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Structure equations on generalized Finsler manifolds

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Abstract. In this paper we generalize the classical structure equations of Riemannian geometry to generalized Finsler manifolds.

1 Introduction

In this paper we deduce structure equations on a manifold which is endowed with a generalized Finsler metric and an Ehresmann connection. In Riemannian geometry, the classical structure equations were adopted by Élie Cartan. However Cartan's formalism was hard to understand for the next generations. In the pull-back formalism of Finsler geometry used by us, it causes a problem that in Grassmann algebra of forms along projection $\tau \colon TM \to M$ we do not have the classical exterior derivative. The vertical and horizontal derivatives, which substitute for exterior derivative, were introduced in 1992 ([8], [14]), and these help us to generalize the structure equations. By using the index-free calculus, it turns out that out of the five partial torsions introduced by Makoto Matsumoto in Finsler geometry only two ones have 'real' torsion property ([7] Chapter II.10, Lemma 1).

2 Preliminaries

We follow the notation and conventions of [14] and [6] as far as feasible. However, for the readers' convenience, in this section we fix some terminology and recall some basic facts.

Throughout this paper, we use the Einstein summation convention. 'Manifold' will always mean a connected, second countable, Hausdorff, smooth manifold of dimension n, $n \geq 1$. If M is a manifold, $C^\infty(M)$ will denote the ring of smooth functions on M. The tangent bundle of M is $\tau \colon TM \to M$, while $\mathring{\tau} \colon \mathring{T}M \to M$ denotes the slit tangent bundle, where $\mathring{T}M$ stands for the set of nonzero tangent vectors to M.

The vertical lift of a function $f \in C^\infty(M)$ is $f^{\mathsf{v}} := f \circ \tau$, the complete lift $f^{\mathsf{c}} \in C^\infty(TM)$ of f is defined by $f^{\mathsf{c}}(v) := v(f)$, $v \in TM$.

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$\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of smooth vector fields on M. Any vector field X on M gives rise canonically two vector fields on TM, the vertical lift X^v of X and the complete lift X^c of X, determined by $X^\mathsf{v} f^\mathsf{c} = (Xf)^\mathsf{v}$, $X^\mathsf{v} f^\mathsf{v} = 0$ and $X^\mathsf{c} f^\mathsf{c} = (Xf)^\mathsf{c}$, $X^\mathsf{c} f^\mathsf{v} = (Xf)^\mathsf{v}$; $f \in C^\infty(M)$.

Let $\mathcal{A}^k(M)$ be C^∞-module of k-forms on M. Then $\mathcal{A}(M) := \bigoplus_{k=0}^n \mathcal{A}^k(M)$ is a graded algebra over $C^\infty(M)$, with multiplication given by the wedge product $\wedge$. If $f \in C^\infty(M)$ then the one-form $\mathrm{d}f$ given by $\mathrm{d}f(X) = Xf$ $(X \in \mathfrak{X}(M))$ is the *differential* of f.

Let $\tau^*TM := TM \times_M TM := \{(u,v) \in TM \times TM \mid \tau(u) = \tau(v)\}$, and let $\tau^*\tau(u,v) := u$ for $(u,v) \in \tau^*TM$. Then $\tau^*\tau$ is a vector bundle with total space τ^*TM and base space TM, the *pull-back* of $\tau\colon TM \to M$ over τ. The $C^\infty(TM)$-module of sections of $\tau^*\tau$ will be denoted by $\mathrm{Sec}(\tau^*\tau)$. Any vector field X on M determines a smooth section

$$\widehat{X}\colon v \in TM \longmapsto \big(v, X \circ \tau(v)\big) \in TM \times_M TM\,,$$

called the *basic section* associated to X. The $C^\infty(TM)$-module $\mathrm{Sec}(\tau^*\tau)$ is generated by the basic sections. Generic sections in $\mathrm{Sec}(\tau^*\tau)$ will be denoted by $\widetilde{X}, \widetilde{Y}, \dots$

The dual of $\mathrm{Sec}(\tau^*\tau)$ will be denoted by $\mathcal{A}^1(\tau^*\tau)$, and its elements is called *one-forms along* τ. $\mathcal{A}(\tau^*\tau)$ is the Grassmann algebra of differential forms along τ.

Starting from the slit tangent bundle $\mathring{\tau}\colon \mathring{T}M \to M$, the pull-back bundle $\mathring{\tau}^*\tau\colon \mathring{T}M \times_M TM \to TM$ is constructed in the same way. Omitting the routine details, we remark that $\mathrm{Sec}(\tau^*\tau)$ may naturally be embedded into the $C^\infty(\mathring{T}M)$-module $\mathrm{Sec}(\mathring{\tau}^*\tau)$.

There exists a canonical injective bundle map $\mathbf{i}\colon TM \times_M TM \to TTM$ given by

$$\mathbf{i}(u,v) := \dot{c}(0)\,, \qquad \text{if} \quad c(t) := u + tv \quad (t \in \mathbb{R})\,,$$

and a canonical surjective bundle map

$$\begin{gathered}\mathbf{j}\colon TTM \to TM \times_M TM\,,\\ w \in T_vTM \longmapsto \mathbf{j}(w) := (v, \tau_*(w)) \in \{v\} \times T_{\tau(v)}M\,.\end{gathered}$$

Then $\mathbf{j} \circ \mathbf{i} = 0$. However, while $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ is a further important canonical object, the *vertical endomorphism* of TTM. The bundle maps $\mathbf{i}$ and $\mathbf{j}$ induce the tensorial maps (denoted by the same symbols)

$$\begin{gathered}\widetilde{X} \in \mathrm{Sec}(\tau^*\tau) \longmapsto \mathbf{i}\widetilde{X} := \mathbf{i} \circ \widetilde{X} \in \mathfrak{X}(TM) \qquad \text{and}\\ \xi \in \mathfrak{X}(TM) \longmapsto \mathbf{j}\xi := \mathbf{j} \circ \xi \in \mathrm{Sec}(\tau^*\tau)\,,\end{gathered}$$

so $\mathbf{J}$ may also be interpreted as a $C^\infty(TM)$-linear endomorphism of $\mathfrak{X}(TM)$. $\mathfrak{X}^\mathsf{v}(TM) := \mathbf{i}\mathrm{Sec}(\tau^*\tau)$ is the module of *vertical vector fields* on TM. The vertical vector fields form a subalgebra of the Lie algebra $\mathfrak{X}(TM)$ at the same time. For any vector field X on M we have $\mathbf{i}\widehat{X} = X^\mathsf{v}$ and $\mathbf{j}X^\mathsf{c} = \widehat{X}$.

An *Ehresmann connection* $\mathcal{H}$ over a manifold M is a right splitting of the canonical exact sequence

$$0 \longrightarrow TM \times_M TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} TM \times_M TM \longrightarrow 0\,,$$

which is smooth only on $\mathring{T}M \times_M TM$, and given on $o(M) \times_M TM$ by $\mathcal{H}\big(o(p), v\big) := (o_*)_p(v)$; $p \in M$, $v \in T_pM$, where $o \in \mathfrak{X}(M)$ is the zero vector field. We associate to any Ehresmann connection $\mathcal{H}$ the *horizontal projector* $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$, the *vertical projector* $\mathbf{v} = 1_{TTM} - \mathbf{h}$ and the *vertical map* $\mathcal{V} := \mathbf{i}^{-1} \circ \mathbf{v}$. The *horizontal lift* of a vector field $X \in \mathfrak{X}(M)$ with respect to $\mathcal{H}$ is $X^{\mathsf{h}} := \mathcal{H}(\widehat{X}) = \mathbf{h}X^{\mathsf{c}} \in \mathfrak{X}(\mathring{T}M)$.

The map $\ell^{\mathsf{h}}\colon X \in \mathfrak{X}(M) \longmapsto \ell^{\mathsf{h}}(X) := X^{\mathsf{h}}$ is said to be the horizontal lifting with respect to $\mathcal{H}$.

An Ehresmann connection $\mathcal{H}$ determines a covariant derivative operator ∇ in the pull-back bundle $\tau^*\tau$ by the rule

$$\nabla_\xi \widetilde{Y} := \mathbf{j}[\mathbf{v}\xi, \mathcal{H}\widetilde{Y}] + \mathcal{V}[\mathbf{h}\xi, \mathbf{i}\widetilde{Y}]\,; \qquad \xi \in \mathfrak{X}(TM), \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau)\,.$$

∇ is said to be the *Berwald derivative* induced by $\mathcal{H}$. Its *v-part* ∇^{v} and *h-part* ∇^{h} are defined by

$$\nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y} := \nabla_{\mathbf{i}\widetilde{X}} \widetilde{Y} = \mathbf{j}[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}]$$

and

$$\nabla^{\mathsf{h}}_{\widetilde{X}} \widetilde{Y} := \nabla_{\mathcal{H}\widetilde{X}} \widetilde{Y} = \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}]$$

($\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau)$). If X and Y are vector fields on M, then $\nabla^{\mathsf{v}}_{\widehat{X}} \widehat{Y} = 0$ and $\mathbf{i}\nabla^{\mathsf{h}}_{\widehat{X}} \widehat{Y} = \big[X^{\mathsf{h}}, Y^{\mathsf{v}}\big]$.

