curvature, etc. Unfortunately, the screen distribution is not unique and there is no preferred one in general, unless some specific geometric conditions are formulated to select and ensure uniqueness in exceptional cases [6], [5], [8], [7]. From above mentioned difficulties and compared to extensive research on global Riemannian and Lorentzian geometries we find out that considerable works are needed in null geometry to fill the gap.

One of the most important and central tools which have been extremely useful in addressing issues on higher-order r-th mean curvature and related topics in Riemannian geometry are Newton transformations [1], [2], [3], [4], [10], [15]. Since any null hypersurface with a fixed rigging do carry two shape operators: with respect to the rigging and the rigged vector fields respectively, we reasonably expect a role of those transformations in the study of null hypersurfaces. Recently in [11], the authors used above transformations of first type (thus, by duality considering the screen structure but not the null hypersurface structure) and examine conditions under which compact null hypersurfaces are totally umbilical in Robertson-Walker (RW) space-times. In the present paper we consider Newton transformations built on both of the two shape operators and establish related curvature properties and derive necessary and sufficient conditions for higher-order umbilicity and maximality, along with general Minkowski-type formulas for null hypersurfaces. The paper is organized as follows. Section 2 sets notations and definitions on riggings (normalizations) and review basic properties on null hypersurfaces, followed by some technical lemmas. Section 3 starts with introducing Newton transformations with respect to the rigged vector field and establishes their basic properties and some characterization results. The behaviour with respect to change in rigging is then examined on these transformations and the section ends with establishing some Minkowski-type integral formulas. In Section 4 we present some physical models in perfect-fluid space-times. The last section is concerned with the Newton transformations with respect to the rigging vector field.

2 Preliminaries

Let $(\bar{M}, \bar{g})$ be an $(n+2)$-dimensional Lorentzian manifold and M a null hypersurface in M. This means that at each $p \in M$, the restriction $\bar{g}_{p|T_pM}$ is degenerate, that is there exists a non-zero vector $U \in T_pM$ such that $\bar{g}(U, X) = 0$ for all

$X \in T_pM$. Hence, in null setting, the normal bundle $TM^\perp$ of the null hypersurface M^{n+1} is a rank 1 vector subbundle of the tangent bundle TM, contrary to the classical theory of non-degenerate hypersurfaces for which the normal bundle has trivial intersection $\{0\}$ with the tangent one and plays an important role in the introduction of the main induced geometric objects on M. Let us start with the usual tools involved in the study of such hypersurfaces according to [12]. They consist in fixing on the null hypersurface a geometric data formed by a lightlike section and a screen distribution. By *screen distribution* on M^{n+1}, we mean a complementary bundle of $TM^\perp$ in TM. It is then a rank n non-degenerate distribution over M. In fact, there are infinitely many possibilities of choices for such a distribution provided the hypersurface M be paracompact, but each of them is canonically isomorphic to the factor vector bundle $TM/TM^\perp$. For reasons that will become obvious in few lines below, let denote such a distribution by $\mathscr{S}(N)$. We then have

$$TM = \mathscr{S}(N) \oplus_{\text{Orth}} TM^\perp, \tag{1}$$

where $\oplus_{\text{Orth}}$ denotes the orthogonal direct sum. From [12], it is known that for a null hypersurface equipped with a screen distribution, there exists a unique rank 1 vector subbundle $\operatorname{tr}(TM)$ of $T\bar{M}$ over M, such that for any non-zero section ξ of $TM^\perp$ on a coordinate neighbourhood $\mathscr{U} \subset M$, there exists a unique section N of $\operatorname{tr}(TM)$ on $\mathscr{U}$ satisfying

$$\bar{g}(N,\xi) = 1, \quad \bar{g}(N,N) = \bar{g}(N,W) = 0, \quad \forall W \in \mathscr{S}(N)|_{\mathscr{U}}. \tag{2}$$

Then $T\bar{M}$ is decomposed as follows:

$$T\bar{M}|_M = TM \oplus \operatorname{tr}(TM) = \{TM^\perp \oplus \operatorname{tr}(TM)\} \oplus_{\text{Orth}} \mathscr{S}(N). \tag{3}$$

We call $\operatorname{tr}(TM)$ a *(null) transversal vector bundle* along M. In fact, from (2) and (3) one shows that, conversely, a choice of a transversal bundle $\operatorname{tr}(TM)$ determines uniquely the screen distribution $\mathscr{S}(N)$. A vector field N as in (2) is called a *null transversal vector field* of M. It is then noteworthy that the choice of a null transversal vector field N along M determines both the null transversal vector bundle, the screen distribution $\mathscr{S}(N)$ and a unique radical vector field, say ξ, satisfying (2). Tangent vector fields to $\mathscr{S}(N)$ (resp. to $TM^\perp$) are called horizontal (resp. vertical). Now, to continue our discussion, we need to clarify the concept of rigging for our null hypersurface.

Definition 1. Let M be a null hypersurface of a Lorentzian manifold. A *rigging* for M is a vector field L defined on some open set containing M such that $L_p \notin T_pM$ for each $p \in M$.

An outstanding property of a rigging is that it allows definition of geometric objects globally on M. We say that we have a *null rigging* in case the restriction of L to the null hypersurface is a null vector field. From now on we fix a null rigging N for M. In particular this rigging fixes a unique null vector field $\xi \in \Gamma(TM^\perp)$ called the *rigged* vector field, all of them defined in an open set containing M (hence globally on M) such that (1), (2) and (3) hold. Whence, from now on, by a *normalized (or rigged) null hypersurface* we mean a triplet (M, g, N) where $g = \bar{g}_{|M}$ is the induced metric on M and N a null rigging for M. In fact, in case the ambient manifold $\bar{M}$ has Lorentzian signature, at an arbitrary point p in M, a real null cone C_p is invariantly defined in the (ambient) tangent space $T_p\bar{M}$ and is tangent to M along a generator emanating from p. This generator is exactly the radical fibre $\Delta_p = T_pM^{\perp}$ and for each null rigging N for M and each $p \in M$ we have $N_p \in C_p \setminus \Delta_p$. Actually, a lightlike hypersurface M of a Lorentzian manifold is a hypersurface which is tangent to the lightlike cone C_p at each point $p \in M$. Recall that a space-time $(\bar{M}, \bar{g})$ is a connected Lorentzian manifold which is "time-oriented", i.e. a causal cone at each $T_p\bar{M}$, $p \in \bar{M}$ (the "future" causal cone) has been continuously chosen. Hence, null hypersurfaces in space-times can be naturally given an orientation by such a continuous districution of causal cones C_p.

Let N be a null rigging of a null hypersurface of a Lorentzian manifold $(\bar{M}, \bar{g})$ and $\theta = \bar{g}(N, \cdot)$ the 1-form metrically equivalent to N defined on $\bar{M}$. Then, take

$$\eta = i^{\star}\theta$$

to be its restriction to M, the map $i\colon M \hookrightarrow \bar{M}$ being the inclusion map. The normalization (M, g, N) will be said to be *closed* if the 1-form η is closed on M. It is easy to check that $\mathscr{S}(N) = \ker(\eta)$ and that the screen distribution $\mathscr{S}(N)$ is integrable whenever η is closed. On a normalized null hypersurface (M, g, N), the Gauss and Weingarten formulas are given by

$$\begin{aligned}
\bar{\nabla}_X Y &= \nabla_X Y + B^N(X, Y)N,\\
\bar{\nabla}_X N &= -A_N X + \tau^N(X)N,\\
\nabla_X PY &= \overset{\star}{\nabla}_X PY + C^N(X, PY)\xi,\\
\nabla_X \xi &= -\overset{\star}{A}_\xi X - \tau^N(X)\xi,
\end{aligned}$$

for any $X, Y \in \Gamma(TM)$, where $\bar{\nabla}$ denotes the Levi-Civita connection on $(\bar{M}, \bar{g})$, ∇ denotes the connection on M induced from $\bar{\nabla}$ through the projection along the rigging N and $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $\mathscr{S}(N)$ induced from ∇ through the projection morphism P of $\Gamma(TM)$ onto $\Gamma\big(\mathscr{S}(N)\big)$ with respect to the decomposition (1). Now the $(0, 2)$ tensors B^N and C^N are the second fundamental forms on TM and $\mathscr{S}(N)$ respectively, A_N and $\overset{\star}{A}_\xi$ are the shape operators on TM and $\mathscr{S}(N)$ respectively and τ^N a 1-form on TM defined by

$$\tau^N(X) = \bar{g}(\bar{\nabla}_X N, \xi).$$

For the second fundamental forms B^N and C^N the following holds

$$B^N(X, Y) = g(\overset{\star}{A}_\xi X, Y), \quad C^N(X, PY) = g(A_N X, Y) \quad \forall X, Y \in \Gamma(TM), \tag{4}$$

and

$$B^N(X, \xi) = 0, \quad \overset{\star}{A}_\xi \xi = 0. \tag{5}$$

It follows from (5) that integral curves of ξ are pregeodesics in both $\bar{M}$ and M, as $\bar{\nabla}_\xi \xi = \nabla_\xi \xi = -\tau^N(\xi)\xi$. Throughout the paper, and without explicit mention, we consider these integral curves to be geodesics which means that

$$\tau^N(\xi) = 0.$$

A null hypersurface M is said to be *totally umbilical* (resp. *totally geodesic*) if there exists a smooth function ρ on M such that at each $p \in M$ and for all $u, v \in T_pM$, $B^N(p)(u,v) = \rho(p)g(u,v)$ (resp. B^N vanishes identically on M). These are intrinsic notions on any null hypersurface in the following way. Note that N being a null rigging for M, a vector field $\widetilde{N} \in \Gamma(T\bar{M})$ is a null rigging for M if and only if it is defined in an open set containing M and there exist a function ψ on $\bar{M}$ and a section ζ of TM such that $\widetilde{N} \circ i = (\psi N) \circ i + \zeta$ with the properties that $\phi = \psi \circ i$ is nowhere vanishing, being i the inclusion map, and $2\phi\eta(\zeta) + \|\zeta\|^2 = 0$ along M. Then we have (see [7] for details on changes in normalizations) $B^{\widetilde{N}} = \frac{1}{\psi \circ i} B^N$ which shows that total umbilicity and total geodesibility are intrinsic properties for M. The total umbilicity and the total geodesibility conditions for M can also be written respectively as $\overset{\star}{A}_\xi = \rho P$ and $\overset{\star}{A}_\xi = 0$. Also, the screen distribution $\mathscr{S}(N)$ is *totally umbilical* (resp. *totally geodesic*) if $C^N(X, PY) = \lambda g(X, Y)$ for all $X, Y \in \Gamma(TM)$ (resp. $C^N = 0$), which is equivalent to $A_N = \lambda P$ (resp. $A_N = 0$). It is noteworthy to mention that the shape operators $\overset{\star}{A}_\xi$ and A_N are $\mathscr{S}(N)$-valued.

The induced connection ∇ is torsion-free, but not necessarily g-metric unless M is totally geodesic. In fact we have for all tangent vector fields X, Y and Z in TM,

$$(\nabla_X g)(Y, Z) = B^N(X, Y)\eta(Z) + B^N(X, Z)\eta(Y). \tag{6}$$

Denote by $\bar{R}$ and R the Riemann curvature tensors of $\bar{\nabla}$ and ∇, respectively. Then the following are the Gauss-Codazzi equations [12, p. 93].

$$\begin{aligned}
\langle \bar{R}(X,Y)Z, \xi \rangle &= (\nabla_X B^N)(Y, Z) - (\nabla_Y B^N)(X, Z) \\
&\quad + \tau^N(X) B^N(Y, Z) - \tau^N(Y) B^N(X, Z), \qquad (7)\\
\langle \bar{R}(X,Y)Z, PW \rangle &= \langle R(X,Y)Z, PW \rangle + B^N(X, Z) C^N(Y, PW) \\
&\quad - B^N(Y, Z) C^N(X, PW), \\
\langle \bar{R}(X,Y)\xi, N \rangle &= \langle R(X,Y)\xi, N \rangle = C^N(Y, \overset{\star}{A}_\xi X) - C^N(X, \overset{\star}{A}_\xi Y) \\
&\quad - 2\mathrm{d}\tau^N(X, Y), \\
\langle \bar{R}(X,Y)PZ, N \rangle &= \langle (\nabla_X A_N)Y, PZ \rangle - \langle (\nabla_Y A_N)X, PZ \rangle \\
&\quad + \tau^N(Y) \langle A_N X, PZ \rangle - \tau^N(X) \langle A_N Y, PZ \rangle \qquad (8)
\end{aligned}$$

for all $X, Y, Z, W \in \Gamma(TM|_{\mathscr{U}})$. The (shape) operator $\overset{\star}{A}_\xi$ is self-adjoint as the second fundamental form B^N is symmetric. However, this is not the case for the operator A_N as shown in the following lemma.

Lemma 1. *For all $X, Y \in \Gamma(TM)$,*

$$\langle A_N X, Y \rangle - \langle A_N Y, X \rangle = \tau^N(X)\eta(Y) - \tau^N(Y)\eta(X) - 2\mathrm{d}\eta(X, Y),$$

where (throughout) $\langle \cdot, \cdot \rangle = \bar{g}$ stands for the Lorentzian metric.

Proof. Recall that $\eta = i^\star\theta$ where $\theta = \langle N, \cdot\rangle$. Taking the differential of θ and using the Weingarten formula, we have for all $X, Y \in \Gamma(TM)$,

$$\begin{aligned} 2\mathrm{d}\eta(X,Y) = 2\mathrm{d}\theta(X,Y) &= \big\langle \bar{\nabla}_X N, Y\big\rangle - \big\langle \bar{\nabla}_Y N, X\big\rangle \\ &= -\big\langle A_N X, Y\big\rangle + \tau^N(X)\eta(Y) + \big\langle A_N Y, X\big\rangle - \tau^N(Y)\eta(X). \end{aligned}$$

Hence,

$$\big\langle A_N X, Y\big\rangle - \big\langle A_N Y, X\big\rangle = \tau^N(X)\eta(Y) - \tau^N(Y)\eta(X) - 2\mathrm{d}\eta(X,Y)$$

as announced. □

In case the normalization is closed the (connection) 1-form τ^N is related to the shape operator of M as follows.