The importance of the Berwald derivative lies, among others, in the fact that the basic geometric data (torsions, curvature, etc.) of an Ehresmann connection $\mathcal{H}$ may conveniently be defined in terms of the Berwald derivative induced by $\mathcal{H}$. In this paper we need the following ($\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau)$):

$$\mathbf{T}(\widetilde{X}, \widetilde{Y}) := \nabla^{\mathsf{h}}_{\widetilde{X}} \widetilde{Y} - \nabla^{\mathsf{h}}_{\widetilde{Y}} \widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}] \quad - \quad \text{the } \textit{torsion} \text{ of } \mathcal{H},$$

$$\mathbf{R}(\widetilde{X}, \widetilde{Y}) := -\mathcal{V}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}] \quad - \quad \text{the } \textit{curvature} \text{ of } \mathcal{H}\,.$$

3 Generalized Finsler manifolds and torsions of a Finsler connection

As in general, by *covariant derivative operator* in the vector bundle $\tau^*\tau$ we mean an $\mathbb{R}$-bilinear map

$$D\colon (\xi, \widetilde{X}) \in \mathfrak{X}(TM) \times \mathrm{Sec}(\tau^*\tau) \longmapsto D_\xi \widetilde{X} \in \mathrm{Sec}(\tau^*\tau)$$

which is tensorial in its first variable and derivation in its second variable.

The *curvature of* D is the

$$R^D(\xi, \eta)\widetilde{X} := D_\xi D_\eta \widetilde{X} - D_\eta D_\xi \widetilde{X} - D_{[\xi,\eta]} \widetilde{X}$$

$C^\infty(TM)$-trilinear map.

A *pseudo-Riemannian metric* on $\tau^*\tau$ is a mapping g that sends a non-degenerate symmetric bilinear form

$$g_v\colon \big(\{v\}\times T_{\tau(v)}M\big)\times\big(\{v\}\times T_{\tau(v)}M\big)\longrightarrow\mathbb{R}$$

(or simply $g_v\colon T_{\tau(v)}M\times T_{\tau(v)}M\to\mathbb{R}$) to every vector $v\in\mathring{T}M$ such that the function

$$g(\widetilde{X},\widetilde{Y})\colon\mathring{T}M\to\mathbb{R},\quad v\longmapsto g(\widetilde{X},\widetilde{Y})(v):=g_v\big(\widetilde{X}(v),\widetilde{Y}(v)\big)$$

is smooth for any two sections $\widetilde{X},\widetilde{Y}\in\mathrm{Sec}(\mathring{\tau}^*\tau)$.

The pair (M,g) is said to be a *generalized Finsler manifold*, if g is a pseudo-Riemannian metric in $\tau^*\tau$. Then we also say that g is a *generalized metric*.

A covariant derivative operator $D\colon\mathfrak{X}(TM)\times\mathrm{Sec}(\tau^*\tau)\to\mathrm{Sec}(\tau^*\tau)$ in (M,g) is said to be *metric* if

$$D_\xi g(\widetilde{X},\widetilde{Y})=\xi g(\widetilde{X},\widetilde{Y})-g(D_\xi\widetilde{X},\widetilde{Y})-g(\widetilde{X},D_\xi\widetilde{Y})=0\,.$$

Let $\mathcal{H}$ be an Ehresmann connection over M and let D be a covariant derivative operator in $\tau^*\tau$. Then the pair $(D,\mathcal{H})$ is called a *Finsler connection*. By the *torsion* of D we mean the map

$$T^D(\xi,\eta):=D_\xi\mathbf{j}\eta-D_\eta\mathbf{j}\xi-\mathbf{j}[\xi,\eta]\,,\qquad(\xi,\eta\in\mathfrak{X}(TM))\,.$$

By the $\mathcal{V}$-*torsion* of D we mean the map

$$T^D_{\mathcal{V}}(\xi,\eta):=D_\xi\mathcal{V}\eta-D_\eta\mathcal{V}\xi-\mathcal{V}[\xi,\eta]\,,\qquad(\xi,\eta\in\mathfrak{X}(TM))\,.$$

It is easy to see that T^D and $T^D_{\mathcal{V}}$ are tensor fields.

We define the following five 'partial torsions' which are introduced by M. Matsumoto ([7] Chapter II.10):

$$\begin{aligned}
\mathcal{T}(\widetilde{X},\widetilde{Y})&:=T^D(\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}) &&\text{h-horizontal torsion,}\\
\mathcal{S}(\widetilde{X},\widetilde{Y})&:=T^D(\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y}) &&\text{h-mixed torsion/Finsler torsion,}\\
\mathbf{R}^1(\widetilde{X},\widetilde{Y})&:=T^D_{\mathcal{V}}(\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y}) &&\text{v-horizontal torsion,}\\
\mathbf{P}^1(\widetilde{X},\widetilde{Y})&:=T^D_{\mathcal{V}}(\mathcal{H}\widetilde{X},\mathbf{i}\widetilde{Y}) &&\text{v-mixed torsion,}\\
\mathbf{Q}^1(\widetilde{X},\widetilde{Y})&:=T^D_{\mathcal{V}}(\mathbf{i}\widetilde{X},\mathbf{i}\widetilde{Y}) &&\text{v-vertical torsion;}
\end{aligned}$$

$(\widetilde{X},\widetilde{Y}\in\mathrm{Sec}(\tau^*\tau))$.

The following formulae can be obtained by a straightforward calculation.

Lemma 1. *Let $(D, \mathcal{H})$ be a Finsler connection over M and let ∇ be the Berwald derivative induced by $\mathcal{H}$. Then for every $\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau)$*

$$\begin{aligned}
\mathcal{T}(\widetilde{X}, \widetilde{Y}) &= D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - D_{\mathcal{H}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]\,,\\
\mathcal{S}(\widetilde{X}, \widetilde{Y}) &= \nabla_{\mathbf{i}\widetilde{Y}}\widetilde{X} - D_{\mathbf{i}\widetilde{Y}}\widetilde{X}\,,\\
\mathbf{R}^1(\widetilde{X}, \widetilde{Y}) &= \mathbf{R}(\widetilde{X}, \widetilde{Y})\,,\\
\mathbf{P}^1(\widetilde{X}, \widetilde{Y}) &= D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y}\,,\\
\mathbf{Q}^1(\widetilde{X}, \widetilde{Y}) &= D_{\mathbf{i}\widetilde{X}}\widetilde{Y} - D_{\mathbf{i}\widetilde{Y}}\widetilde{X} - \mathbf{i}^{-1}[\mathbf{i}\widetilde{X}, \mathbf{i}\widetilde{Y}]\,.
\end{aligned}$$

We have an important remark that among the above mentioned five partial torsions only two ones have ‘real’ torsion property: the h-horizontal torsion $\mathcal{T}$ and the v-vertical torsion $\mathbf{Q}^1$.

Proposition 1. *Let (M, g) be a generalized Finsler manifold endowed with an Ehresmann connection $\mathcal{H}$. Then exists a unique covariant derivative operator D such that*

(i) D *is metric,*

(ii) $\mathcal{T}(\widetilde{X}, \widetilde{Y}) = \mathbf{T}(\widetilde{X}, \widetilde{Y})\,,$

(iii) $\mathbf{Q}^1(\widetilde{X}, \widetilde{Y}) = 0\,,$

for any $\widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\tau^*\tau)\,.$

For a proof we refer to [6].

We say that D is the *canonical covariant derivative* for the structure $(M, g, \mathcal{H})$.

4 Structure equations

The following concepts and results can be found in [14] Chapter 2, Section E.

Lemma and Definition 2. *There is a unique graded derivation* $\mathrm{d}^{\mathsf{v}} \colon \mathcal{A}(\tau^*\tau) \to \mathcal{A}(\tau^*\tau)$ *of degree* 1 *such that*

$$(\mathrm{d}^{\mathsf{v}} f)(\widetilde{X}) := \mathrm{d}f(\mathbf{i}\widetilde{X})\,, \qquad \textit{and}$$

$$\begin{aligned}
\mathrm{d}^{\mathsf{v}}\widetilde{\alpha}(\widetilde{X}_1, \dots, \widetilde{X}_{k+1}) := {}& \sum_{i=1}^{k+1}(-1)^{i+1}(\mathbf{i}\widetilde{X}_i)\widetilde{\alpha}(\widetilde{X}_1, \dots, \hat{\widetilde{X}}_i, \dots, \widetilde{X}_{k+1})\\
&+ \sum_{1\le i<j\le k+1}(-1)^{i+j}\widetilde{\alpha}(\mathbf{i}^{-1}[\mathbf{i}\widetilde{X}_i, \mathbf{i}\widetilde{X}_j], \dots, \hat{\widetilde{X}}_i, \dots, \hat{\widetilde{X}}_j, \dots, \widetilde{X}_{k+1})
\end{aligned}$$

for all $f \in C^\infty(TM)$, $\widetilde{X}, \widetilde{X}_i \in \mathrm{Sec}(\tau^*\tau)$ $(i = 1, \dots, k+1)$ *and* $\widetilde{\alpha} \in \mathcal{A}^k(\tau)$. d^{v} *is said to be the* vertical exterior derivative *on* $\mathcal{A}(\tau^*\tau)$.

Lemma and Definition 3. *Let $\mathcal{H}$ be an Ehresmann connection. There is a unique graded derivation* $\mathrm{d}^{\mathsf{h}}\colon \mathcal{A}(\tau^*\tau) \to \mathcal{A}(\tau^*\tau)$ *of degree 1 such that*

$$(\mathrm{d}^{\mathsf{h}} f)(\widetilde{X}) := \mathrm{d}f(\mathcal{H}\widetilde{X}), \qquad \text{and}$$

$$\mathrm{d}^{\mathsf{h}}\widetilde{\alpha}(\widetilde{X}_1,\dots,\widetilde{X}_{k+1}) := \sum_{i=1}^{k+1}(-1)^{i+1}(\mathcal{H}\widetilde{X}_i)\widetilde{\alpha}(\widetilde{X}_1,\dots,\hat{\widetilde{X}}_i,\dots,\widetilde{X}_{k+1}) + \sum_{1\le i<j\le k+1}(-1)^{i+j}\widetilde{\alpha}(\mathbf{j}[\mathcal{H}\widetilde{X}_i,\mathcal{H}\widetilde{X}_j],\dots,\hat{\widetilde{X}}_i,\dots,\hat{\widetilde{X}}_j,\dots,\widetilde{X}_{k+1})$$

for all $f \in C^\infty(TM)$, $\widetilde{X},\widetilde{X}_i \in \mathrm{Sec}(\tau^*\tau)$ $(i=1,\dots,k+1)$ and $\widetilde{\alpha}\in\mathcal{A}^k(\tau)$. d^{h} is called the horizontal exterior derivative *on $\mathcal{A}(\tau^*\tau)$ with respect to $\mathcal{H}$.*

In the above formulas the notation $\hat{\widetilde{X}}$ means that the argument $\widetilde{X}$ is deleted. If $k=1$, we obtain

$$\mathrm{d}^{\mathsf{v}}\widetilde{\alpha}(\widetilde{X}_1,\widetilde{X}_2) = (\mathbf{i}\widetilde{X}_1)\widetilde{\alpha}(\widetilde{X}_2) - (\mathbf{i}\widetilde{X}_2)\widetilde{\alpha}(\widetilde{X}_1) - \widetilde{\alpha}(\mathcal{V}[\mathbf{i}\widetilde{X}_1,\mathbf{i}\widetilde{X}_2]), \tag{1}$$

$$\mathrm{d}^{\mathsf{h}}\widetilde{\alpha}(\widetilde{X}_1,\widetilde{X}_2) = (\mathcal{H}\widetilde{X}_1)\widetilde{\alpha}(\widetilde{X}_2) - (\mathcal{H}\widetilde{X}_2)\widetilde{\alpha}(\widetilde{X}_1) - \widetilde{\alpha}(\mathbf{j}[\mathcal{H}\widetilde{X}_1,\mathcal{H}\widetilde{X}_2]). \tag{2}$$

Let (M,g) be a generalized Finsler manifold. Let $(\widetilde{E}_i)_{i=1}^n$ be a family of g-orthonormal sections in $\mathrm{Sec}(\tau^*\tau)$ on open subset $\mathcal{U}\subset TM$:

$$\widetilde{E}_i\colon v\in\mathcal{U}\longmapsto \widetilde{E}_i(v)\in T_{\tau(v)}M,$$
$$g(\widetilde{E}_i,\widetilde{E}_j) = \delta_{ij} \quad (1\le i,j\le n).$$

Let $(\widetilde{\Theta}^i)_{i=1}^n$ be denote the family of dual 1-forms of $(\widetilde{E}_i)_{i=1}^n$. Then

$$\widetilde{\Theta}^i(\widetilde{E}_j) = \delta^i_j, \quad 1\le i,j\le n.$$

Using these local frame fields, every section $\widetilde{X}$ of $\mathring{\tau}^*\tau$ over $\mathcal{U}$ can be expressed as

$$\widetilde{X} = \widetilde{\Theta}^i(\widetilde{X})\widetilde{E}_i. \tag{3}$$

Indeed,

$$\widetilde{\Theta}^i(\widetilde{X})\widetilde{E}_i = \widetilde{\Theta}^i(\widetilde{X}^j\widetilde{E}_j)\widetilde{E}_i = \widetilde{X}^j\widetilde{\Theta}^i(\widetilde{E}_j)\widetilde{E}_i = \widetilde{X}^j\delta^i_j\widetilde{E}_i = \widetilde{X}^j\widetilde{E}_j = \widetilde{X}.$$

If $\mathcal{H}$ is an Ehresmann connection on M, then there exist 2-forms $\widetilde{\vartheta}^i$ along τ (on $\mathcal{U}$) such that

$$\mathbf{T}(\widetilde{X},\widetilde{Y}) = \widetilde{\vartheta}^i(\widetilde{X},\widetilde{Y})\widetilde{E}_i, \tag{4}$$

for any sections $\widetilde{X},\widetilde{Y}$ of $\tau^*\tau$ over $\mathcal{U}$.

Let R^D be the curvature tensor of D. Then there exist 2-forms $\widetilde{\Omega}^i_j$ along τ such that

$$R^D(\xi,\eta)\widetilde{E}_j = \widetilde{\Omega}^i_j(\xi,\eta)\widetilde{E}_i. \tag{5}$$

We say that $\widetilde{\vartheta}^i$ are the *torsion two-forms*, $\widetilde{\Omega}^i_j$ are the *curvature two-forms* of the Ehresmann connection with respect to $(\widetilde{E}_i)_{i=1}^n$.

Theorem and Definition 1. *Let (M,g) be a generalized Finsler manifold. Let $\mathcal{H}$ be an Ehresmann connection and let D be the canonical covariant derivative for $(M,g,\mathcal{H})$. Suppose that g is positive definite and let $\mathcal{U}$ be an open subset of TM. Define $(\widetilde{E}_i)_{i=1}^n$ and $(\widetilde{\Theta}^i)_{i=1}^n$ as above. Then there exists a unique family $\big(\widetilde{\omega}^i_j\big)_{1\leq i,j\leq n}$ of 1-forms on $\mathcal{U}$ such that*

$$\widetilde{\omega}^i_j = -\widetilde{\omega}^j_i\,, \tag{6}$$

$$\mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^i = -(\widetilde{\omega}^i_j \circ \mathbf{i}) \wedge \widetilde{\Theta}^j \qquad (1\leq i\leq n)\,, \tag{7}$$

$$\mathrm{d}^{\mathsf{h}}\widetilde{\Theta}^i = -(\widetilde{\omega}^i_j \circ \mathcal{H}) \wedge \widetilde{\Theta}^j - \widetilde{\vartheta}^i \qquad (1\leq i\leq n)\,, \tag{8}$$

$$\widetilde{\Omega}^i_j = d\widetilde{\omega}^i_j + \widetilde{\omega}^i_k \wedge \widetilde{\omega}^k_j\,. \tag{9}$$

The 1-forms $\widetilde{\omega}^i_j$ are said to be the connection forms. *Relations* (7) *and* (8) *are called the* first structure equations. *Relations* (9) *are mentioned as the* second structure equations.

Remark 1. Owing to Proposition 1, the structure equations of v-vertical torsion $\mathbf{Q}^1$ are not relevant.

Proof. Define the 1-forms $\widetilde{\omega}^i_j$ by

$$\widetilde{\omega}^i_j(\xi) := \widetilde{\Theta}^i(D_\xi \widetilde{E}_j) \qquad (\xi \in \mathfrak{X}(TM))\,.$$

(1) Since D is metric, we have

$$\begin{aligned}
0 &= (D_\xi g)(\widetilde{E}_i, \widetilde{E}_j)\\
&= \xi g(\widetilde{E}_i, \widetilde{E}_j) - g(D_\xi \widetilde{E}_i, \widetilde{E}_j) - g(D_\xi \widetilde{E}_j, \widetilde{E}_i)\\
&\overset{(3)}{=} \xi\delta_{ij} - g(\widetilde{\Theta}^k(D_\xi\widetilde{E}_i)\widetilde{E}_k, \widetilde{E}_j) - g(\widetilde{\Theta}^k(D_\xi\widetilde{E}_j)\widetilde{E}_k, \widetilde{E}_i)\\
&= -g(\widetilde{\omega}^k_i\widetilde{E}_k, \widetilde{E}_j) - g(\widetilde{\omega}^k_j\widetilde{E}_k, \widetilde{E}_i)\\
&= -\widetilde{\omega}^k_i g(\widetilde{E}_k, \widetilde{E}_j) - \widetilde{\omega}^k_j g(\widetilde{E}_k, \widetilde{E}_i)\\
&= -\widetilde{\omega}^j_i - \widetilde{\omega}^i_j\,,
\end{aligned}$$

whence (6).