Lemma 2. *Let (M, g, N) be a closed normalization of a null hypersurface M in a Lorentzian manifold such that $\tau^N(\xi) = 0$. Then*

$$\tau^N = -\big\langle A_N\xi, \cdot\big\rangle.$$

Proof. Assume $\eta = i^\star\theta$ closed and let X, Y be tangent vector fields to M. The condition $X\cdot\eta(Y) - Y\cdot\eta(X) - \eta([X,Y]) = 0$ is equivalent to $\big\langle \bar{\nabla}_X N, Y\big\rangle = \big\langle \bar{\nabla}_Y N, X\big\rangle$. Then by the Weingarten formula, we get

$$\big\langle -A_N X, Y\big\rangle + \tau^N(X)\eta(Y) = \big\langle -A_N Y, X\big\rangle + \tau^N(Y)\eta(X).$$

In this relation, take $Y = \xi$ to get

$$\tau^N(X) = -\big\langle A_N\xi, X\big\rangle + \tau^N(\xi)\eta(X)$$

which gives the desired formula as $\tau^N(\xi) = 0$. □

The following relations (see a detailed proof in [6]) account for effects of the rigging change $N \longrightarrow \widetilde{N}_{|_M} = \phi N + \zeta$ on the induced geometric objects described in Section 2. Throughout, items with the symbol $\sim$ apply to $\widetilde{N}$.

$$\widetilde{\xi} = \frac{1}{\phi}\xi, \quad B^{\widetilde{N}}(X,Y) = \frac{1}{\phi}B^N(X,Y), \quad \widetilde{P} = P - \frac{1}{\phi}g(\zeta,\cdot)\xi$$

$$\begin{aligned} C^{\widetilde{N}}(X, \widetilde{P}Y) &= \phi C^N(X, PY) - g(\nabla_X\zeta, PY) \\ &\quad + \left[\tau^N(X) + \frac{X\cdot\phi}{\phi} + \frac{1}{\phi}B^N(\zeta, X)\right]g(\zeta, Y) \end{aligned} \tag{9}$$

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{\phi}B^N(X,Y)\zeta, \quad \overset{\star}{A}_{\widetilde{\xi}} = \frac{1}{\phi}\overset{\star}{A}_\xi - \frac{1}{\phi^2}B^N(\zeta,\cdot)\xi \tag{10}$$

$$A_{\widetilde{N}} = \phi A_N - \nabla_{.}\zeta + \left[\tau^N + d\ln|\phi| + \frac{1}{\phi}B^N(\zeta,\cdot)\right]\zeta$$

for all tangent vector fields X and Y. Throughout the following ranges of indices is used: $i, j, l = 1, \ldots, n, \quad \alpha, \beta = 0, 1, \ldots, n, \quad a, b = 0, 1, \ldots, n+1$.

3 Newton transformations and Minkowski integral formulas with respect to the rigged section

Due to the first relation in (4), it is noteworthy that among the two shape operators carried out by the rigged null hypersurface M, $\overset{\star}{A}_\xi$ is actually the one that encodes at best its null geometry. We introduce in this section the Newton transformations corresponding to it. The second one A_N is instead more concerned with the screen structure $\mathscr{S}(N)$ and will be considered subsequently.

3.1 Newton transformations of $\overset{\star}{A}_\xi$

Let (M, g, N) be an $(n+1)$-dimensional normalized null hypersurface with rigged vector field ξ. Relation (4) shows that $\overset{\star}{A}_\xi$ is a self-adjoint linear operator on each fibre T_pM $(p \in M)$ and $\overset{\star}{A}_\xi \xi = 0$. Then, $\overset{\star}{A}_\xi$ is diagonalizable and have $(n+1)$ real-valued eigenfunctions $\overset{\star}{k}_0 = 0$, $\overset{\star}{k}_1, \dots, \overset{\star}{k}_n$ called principal curvatures of the null hypersurface with respect the shape operator $\overset{\star}{A}_\xi$. With respect to a quasi-orthonormal frame field $\{\overset{\star}{E}_0 = \xi, \overset{\star}{E}_1, \dots, \overset{\star}{E}_n\}$ of corresponding eigenvector fields the matrix of $\overset{\star}{A}_\xi$ take the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \overset{\star}{k}_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overset{\star}{k}_n \end{pmatrix}.$$

The function $\overset{\star}{H}_1 = \frac{1}{n+1}\operatorname{tr}(\overset{\star}{A}_\xi)$ is the mean curvature function of the null hypersurface and is a member of a familly of $n+1$ similar invariants $(\overset{\star}{H}_r)_{0 \le r \le n}$ called r-th mean curvature given by

$$\overset{\star}{H}_r = \binom{n+1}{r}^{-1} \sigma_r(\overset{\star}{k}_0, \dots, \overset{\star}{k}_n) \text{ and } \overset{\star}{H}_0 = 1 \text{ (constant function 1)},$$

where for $1 \le r \le n$, the algebraic invariant σ_r is the r-th elementary symmetric polynomial given by

$$\sigma_r(\overset{\star}{k}_0, \dots, \overset{\star}{k}_n) = \sum_{0 \le i_1 < \cdots < i_r \le n} \overset{\star}{k}_{i_1} \cdots \overset{\star}{k}_{i_r}.$$

It follows that the characteristic polynomial of $\overset{\star}{A}_\xi$ is given by

$$P(t) = \det(\overset{\star}{A}_\xi - tI) = \sum_{a=0}^{n+1} (-1)^a \binom{n+1}{a} \overset{\star}{H}_r t^{n+1-a}.$$

Set $\overset{\star}{S}_r = \sigma_r(\overset{\star}{k}_0, \dots, \overset{\star}{k}_n)$ and $\overset{\star}{S}{}_r^\alpha = \sigma_r(\overset{\star}{k}_0, \dots, \overset{\star}{k}_{\alpha-1}, \overset{\star}{k}_{\alpha+1}, \dots, \overset{\star}{k}_n)$.

Definition 2. Let r be an integer such that $1 \leq r \leq n$. The null hypersurface M is r-umbilical (resp. r-maximal) if

$$\overset{\star}{S}{}_r^i = \overset{\star}{S}{}_r^j \quad \forall i, j \in \{1, \ldots, n\} \quad (\text{resp. } \overset{\star}{H}_r = 0).$$

Remark 1.

1. As we show below (18) both r-maximality and r-total umbilicity are independent of the rigging.
2. The r-total umbilicity (respectively, r-maximality) generalize the totally umbilical (respectively, maximal) obtained when $r = 1$. But, it is easy to check that any totally umbilical hypersurface is r-totally umbilical for all r.
3. For a 4-dimensional null hypersurface (i.e. $n = 3$), total umbilicity and 2-total umbilicity are equivalent.

Example 1. Consider the 6-dimensional space $\bar{M} = \mathbb{R}^6$ endowed with the Lorentzian metric

$$\bar{g} = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + \exp 2x^0[(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2] + \exp 2x^1[(\mathrm{d}x^4)^2 + (\mathrm{d}x^5)^2],$$

$(x^0, \ldots, x^5)$ being the usual rectangular coordinates on $\bar{M}$. The only non-zero Christoffel coefficients of the Levi-Civita connection of $\bar{g}$ are

$$\Gamma_{02}^2 = \Gamma_{03}^3 = \Gamma_{14}^4 = \Gamma_{15}^5 = 1, \quad \Gamma_{22}^0 = \Gamma_{33}^0 = -\exp 2x^0, \quad \Gamma_{44}^1 = \Gamma_{55}^1 = \exp 2x^1.$$

Now, consider the hypersurface M of $\bar{M}$ define by

$$M = \{(x^0, \ldots, x^5) \in \mathbb{R}^6 ; \; x^0 + x^1 = 0\}.$$

Then, M is a null hypersurface of $(\bar{M}, \bar{g})$ and the vector field $N = -\frac{1}{2}\left(\frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^1}\right)$ is a null rigging for M with rigged vector field $\xi = \frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}$ and we have $\mathscr{S}(N) = \operatorname{span}\{\overset{\star}{E}_1, \overset{\star}{E}_2, \overset{\star}{E}_3, \overset{\star}{E}_4\}$ with

$$\overset{\star}{E}_1 = e^{-2x^0} \frac{\partial}{\partial x^2}, \quad \overset{\star}{E}_2 = e^{-2x^0} \frac{\partial}{\partial x^3}, \quad \overset{\star}{E}_3 = e^{-2x^1} \frac{\partial}{\partial x^4}, \quad \overset{\star}{E}_4 = e^{-2x^1} \frac{\partial}{\partial x^5}.$$

Then it is easy to check that

$$\begin{aligned}
\nabla_{\overset{\star}{E}_1} \xi &= \overset{\star}{E}_1 \Rightarrow \overset{\star}{k}_1 = -1,\\
\nabla_{\overset{\star}{E}_2} \xi &= \overset{\star}{E}_2 \Rightarrow \overset{\star}{k}_2 = -1,\\
\nabla_{\overset{\star}{E}_3} \xi &= -\overset{\star}{E}_3 \Rightarrow \overset{\star}{k}_3 = 1,\\
\nabla_{\overset{\star}{E}_4} \xi &= -\overset{\star}{E}_4 \Rightarrow \overset{\star}{k}_4 = 1.
\end{aligned}$$

Hence, M is 2-totally umbilical but it is not totally umbilical.

For each $r = 0, \ldots, n+1$, the r-th Newton transformation $\overset{\star}{T}_r \colon \Gamma(TM) \to \Gamma(TM)$ of the endomorphism $\overset{\star}{A}_\xi$, is given by

$$\overset{\star}{T}_r = \sum_{a=0}^{r} (-1)^a \overset{\star}{S}_a \overset{\star}{A}{}_\xi^{r-a}.$$

Inductively,

$$\overset{\star}{T}_0 = I \quad \text{and} \quad \overset{\star}{T}_r = (-1)^r \overset{\star}{S}_r I + \overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1},$$

where I denotes the identity map in $\Gamma(TM)$. According to the Cayley-Hamilton theorem, we have $\overset{\star}{T}_{n+1} = 0$. By elementary algebraic computations, the following is straightforward.

Proposition 1. *(a) $\overset{\star}{T}_r$ is self-adjoint and commute with $\overset{\star}{A}_\xi$;*

(b) $\overset{\star}{T}_r \overset{\star}{E}_\alpha = (-1)^r \overset{\star}{S}{}_r^\alpha \overset{\star}{E}_\alpha$;

(c) $\operatorname{tr}(\overset{\star}{T}_r) = (-1)^r (n+1-r) \overset{\star}{S}_r$;

(d) $\operatorname{tr}\left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}\right) = (-1)^{r-1} r \overset{\star}{S}_r$;

(e) $\operatorname{tr}\left(\overset{\star}{A}{}_\xi^2 \circ \overset{\star}{T}_{r-1}\right) = (-1)^r \left(\overset{\star}{S}_1 \overset{\star}{S}_r + (r+1) \overset{\star}{S}_{r+1}\right)$;

(f) $\operatorname{tr}(\overset{\star}{T}_{r-1} \circ \nabla_X \overset{\star}{A}_\xi) = (-1)^r X \cdot \overset{\star}{S}_r$.

Proof. The first item is due to the fact that $\overset{\star}{A}_\xi$ is self-adjoint. We show (b) inductively. In item (b) observe that the equality is trivial for $r = 0$. Assume that (b) holds for $r-1$ and observe that $\overset{\star}{S}{}_r^\alpha = \overset{\star}{S}_r - \overset{\star}{k}_\alpha \overset{\star}{S}{}_{r-1}^\alpha$. Then using the above and the well-known iterative relation characterizing the $\overset{\star}{T}_r$, we get,

$$\begin{aligned} \overset{\star}{T}_r \overset{\star}{E}_\alpha &= (-1)^r \overset{\star}{S}_r \overset{\star}{E}_\alpha + \overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1} \overset{\star}{E}_\alpha \\ &= (-1)^r \left(\overset{\star}{S}_r - \overset{\star}{k}_\alpha \overset{\star}{S}{}_{r-1}^\alpha\right) \overset{\star}{E}_\alpha \\ &= (-1)^r \overset{\star}{S}{}_r^\alpha \overset{\star}{E}_\alpha \end{aligned}$$

which shows (b). Through the above proof of b we see that $(-1)^r \overset{\star}{S}{}_r^\alpha$ are eigenfunctions associated to $\overset{\star}{E}_\alpha$ for each α and then we have

$$\operatorname{tr}(\overset{\star}{T}_r) = (-1)^r \sum_{\alpha=0}^{n} \overset{\star}{S}{}_r^\alpha$$

and each of the $\binom{n+1}{r}$ degree r monomials of $\overset{\star}{S}_r$ can be counted $(n+1)\binom{n}{r}$ times in the above summation. Thus

$$\sum_{\alpha=0}^{n} \overset{\star}{S}{}_r^\alpha = \frac{(n+1)\binom{n}{r}}{\binom{n+1}{r}} \overset{\star}{S}_r = (n+1-r) \overset{\star}{S}_r$$

and (c) is proved. By using the iterative formula of $\overset{\star}{T}_r$,

$$\begin{aligned}\operatorname{tr}\left(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}\right) &= \operatorname{tr}(\overset{\star}{T}_r) + (-1)^r \overset{\star}{S}_r \operatorname{tr}(I)\\ &= (-1)^r \left((n+1-r)\overset{\star}{S}_r + (n+1)\overset{\star}{S}_r\right)\\ &= (-1)^{r-1} r\overset{\star}{S}_r,\end{aligned}$$

that is (d). Item (e) is immediate as

$$\begin{aligned}\operatorname{tr}\left(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}\right) &= \operatorname{tr}(\overset{\star}{A}_\xi \circ \overset{\star}{T}_r) + (-1)^r \overset{\star}{S}_r \operatorname{tr}(\overset{\star}{A}_\xi)\\ &= (-1)^r \left(\overset{\star}{S}_1\overset{\star}{S}_r + (r+1)\overset{\star}{S}_{r+1}\right).\end{aligned}$$

Finally,

$$\begin{aligned}g(\overset{\star}{T}_{r-1}(\nabla_X \overset{\star}{A}_\xi)\overset{\star}{E}_i, \overset{\star}{E}_i) &= g(\overset{\star}{T}_{r-1}\nabla_X \overset{\star}{k}_i \overset{\star}{E}_i, \overset{\star}{E}_i) - g(\overset{\star}{T}_{r-1} \circ \overset{\star}{A}_\xi \nabla_X \overset{\star}{E}_i, \overset{\star}{E}_i)\\ &= X(\overset{\star}{k}_i) g(\overset{\star}{T}_{r-1}\overset{\star}{E}_i, \overset{\star}{E}_i)\\ &= (-1)^{r-1} X(\overset{\star}{k}_i)\overset{\star}{S}{}^i_{r-1}\end{aligned}$$

and $\eta(\overset{\star}{T}_{r-1}(\nabla_X \overset{\star}{A}_\xi)\xi) = 0$. Hence

$$\operatorname{tr}(\overset{\star}{T}_{r-1} \circ \nabla_X \overset{\star}{A}_\xi) = (-1)^{r-1} \sum_{i=1}^{n} X(\overset{\star}{k}_i)\overset{\star}{S}{}^i_{r-1} = (-1)^{r-1} X(\overset{\star}{S}_r),$$

which completes the proof. □

Now, we get the following.