(2) *Equations* (7). The left-hand side of (7) can be manipulated as follows:

$$\begin{aligned}
\mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^i(\widetilde{E}_k, \widetilde{E}_l) &\overset{(1)}{=} (\mathbf{i}\widetilde{E}_k)\widetilde{\Theta}^i\widetilde{E}_l - (\mathbf{i}\widetilde{E}_l)\widetilde{\Theta}^i\widetilde{E}_k - \widetilde{\Theta}^i(\mathcal{V}[\mathbf{i}\widetilde{E}_k, \mathbf{i}\widetilde{E}_l])\\
&= (\mathbf{i}\widetilde{E}_k)\delta^i_l - (\mathbf{i}\widetilde{E}_l)\delta^i_k - \widetilde{\Theta}^i(\mathcal{V}[\mathbf{i}\widetilde{E}_k, \mathbf{i}\widetilde{E}_l])\\
&= -\widetilde{\Theta}^i(\mathcal{V}[\mathbf{i}\widetilde{E}_k, \mathbf{i}\widetilde{E}_l])\,.
\end{aligned}$$

Evaluating the right-hand side at $(\widetilde{E}_k, \widetilde{E}_l)$ we find

$$\begin{aligned}
\big((\widetilde{\omega}^i_j \circ \mathbf{i}) \wedge \widetilde{\Theta}^j\big)(\widetilde{E}_k, \widetilde{E}_l) &= \widetilde{\omega}^i_j(\mathbf{i}\widetilde{E}_k)\widetilde{\Theta}^j\widetilde{E}_l - \widetilde{\omega}^i_j(\mathbf{i}\widetilde{E}_l)\widetilde{\Theta}^j\widetilde{E}_k = \widetilde{\omega}^i_l(\mathbf{i}\widetilde{E}_k)\widetilde{\omega}^i_k(\mathbf{i}\widetilde{E}_l)\\
&= \widetilde{\Theta}^i(D_{\mathbf{i}\widetilde{E}_k}\widetilde{E}_l) - \widetilde{\Theta}^i(D_{\mathbf{i}\widetilde{E}_l}\widetilde{E}_k)\\
&= \widetilde{\Theta}^i(D_{\mathbf{i}\widetilde{E}_k}\widetilde{E}_l - D_{\mathbf{i}\widetilde{E}_l}\widetilde{E}_k) = \widetilde{\Theta}^i(\mathcal{V}[\mathbf{i}\widetilde{E}_k, \mathbf{i}\widetilde{E}_l])\,,
\end{aligned}$$

taking into account in the last step that $\mathbf{Q}^1 = 0$ by Proposition 1, and hence $0 = \mathbf{Q}^1(\widetilde{E}_k, \widetilde{E}_l) = D_{\mathbf{i}\widetilde{E}_k}\widetilde{E}_l - D_{\mathbf{i}\widetilde{E}_l}\widetilde{E}_k - \mathcal{V}[\mathbf{i}\widetilde{E}_k, \mathbf{i}\widetilde{E}_l]$.

(3) *Equations* (8).

$$\begin{aligned} \mathrm{d}^{\mathsf{h}}\widetilde{\Theta}^i(\widetilde{E}_k, \widetilde{E}_l) &\overset{(2)}{=} (\mathcal{H}\widetilde{E}_k)\widetilde{\Theta}^i\widetilde{E}_l - (\mathcal{H}\widetilde{E}_l)\widetilde{\Theta}^i\widetilde{E}_k - \widetilde{\Theta}^i(\mathbf{j}[\mathcal{H}\widetilde{E}_k, \mathcal{H}\widetilde{E}_l]) \\ &= (\mathcal{H}\widetilde{E}_k)\delta^i_l - (\mathcal{H}\widetilde{E}_l)\delta^i_k - \widetilde{\Theta}^i(\mathbf{j}[\mathcal{H}\widetilde{E}_k, \mathcal{H}\widetilde{E}_l]) \\ &= -\widetilde{\Theta}^i(\mathbf{j}[\mathcal{H}\widetilde{E}_k, \mathcal{H}\widetilde{E}_l]) \end{aligned}$$

Since $\mathbf{T}(\widetilde{X}, \widetilde{Y}) \overset{\text{Prop. 1 (ii)}}{=} D_{\mathcal{H}\widetilde{X}}\widetilde{Y} - D_{\mathcal{H}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]$, we get

$$\begin{aligned} \big((\widetilde{\omega}^i_j \circ \mathcal{H}) \wedge \widetilde{\Theta}^j - \widetilde{\vartheta}^i\big)(\widetilde{E}_k, \widetilde{E}_l) &= \widetilde{\omega}^i_j(\mathcal{H}\widetilde{E}_k)\widetilde{\Theta}^j\widetilde{E}_l - \widetilde{\omega}^i_j(\mathcal{H}\widetilde{E}_l)\widetilde{\Theta}^j\widetilde{E}_k - \widetilde{\vartheta}^i(\widetilde{E}_k, \widetilde{E}_l) \\ &= \widetilde{\omega}^i_l(\mathcal{H}\widetilde{E}_k) - \widetilde{\omega}^i_k(\mathcal{H}\widetilde{E}_l) - \widetilde{\vartheta}^i(\widetilde{E}_k, \widetilde{E}_l) \\ &= \widetilde{\Theta}^i(D_{\mathcal{H}\widetilde{E}_k}\widetilde{E}_l) - \widetilde{\Theta}^i(D_{\mathcal{H}\widetilde{E}_l}\widetilde{E}_k) - \widetilde{\vartheta}^i(\widetilde{E}_k, \widetilde{E}_l) \\ &= \widetilde{\Theta}^i(D_{\mathcal{H}\widetilde{E}_k}\widetilde{E}_l - D_{\mathcal{H}\widetilde{E}_l}\widetilde{E}_k) - \widetilde{\vartheta}^i(\widetilde{E}_k, \widetilde{E}_l) \\ &= \widetilde{\Theta}^i(\mathbf{T}(\widetilde{E}_k, \widetilde{E}_l) + \mathbf{j}[\mathcal{H}\widetilde{E}_k, \mathcal{H}\widetilde{E}_l]) - \widetilde{\vartheta}^i(\widetilde{E}_k, \widetilde{E}_l) \\ &\overset{(4)}{=} \widetilde{\Theta}^i(\widetilde{\vartheta}^s(\widetilde{E}_k, \widetilde{E}_l)\widetilde{E}_s) + \widetilde{\Theta}^i(\mathbf{j}[\mathcal{H}\widetilde{E}_k, \mathcal{H}\widetilde{E}_l]) - \widetilde{\vartheta}^i(\widetilde{E}_k, \widetilde{E}_l) \\ &= \widetilde{\Theta}^i(\mathbf{j}[\mathcal{H}\widetilde{E}_k, \mathcal{H}\widetilde{E}_l])\,. \end{aligned}$$

(4) *Equations* (9). By using the definition of D and R^D, relation (3), we find

$$\begin{aligned} \widetilde{\Omega}^i_j(\xi, \eta)\widetilde{E}_i &\overset{(5)}{=} R^D(\xi, \eta)\widetilde{E}_j = D_\xi D_\eta \widetilde{E}_j - D_\eta D_\xi \widetilde{E}_j - D_{[\xi,\eta]}\widetilde{E}_j \\ &= D_\xi(\widetilde{\Theta}^k(D_\eta\widetilde{E}_j)\widetilde{E}_k) - D_\eta(\widetilde{\Theta}^k(D_\xi\widetilde{E}_j)\widetilde{E}_k) - \widetilde{\Theta}^i(D_{[\xi,\eta]}\widetilde{E}_j)\widetilde{E}_i \\ &= \xi(\widetilde{\Theta}^k(D_\eta\widetilde{E}_j))\widetilde{E}_k + \widetilde{\Theta}^k(D_\eta\widetilde{E}_j)D_\xi\widetilde{E}_k \\ &\quad - \eta(\widetilde{\Theta}^k(D_\xi\widetilde{E}_j))\widetilde{E}_k - \widetilde{\Theta}^k(D_\xi\widetilde{E}_j)D_\eta\widetilde{E}_k - \widetilde{\Theta}^i(D_{[\xi,\eta]}\widetilde{E}_j)\widetilde{E}_i \\ &= \xi(\widetilde{\Theta}^i(D_\eta\widetilde{E}_j))\widetilde{E}_i - \eta(\widetilde{\Theta}^i(D_\xi\widetilde{E}_j))\widetilde{E}_i - \widetilde{\Theta}^i(D_{[\xi,\eta]}\widetilde{E}_j)\widetilde{E}_i \\ &\quad + \widetilde{\omega}^k_j(\eta)D_\xi\widetilde{E}_k - \widetilde{\omega}^k_j(\xi)D_\eta\widetilde{E}_k \\ &= \xi(\widetilde{\Theta}^i(D_\eta\widetilde{E}_j))\widetilde{E}_i - \eta(\widetilde{\Theta}^i(D_\xi\widetilde{E}_j))\widetilde{E}_i - \widetilde{\Theta}^i(D_{[\xi,\eta]}\widetilde{E}_j)\widetilde{E}_i \\ &\quad + \widetilde{\omega}^k_j(\eta)\widetilde{\Theta}^i(D_\xi\widetilde{E}_k)\widetilde{E}_i - \widetilde{\omega}^k_j(\xi)\widetilde{\Theta}^i(D_\eta\widetilde{E}_k)\widetilde{E}_i \\ &= \xi(\widetilde{\omega}^i_j(\eta))\widetilde{E}_i - \eta(\widetilde{\omega}^i_j(\xi))\widetilde{E}_i - \widetilde{\omega}^i_j([\xi,\eta])\widetilde{E}_i \\ &\quad + \widetilde{\omega}^k_j(\eta)\widetilde{\omega}^i_k(\xi)\widetilde{E}_i - \widetilde{\omega}^k_j(\xi)\widetilde{\omega}^i_k(\eta)\widetilde{E}_i \\ &= \big(\xi(\widetilde{\omega}^i_j(\eta)) - \eta(\widetilde{\omega}^i_j(\xi)) - \widetilde{\omega}^i_j([\xi,\eta]) + \widetilde{\omega}^i_k(\xi)\widetilde{\omega}^k_j(\eta) - \widetilde{\omega}^i_k(\eta)\widetilde{\omega}^k_j(\xi)\big)\,\widetilde{E}_i\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} (\mathrm{d}\widetilde{\omega}^i_j + \widetilde{\omega}^i_k \wedge \widetilde{\omega}^k_j)(\xi, \eta) &= d\widetilde{\omega}^i_j(\xi, \eta) + \widetilde{\omega}^i_k(\xi)\widetilde{\omega}^k_j(\eta) - \widetilde{\omega}^i_k(\eta)\widetilde{\omega}^k_j(\xi) \\ &= \xi(\widetilde{\omega}^i_j(\eta)) - \eta(\widetilde{\omega}^i_j(\xi)) - \widetilde{\omega}^i_j([\xi,\eta]) \\ &\quad + \widetilde{\omega}^i_k(\xi)\widetilde{\omega}^k_j(\eta) - \widetilde{\omega}^i_k(\eta)\widetilde{\omega}^k_j(\xi)\,, \end{aligned}$$

which concludes the proof of (9).

(5) *Uniqueness of the family* $(\widetilde{\omega}^i_j)$. We use the fact that any 1-form of an open subset of TM is completely determined by its action over vertical and horizontal vector fields.

First we prove that the effect of the connection forms on vertical vector fields is well-defined. We start on (7) and paragraph 2 of this proof.

$$\begin{aligned}
\mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^i(\widetilde{E}_j,\widetilde{E}_k) &= \widetilde{\omega}^i_j(\mathbf{i}\widetilde{E}_k) - \widetilde{\omega}^i_k(\mathbf{i}\widetilde{E}_j)\,,\\
\mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^j(\widetilde{E}_k,\widetilde{E}_i) &= \widetilde{\omega}^j_k(\mathbf{i}\widetilde{E}_i) - \widetilde{\omega}^j_i(\mathbf{i}\widetilde{E}_k)\,,\\
\mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^k(\widetilde{E}_i,\widetilde{E}_j) &= \widetilde{\omega}^k_i(\mathbf{i}\widetilde{E}_j) - \widetilde{\omega}^k_j(\mathbf{i}\widetilde{E}_i)\,.
\end{aligned}$$

Now we add the first two equalities, and subtract the third. Taking into account (6), we obtain

$$\widetilde{\omega}^i_j(\mathbf{i}\widetilde{E}_k) = \frac{1}{2}\left(\mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^i(\widetilde{E}_j,\widetilde{E}_k) + \mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^j(\widetilde{E}_k,\widetilde{E}_i) - \mathrm{d}^{\mathsf{v}}\widetilde{\Theta}^k(\widetilde{E}_i,\widetilde{E}_j)\right)\,,$$

and this relation proves the statement.

Similarly, we have

$$\begin{aligned}
\mathrm{d}^{\mathsf{h}}\widetilde{\Theta}^i(\widetilde{E}_j,\widetilde{E}_k) &= \widetilde{\omega}^i_j(\mathcal{H}\widetilde{E}_k) - \widetilde{\omega}^i_k(\mathcal{H}\widetilde{E}_j) + \widetilde{\vartheta}^i(\widetilde{E}_j,\widetilde{E}_k)\,,\\
\mathrm{d}^{\mathsf{h}}\widetilde{\Theta}^j(\widetilde{E}_k,\widetilde{E}_i) &= \widetilde{\omega}^j_k(\mathcal{H}\widetilde{E}_i) - \widetilde{\omega}^j_i(\mathcal{H}\widetilde{E}_k) + \widetilde{\vartheta}^j(\widetilde{E}_k,\widetilde{E}_i)\,,\\
\mathrm{d}^{\mathsf{h}}\widetilde{\Theta}^k(\widetilde{E}_i,\widetilde{E}_j) &= \widetilde{\omega}^k_i(\mathcal{H}\widetilde{E}_j) - \widetilde{\omega}^k_j(\mathcal{H}\widetilde{E}_i) + \widetilde{\vartheta}^k(\widetilde{E}_i,\widetilde{E}_j)\,.
\end{aligned}$$

Adding the first two equalities, and subtracting the third, by using (6) we find

$$\begin{aligned}
\widetilde{\omega}^i_j(\mathcal{H}\widetilde{E}_k) =& \frac{1}{2}\left(\mathrm{d}^{\mathsf{h}}\widetilde{\omega}^i(\widetilde{E}_j,\widetilde{E}_k) + \mathrm{d}^{\mathsf{h}}\widetilde{\omega}^j(\widetilde{E}_k,\widetilde{E}_i) - \mathrm{d}^{\mathsf{h}}\widetilde{\omega}^k(\widetilde{E}_i,\widetilde{E}_j)\right)\\
&- \frac{1}{2}\left(\widetilde{\vartheta}^i(\widetilde{E}_j,\widetilde{E}_k) + \widetilde{\vartheta}^j(\widetilde{E}_k,\widetilde{E}_i) - \widetilde{\vartheta}^k(\widetilde{E}_i,\widetilde{E}_j)\right)\,.
\end{aligned}$$

□

References

[1] M. Crampin: On horizontal distributions on the tangent bundle of a differentiable manifold. *J. London Math. Soc.* 3 (2) (1971) 178–182.

[2] M. Crampin: Connections of Berwald type. *Publ. Math. Debrecen* 57 (2000) 455–473.

[3] M. de León, P.R. Rodrigues: *Methods of Differential Geometry in Analytical Mechanics.* North-Holland (Amsterdam) (1989).

[4] W. Greub, S. Halperin, R. Vanstone: *Connections, Curvature, and Cohomology Vol. I.* Academic Press, New York (1972).

[5] J. Grifone: Structure presque tangente et connexions I.. *Ann. Inst. Fourier, Grenoble* 22 (1) (1972) 287–334.

[6] R.L. Lovas, J. Pék, J. Szilasi: Ehresmann connections, metrics and good metric derivatives. *Advanced Studies in Pure Mathematics* 48 (2007) 263–308. Finsler Geometry, Sapporo 2005 – In Memory of Makoto Matsumoto.

[7] M. Matsumoto: *Foundations of Finsler Geometry and Special Finsler Spaces.* Kaiseisha Press, Otsu (1986).

[8] E. Martinez, J.F. Cariñena, W. Sarlet: Derivations of differential forms along the tangent bundle projection. *Diff. Geom. Appl.* 2 (1992) 17–43.

[9] R. Miron: Metrical Finsler structures and metrical Finsler connections. *J. Math. Kyoto Univ.* 23 (1983) 219–224.

[10] T. Mestdag, J. Szilasi, V. Tóth: On the geometry of generalized metrics. *Publ. Math. Debrecen* 62 (2003) 511–545.

[11] A. Moór: Entwicklung einer Geometrie der allgemeinen metrischen Linienelementräume. *Acta Sci. Math. Szeged* 17 (1956) 85–120.

[12] A. Moór: Eine Verallgemeinerung der metrischen Übertragung in allgemeinen metrischen Räumen. *Publ. Math. Debrecen* 10 (1963) 145–150.

[13] P. Petersen: *Riemannian Geometry*. Springer, Berlin (1968).

[14] J. Szilasi: A Setting for Spray and Finsler Geometry. In: P. L. Antonelli (ed.): *Handbook of Finsler Geometry, Vol. 2.* Kluwer Academic Publishers, Dordrecht (2003) 1182–1426.

[15] J.R. Vanstone: A generalization of Finsler Geometry. *Canad. J. Math.* 14 (1962) 87–112.

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Control Systems on the Orthogonal Group SO(4)

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Abstract. We classify the left-invariant control affine systems evolving on the orthogonal group $\mathsf{SO}(4)$. The equivalence relation under consideration is detached feedback equivalence. Each possible number of inputs is considered; both the homogeneous and inhomogeneous systems are covered. A complete list of class representatives is identified and controllability of each representative system is determined.