Proposition 2. *Let r be an integer such that $1 \le r \le n$. A non-maximal point $p \in M$ is r-umbilical if and only if*

$$\forall i \in \{1, \dots, n\}, \quad \overset{\star}{S}{}^i_r(p) = (r+1)\frac{\overset{\star}{S}_{r+1}(p)}{\overset{\star}{S}_1(p)}.$$

Proof. Just observe that $\overset{\star}{S}_{r+1} = \overset{\star}{S}{}^i_{r+1} + \overset{\star}{k}_i \overset{\star}{S}{}^i_r$. □

Remark 2. For a large class of null hypersurfaces, namely closed null hypersurfaces, the above proposition cannot be applied globally as they do admit (at least) one maximal point [14, Remark 10, page 7].

From now on, only Lorentzian ambient manifolds will be in consideration. Recall that to a normalized null hypersurface (M^{n+1}, g, N) is associated a (nondegenerate) metric $g_\eta = g + \eta \otimes \eta$ [9]. The ambient manifold being Lorentzian, the induced metric g on M has signature $(0, n)$. It follows that the hypersurface M equipped with the associated metric g_η is a Riemannian manifold. Let $(e_0 = \xi, e_1, \dots, e_n)$

be a g_η-orthonormal basis of $\Gamma(TM)$ with $\mathscr{S}(N) = \mathrm{span}\{e_1, \dots, e_n\}$. The divergence of the operator $\overset{\star}{T}_r \colon \Gamma(TM) \to \Gamma(TM)$ is the vector field $\mathrm{div}^\nabla(\overset{\star}{T}_r) \in \Gamma(TM)$ defined as the trace of the $\mathrm{End}(TM)$-valued operator $\nabla\overset{\star}{T}_r$ and given by

$$\mathrm{div}^\nabla(\overset{\star}{T}_r) = \mathrm{tr}(\nabla\overset{\star}{T}_r) = \sum_{\alpha,\beta=0}^{n} g_\eta^{\alpha,\beta}(\nabla\overset{\star}{T}_r)(e_\alpha, e_\beta) = \sum_{\alpha=0}^{n}(\nabla_{e_\alpha}\overset{\star}{T}_r)e_\alpha.$$

By using the definition of the covariant derivative of a tensor and using (6),

$$g((\nabla_{e_\alpha}\overset{\star}{A}_\xi)\overset{\star}{T}_{r-1}e_\alpha, X) = g(\overset{\star}{T}_{r-1}e_\alpha, (\nabla_{e_\alpha}\overset{\star}{A}_\xi)X) - \eta(X)B^N(e_\alpha, \overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}e_\alpha).$$

Hence

$$\begin{aligned}\sum_{\alpha=0}^{n} g((\nabla_{e_\alpha}\overset{\star}{A}_\xi)\overset{\star}{T}_{r-1}e_\alpha, X) = \sum_{\alpha=0}^{n} g(\overset{\star}{T}_{r-1}e_\alpha, (\nabla_{e_\alpha}\overset{\star}{A}_\xi)X) \\ - \eta(X)\,\mathrm{tr}\left(\overset{\star}{A}{}_\xi^2 \circ \overset{\star}{T}_{r-1}\right).\end{aligned} \tag{11}$$

Proposition 3. *For all $X \in \Gamma(TM)$,*

$$\begin{aligned} g(\mathrm{div}\,\overset{\star}{T}_r, X) &= \sum_{a=0}^{r-1}\sum_{i=1}^{n} \bar{g}\left(\bar{R}(e_i,\xi)\overset{\star}{T}_a e_i, \overset{\star}{A}{}_\xi^{r-1-a}X\right) \\ &+ \sum_{a=0}^{r-1}\left(\tau^N(\overset{\star}{A}{}_\xi^{r-1-a}X)\,\mathrm{tr}(\overset{\star}{A}_\xi \circ \overset{\star}{T}_a) - \tau^N(P(\overset{\star}{A}_\xi \circ \overset{\star}{T}_a X))\right) \\ &+ (-1)^r\eta(X)\left(\sum_{i=1}^{n}\overset{\star}{S}{}^i_{r-1}\overset{\star}{k}{}_i^2 - \xi(\overset{\star}{S}_r)\right)\end{aligned} \tag{12}$$

Proof. Using iterative formula,

$$\begin{aligned}\mathrm{div}^\nabla(\overset{\star}{T}_r) &= (-1)^r\,\mathrm{div}(\overset{\star}{S}_r I) + \mathrm{div}(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}) \\ &= (-1)^r\sum_{\alpha=0}^{n}\big((e_\alpha \cdot \overset{\star}{S}_r)e_\alpha + (\nabla_{e_\alpha}\overset{\star}{A}_\xi)\overset{\star}{T}_{r-1}e_\alpha\big) + \overset{\star}{A}_\xi(\mathrm{div}\,\overset{\star}{T}_{r-1}).\end{aligned}$$

Hence by using (11) we get

$$\begin{aligned} g(\mathrm{div}^\nabla(\overset{\star}{T}_r), X) = g(\mathrm{div}\,\overset{\star}{T}_{r-1}, \overset{\star}{A}_\xi X) + (-1)^r PX(\overset{\star}{S}_r) - \eta(X)\,\mathrm{tr}\left(\overset{\star}{A}{}_\xi^2 \circ \overset{\star}{T}_{r-1}\right) \\ + \sum_{\alpha=0}^{n} g\big(\overset{\star}{T}_{r-1}e_\alpha, (\nabla_{e_\alpha}\overset{\star}{A}_\xi)X\big).\end{aligned} \tag{13}$$

By using the Gauss-Codazzi equation (8) with the substitutions

$$X \longleftarrow e_\alpha, \quad Y \longleftarrow X, \quad Z \longleftarrow \overset{\star}{T}_{r-1}e_\alpha,$$

we get

$$
\begin{aligned}
g\big(\overset{\star}{T}_{r-1}e_\alpha, (\nabla_{e_\alpha}\overset{\star}{A}_\xi)X\big) = \bar{g}\big(\bar{R}(e_\alpha, X)\overset{\star}{T}_{r-1}e_\alpha, \xi\big) + g\big(\overset{\star}{T}_{r-1}e_\alpha, (\nabla_X \overset{\star}{A}_\xi)e_\alpha\big) \\
+ B^N(e_\alpha, \overset{\star}{T}_{r-1}e_\alpha)\tau^N(X) \\
- B^N(X, \overset{\star}{T}_{r-1}e_\alpha)\tau^N(e_\alpha).
\end{aligned}
\tag{14}
$$

Observe that

$$
\sum_{\alpha=0}^{n} g\big(\overset{\star}{T}_{r-1}e_\alpha, (\nabla_X \overset{\star}{A}_\xi)e_\alpha\big) = \operatorname{tr}(\overset{\star}{T}_{r-1} \circ \nabla_X \overset{\star}{A}_\xi), \tag{15}
$$

and using this along with (13), (14), (15) and Proposition 1, we obtain

$$
\begin{aligned}
g(\operatorname{div}^\nabla(\overset{\star}{T}_r), X) &= g(\operatorname{div}\overset{\star}{T}_{r-1}, \overset{\star}{A}_\xi X) + (-1)^{r-1}\eta(X)\xi(\overset{\star}{S}_r) \\
&\quad + \sum_{\alpha=0}^{n}\Big(\bar{g}(\bar{R}(e_\alpha, X)\overset{\star}{T}_{r-1}e_\alpha, \xi) - B^N(X, \overset{\star}{T}_{r-1}e_\alpha)\tau^N(e_\alpha)\Big) \\
&\quad + \tau^N(X)\operatorname{tr}\Big(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}\Big) - \eta(X)\operatorname{tr}\Big(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}\Big) \\
&= g(\operatorname{div}\overset{\star}{T}_{r-1}, \overset{\star}{A}_\xi X) + (-1)^{r-1}\eta(X)\xi(\overset{\star}{S}_r) \\
&\quad + \sum_{\alpha=0}^{n}\bar{g}(\bar{R}(e_\alpha, X)\overset{\star}{T}_{r-1}e_\alpha, \xi) \\
&\quad - \tau^N\Big(P(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}X)\Big) \\
&\quad + \tau^N(X)\operatorname{tr}\Big(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}\Big) - \eta(X)\operatorname{tr}\Big(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}\Big).
\end{aligned}
$$

By using the above iterative formula and Proposition 1, we deduce (12). □

Remark 3. Taking $r = 1$ in (12) and $X = \xi$, we get

$$
\overline{\operatorname{Ric}}(\xi) = \xi(\overset{\star}{S}_1) + \tau^N(\xi)\overset{\star}{S}_1 - \sum_{i=1}^{n}\overset{\star}{k}{}_i^2. \tag{16}
$$

In case the ambient manifold $\bar{M}$ is a space form and $\tau^N = 0$, the vector field $\operatorname{div}\overset{\star}{T}_r$ is $TM^\perp$-valued, that is $g(\operatorname{div}^\nabla(\overset{\star}{T}_r), X) = 0$ for all $X \in TM$, and

$$
\xi(\overset{\star}{S}_r) = (-1)^{r-1}\operatorname{tr}\Big(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}\Big).
$$

Also (setting $X = \xi$) the following partial differential equation holds for each $r = 1, \dots, n+1$

$$
(-1)^{r-1}\xi(\overset{\star}{S}_r) + \tau^N(\xi)\operatorname{tr}(\overset{\star}{A}_\xi \circ \overset{\star}{T}_{r-1}) - \operatorname{tr}(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}) = 0; \tag{17}
$$

or equivalently

$$
\xi(\overset{\star}{S}_r) + r\overset{\star}{S}_r\tau^N(\xi) - \sum_{i=1}^{n}\overset{\star}{k}{}_i^2\overset{\star}{S}{}^\alpha_{r-1} = 0.
$$

From the above equation, we recover the well-known fact that for totally umbilical null hypersurfaces with principal curvature (umbilicity factor) ρ in a space form, the following partial differential equation holds [12, p. 108]:

$$\xi(\rho) + \rho\tau^N(\xi) - \rho^2 = 0.$$

We also derive the following.

Theorem 1. *Let (M^{n+1}, g, N) be a normalized null hypersurface of a Lorentzian space form $(\bar{M}(c)^{n+2}, \bar{g})$ with rigged vector field ξ and $\tau^N = 0$. Then*

(a) For each $r \in \{1, \ldots, n\}$, M is r-maximal if and only if the endomorphism $\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}$ is trace-free.

(b) M is maximal if and only if M is totally geodesic.

(c) If M is r-maximal for some $r = 1, \ldots, n$, then M is s-maximal for all $s \geq r$.

Proof. From (17),

$$(-1)^{r-1}\xi(\overset{\star}{S}_r) - \operatorname{tr}(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}) = 0,$$

as $\tau^N = 0$. Then the first item is immediate. Now, take $r = 1$ in the same equation (17) to get (b). Finally, if M is r-maximal then by the first item, $\operatorname{tr}\big(\overset{\star}{A}{}^2_\xi \circ \overset{\star}{T}_{r-1}\big) = 0$. Hence, Proposition 1 leads to

$$\overset{\star}{S}_1\overset{\star}{S}_r + (r+1)\overset{\star}{S}_{r+1} = 0,$$

which shows that $\overset{\star}{S}_r = 0$ implies $\overset{\star}{S}_{r+1} = 0$ and the proof is complete. □

Recall that a pseudo-Riemannian manifold satisfies the null (resp. the reverse null) convergence condition if $\overline{\operatorname{Ric}}(V) \geq 0$ (resp. $\overline{\operatorname{Ric}}(V) \leq 0$) for any null vector field V.

Theorem 2. *Let $(\bar{M}, \bar{g})$ be a Lorentzian manifold. If for $\bar{M}$ the null convergence condition holds, then for any null hypersurface M of $\bar{M}$, M is maximal if and only if M is totally geodesic.*

Proof. Assume M is maximal. From (16) we have

$$\overline{\operatorname{Ric}}(\xi) = -\sum_{i=1}^{n} \overset{\star}{k}{}_i^2 \geq 0 \quad \text{as} \quad \overset{\star}{S}_1 = 0\,.$$

Hence each $\overset{\star}{k}_i$ vanishes and M is totally geodesic. The converse is immediate. □

3.2 Newton transformations and change of rigging

As stated above, N being a null rigging for M, a vector field $\widetilde{N} \in \Gamma(T\bar{M})$ is a null rigging for M if and only if it is defined in an open set containing M and there exist a smooth function ϕ on M and a section ζ of TM such that $\widetilde{N} \circ i = \phi N + \zeta$ with the properties that ϕ is nowhere vanishing, being i the inclusion map, and $2\phi\eta(\zeta) + \|\zeta\|^2 = 0$ along M (see [7] for details on changes in normalizations). For each i, set

$$\overset{\star}{\widetilde{E}}_i = \widetilde{P}\overset{\star}{E}_i = \overset{\star}{E}_i - \frac{1}{\phi} g(\zeta, \overset{\star}{E}_i)\xi \quad \text{and} \quad \overset{\star}{\widetilde{E}}_0 = \widetilde{\xi} := \frac{1}{\phi}\xi.$$

Lemma 3. *$(\overset{\star}{\widetilde{E}}_0, \dots, \overset{\star}{\widetilde{E}}_n)$ is a quasi-orthonormal basis of $\Gamma(TM)$ which diagonalizes $\overset{\star}{A}_{\widetilde{\xi}}$ with eigenfunctions $\overset{\star}{\widetilde{k}}_\alpha = \frac{1}{\phi}\overset{\star}{k}_\alpha$.*

Proof. $g(\overset{\star}{\widetilde{E}}_0, \overset{\star}{\widetilde{E}}_\alpha) = \frac{1}{\phi} g(\xi, \overset{\star}{\widetilde{E}}_\alpha) = 0$ and $g(\overset{\star}{\widetilde{E}}_i, \overset{\star}{\widetilde{E}}_j) = g(\overset{\star}{E}_i, \overset{\star}{E}_j) = \delta_{ij}$, $\overset{\star}{A}_{\widetilde{\xi}}\widetilde{\xi} = 0$ and

$$\begin{aligned}
\overset{\star}{A}_{\widetilde{\xi}}\overset{\star}{\widetilde{E}}_i &= \overset{\star}{A}_\xi \overset{\star}{\widetilde{E}}_i - \frac{1}{\phi^2} B^N(\zeta, \overset{\star}{\widetilde{E}}_i)\xi \\
&= \overset{\star}{A}_\xi \overset{\star}{E}_i - \frac{1}{\phi^2} g(\zeta, \overset{\star}{A}_\xi \overset{\star}{E}_i)\xi \\
&= \frac{1}{\phi}\overset{\star}{k}_i \left(\overset{\star}{E}_i - \frac{1}{\phi} g(\zeta, \overset{\star}{E}_i)\xi \right) \\
&= \frac{1}{\phi}\overset{\star}{k}_i \overset{\star}{\widetilde{E}}_i.
\end{aligned}$$

□

Hence, through the change $\widetilde{N} = \phi N + \zeta$,

$$\overset{\star}{\widetilde{k}}_\alpha = \frac{1}{\phi}\overset{\star}{k}_\alpha, \quad \overset{\star}{\widetilde{H}}_r = \frac{1}{\phi^r}\overset{\star}{H}_r, \quad \overset{\star}{\widetilde{S}}_r = \frac{1}{\phi^r}\overset{\star}{S}_r, \quad \overset{\star}{\widetilde{S}}{}^i_r = \frac{1}{\phi^r}\overset{\star}{S}{}^i_r \tag{18}$$

and we have the next lemma.

Lemma 4. *Let (M^{n+1}, g, N) be a normalized null hypersurface of a Lorentzian manifold $(\bar{M}^{n+2}, \bar{g})$. Consider the change of normalization $\widetilde{N} = \phi N + \zeta$. Then*

$$\overset{\star}{\widetilde{T}}_r = \frac{1}{\phi^r}\overset{\star}{T}_r - \frac{1}{\phi^{r+1}} \sum_{a=0}^{r-1} (-1)^a \overset{\star}{S}_a g\left(\zeta, \overset{\star}{A}{}^{r-a}_{\xi}\right)\xi.$$

Proof. By use of second relation in (10) we have

$$\begin{aligned}
\overset{\star}{\widetilde{T}}_r &= \sum_{a=0}^{r} (-1)^a \overset{\star}{\widetilde{S}}_a \overset{\star}{A}{}^{r-a}_{\widetilde{\xi}} \\
&= (-1)^r \overset{\star}{\widetilde{S}}_r I + \sum_{a=0}^{r-1} (-1)^a \overset{\star}{\widetilde{S}}_a \left(\frac{1}{\phi} A_\xi - \frac{1}{\phi^2} B^N(\zeta, \cdot)\xi \right)^{r-a}.
\end{aligned}$$

As $r-a\geq 1$ and $\overset{\star}{A}_\xi\xi=0$,

$$\left(\frac{1}{\phi}A_\xi-\frac{1}{\phi^2}B^N(\zeta,\cdot)\xi\right)^{r-a}=\frac{1}{\phi^{r-a}}A_\xi^{r-a}-\frac{1}{\phi^{r+1-a}}B^N(\zeta,\overset{\star}{A}{}_\xi^{r-a-1})\xi.$$

This completes the proof. □

For each i, in view of (9) we get

$$\begin{aligned}\widetilde{\nabla}_{\tilde{\overset{\star}{E}}_i}\tilde{\overset{\star}{E}}_i&=\nabla_{\tilde{\overset{\star}{E}}_i}\tilde{\overset{\star}{E}}_i-\frac{1}{\phi}B^N(\tilde{\overset{\star}{E}}_i,\tilde{\overset{\star}{E}}_i)\zeta\\&=\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i-\overset{\star}{E}_i\left(\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\right)\xi-\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\nabla_{\overset{\star}{E}_i}\xi-\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\nabla_\xi\overset{\star}{E}_i\\&\quad+\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\nabla_\xi\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\xi-\frac{1}{\phi}B^N(\overset{\star}{E}_i,\overset{\star}{E}_i)\zeta\\&=\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i-\frac{\overset{\star}{k}_i}{\phi}g(\zeta,\overset{\star}{E}_i)\overset{\star}{E}_i-\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)P\nabla_\xi\overset{\star}{E}_i-\frac{\overset{\star}{k}_i}{\phi}P\zeta\\&\quad+\left[\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\tau^N(\overset{\star}{E}_i)-\overset{\star}{E}_i\left(\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\right)\right.\\&\qquad\left.+\frac{1}{2}\xi\left(\frac{1}{\phi^2}g(\zeta,\overset{\star}{E}_i)^2\right)-\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\eta(\nabla_\xi\overset{\star}{E}_i)\right]\xi\end{aligned}$$

Hence

$$\begin{aligned}\widetilde{\nabla}_{\tilde{\overset{\star}{E}}_i}\tilde{\overset{\star}{E}}_i&=\frac{1}{\phi}\left(\overset{\star}{k}_ig(\zeta,\overset{\star}{E}_i)\overset{\star}{E}_i-g(\zeta,\overset{\star}{E}_i)\sum_{j=1}^n g(\nabla_\xi\overset{\star}{E}_i,\overset{\star}{E}_j)\overset{\star}{E}_j-\overset{\star}{k}_i\sum_{j=1}^n g(\zeta,\overset{\star}{E}_j)\overset{\star}{E}_j\right)\\&\quad+\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i+\eta\big(\widetilde{\nabla}_{\tilde{\overset{\star}{E}}_i}\tilde{\overset{\star}{E}}_i-\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i\big)\xi,\end{aligned}\tag{19}$$

and

$$\begin{aligned}\eta(\widetilde{\nabla}_{\tilde{\overset{\star}{E}}_i}\tilde{\overset{\star}{E}}_i-\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i)&=\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\tau^N(\overset{\star}{E}_i)-\overset{\star}{E}_i\left(\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\right)\\&\quad-\frac{1}{\phi}g(\zeta,\overset{\star}{E}_i)\eta(\nabla_\xi\overset{\star}{E}_i)+\frac{1}{2}\xi\left(\frac{1}{\phi^2}g(\zeta,\overset{\star}{E}_i)^2\right).\end{aligned}$$

Lemma 5. *Let (M^{n+1},g,N) be a normalized null hypersurface of a Lorentzian manifold $(\bar{M}^{n+2},\bar{g})$ such that for a fixed r, $\xi\cdot\overset{\star}{S}{}_r^i=0$ for $i=1,\dots,n$. Consider the change of normalization $\widetilde{N}=\phi N+\zeta$, $\zeta|_M\in\Gamma(TM)$, $\phi\in\mathbb{R}$. Then*

$$\begin{aligned}\operatorname{div}^{\widetilde{\nabla}}(\tilde{\overset{\star}{T}}_r)&=\frac{1}{\phi^r}\operatorname{div}^{\nabla}(\overset{\star}{T}_r)+\eta(\operatorname{div}^{\widetilde{\nabla}}(\tilde{\overset{\star}{T}}_r)-\frac{1}{\phi^r}\operatorname{div}^{\nabla}(\overset{\star}{T}_r))\xi\\&\quad+\frac{(-1)^r}{\phi^{r+1}}\sum_{j=1}^n\sum_{i=1}^n(\overset{\star}{S}{}_r^j-\overset{\star}{S}{}_r^i)\left(g(\nabla_\xi\overset{\star}{E}_i,\overset{\star}{E}_j)g(\zeta,\overset{\star}{E}_i)+\overset{\star}{k}_ig(\zeta,\overset{\star}{E}_j)\right)\overset{\star}{E}_j.\end{aligned}\tag{20}$$

In particular for $r = 0, \ldots, n$, $\operatorname{div}^{\widetilde{\nabla}}\big(\overset{\star}{\widetilde{T}}_r\big) - \frac{1}{\phi^r}\operatorname{div}^{\nabla}(\overset{\star}{T}_r)$ is $TM^{\perp}$-valued if and only if for each $j = 1, \ldots, n$,

$$\sum_{i=1}^{n}(\overset{\star}{S}{}^{j}_{r} - \overset{\star}{S}{}^{i}_{r})\Big(g(\nabla_{\xi}\overset{\star}{E}_i, \overset{\star}{E}_j)g(\zeta, \overset{\star}{E}_i) + \overset{\star}{k}_i g(\zeta, \overset{\star}{E}_j)\Big) = 0. \tag{21}$$

Proof. Observe that $(\widetilde{\nabla}_{\widetilde{\xi}}\overset{\star}{\widetilde{T}}_r)\widetilde{\xi} = (-1)^r\widetilde{\xi}(\widetilde{S}_r)\widetilde{\xi}$. Then

$$\begin{aligned}
\operatorname{div}^{\widetilde{\nabla}}(\overset{\star}{\widetilde{T}}_r) &= \sum_{i=1}^{n}(\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{T}}_r)\overset{\star}{\widetilde{E}}_i \\
&= \sum_{i=1}^{n}(\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{T}}_r\overset{\star}{\widetilde{E}}_i - \overset{\star}{\widetilde{T}}_r\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{E}}_i) + (-1)^r\widetilde{\xi}(\widetilde{S}_r)\widetilde{\xi}.
\end{aligned}$$

By the second item in Proposition 1,

$$\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{T}}_r\overset{\star}{\widetilde{E}}_i = (-1)^r\overset{\star}{\widetilde{S}}{}^{i}_{r}\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{E}}_i + (-1)^r\big(\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{S}}{}^{i}_{r}\big)\overset{\star}{\widetilde{E}}_i.$$

Then thanks to (19) and by direct calculation, we get

$$\begin{aligned}
\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{T}}_r\overset{\star}{\widetilde{E}}_i = \frac{(-1)^r}{\phi^{r+1}}\overset{\star}{S}{}^{i}_{r}\Big(\overset{\star}{k}_i g(\zeta, \overset{\star}{E}_i)\overset{\star}{E}_i - \sum_{j=1}^{n}\big(g(\zeta, \overset{\star}{E}_i)g(\nabla_{\xi}\overset{\star}{E}_i, \overset{\star}{E}_j) + \overset{\star}{k}_i g(\zeta, \overset{\star}{E}_j)\big)\overset{\star}{E}_j\Big) \\
+ \frac{1}{\phi^r}\nabla_{\overset{\star}{E}_i}\overset{\star}{T}_r\overset{\star}{E}_i + \eta(\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{T}}_r\overset{\star}{\widetilde{E}}_i - \frac{1}{\phi^r}\nabla_{\overset{\star}{E}_i}\overset{\star}{T}_r\overset{\star}{E}_i)\xi \\
+ (-1)^r\Big(\overset{\star}{S}{}^{i}_{r}\overset{\star}{E}_i(1/\phi^r) - \frac{1}{\phi}g(\zeta, \overset{\star}{E}_i)\xi(\overset{\star}{S}{}^{i}_{r}/\phi^r)\Big)\overset{\star}{E}_i
\end{aligned}$$

in which the last term vanishes due to $\xi \cdot \overset{\star}{S}{}^{i}_{r} = 0$ and $\phi \in \mathbb{R}$. Now (19) and Lemma 4 yield

$$\begin{aligned}
\overset{\star}{\widetilde{T}}_r\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{E}}_i &= \frac{1}{\phi^r}\overset{\star}{T}_r\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{E}}_i - \frac{1}{\phi^{r+1}}\sum_{a=0}^{r-1}(-1)^a\overset{\star}{S}_a g(\overset{\star}{A}{}^{r-a}_{\xi}, \widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{E}}_i)\xi \\
&= \frac{(-1)^r}{\phi^{r+1}}\overset{\star}{k}_i\overset{\star}{S}{}^{i}_{r}g(\zeta, \overset{\star}{E}_i)\overset{\star}{E}_i \\
&\quad + \frac{(-1)^{r+1}}{\phi^{r+1}}\sum_{j=1}^{n}\overset{\star}{S}{}^{j}_{r}\big(g(\zeta, \overset{\star}{E}_i)g(\nabla_{\xi}\overset{\star}{E}_i, \overset{\star}{E}_j) + \overset{\star}{k}_i g(\zeta, \overset{\star}{E}_j)\big)\overset{\star}{E}_j \\
&\quad + \frac{1}{\phi^r}\overset{\star}{T}_r\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i + \eta\big(\overset{\star}{\widetilde{T}}_r\widetilde{\nabla}_{\overset{\star}{\widetilde{E}}_i}\overset{\star}{\widetilde{E}}_i - \frac{1}{\phi^r}\overset{\star}{T}_r\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i\big)\xi.
\end{aligned}$$

The desired expression follows from direct substitution. The last claim is immediate by cancelling the screen term represented by the last summation in (20). □

Theorem 3. *Let (M^{n+1}, g, N) be a normalized null hypersurface of a Lorentzian manifold $(\bar{M}^{n+2}, \bar{g})$, and $r \in \{1, \ldots, n\}$ such that $\xi \cdot \overset{\star}{S}{}^i_r = 0$ for $i = 1, \ldots, n$. Then $\operatorname{div}^{\widetilde{\nabla}}(\widetilde{\overset{\star}{T}}_r) - \frac{1}{\phi^r} \operatorname{div}^{\nabla}(\overset{\star}{T}_r)$ is $TM^{\perp}$-valued for any change of normalization $\widetilde{N} = \phi N + \zeta$ with $\phi \in \mathbb{R}$, if and only if any point of M is r-umbilical or both maximal and $(r+1)$-maximal.*