1 Introduction

A control system is given by a dynamical polysystem together with a class of "admissible inputs" (also called controls). More precisely, a (smooth) *control system* Σ on M consists of a family $\mathcal{X} = (\Xi_u)_{u\in U}$ of smooth vector fields on the state space M and an input class $\mathcal{U}$. M is a smooth (real, finite-dimensional) manifold, and an element of $\mathcal{U}$ is a U-valued map (defined on some interval of $\mathbb{R}$) which is (Lebesgue) measurable or piecewise constant, or of some regularity type between these two possibilities. The input set U is usually equipped with a separable metric space structure. For the purposes of this paper, we shall assume that $U = \mathbb{R}^\ell$. In classical notation, a control system Σ on M is written as

$$\Sigma : \quad \dot{x} = \Xi(x, u)\,, \qquad x \in \mathsf{M}\,,\ u \in U\,.$$

Here $\Xi\colon \mathsf{M} \times U \to T\mathsf{M}, (x, u) \mapsto \Xi(x, u) = \Xi_u(x) \in T_x\mathsf{M}$ is the map describing the dynamics (i.e., the vector fields) of the system. We assume that Ξ is a smooth map. Standard references for nonlinear control systems are [16], [24]. When the state space is a (real, finite-dimensional) Lie group G and the dynamics $\Xi_u = \Xi(\cdot, u)$ are left invariant, the control system is termed as *left-invariant*. Such control systems have been studied by a number of authors over the past few decades (see, e.g., [3], [19], [20], [26], [28]).

A *trajectory* of Σ (corresponding to an admissible input $u(\cdot) \in \mathcal{U}$) is an absolutely continuous curve γ in M such that $\dot{\gamma}(t) = \Xi_{u(t)}(\gamma(t))$ for almost all t.

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Carathéodory's existence and uniqueness theorem guarantees the local existence and global uniqueness of trajectories. The initial condition (initial state) is just a starting point for the trajectory; different admissible inputs provide, generally speaking, different trajectories starting from a fixed state. All these trajectories fill the set attainable from the given initial state. To characterize such sets is the first natural problem in control theory: the *controllability problem*. As soon as the possibility to attain a certain state is established, we try to do it in the best possible way. This is the *optimal control problem* (see, e.g., [3], [19]).

The most natural equivalence relation for control systems is equivalence up to coordinate changes in the state space. This is called *state space equivalence*. We say that two control systems Σ and $\widetilde{\Sigma}$ are state space equivalent if there exists a diffeomorphism ϕ between the state spaces which transforms the dynamics Ξ_u to $\widetilde{\Xi}_u$. State space equivalence is well understood ([17]). It establishes a one-to-one correspondence between the trajectories of the equivalent systems (corresponding to the same admissible inputs). This equivalence relation is very strong; any general classification appears to be very difficult if not impossible. However, some reasonable classification in low dimensions is possible (see [2], [11]).

Another important equivalence relation for control systems is that of *feedback equivalence*. Applying feedback transformations means that we also modify the controls (which remain unchanged for state space equivalence) in a way that is state dependent. (Feedback control may be used to achieve desired dynamical properties of the system, like stabilizability.) We say that two control systems Σ and $\widetilde{\Sigma}$ are feedback equivalent if there exists a diffeomorphism $\widetilde{x} = \phi(x)$ between the state spaces and an invertible transformation $\widetilde{u} = \varphi(x, u)$ of controls such that the diffeomorphism $\Phi(x, u) = (\phi(x), \varphi(x, u))$ brings Σ into $\widetilde{\Sigma}$. Feedback equivalent systems have geometrically the same set of trajectories which are parametrized differently by admissible inputs. Feedback equivalence has been extensively studied in the last few decades (see [25] and the references therein). Many problems concerning feedback equivalence are studied and solved for control affine systems (i.e., control systems with dynamics affine in controls) and then extended to the general case (for details, see [17], [25]).

In the context of *left-invariant* control systems, feedback equivalence is specialized by requiring that the feedback transformations are independent of the state variable. Such transformations are precisely those that are compatible with the Lie group structure. This is called *detached feedback equivalence*. It turns out that two (full-rank) left-invariant control systems are detached feedback equivalent if and only if there exists a Lie group isomorphism between the state spaces, relating their dynamics. Several classes of left-invariant control affine systems have recently been classified (cf. [7], [9]).

In this paper we consider left-invariant control affine systems, evolving on the (six-dimensional) orthogonal group $\mathsf{SO}(4)$. These systems have the form

$$\Sigma: \quad \dot{g} = g(A + u_1 B_1 + \cdots + u_\ell B_\ell), \qquad g \in \mathsf{SO}(4),\ u \in \mathbb{R}^\ell$$

where $A, B_1, \ldots, B_\ell \in \mathfrak{so}(4)$. (The elements $B_1, \ldots, B_\ell$ are assumed to be linearly independent.) The aim is to classify, under detached feedback equivalence, all such systems; a list of class representatives will be produced. In addition, we identify precisely those systems which are controllable. The homogeneous systems are considered first. The single-input, two-input, and three-input systems are classified by exploiting the singular value decomposition. The classification of the four-input and five-input systems follow as corollaries. For the inhomogeneous systems, the classification is based, in each case, on its homogeneous counterpart.

We conclude the paper with a few remarks. Moreover, we refer briefly to other works on $\mathsf{SO}(4)$ (and its Lie algebra) dealing with some interesting variational problems as well as integrable Hamiltonian systems (and their applications).

A tabulation of the classification in matrix form is appended.

2 Invariant control systems

An (ℓ-input) left-invariant control affine system Σ on G is a control system of the form

$$\Sigma: \quad \dot{g} = g\,\Xi(\mathbf{1}, u) = g(A + u_1 B_1 + \cdots + u_\ell B_\ell)\,, \qquad g \in \mathsf{G}\,,\ u \in \mathbb{R}^\ell.$$

Here G is a (real, finite-dimensional) connected matrix Lie group with Lie algebra $\mathfrak{g}$. The *parametrization map* $\Xi(\mathbf{1}, \cdot) \colon \mathbb{R}^\ell \to \mathfrak{g}$ is an injective affine map (i.e., $B_1, \ldots, B_\ell$ are linearly independent). Note that the dynamics $\Xi_u = \Xi(\cdot, u)$ are invariant under left translations, i.e., $\Xi(g, u) = g\,\Xi(\mathbf{1}, u)$. Such a system is denoted by $\Sigma = (\mathsf{G}, \Xi)$ (cf. [6]). Σ is completely determined by the specification of its state space G and its parametrization map $\Xi(\mathbf{1}, \cdot)$. Hence, for a fixed G, we shall specify Σ by simply writing

$$\Sigma: \quad A + u_1 B_1 + \cdots + u_\ell B_\ell\,.$$

The *trace* $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot) = A + \Gamma^0 = A + \langle B_1, \ldots, B_\ell \rangle$ is an affine subspace of $\mathfrak{g}$. A system Σ is called *homogeneous* if $A \in \Gamma^0$, and *inhomogeneous* otherwise. Σ has *full rank* if the Lie algebra generated by its trace coincides with $\mathfrak{g}$.

The admissible inputs are piecewise-continuous maps $u(\cdot) \colon [0, T] \to \mathbb{R}^\ell$. A *trajectory* for an admissible input $u(\cdot)$ is an absolutely continuous curve $g(\cdot) \colon [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\,\Xi(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$. A system Σ is said to be controllable if, given any pair of points $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot)$ such that $g(0) = g_0$ and $g(T) = g_1$. If Σ is controllable, then it has full rank. Moreover, if Σ is homogeneous or if G is compact, then the full-rank condition implies controllability. For more details on invariant control systems see, e.g., [19], [20], [26].

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}, \Xi')$ be two systems on G. We say that Σ and Σ' are (locally) *detached feedback equivalent* if there exist open neighbourhoods N and N' of (the unit element) $\mathbf{1}$ and a (local) diffeomorphism $\Phi = \phi \times \varphi \colon N \times \mathbb{R}^\ell \to N' \times \mathbb{R}^\ell$ such that $\phi(\mathbf{1}) = \mathbf{1}$ and $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \varphi(u))$ for $g \in N$ and $u \in \mathbb{R}^\ell$. (Here $T_g\phi$ denotes the tangent map of ϕ at g.)