Proof. Let $r \in \{1, \ldots, n\}$ and $\operatorname{div}^{\widetilde{\nabla}}(\widetilde{\overset{\star}{T}}_r) - \frac{1}{\phi^r} \operatorname{div}^{\nabla}(\overset{\star}{T}_r) \in \Gamma(\operatorname{Rad} TM)$ for any change of normalization $\widetilde{N} = \phi N + \zeta$. Then by (21),

$$\sum_{i=1}^{n} (\overset{\star}{S}{}^j_r - \overset{\star}{S}{}^i_r) \Big(g(\nabla_\xi \overset{\star}{E}_i, \overset{\star}{E}_j) g(\zeta, \overset{\star}{E}_i) + \overset{\star}{k}_i g(\zeta, \overset{\star}{E}_j) \Big) = 0, \quad \forall j = 1, \ldots, n.$$

Consider the particular changes $\widetilde{N} = N + \overset{\star}{E}_l$ for $l = 1, \ldots, n$. Then for each l,

$$(\overset{\star}{S}{}^j_r - \overset{\star}{S}{}^l_r) g(\nabla_\xi \overset{\star}{E}_l, \overset{\star}{E}_j) + \sum_{i=1}^{n} (\overset{\star}{S}{}^j_r - \overset{\star}{S}{}^i_r) \overset{\star}{k}_i \delta_{lj} = 0, \quad \forall j = 1, \ldots, n,$$

and setting $j = l$ yields

$$\overset{\star}{S}_1 \overset{\star}{S}{}^l_r - (r+1) \overset{\star}{S}_{r+1} = 0.$$

By Proposition 2 we deduce that any non-maximal point of M is r-umbilical. The converse is straightforward. □

3.3 Minkowski integral formulas

Using Newton transformations with respect to the shape operator $\overset{\star}{A}_\xi$ we introduce some Minkowski-type integral formulas on null hypersurfaces of Lorentzian manifolds carrying some conformal Killing vector field.

Recall that when a manifold M is provided with a linear connection D and X is a section of the tangent bundle of M, the map $DX \colon \Gamma(TM) \to \Gamma(TM)$ given by $T_pM \ni Y_p \mapsto D_{Y_p} X_p$ is an endomorphism at each point $p \in M$. The divergence of X (with respect to D) is defined as the trace of DX, that is

$$\operatorname{div}^D(X) = \operatorname{tr}(DX).$$

In particular on semi-Riemannian manifolds the default (natural) connection used in calculating the divergence is the Levi-Civita connection.

Let (M^{n+1}, g, N) be a normalized null hypersurface of a Lorentzian manifold $(\bar{M}^{n+2}, \bar{g})$ with rigged vector field ξ and $\tau^N = 0$. Let ∇ denote the linear connection induced by the rigging N and assume $K \in \Gamma(TM)$ is a conformal Killing vector field with smooth conformal factor 2Φ. For each $r \in \{0, \ldots, n+1\}$ we have

$$\operatorname{div}^{\nabla}(\overset{\star}{T}_r K) = \operatorname{tr}(\nabla \overset{\star}{T}_r K) = \sum_{i=1}^{n} g(\nabla_{\overset{\star}{E}_i} \overset{\star}{T}_r K, \overset{\star}{E}_i) + \bar{g}(\nabla_\xi \overset{\star}{T}_r K, N).$$

But

$$
\begin{aligned}
g(\nabla_{\overset{\star}{E}_i}\overset{\star}{T}_r K,\overset{\star}{E}_i) &= \overset{\star}{E}_i\cdot g(\overset{\star}{T}_r K,\overset{\star}{E}_i) - g(K,\overset{\star}{T}_r\nabla_{\overset{\star}{E}_i}\overset{\star}{E}_i) - \eta(\overset{\star}{T}_r K)B^N(\overset{\star}{E}_i,\overset{\star}{E}_i)\\
&= \overset{\star}{E}_i\cdot g(K,\overset{\star}{T}_r\overset{\star}{E}_i) + g(K,(\nabla_{\overset{\star}{E}_i}\overset{\star}{T}_r)\overset{\star}{E}_i) - g(K,\nabla_{\overset{\star}{E}_i}\overset{\star}{T}_r\overset{\star}{E}_i)\\
&\quad - \eta(\overset{\star}{T}_r K)B^N(\overset{\star}{E}_i,\overset{\star}{E}_i)\\
&= g(K,(\nabla_{\overset{\star}{E}_i}\overset{\star}{T}_r)\overset{\star}{E}_i) + (-1)^r\overset{\star}{S}{}^i_r g(\nabla_{\overset{\star}{E}_i}K,\overset{\star}{E}_i)\\
&\quad + \eta(K)B^N(\overset{\star}{E}_i,\overset{\star}{T}_r\overset{\star}{E}_i) - \eta(\overset{\star}{T}_r K)B^N(\overset{\star}{E}_i,\overset{\star}{E}_i).
\end{aligned}
$$

As $L_K g = 2\varphi g$ we have $g(\nabla_{\overset{\star}{E}_i}K,\overset{\star}{E}_i) = \varphi g(\overset{\star}{E}_i,\overset{\star}{E}_i)$. Hence

$$
\begin{aligned}
\mathrm{div}^{\nabla}(\overset{\star}{T}_r K) &= g(\mathrm{div}^{\nabla}(\overset{\star}{T}_r),K) + \varphi\big((-1)^{r-1}\overset{\star}{S}_r + \mathrm{tr}(\overset{\star}{T}_r)\big)\\
&\quad + \eta(K)\,\mathrm{tr}\left(\overset{\star}{A}_\xi\circ\overset{\star}{T}_r - (-1)^r\overset{\star}{S}_r\overset{\star}{A}_\xi\right) + \eta(\nabla_\xi\overset{\star}{T}_r K)\\
&= g(\mathrm{div}^{\nabla}(\overset{\star}{T}_r),K) + (-1)^r(n-r)\overset{\star}{S}_r\varphi\\
&\quad + \eta(K)\,\mathrm{tr}\left(\overset{\star}{A}{}^2_\xi\circ\overset{\star}{T}_{r-1}\right) + \eta(\nabla_\xi\overset{\star}{T}_r K).
\end{aligned}
$$

Now using Proposition 1 leads to

$$
\begin{aligned}
\mathrm{div}^{\nabla}(\overset{\star}{T}_r K) &= g(\mathrm{div}^{\nabla}(\overset{\star}{T}_r),K) + \eta(\nabla_\xi\overset{\star}{T}_r K)\\
&\quad + (-1)^r\left(c_r\overset{\star}{H}_r\varphi + c'_r\overset{\star}{H}_{r+1}\eta(K) - c''_r\overset{\star}{H}_1\overset{\star}{H}_r\eta(K)\right).
\end{aligned}
\tag{22}
$$

where

$$
c_r = (n-r)\binom{n+1}{r},\qquad c'_r = (n+1)\binom{n}{r},\quad c''_r = (n+1)\binom{n+1}{r}.
$$

Also, a straightforward computation gives

$$
\eta(\nabla_\xi\overset{\star}{T}_r K) = (-1)^r\xi(\overset{\star}{S}_r\eta(K)) + (-1)^r\overset{\star}{S}_r\tau^N(\xi)\eta(K) - \bar{g}(\overset{\star}{T}_r K, A_N\xi).
$$

We deduce the following.

Theorem 4. *Let (M^{n+1},g,N) be a normalized null hypersurface of a space-time $(\bar{M}^{n+2},\bar{g})$ with rigged vector field ξ and $\tau^N=0$, carrying a compactly supported conformal Killing vector field K with smooth conformal factor 2Φ. Then, for each $r=1,\dots,n+1$, the following holds*

$$
\begin{aligned}
&\int_M\left(g(\mathrm{div}\,\overset{\star}{T}_{r-1},K) + \eta(\nabla_\xi\overset{\star}{T}_{r-1}K)\right)\mathrm{d}V\\
&\qquad = (-1)^r\int_M\left(c_{r-1}\overset{\star}{H}_{r-1}\varphi + \eta(K)(c'_{r-1}\overset{\star}{H}_r - c''_{r-1}\overset{\star}{H}_1\overset{\star}{H}_{r-1})\right)\mathrm{d}V,
\end{aligned}
\tag{23}
$$

where $\mathrm{d}V = i_N\mathrm{d}\bar{V}$ and $\mathrm{d}\bar{V}$ is the (fixed) volume element on $\bar{M}$ with respect to $\bar{g}$ and the given orientation.

In particular for horizontal conformal Killing vector fields K on M we have

$$\int_M \Big(g(\operatorname{div} \overset{\star}{T}_{r-1}, K)\Big)\,\mathrm{d}V = (-1)^r \int_M \Big(c_{r-1}\overset{\star}{H}_{r-1}\varphi\Big)\,\mathrm{d}V. \tag{24}$$

Proof. Since K is compactly supported, by Stoke's Theorem,

$$\int_M \operatorname{div}^{\nabla}(\overset{\star}{T}_r K)\,\mathrm{d}V = 0$$

and (23) is straightforward from (22). Now, assume K to be tangent to the screen structure $\mathscr{S}(N)$. Then $\eta(K) = 0$. Also, as $\tau^N = 0$ we have from Lemma 2 and (4) that $C^N(\xi, \overset{\star}{T}_{r-1}K) = 0$. Therefore

$$\nabla_\xi(\overset{\star}{T}_{r-1}K) = \overset{\star}{\nabla}_\xi(\overset{\star}{T}_{r-1}K) + C^N(\xi, \overset{\star}{T}_{r-1}K)\xi = \overset{\star}{\nabla}_\xi \overset{\star}{T}_{r-1}K \in \mathscr{S}(N).$$

Hence $\eta\big(\nabla_\xi \overset{\star}{T}_{r-1}K\big) = 0$ and the relation (24) follows (23). □

Remark 4. In Theorem 4 and below, the condition *compactly supported* may be removed and replaced by *compact null hypersurface without boundary*.

Corollary 1. *Let (M^{n+1}, g, N) be a normalized null hypersurface of a space-time $(\bar{M}^{n+2}, \bar{g})$ with rigged vector field ξ and $\tau^N = 0$, carrying a compactly supported conformal Killing vector field K with smooth conformal factor 2Φ. Suppose that for some $r = 1, \dots, n+1$ the following condition holds*

$$\int_M g(\operatorname{div} \overset{\star}{T}_{r-1}, K)\,\mathrm{d}V = 0.$$

Then

$$\int_M \Big(c_{r-1}\overset{\star}{H}_{r-1}\Phi + c'_{r-1}\bar{g}(K,N)\overset{\star}{H}_r - c''_{r-1}\bar{g}(K,N)\overset{\star}{H}_1\overset{\star}{H}_{r-1}\Big)\,\mathrm{d}V$$
$$= (-1)^r \int_M \eta(\nabla_\xi \overset{\star}{T}_{r-1}K)\,\mathrm{d}V. \tag{25}$$

In particular, (25) always holds when the ambient space-time $(\bar{M}^{n+2}, \bar{g})$ has constant sectional curvature and for the conformal factor 2Φ we have

$$\int_M \Phi\,\mathrm{d}V = -\frac{1}{n}\int_M \eta(\nabla_\xi K)\,\mathrm{d}V. \tag{26}$$

Moreover, if the conformal Killing vector field K is horizontal then $\int_M \Phi\,\mathrm{d}V = 0$.

Formula (25) is the r-th Minkowski-type formula of the null hypersurface M, with respect to the shape operator $\overset{\star}{A}_\xi$.

Proof. Setting $\int_M g(\operatorname{div} \overset{\star}{T}_{r-1}, K)\,\mathrm{d}V$ to 0 in (23) leads to (25). From Remark 3 we know that when the ambient manifold has constant sectional curvature, $\operatorname{div}^{\nabla}(\overset{\star}{T}_{r-1})$ is $TM^{\perp}$-valued and the vanishing condition is fulfilled. Finally, set $r = 1$ in (25) to get (26), using the fact that $c_0 = n$, $c_0' = c_0''$ and $\overset{\star}{H}_0 = 1$. If in addition the conformal Killing vector field is tangent to the screen structure then by the screen Gauss formula and and $\tau^N = 0$ we have $C^N(\xi, \cdot) = 0$ and then $\nabla_\xi K \in \mathscr{S}(N)$, that is $\eta(\nabla_\xi K) = 0$ and the last claim follows. □

Corollary 2. *Let (M^{n+1}, g, N) be a normalized null hypersurface of a space-time $(\bar{M}^{n+2}, \bar{g})$ with rigged vector field ξ and $\tau^N = 0$, carrying a compactly supported conformal Killing vector field K with smooth conformal factor 2Φ. If M is r-totally umbilical for some $r = 1, \ldots, (n+1)$ and satisfies both $\xi \cdot \overset{\star}{S}{}^i_r = 0$ for $i = 1, \ldots, n$ and the r-th Minkowski-type formula* (25)*, then the same is true for all rigging of the form $\widetilde{N} = \psi N + \zeta$ with constant ψ.*

Proof. Consider from N a rigging $\widetilde{N} = \psi N + \zeta$. We pointed out in Theorem 3 that

$$\operatorname{div}^{\widetilde{\nabla}} \overset{\star}{\widetilde{T}}_{r-1} - \frac{1}{\psi^{r-1}} \operatorname{div}^{\nabla} \overset{\star}{T}_{r-1} \in \Gamma(\operatorname{Rad} TM).$$

Thus, $g(\operatorname{div}^{\widetilde{\nabla}} \overset{\star}{\widetilde{T}}_{r-1}, K) = \frac{1}{\psi^{r-1}} g(\operatorname{div}^{\nabla} \overset{\star}{T}_{r-1}, K)$. It follows that for constant ψ we have

$$\int_M g(\operatorname{div}^{\widetilde{\nabla}} \overset{\star}{\widetilde{T}}_{r-1}, K) = \frac{1}{\psi^{r-1}} \int_M g(\operatorname{div}^{\nabla} \overset{\star}{T}_{r-1}, K),$$

which shows that integrals from both sides vanish or not, simultaneously. □

4 Physical models

As usual, stationary and axisymmetric perfect fluid metrics are studied under the assumption of the existence of a conformal Killing vector field. Let $(\bar{M}^4, \bar{g})$ be the Einstein static fluid space-time with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (1 - \varrho^2)^{-1}\mathrm{d}\varrho^2 + \varrho^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big),$$

with the fluid four-velocity vector $u^a = \delta_0^a$ $(a = 0, 1, 2, 3)$. This space-time admits a conformal Killing vector field

$$K^a = (1 - \varrho^2)^{1/2} \cos t\, \delta_0^a - \varrho(1 - \varrho^2)^{1/2} \sin t\, \delta_1^a.$$

In fact in this space-time, the relation

$$\varrho = \cos t, \quad t \in \left]0, \frac{\pi}{2}\right[$$

defines a compact null hypersurface M for which the kernel of the degenerate induced metric g is spanned by the null conformal Killing vector field K. In other words, (M, g) is a compact totally umbilical null hypersurface. Indeed, consider the vector field

$$N^a = -\frac{1}{2\varrho^2}(1-\varrho^2)^{-1/2}\big[\cos t\,\delta^a_0 + \varrho\sin t\,\delta^a_1\big].$$

This is a null rigging with associated screen structure $\mathscr{S}(N) = \operatorname{span}(\partial_\theta, \partial_\phi)$. It is easy to check that $\tau^N = 0$ and $\overset{\star}{A}_K = \big[(1-\varrho^2)^{1/2}\sin t\big]P$ where P denotes the projection morphism of TM onto $\mathscr{S}(N)$. The Newton transformations are given by

$$\overset{\star}{T}_0 = I, \quad \overset{\star}{T}_r = (1-\varrho^2)^{r/2}\sin^r t\left[\sum_{a=0}^{r}(-1)^a\binom{3}{a}\right]P, \quad r = 1, 2, 3.$$

The scale factor is given by $\Phi = -(1-\varrho^2)^{1/2}\sin t$. Also, for all $r \geq 1$,

$$\int_M g(\operatorname{div}\overset{\star}{T}_{r-1}, K)\,\mathrm{d}V = 0$$

and $\bar{g}(K, N) = 1$ and by direct calculation, we get $\int_M \Phi\,\mathrm{d}V = -2\pi$ which is non-zero. Observe that the conformal Killing vector field K is not compactly supported in M.

In general, when interested by perfect-fluid solutions of Einstein's field equations, it is well-known that there exist coordinates $\{t, x, y, z\}$ such that $U = \partial_y$ and $T = \partial_z$ are two Killing vector fields and in which the metric takes the form

$$\mathrm{d}s^2 = \frac{1}{S^2(t,x)}\Big[-\mathrm{d}t^2 + \mathrm{d}x^2 + F(t,x)\big(P^{-1}(t,x)\,\mathrm{d}y^2 + P(t,x)(\mathrm{d}z + W(t,x)\,\mathrm{d}y)^2\big)\Big].$$

Let us consider the 1-forms $\{\theta^a\}$ such that

$$\theta^0 = \frac{1}{S(t,x)}\,\mathrm{d}t \qquad \theta^1 = \frac{1}{S(t,x)}\,\mathrm{d}x$$

$$\theta^2 = \frac{1}{S(t,x)}\sqrt{\frac{F(t,x)}{P(t,x)}}\,\mathrm{d}y \qquad \theta^3 = \frac{1}{S(t,x)}\sqrt{F(t,x)P(t,x)}(\mathrm{d}z + W(t,x)\,\mathrm{d}y),$$

and let $S_{\alpha\beta}$ stand for the components of the Einstein tensor in the $\{\theta^a\}$ cobasis. Then the Einstein field equations can be written in terms of the $S_{\alpha\beta}$ and due to the symmetries inherent to this setting we are led to three inequivalent Lie algebras [16]. The Lie algebra A is given by

$$[U, T] = 0, \quad [U, K] = \frac{1}{2}(c+b)U, \quad [T, K] = \frac{1}{2}(c-b)T,$$

where b and c are arbitrary (possibly vanishing) constants and K is a conformal Killing vector field given by

$$K = \partial_t + \frac{1}{2}(c+b)yU + \frac{1}{2}(c-b)zT.$$

The line element in this case has the form

$$\mathrm{d}s^2 = \frac{1}{S^2(t,x)}\Big[-\mathrm{d}t^2 + \mathrm{d}x^2 + F(x)P^{-1}(x)e^{-(b+c)t}\,\mathrm{d}y^2 + F(x)P(x)e^{(b-c)t}\big(\mathrm{d}z + W(x)e^{-bt}\,\mathrm{d}y\big)^2\Big].$$

Similarly, for the Lie Algebra B we have

$$[U,T]=0, \quad [U;K] = \frac{1}{2}cU + aT, \quad [T,K] = \frac{1}{2}cT,$$

where a is a non-vanishing constant and K is a conformal Killing vector field given by

$$K = \partial_t + \frac{1}{2}cyU + \big(ay + \frac{1}{2}cz\big)T.$$

The corresponding line element has the form

$$\mathrm{d}s^2 = \frac{1}{S^2(t,x)}\Big[-\mathrm{d}t^2 + \mathrm{d}x^2 + F(x)e^{-ct}\big(P^{-1}(x)\,\mathrm{d}y^2 + P(x)(\mathrm{d}z + [W(x)+at]\,\mathrm{d}y)^2\big)\Big].$$

Finally for the Lie algebra VII (so named because it corresponds to the Bianchi type VII in Bianchi's classification of three-dimensional Lie algebra) the product is defined by

$$[U,T]=0, \quad [U;K] = \frac{1}{2}cU - aT, \quad [T,K] = aU + \frac{1}{2}cT,$$

where $a \neq 0$ and c are constant and K is a conformal Killing vector field given by

$$K = \partial_t + \left(\frac{1}{2}cy + az\right)U + \left(-ay + \frac{1}{2}cz\right)T.$$

For each conformal Killing vector field K in above three (non equivalent) Lie algebras, the scale factor Φ is given by

$$\Phi = -\frac{S_{,t}}{S}.$$

Now, for each Lie algebra, consider the two distributions

$$\mathscr{D}_{U,K} = \mathrm{span}\{U,K\}, \quad \mathscr{D}_{T,K} = \mathrm{span}\{T,K\}$$

involving the conformal Killing vector field K. For the Lie algebra A the two distributions $\mathscr{D}_{U,K}$ and $\mathscr{D}_{T,K}$ are both integrable. For the Lie algebra B, only $\mathscr{D}_{T,K}$ is integrable and for the Lie algebra VII, none of them is integrable. Let M be any compact null hypersurface without boundary in the perfect-fluid space-time. Assume N is a rigging for M with screen structure $\mathscr{S}(N) = \mathscr{D}_{U,K}$ or $\mathscr{D}_{T,K}$ according to the Lie algebra A or $\mathscr{S}(N) = \mathscr{D}_{T,K}$ when dealing with the Lie algebra B. Then K is a horizontal conformal Killing vector field in the rigged null hypersurface M. If the $\mathrm{d}s^2$ have constant sectional curvature and τ^N is vanishing, we get thanks to Corollary 1 that

$$\int_M \frac{S_{,t}}{S}\,\mathrm{d}V = 0.$$

5 Newton transformation of the null hypersurface with respect to the shape operator A_N

Throughout this section the normalization is assumed to be closed. In this case Lemma 1 asserts that

$$\langle A_N X, Y\rangle - \langle A_N Y, X\rangle = \tau^N(X)\eta(Y) - \tau^N(Y)\eta(X) \tag{27}$$

for all $X, Y \in \Gamma(TM)$. It follows that the operator A_N is symmetric when restricted to the screen structure $\mathscr{S}(N)$. The ambient manifold will also considered to be Lorentzian which implies that the screen structure is Riemannian. Let $(E_0 = \xi, E_1, \ldots, E_n)$ be a quasi-orthonormal frame field of TM with $\mathscr{S}(N) = \operatorname{span}\{E_1, \ldots, E_n\}$. Then the matrix of A_N has the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \star & k_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \star & 0 & \cdots & k_n \end{pmatrix} \tag{28}$$

where $k_0, k_1, \ldots, k_n$ are the principal curvatures of the null hypersurface M with respect to the shape operator A_N. The scalar function $H_1 = \frac{1}{n+1}\operatorname{tr}(A_N)$ is the mean curvature of the null hypersurface with respect to A_N. For $0 \leq r \leq n+1$, the r-th mean curvature of the null hypersurface with respect to the shape operator A_N is defined by

$$H_r = \binom{n+1}{r}^{-1} \sigma_r(k_0, \ldots, k_n) \quad \text{and} \quad H_0 = 1,$$

and $S_r = \binom{n+1}{r} H_r$.

The characteristic polynomial of A_N is given by

$$P(t) = \det(A_N - tI) = \sum_{a=0}^{n+1} (-1)^a \binom{n+1}{a} H_a t^{n+1-a}.$$

In a similar way as for the operator $\overset{\star}{A}_\xi$ the Newton transformations T_r $(0 \leq r \leq n+1)$ of the null hypersurface M with respect to A_N are given by

$$T_r = \sum_{a=0}^{r} (-1)^a \binom{n+1}{a} H_a A_N^{r-a}.$$

Inductively,

$$T_0 = I \quad \text{and} \quad T_r = (-1)^r \binom{n+1}{r} H_r I + A_N \circ T_{r-1},$$

and the following items are straightforward.

Proposition 4. *(a) The transformations T_r $(0 \leq r \leq n+1)$ are self-adjoint on $\mathscr{S}(N)$ and commute with A_N.*

(b) $T_r E_i = (-1)^r S_r^i E_i$.

(c) $\operatorname{tr}(T_r) = (-1)^r (n+1-r) S_r$.

(d) $\operatorname{tr}(A_N \circ T_{r-1}) = (-1)^{r-1} r S_r$.

(e) $\operatorname{tr}(A_N^2 \circ T_{r-1}) = (-1)^r (S_1 S_r + (r+1) S_{r+1})$.

We prove the following.

Proposition 5. *For all $X \in \Gamma(TM)$,*

$$\begin{aligned} g(\operatorname{div}^\nabla(T_r), X) = {} & g\big(\operatorname{div}^\nabla(T_{r-1}), A_N X\big) - (-1)^r \big(g(X,N)\xi(S_r) - S_{r-1}^0 X(k_0)\big) \\ & + g((\nabla_\xi A_N) T_{r-1}\xi, X) \\ & + \sum_{i=1}^n \Big\{ \bar{g}(\bar{R}(E_i, X) T_{r-1} E_i, N) + \eta(X) B^N(E_i, A_N \circ T_{r-1} E_i) \\ & + g(A_N X, E_i) \tau^N(E_i) - k_i \tau^N(\xi) \\ & + E_i \big(\tau^N(T_{r-1}E_i)\eta(X) - \tau^N(X)\eta(T_{r-1}E_i)\big) \\ & - \tau^N(T_{r-1}E_i)\eta(\nabla_{E_i} X) + \tau^N(\nabla_{E_i} X)\eta(T_{r-1}E_i) \\ & - \tau^N(\nabla_{E_i} T_{r-1} E_i)\eta(X) + \tau^N(X)\eta(\nabla_{E_i} T_{r-1} E_i) \Big\}. \end{aligned} \tag{29}$$

Proof. Using iterative formula,

$$\begin{aligned} \operatorname{div}^\nabla(T_r) &= (-1)^r \operatorname{div}^\nabla(S_r I) + \operatorname{div}^\nabla(A_N \circ T_{r-1}) \\ &= \sum_{\alpha=0}^n \big((-1)^r (e_\alpha \cdot S_r) e_\alpha + (\nabla_{e_\alpha} A_N) T_{r-1} e_\alpha\big) + A_N(\operatorname{div}^\nabla(T_{r-1})). \end{aligned}$$

Hence, using (27),

$$\begin{aligned} g(\operatorname{div}^\nabla(T_r), X) = {} & (-1)^r PX(S_r) + g\big(\operatorname{div}^\nabla(T_{r-1}), A_N X\big) - \tau^N(X)\eta(\operatorname{div}^\nabla(T_{r-1})) \\ & + \tau^N(\operatorname{div}^\nabla(T_{r-1}))\eta(X) + \sum_{\alpha=0}^n g\big((\nabla_{e_\alpha} A_N) T_{r-1} e_\alpha, X\big). \end{aligned} \tag{30}$$

Also

$$\begin{aligned} g\big((\nabla_{E_i} A_N) T_{r-1} E_i, X\big) = {} & g\big(T_{r-1}E_i, (\nabla_{E_i} A_N) X\big) + (-1)^r \eta(X) k_i S_{r-1}^i B^N(E_i, E_i) \\ & + \tau^N(\nabla_{E_i} X)\eta(T_{r-1}E_i) - \tau^N(T_{r-1}E_i)\eta(\nabla_{E_i} X) \\ & + \tau^N(X)\eta(\nabla_{E_i} T_{r-1}E_i) - \tau^N(\nabla_{E_i} T_{r-1} E_i)\eta(X) \\ & + E_i\big(\tau^N(T_{r-1}E_i)\eta(X) - \tau^N(X)\eta(T_{r-1}E_i)\big). \end{aligned} \tag{31}$$

Apply the Gauss-Codazzi equation (7) with the substitutions

$$X \to E_i, \quad Y \to X, \quad Z \to T_{r-1} E_i,$$

to get

$$g\big(T_{r-1}E_i,(\nabla_{E_i}A_N)X\big)=\bar g\big(\bar R(E_i,X)T_{r-1}E_i,N\big)-k_i\tau^N(X)\\ +g\big((\nabla_X A_N)E_i,T_{r-1}E_i\big)+g(A_NX,E_i). \tag{32}$$

Also, we have

$$\sum_{i=1}^{n} g\big(T_{r-1}E_i,(\nabla_X A_N)E_i\big)=(-1)^{r-1}\sum_{i=1}^{n} S^i_{r-1}X(k_i)\\ =(-1)^{r-1}\big(X(S_r-S^0_{r-1}X(k_0)\big). \tag{33}$$

Now, feeding back (33) into (32) and then the resulting expression into (31) we obtain by substitution in (30) the desired expression (29). □