Proposition 1 ([12]). *Two full-rank systems Σ and Σ' are detached feedback equivalent if and only if there exists a Lie algebra automorphism $\psi \colon \mathfrak{g} \to \mathfrak{g}$ such that $\psi \cdot \Gamma = \Gamma'$.*

Proof. (Sketch) Suppose Σ and Σ' are detached feedback equivalent. Then

$$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'\big(\mathbf{1}, \varphi(u)\big)$$

and so $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. Let $u, v \in \mathbb{R}^\ell$, and let $\Xi_u = \Xi(\cdot, u)$ and $\Xi_v = \Xi(\cdot, v)$ denote the corresponding vector fields. Then $\phi_*[\Xi_u, \Xi_v] = [\phi_*\Xi_u, \phi_*\Xi_v]$ and so

$$T_{\mathbf{1}}\phi \cdot [\Xi_u(\mathbf{1}), \Xi_v(\mathbf{1})] = [\Xi'_{\varphi(u)}(\mathbf{1}), \Xi'_{\varphi(v)}(\mathbf{1})] = [T_{\mathbf{1}}\phi \cdot \Xi_u(\mathbf{1}), T_{\mathbf{1}}\phi \cdot \Xi_v(\mathbf{1})] \,.$$

As the elements $\Xi_u(\mathbf{1})$, $u \in \mathbb{R}^\ell$, generate the Lie algebra, it follows that $T_{\mathbf{1}}\phi$ is a Lie algebra isomorphism. Conversely, suppose we have a Lie algebra isomorphism ψ such that $\psi \cdot \Gamma = \Gamma'$. Then there exist neighbourhoods N and N' of $\mathbf{1}$ and a (local) group isomorphism $\phi\colon N \to N'$ such that $T_{\mathbf{1}}\phi = \psi$ (see, e.g., [21]). The equation $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$ defines an affine isomorphism $\varphi\colon \mathbb{R}^\ell \to \mathbb{R}^{\ell'}$. Consequently

$$T_g\phi \cdot \Xi(g, u) = T_{\mathbf{1}}L_{\phi(g)} \cdot \psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\phi(g), \varphi(u)) \,.$$

Hence Σ and Σ' are detached feedback equivalent. $\square$

In this paper, we shall find it convenient to use the above characterization as the definition of equivalence. More precisely, we say that two (not necessarily full-rank) systems Σ and Σ' are *equivalent* if there exists $\psi \in \mathsf{Aut}(\mathfrak{g})$ such that $\psi \cdot \Gamma = \Gamma'$. In particular, if $\Gamma = \Gamma'$, then we say that Σ' is a *reparametrization* of Σ. Notice that if two systems are equivalent, then they are detached feedback equivalent. (The converse, however, does not hold.) Any two equivalent systems are either both controllable or neither is controllable whenever the full-rank condition is equivalent to controllability.

3 The orthogonal group SO(4)

The orthogonal group

$$\mathsf{SO}(4) = \left\{ g \in \mathsf{GL}(4, \mathbb{R}) : g^\top g = \mathbf{1},\ \det g = 1 \right\}$$

is a six-dimensional semisimple compact connected Lie group. Its Lie algebra

$$\mathfrak{so}(4) = \left\{ A \in \mathbb{R}^{4\times 4} : A^\top + A = \mathbf{0} \right\}$$

is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Let

$$\mathbf{E}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \mathbf{E}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad \mathbf{E}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be the standard (ordered) basis for $\mathfrak{so}(3)$. The map $\varsigma\colon \mathfrak{so}(3) \oplus \mathfrak{so}(3) \to \mathfrak{so}(4)$, given by

$$\left(\begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix} \right)$$
$$\mapsto \ \frac{1}{2} \begin{bmatrix} 0 & x_3 - y_3 & x_2 - y_2 & x_1 - y_1 \\ -x_3 + y_3 & 0 & x_1 + y_1 & -x_2 - y_2 \\ -x_2 + y_2 & -x_1 - y_1 & 0 & x_3 + y_3 \\ -x_1 + y_1 & x_2 + y_2 & -x_3 - y_3 & 0 \end{bmatrix}$$

is a Lie algebra isomorphism. The *natural basis* of $\mathfrak{so}(4)$ is given by

$$\begin{aligned} E_i &= \varsigma \cdot (\mathbf{E}_i, \mathbf{0}) && i = 1,2,3 \\ E_j &= \varsigma \cdot (\mathbf{0}, \mathbf{E}_{j-3}) && j = 4,5,6. \end{aligned}$$

(This choice of basis proves to be the most convenient, especially for expressing the group of automorphisms.) The commutator table for $\mathfrak{so}(4)$ is given below.

	E_1	E_2	E_3	E_4	E_5	E_6
E_1	0	E_3	$-E_2$	0	0	0
E_2	$-E_3$	0	E_1	0	0	0
E_3	E_2	$-E_1$	0	0	0	0
E_4	0	0	0	0	E_6	$-E_5$
E_5	0	0	0	$-E_6$	0	E_4
E_6	0	0	0	E_5	$-E_4$	0

The group of inner automorphisms of $\mathfrak{so}(4)$ is given by

$$\mathsf{Int}(\mathfrak{so}(4)) = \left\{ \begin{bmatrix} \psi_1 & \mathbf{0} \\ \mathbf{0} & \psi_2 \end{bmatrix} : \psi_1, \psi_2 \in \mathsf{SO}(3) \right\}.$$

Proposition 2 ([1]). *The group of automorphisms* $\mathsf{Aut}(\mathfrak{so}(4))$ *is generated by* $\mathsf{Int}(\mathfrak{so}(4))$ *and the swap automorphism* $\zeta = \begin{bmatrix} \mathbf{0} & I_3 \\ I_3 & \mathbf{0} \end{bmatrix}$.

Moreover, the group of automorphisms decomposes as a semi-direct product:

$$\mathsf{Aut}(\mathfrak{so}(4)) = \mathsf{Int}(\mathfrak{so}(4)) \rtimes \{\mathbf{1}, \zeta\}.$$

4 Homogeneous systems

In this section we classify the homogeneous systems on $\mathsf{SO}(4)$. We may assume that $\Xi(\mathbf{1}, 0) = \mathbf{0}$; indeed any homogeneous system is equivalent to one for which this is the case (by use of some reparametrization). We distinguish between the number ℓ of controls involved; this yields six types of systems. For each of these types we simplify an arbitrary system by successively applying automorphisms (as well as considering reparametrizations of the system). Finally, we verify that all the candidates for class representatives are distinct and non-equivalent. Families of representatives are typically parametrized by some vector $\boldsymbol{\alpha} = (\alpha_i)$ or some scalar β.

Any automorphism of $\mathfrak{so}(4)$ preserves the dot product $A \bullet B = \sum_{i=1}^{6} a_i b_i$. (Here $A = \sum_{i=1}^{6} a_i E_i$ and $B = \sum_{i=1}^{6} b_i E_i$.) Let $\Gamma^\perp$ denote the orthogonal complement of a subspace $\Gamma \subset \mathfrak{so}(4)$.

Lemma 1. *Suppose* $\Gamma, \widetilde{\Gamma}$ *are subspaces of* $\mathfrak{so}(4)$ *and* $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$. *Then* $\psi \cdot \Gamma = \widetilde{\Gamma}$ *if and only if* $\psi \cdot \Gamma^\perp = \widetilde{\Gamma}^\perp$.

The classification of the $(6-\ell)$-input systems therefore follows from the classification of the ℓ-input systems. Hence, we need only classify the single-input, two-input, and three-input systems. The results for the four-input and five-input systems then follow as corollaries. (The classification for the six-input systems is trivial.)

When convenient, an ℓ-input homogeneous system

$$\Sigma\colon\; u_1 \sum_{i=1}^{6} b_1^i E_i + \cdots + u_\ell \sum_{i=1}^{6} b_\ell^i E_i$$

will be written (in matrix form) as

$$\Sigma\colon\; \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} b_1^1 & \dots & b_\ell^1 \\ \vdots & & \vdots \\ b_1^6 & \dots & b_\ell^6 \end{bmatrix}.$$

Here $M_1, M_2 \in \mathbb{R}^{3\times\ell}$.

The evaluation $\psi \cdot \Xi(1, \boldsymbol{u})$ then becomes a matrix multiplication. Accordingly, two ℓ-input homogeneous systems $\Sigma\colon \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ and $\Sigma'\colon \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}$ are equivalent if and only if there exist an automorphism $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$ and $K \in \mathsf{GL}(\ell, \mathbb{R})$ such that

$$\psi \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} K = \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}.$$

(K corresponds to a reparametrization $\Xi(\mathbf{1}, Ku)$ of the system Σ.) More precisely, Σ and Σ' are equivalent if and only if there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(\ell, \mathbb{R})$ such that

$$\begin{aligned} & (R_1 M_1 K = M_1' \quad \text{and} \quad R_2 M_2 K = M_2') \\ \text{or} \quad & (R_1 M_2 K = M_1' \quad \text{and} \quad R_2 M_1 K = M_2'). \end{aligned}$$

The *singular value decomposition* (SVD) turns out to be useful in classifying systems. For any matrix $M \in \mathbb{R}^{m\times n}$ of rank r, there exist orthogonal matrices $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ and a diagonal matrix $D \in \mathbb{R}^{r\times r} = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$ such that $M = U \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V^\top$ with $\sigma_1 \geq \cdots \geq \sigma_r > 0$. Specialized forms of the SVD (stated as lemmas) will be used in classifying the two-input and three-input homogeneous systems.