For the rest of the section we assume $\tau^N=0$ which is equivalent to saying that the starred entries in the matrix of A_N (see (28)) are zero, that is $A_N\xi=0$. Then

$$g(\operatorname{div}^{\nabla}(T_r),X)=\sum_{a=0}^{r-1}\sum_{i=1}^{n}\bar g\big(\bar R(E_i,N)T_aE_i,A_N^{r-1-a}X\big)\\ -\eta(X)\Big(\operatorname{tr}(\overset{\star}{A}_\xi\circ A_N\circ T_{r-1})+(n+1-r)^{-1}\xi\big(\operatorname{tr}(T_r)\big)\Big).$$

In particular when the ambient manifold is Lorentzian with constant sectional curvature c we have

$$g(\operatorname{div}^{\nabla}(T_r),X)=\eta(X)\big(c\operatorname{tr}(T_{r-1})+(-1)^r cS_{r-1}-\operatorname{tr}(\overset{\star}{A}_\xi\circ A_N\circ T_{r-1})-(-1)^r\xi(S_r)\big). \tag{34}$$

Now we state the following

Theorem 5. *Let (M^{n+1},g,N) be a closed normalization of a null hypersurface of a Lorentzian space form $(\bar M(c)^{n+2},\bar g)$ with rigged vector field ξ and $\tau^N=0$. Then, for all $r=0,\dots,n+1$, $\operatorname{div}^{\nabla}(T_r)$ is $TM^{\perp}$-valued and*

$$\xi(S_r)+c(n+1-r)S_{r-1}=(-1)^{r-1}\operatorname{tr}(\overset{\star}{A}_\xi\circ A_N\circ T_{r-1}). \tag{35}$$

Proof. Let $X\in\chi(M)$. We have

$$\begin{aligned}g(\operatorname{div}^{\nabla}(T_r),X)&=g(\operatorname{div}^{\nabla}(T_r),PX)\\ &\overset{(34)}{=}\eta(PX)\Big(c\operatorname{tr}(T_{r-1})+(-1)^r cS_{r-1}\\ &\qquad-\operatorname{tr}(\overset{\star}{A}_\xi\circ A_N\circ T_{r-1})-(-1)^r\xi(S_r)\Big)\\ &=0\end{aligned}$$

as $\eta(PX)=0$, which shows that $\operatorname{div}^{\nabla}(T_r)$ is $TM^{\perp}$-valued. It follows from the same equation (34) setting $X:=\xi$ and using the third item in Proposition 4 that (35) holds. □

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The gap theorems for some extremal submanifolds in a unit sphere

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Abstract. Let M be an n-dimensional submanifold in the unit sphere S^{n+p}, we call M a k-extremal submanifold if it is a critical point of the functional $\int_M \rho^{2k}\,\mathrm{d}v$. In this paper, we can study gap phenomenon for these submanifolds.

1 Introduction and theorems

Let $x\colon M^n \hookrightarrow S^{n+p}(1)$ be an n-dimensional compact submanifold in a unit sphere, and let

- $e_1, \dots, e_n$ be a local orthonormal frame of tangent vector field on M,
- $e_{n+1}, \dots, e_{n+p}$ be a local orthonormal frame of normal vector field on M,
- $\omega_1, \dots, \omega_n,\ \omega_{n+1}, \dots, \omega_{n+p}$ be its dual coframe field.

Then the second fundamental form and the mean curvature vector of M are

$$A = \sum_{i,j,\alpha} h^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha, \quad \mathbf{H} = \sum_\alpha H^\alpha e_\alpha = \frac{1}{n}\sum_{i,\alpha} h^\alpha_{ii} e_\alpha. \tag{1}$$

We can define trace-free linear maps $\phi_\alpha\colon T_qM \to T_qM$ by

$$\langle \phi^\alpha X, Y\rangle = \langle A^\alpha X, Y\rangle - \langle X, Y\rangle\langle \mathbf{H}, e_\alpha\rangle,$$

where $q \in M$, A^α is the shape operator of e_α,

$$A^\alpha(e_i) = -\sum_j \langle \overline{\nabla}_{e_i} e_\alpha, e_j\rangle e_j = \sum_j h^\alpha_{ij} e_j\,,$$

and we define a bilinear map $\phi\colon T_qM \times T_qM \to T_qM^\perp$ by

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$$\phi(X,Y) = \sum_{\alpha=n+1}^{n+p} \langle \phi^\alpha X, Y\rangle e_\alpha. \tag{2}$$

It's easy to check that $|\phi|^2 = |A|^2 - nH^2$, where $H^2 = |\mathbf{H}|^2 = \sum_\alpha (H^\alpha)^2$, and we denote $\rho = |\phi|$. For any fixed number k with $k \geq 1$, we can define the following functional

$$F_k(x) = \int_M \rho^{2k}\, \mathrm{d}v. \tag{3}$$

When $k = \frac{n}{2}$, it is the Willmore functional. We say $x\colon M \to S^{n+p}$ is a k-extremal submanifold if it is a critical point of the functional $F_k(x)$.

It seems very interesting to study the gap phenomenon for submanifolds, and there are some results about compact minimal submanifolds in $S^{n+p}(1)$, such as in [7]. For Willmore submanifolds, H. Li proved:

Theorem 1. *[6] Let M be an n-dimensional compact Willmore submanifold in S^{n+p}, then*

$$\int_M \left[n - \left(2 - \frac{1}{p}\right)\rho^2\right]\rho^n\, \mathrm{d}v \leq 0. \tag{4}$$

In particular, if $\rho^2 \leq \frac{n}{2-1/p}$, then either $\rho = 0$ and M is a totally umbilical submanifold, or $\rho^2 = \frac{n}{2-1/p}$. In the latter case, either $p = 1$ and M is a Willmore torus $W_{m,n-m} = S^m\big(\sqrt{\frac{n-m}{n}}\big) \times S^{n-m}\big(\sqrt{\frac{m}{n}}\big)$; or $n = 2$, $p = 2$ and M is the Veronese surface.

And for k-extremal submanifolds, Z. Guo and H. Li, the second author proved:

Theorem 2. *[1], [9] Let M be an n-dimensional compact k-extremal submanifold in S^{n+p}, $1 \leq k < \frac{n}{2}$, then*

$$\int_M \left[n - \left(2 - \frac{1}{p}\right)\rho^2\right]\rho^{2k}\, \mathrm{d}v \leq 0. \tag{5}$$

In particular, if $\rho^2 \leq \frac{n}{2-1/p}$, then either $\rho = 0$ and M is a totally umbilical submanifold, or $\rho^2 = \frac{n}{2-1/p}$. In the latter case, either $p = 1, n = 2m$ and M is a Clifford torus $C_{m,m} = S^m\left(\sqrt{\frac{1}{2}}\right) \times S^m\left(\sqrt{\frac{1}{2}}\right)$; or $n = 2$, $p = 2$ and M is the Veronese surface.

In 2011, H. Xu and D. Yang proved the following pinching theorem for submanifold which is a critical point of the functional $F_1(x)$.

Theorem 3. *[8] Let M be an n-dimensional compact 1-extremal submanifold in S^{n+p}, then there exists an explicit positive constant A_n depending only on n such that if*

$$\left(\int_M \rho^n\, \mathrm{d}v\right)^{\frac{2}{n}} < A_n, \tag{6}$$

$$A_n = \begin{cases} \min\Big\{\dfrac{n(n-2)^2}{4n(n-1)^2+(n-2)^2}, \\ \qquad \dfrac{(n-2)^2(\frac{n}{2}-n)}{4(\frac{n}{2}-n)(n-1)^2+(n-2)^2}\Big\}C(n)^{-2} \quad (p=1); \\ \dfrac{2}{3}\min\Big\{\dfrac{n(n-2)^2}{4n(n-1)^2+(n-2)^2}, \\ \qquad \dfrac{(n-2)^2(\frac{n}{2}-n)}{4(\frac{n}{2}-n)(n-1)^2+(n-2)^2}\Big\}C(n)^{-2} \quad (p\geq 2), \end{cases}$$

then M is a totally umbilical submanifold, where $C(n)$ is a positive constant depending on n which satisfies:

$$\Big(\int_M f^{\frac{n}{n-1}}\,\mathrm{d}v\Big)^{\frac{n-1}{n}} \leq C(n)\int_M \big(|\nabla f| + (1+H^2)f\big)\,\mathrm{d}v \tag{7}$$

holds for any $f \in C^1(M)$.

In this paper, we prove the following theorems for the k-extremal submanifold when $1 \leq k < \frac{n}{2}$:

Theorem 4. *Let M be an n-dimensional compact k-extremal submanifold in S^{n+p} ($n \geq 3$), $1 \leq k < \frac{n}{2}$, then there exists an explicit positive constant $A_{n,k}$ depending only on n and k such that if*

$$\Big(\int_M \rho^n\,\mathrm{d}v\Big)^{\frac{2}{n}} < A_{n,k}, \tag{8}$$

where

$$A_{n,k} = \begin{cases} C(n)^{-2}\min\Big\{\dfrac{n(n-2)^2(2k-1)}{4n(n-1)^2k^2+(2k-1)(n-2)^2}, \\ \qquad \dfrac{(2k-1)(n-2)^2(\frac{n^2}{2k}-n)}{4(\frac{n^2}{2k}-n)(n-1)^2k^2+(2k-1)(n-2)^2}\Big\} \quad (p=1); \\ \dfrac{2}{3}C(n)^{-2}\min\Big\{\dfrac{n(n-2)^2(2k-1)}{4n(n-1)^2k^2+(2k-1)(n-2)^2}, \\ \qquad \dfrac{(2k-1)(n-2)^2(\frac{n^2}{2k}-n)}{4(\frac{n^2}{2k}-n)(n-1)^2k^2+(2k-1)(n-2)^2}\Big\} \quad (p\geq 2), \end{cases}$$

then M is a totally umbilical submanifold, where $C(n)$ is the same constant as above.

Theorem 5. *Let M be an n-dimensional ($n \geq 3$) compact k-extremal submanifold with flat normal bundle in S^{n+p}, $1 \leq k < \frac{n}{2}$. If $\rho^2 \leq n$, then either $\rho = 0$ and M is a totally umbilical submanifold, or $p = 1, n = 2m$ and M is a Clifford torus $C_{m,m} = S^m\Big(\sqrt{\frac{1}{2}}\Big) \times S^m\Big(\sqrt{\frac{1}{2}}\Big)$.*

Remark 1. If $k = \frac{n}{2}$, then $A_{n,k} = 0$, so our Theorem 4 is trivial when $k = \frac{n}{2}$. If $k = 1$, $A_{n,1} = A_n$, our Theorem 4 reduces to Xu-Yang's Theorem 3.

2 Preliminaries and lemmas

We shall make use of the following convention on the range of indices:

$$1 \le A, B, C \le n+p, \quad 1 \le i, j, k \le n, \quad n+1 \le \alpha, \beta, \gamma \le n+p.$$

We choose a local orthonormal frame field $\{e_1, \dots, e_n, e_{n+1}, \dots e_{n+p}\}$ along M, with $\{e_i\}_{i=1,2,\dots,n}$ tangent to M and $\{e_\alpha\}_{\alpha=n+1,n+2,\dots,n+p}$ normal to M. Let $\{\omega_A\}$ be the corresponding dual coframe, and $\{\omega_{AB}\}$ be the connection 1-form on S^{n+p}. Restricted on M, the curvature tensor, the normal curvature tensor can be given by

$$\mathrm{d}\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{9}$$

$$\mathrm{d}\omega_{\alpha\beta} - \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} = -\frac{1}{2} \sum_{k,l} R^\perp_{\alpha\beta kl} \omega_k \wedge \omega_l. \tag{10}$$

and the mean curvature $\mathbf{H} = \sum_\alpha H^\alpha e_\alpha$, where $H^\alpha = \frac{1}{n} \sum_i h^\alpha_{ii}$.

The covariant derivative of the second fundamental form is given by

$$\sum_k h^\alpha_{ij,k} \omega_k = \mathrm{d}h^\alpha_{ij} + \sum_k h^\alpha_{ki} \omega_{kj} + \sum_k h^\alpha_{kj} \omega_{ki} + \sum_\beta h^\beta_{ij} \omega_{\beta\alpha}, \tag{11}$$

$$\sum_l h^\alpha_{ij,kl} \omega_l = \mathrm{d}h^\alpha_{ij,k} + \sum_l h^\alpha_{lj,k} \omega_{li} + \sum_l h^\alpha_{ij,l} \omega_{lk} + \sum_l h^\alpha_{il,k} \omega_{lj} + \sum_\beta h^\beta_{ij,k} \omega_{\beta\alpha}. \tag{12}$$

In [9], the second author calculated the Euler-Lagrangian equation of $F_k(x)$:

Lemma 1. *[9] If $x\colon M \to R^{n+p}(c)$ be an n-dimensional submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. Then for $k \ge 1$, M is an extremal submanifold of $F_k(x)$ if and only if for $n+1 \le \alpha \le n+p$,*

$$\begin{aligned} 0 = &-\Delta(\rho^{2k-2}) H^\alpha + 2(n-1) \sum_i (\rho^{2k-2})_{,i} H^\alpha_{,i} \\ &+ \sum_{i,j} (\rho^{2k-2})_{,ij} h^\alpha_{ij} + (n-1)\rho^{2k-2} \Delta^\perp H^\alpha \\ &+ \rho^{2k-2} \Bigg[\sum_{i,j,k,\beta} h^\alpha_{ij} h^\beta_{jk} h^\beta_{ki} - \sum_{i,j,\beta} H^\beta h^\alpha_{ij} h^\beta_{ij} - \frac{n}{2k} \rho^2 H^\alpha \Bigg]. \end{aligned} \tag{13}$$

Using the above lemma, we can get that:

Lemma 2. *If M is an extremal submanifold of $F_k(x)$, then*

$$\begin{aligned} &\int_M \rho^{2k-2} \Bigg(\Delta H^2 - 2 \sum_{i,j,\alpha} h^\alpha_{ij} H^\alpha_{,ij} \Bigg) \mathrm{d}v \\ &\qquad = 2 \int_M \rho^{2k-2} |\nabla^\perp \mathbf{H}|^2 \, \mathrm{d}v + 2 \int_M \rho^{2k-2} F \, \mathrm{d}v, \end{aligned} \tag{14}$$

where $\nabla^\perp$ is the normal connection on M, and

$$F := \sum_{i,j,k,\alpha,\beta} H^\alpha h^\alpha_{ij} h^\beta_{jk} h^\beta_{ji} - \sum_{j,k,\alpha,\beta} H^\alpha H^\beta h^\alpha_{jk} h^\beta_{jk} - \frac{n}{2k} \rho^2 H^2 .$$

Proof. Multiplying the equation (13) by H^α and integrating over M we obtain

$$\begin{aligned}0 = &-\int_M \Delta(\rho^{2k-2})H^2\,\mathrm{d}v + 2(n-1)\int_M \sum_{i,\alpha}(\rho^{2k-2})_{,i}H^\alpha_{,i}H^\alpha\,\mathrm{d}v\\ &+\int_M \sum_{i,j,\alpha}(\rho^{2k-2})_{,ij}h^\alpha_{ij}H^\alpha\,\mathrm{d}v + (n-1)\int_M\sum_{i,\alpha}\rho^{2k-2}H^\alpha_{,ii}H^\alpha\,\mathrm{d}v\\ &+\int_M \rho^{2k-2}F\,\mathrm{d}v,\end{aligned}\tag{15}$$

and integrating by parts, we can get

$$\int_M\sum_{i,\alpha}(\rho^{2k-2})_{,i}H^\alpha_{,i}H^\alpha\,\mathrm{d}v = -\int_M\sum_i\rho^{2k-2}_{,ii}H^2\,\mathrm{d}v - \int_M\sum_{i,\alpha}\rho^{2k-2}_{,i}H^\alpha_{,i}H^\alpha\,\mathrm{d}v,$$

so

$$2\int_M\sum_{i,\alpha}(\rho^{2k-2})_{,i}H^\alpha_{,i}H^\alpha\,\mathrm{d}v = -\int_M\Delta\rho^{2k-2}H^2\,\mathrm{d}v = -\int_M\rho^{2k-2}\Delta H^2\,\mathrm{d}v.\tag{16}$$

Thus we have the following calculations:

$$\begin{aligned}\int_M\sum_{i,j,\alpha}(\rho^{2k-2})_{,ij}h^\alpha_{ij}H^\alpha\,\mathrm{d}v &= -\int_M\sum_{i,j,\alpha}(\rho^{2k-2})_{,i}h^\alpha_{ij,j}H^\alpha\,\mathrm{d}v - \int_M\sum_{i,j,\alpha}(\rho^{2k-2})_{,i}h^\alpha_{ij}H^\alpha_{,j}\,\mathrm{d}v\\ &= -n\int_M\sum_{i,\alpha}(\rho^{2k-2})_{,i}H^\alpha_{,i}H^\alpha\,\mathrm{d}v + \int_M\sum_{i,j,\alpha}\rho^{2k-2}h^\alpha_{ij,i}H^\alpha_{,j}\,\mathrm{d}v\\ &\quad+\int_M\sum_{i,j,\alpha}\rho^{2k-2}h^\alpha_{ij}H^\alpha_{,ji}\,\mathrm{d}v\\ &= \frac{n}{2}\int_M\rho^{2k-2}\Delta H^2\,\mathrm{d}v + n\int_M\rho^{2k-2}|\nabla^\perp\mathbf{H}|^2\,\mathrm{d}v\\ &\quad+\int_M\sum_{i,j,\alpha}\rho^{2k-2}h^\alpha_{ij}H^\alpha_{,ij}\,\mathrm{d}v,\end{aligned}\tag{17}$$

$$\int_M\sum_{i,\alpha}\rho^{2k-2}H^\alpha_{,ii}H^\alpha\,\mathrm{d}v = \frac{1}{2}\int_M\rho^{2k-2}\Delta H^2\,\mathrm{d}v - \int_M\rho^{2k-2}|\nabla^\perp\mathbf{H}|^2\,\mathrm{d}v.\tag{18}$$

Then (15) becomes

$$\begin{aligned}0 = &-\frac{1}{2}\int_M\rho^{2k-2}\Delta H^2\,\mathrm{d}v + \int_M\rho^{2k-2}|\nabla^\perp\mathbf{H}|^2\,\mathrm{d}v\\ &+\int_M\sum_{i,j,\alpha}\rho^{2k-2}h^\alpha_{ij}H^\alpha_{,ij}\,\mathrm{d}v + \int_M\rho^{2k-2}F\,\mathrm{d}v,\end{aligned}\tag{19}$$

so (14) holds. □

We also need the following inequalities:

Lemma 3. *[8] Let M be an n-dimensional ($n \geq 3$) compact submanifold in the unit sphere S^{n+p}. Then for any $f \in C^1(M)$, $f \geq 0$, $t > 0$, f satisfies the following inequality*

$$\int_M |\nabla f|^2 \,\mathrm{d}v \geq c_1(n,t)\Big(\int_M f^{\frac{2n}{n-2}} \,\mathrm{d}v\Big)^{\frac{n-2}{n}} - c_2(n,t)\int_M (1+H^2)f^2 \,\mathrm{d}v, \tag{20}$$

where $c_1(n,t) = \frac{(n-2)^2}{4C(n)^2(1+t)(n-1)^2}$, $c_2(n,t) = \frac{(n-2)^2}{4t(n-1)^2}$.

Lemma 4. *[4] Let $B^1, B^2, \ldots, B^m$ be symmetric ($n \times n$)-matrices, Set $S_{\alpha\beta} = \operatorname{tr}(B^\alpha B^\beta)$, $S_\alpha = S_{\alpha\alpha}$, $S = \sum_\alpha S_\alpha$, then*

$$\sum_{\alpha,\beta} |B^\alpha B^\beta - B^\beta B^\alpha|^2 + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leq \frac{3}{2}\Big(\sum_\alpha S_\alpha\Big)^2, \tag{21}$$

where $|B|^2 = \operatorname{tr} B^t B$.

3 Proof of the theorems

We also need a Simons' type formula, which can be found in [6]:

Lemma 5. *If $x\colon M \to S^{n+m}$ be an n-dimensional submanifold, then*

$$\begin{aligned}\frac{1}{2}\Delta\rho^2 = |\nabla A|^2 - n^2|\nabla^\perp \mathbf{H}|^2 + \sum_{i,j,k,\alpha}(h_{ij}^\alpha h_{kk,i}^\alpha)_{,j} \\ + n\sum_{\alpha,\beta,i,j,k} H^\beta \phi_{ij}^\beta \phi_{jk}^\alpha \phi_{ki}^\alpha + n\rho^2 + n^2H^2\rho^2 \\ - \sum_{\alpha,\beta}\sigma_{\alpha\beta}^2 - \sum_{\alpha,\beta,i,j}(R_{\alpha\beta ij}^\perp)^2 - \frac{1}{2}\Delta(nH^2),\end{aligned} \tag{22}$$

where ϕ is the trace-free tensor which defined above, $\sigma_{\alpha\beta} = \sum_{i,j}\phi_{ij}^\alpha\phi_{ij}^\beta$.

From

$$0 = \int_M \Delta\rho^{2k} \,\mathrm{d}v = 2\int_M \Delta\rho^2\rho^{2k-2} \,\mathrm{d}v + 2\int_M \langle\nabla\rho^2, \nabla\rho^{2k-2}\rangle \,\mathrm{d}v, \tag{23}$$

and (22), we get that

$$\begin{aligned}\frac{1}{2}\int_M \Delta\rho^2\rho^{2k-2} \,\mathrm{d}v = \int_M |\nabla A|^2\rho^{2k-2} \,\mathrm{d}v + n\int_M (\sum_{\alpha,i,j} h_{ij}^\alpha H_{,ij}^\alpha - \frac{1}{2}\Delta H^2)\rho^{2k-2} \,\mathrm{d}v \\ + \int_M E\rho^{2k-2} \,\mathrm{d}v,\end{aligned} \tag{24}$$

where

$$E := n\sum_{\alpha,\beta,i,j,k} H^\beta\phi_{ij}^\beta\phi_{jk}^\alpha\phi_{ki}^\alpha + n\rho^2 + n^2H^2\rho^2 - \sum_{\alpha,\beta}\sigma_{\alpha\beta}^2 - \sum_{\alpha,\beta,i,j}(R_{\alpha\beta ij}^\perp)^2\,.$$

Using (14) and (23),

$$\begin{aligned}0 = &\int_M (|\nabla A|^2 - n|\nabla^\perp \mathbf{H}|^2)\rho^{2k-2}\,\mathrm{d}v \\ &+ \int_M (E - nF)\rho^{2k-2}\,\mathrm{d}v + (2k-2)\int_M |\nabla\rho|^2\rho^{2k-2}\,\mathrm{d}v,\end{aligned} \tag{25}$$

from Lemma 2.1 in [8] we know that

$$|\nabla A|^2 - n|\nabla^\perp \mathbf{H}|^2 = \sum_{\alpha,i,j,k} (\phi^\alpha_{ij,k})^2 \geq |\nabla\rho|^2. \tag{26}$$

By a direct computation, we have that

$$E - nF = n\rho^2 + \frac{n^2}{2k}\rho^2 H^2 - n\sum_{\alpha,\beta,i,j} H^\alpha H^\beta \phi^\alpha_{ij}\phi^\beta_{ij} - \sum_{\alpha,\beta}\sigma^2_{\alpha\beta} - \sum_{\alpha,\beta,i,j}(R^\perp_{\alpha\beta ij})^2, \tag{27}$$

for

$$\sum_{\alpha,\beta,i,j} H^\alpha H^\beta \phi^\alpha_{ij}\phi^\beta_{ij} = \sum_{i,j}\Big(\sum_\alpha H^\alpha\phi^\alpha_{ij}\Big)^2 \leq \bigg(\sum_{i,j}\Big(\sum_\alpha \phi^\alpha_{ij}\Big)^2\bigg)\bigg(\Big(\sum_\alpha H^\alpha\Big)^2\bigg) = \rho^2 H^2, \tag{28}$$

then

$$\begin{aligned}0 \geq &\frac{2k-1}{k^2}\int_M |\nabla\rho^k|^2\,\mathrm{d}v \\ &+ \int_M \bigg[n\rho^2 + \Big(\frac{n^2}{2k} - n\Big)H^2\rho^2 - \sum_{\alpha,\beta}\sigma^2_{\alpha\beta} - \sum_{\alpha,\beta,i,j}\big(R^\perp_{\alpha\beta ij}\big)^2\bigg]\rho^{2k-2}\,\mathrm{d}v.\end{aligned} \tag{29}$$

Proof. (Theorem 4) From Lemma 4,

$$E - nF \geq n\rho^2 + (\frac{n^2}{2k} - n)\rho^2 H^2 - \eta\rho^4, \tag{30}$$

where $\eta = \min(\frac{3}{2}, 2 - \frac{1}{p})$.

From (25), (26) and (30), we know that the following inequality holds,

$$\frac{2k-1}{k^2}\int_M |\nabla\rho^k|^2\,\mathrm{d}v + \int_M \bigg[n + \Big(\frac{n^2}{2k} - n\Big)H^2 - \eta\rho^2\bigg]\rho^{2k}\,\mathrm{d}v \leq 0, \tag{31}$$

and with Lemma 3 and (31), we can get:

$$\begin{aligned}0 \geq &\frac{2k-1}{k^2}c_1(n,t)\Big(\int_M \rho^{\frac{2n}{n-2}k}\,\mathrm{d}v\Big)^{\frac{n-2}{n}} + \Big(n - \frac{2k-1}{k^2}c_2(n,t)\Big)\Big(\int_M \rho^{2k}\,\mathrm{d}v\Big) \\ &+ \Big(\frac{n^2}{2k} - n - \frac{2k-1}{k^2}c_2(n,t)\Big)\Big(\int_M H^2\rho^{2k}\,\mathrm{d}v\Big) - \eta\int_M \rho^{2k+2}\,\mathrm{d}v.\end{aligned} \tag{32}$$

Using the Hölder's inequality, we have

$$
\begin{aligned}
0 \geq & \left[\frac{2k-1}{k^2} c_1(n,t) - \eta\Big(\int_M \rho^n \,\mathrm{d}v\Big)^{\frac{2}{n}}\right]\Big(\int_M \rho^{\frac{2n}{n-2}k} \,\mathrm{d}v\Big)^{\frac{n-2}{n}} \\
& + \left(n - \frac{2k-1}{k^2} c_2(n,t)\right)\Big(\int_M \rho^{2k} \,\mathrm{d}v\Big) \\
& + \left[\frac{n^2}{2k} - n - \frac{2k-1}{k^2} c_2(n,t)\right]\Big(\int_M H^2 \rho^{2k} \,\mathrm{d}v\Big),
\end{aligned}
$$

let $t = \frac{(n-2)^2(2k-1)}{4(n-1)^2 k^2} \max\left(\frac{2k}{n^2-2kn}, \frac{1}{n}\right)$, then Theorem 4 follows. □

Proof. (Theorem 5) If M has normal flat bundle, then (29) become

$$
\begin{aligned}
0 \geq & \frac{2k-1}{k^2} \int_M |\nabla \rho^k|^2 \,\mathrm{d}v \\
& + \int_M \left[n\rho^2 + \Big(\frac{n^2}{2k} - n\Big) H^2 \rho^2 - \sum_{\alpha,\beta} \sigma_{\alpha\beta}^2\right] \rho^{2k-2} \,\mathrm{d}v \\
\geq & \int_M \left[n\rho^2 + \Big(\frac{n^2}{2k} - n\Big) H^2 \rho^2 - \rho^4\right] \rho^{2k-2} \,\mathrm{d}v \\
\geq & \int_M (n - \rho^2) \rho^{2k} \,\mathrm{d}v. \qquad (33)
\end{aligned}
$$

So if $\rho \leq n$, then either $\rho = 0$ and M is a totally umbilical submanifold, or $\rho^2 = n$, for $k < \frac{n}{2}$, from (33), we know that $H = 0$, with the Theorem 3 in [3], we know that M lies in a $(n+1)$-dimensional unit sphere, so the Theorem 5 follows from the Theorem 2. □

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