Theorem 1. *Any single-input homogeneous system is equivalent to*

$$\Sigma_\beta^{(1,0)} :\; u_1(E_1 + \beta E_4)$$

for some $0 \leq \beta \leq 1$. Here β parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

Remark 1. Clearly, no single-input homogeneous system is controllable.

Proof. Let $\Sigma\colon \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ be a single-input system. (Here $M_1, M_2 \in \mathbb{R}^{3\times 1}$.) We may assume that $M_1 \neq \mathbf{0}$. (If not, consider $\Sigma\colon \zeta \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$.) There exist $R_1, R_2 \in \mathsf{SO}(3)$ such that

$$R_1 M_1 \frac{1}{\|M_1\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad R_2 M_2 \frac{1}{\|M_1\|} = \begin{bmatrix} \frac{\|M_2\|}{\|M_1\|} \\ 0 \\ 0 \end{bmatrix}.$$

Thus Σ is equivalent to $\Sigma'\colon u_1(E_1 + \frac{\|M_2\|}{\|M_1\|}E_4)$. If $\frac{\|M_2\|}{\|M_1\|} > 1$, then we have

$$\zeta \cdot \left\langle E_1 + \frac{\|M_2\|}{\|M_1\|}E_4 \right\rangle = \left\langle E_1 + \frac{\|M_1\|}{\|M_2\|}E_4 \right\rangle,$$

and so Σ is equivalent to $\Sigma''\colon u_1(E_1 + \frac{\|M_1\|}{\|M_2\|}E_4)$. Hence Σ is equivalent to $\Sigma^{(1,0)}_\beta$ for some $0 \leq \beta \leq 1$.

Suppose $\Sigma^{(1,0)}_\beta$ and $\Sigma^{(1,0)}_{\beta'}$ are equivalent. Then there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $k \neq 0$ such that

$$\begin{aligned} &\left(R_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad R_2 \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} \beta' \\ 0 \\ 0 \end{bmatrix} \right) \\ \text{or} \quad &\left(R_1 \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad R_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} \beta' \\ 0 \\ 0 \end{bmatrix} \right). \end{aligned}$$

Therefore $|\beta| = |\beta'|$ or $|\beta\beta'| = 1$. Thus, as $0 \leq \beta, \beta' \leq 1$, we get $\beta = \beta'$. □

Corollary 1. *Any five-input homogeneous system is equivalent to*

$$\Sigma^{(5,0)}_\beta : \; u_1(E_4 - \beta E_1) + u_2 E_2 + u_3 E_3 + u_4 E_5 + u_5 E_6$$

for some $0 \leq \beta \leq 1$. Here β parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

Remark 2. Every five-input homogeneous system is controllable.

Lemma 2. *For any $M \in \mathbb{R}^{3\times 2}$ there exist orthogonal matrices $R_1 \in \mathsf{SO}(3)$ and $R_2 \in \mathsf{O}(2)$ such that $R_1 M R_2 = \begin{bmatrix} D \\ 0 \;\; 0 \end{bmatrix}$, where $D = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ and $a_1 \geq a_2 \geq 0$. If*

$$R_1 \begin{bmatrix} D \\ 0 \;\; 0 \end{bmatrix} R_2 = \begin{bmatrix} D' \\ 0 \;\; 0 \end{bmatrix}$$

for some $R_1 \in \mathsf{SO}(3)$ and $R_2 \in \mathsf{O}(2)$, then $D = D'$ (provided that D and D' are diagonal matrices such that $a_1 \geq a_2 \geq 0$ and $a_1' \geq a_2' \geq 0$).

Theorem 2. *Any two-input homogeneous system is equivalent to exactly one of the systems*

$$\Sigma_1^{(2,0)} : u_1 E_1 + u_2 E_4$$
$$\Sigma_{2,\boldsymbol{\alpha}}^{(2,0)} : u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5)$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$, where $0 = \alpha_2 \leq \alpha_1$ or $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ or $0 < \alpha_2 \leq \alpha_1 < 1$. Here $\boldsymbol{\alpha}$ parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

Remark 3. $\Sigma_1^{(2,0)}$ is not controllable. $\Sigma_{2,\boldsymbol{\alpha}}^{(2,0)}$ is not controllable exactly when $\alpha_2 = 0$ or $\alpha_1 = \alpha_2 = 1$.

Proof. Let $\Sigma\colon \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ be a two-input system. (Here $M_1, M_2 \in \mathbb{R}^{3\times 2}$.) Now either $\operatorname{rank}(M_1) = \operatorname{rank}(M_2) = 1$ or $\max\{\operatorname{rank}(M_1), \operatorname{rank}(M_2)\} = 2$. Suppose $\operatorname{rank}(M_1) = \operatorname{rank}(M_2) = 1$. Then there exists $K \in \mathsf{GL}(2,\mathbb{R})$ such that

$$M_1 K = \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \\ b_3 & 0 \end{bmatrix} \qquad \text{and} \qquad M_2 K = \begin{bmatrix} 0 & b_4 \\ 0 & b_5 \\ 0 & b_6 \end{bmatrix}.$$

Hence there exists $R_1, R_2 \in \mathsf{SO}(3)$ such that

$$R_1 \begin{bmatrix} b_1 & 0 \\ b_2 & 0 \\ b_3 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{b_1^2+b_2^2+b_3^2}} & 0 \\ 0 & \frac{1}{\sqrt{b_4^2+b_5^2+b_6^2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\text{and} \qquad R_2 \begin{bmatrix} 0 & b_4 \\ 0 & b_5 \\ 0 & b_6 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{b_1^2+b_2^2+b_3^2}} & 0 \\ 0 & \frac{1}{\sqrt{b_4^2+b_5^2+b_6^2}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore Σ is equivalent to $\Sigma_1^{(2,0)}$.

On the other hand, suppose $\operatorname{rank}(M_1) = 2$ or $\operatorname{rank}(M_2) = 2$. We may assume $\operatorname{rank}(M_1) = 2$. (If not, consider $\Sigma\colon \zeta \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$.) There exists $R_1 \in \mathsf{SO}(3)$ such that $R_1 M_1 = \begin{bmatrix} M_1' \\ 0 \quad 0 \end{bmatrix}$. Hence, there exists $K \in \mathsf{GL}(2,\mathbb{R})$ such that $R_1 M_1 K = I_{2,0}$, where $I_{2,0} = \begin{bmatrix} I_2 \\ 0 \quad 0 \end{bmatrix}$. Thus Σ is equivalent to $\Sigma'\colon \begin{bmatrix} I_{2,0} \\ M_2' \end{bmatrix}$. By lemma 2, there exist $R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{O}(2)$ such that

$$\begin{bmatrix} K^{-1} & & 0 \\ & & 0 \\ 0 & 0 & \det K \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad R_2 M_2' K = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix}$$

for some $\alpha_1 \geq \alpha_2 \geq 0$. If $\alpha_2 = 0$ or $0 \leq \alpha_2 \leq \alpha_1 < 1$, then Σ is equivalent to $\Sigma_{2,\boldsymbol{\alpha}}^{(2,0)}$.

Suppose $1 < \alpha_2 \leq \alpha_1$. Then

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\alpha_1} \\ \frac{1}{\alpha_2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{and} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\alpha_1} \\ \frac{1}{\alpha_2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha_2} & 0 \\ 0 & \frac{1}{\alpha_1} \\ 0 & 0 \end{bmatrix}$$

with $0 < \frac{1}{\alpha_1} \leq \frac{1}{\alpha_2} < 1$. Thus Σ is equivalent to $\Sigma^{(2,0)}_{2,\boldsymbol{\alpha}'}$ for some $0 < \alpha_2' \leq \alpha_1' < 1$. Suppose $\alpha_2 \leq 1 \leq \alpha_1$. If $\frac{1}{\alpha_2} \leq \alpha_1$, then we are done. If $\frac{1}{\alpha_2} > \alpha_1$, then Σ is likewise equivalent to $\Sigma^{(2,0)}_{2,\boldsymbol{\alpha}'}$ for some $1 \leq \frac{1}{\alpha_2'} \leq \alpha_1'$.

We now verify that none of the class representatives are equivalent. As the traces of $\Sigma^{(2,0)}_1$ and $\Sigma^{(2,0)}_{2,\boldsymbol{\alpha}}$, respectively, do not generate the same subalgebra (for any $\alpha_1, \alpha_2 \in \mathbb{R}$), they cannot be equivalent. We claim that $\Sigma^{(2,0)}_{2,\boldsymbol{\alpha}}$ and $\Sigma^{(2,0)}_{2,\boldsymbol{\alpha}'}$ are equivalent only if $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$. Indeed, assume there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(2,\mathbb{R})$ such that

$$R_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad R_2 \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} \alpha_1' & 0 \\ 0 & \alpha_2' \\ 0 & 0 \end{bmatrix}.$$

Then $K \in \mathsf{O}(2)$ and so, by lemma 2, it follows that $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$. On the other hand, assume there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(2,\mathbb{R})$ such that

$$R_1 \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad R_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} \alpha_1' & 0 \\ 0 & \alpha_2' \\ 0 & 0 \end{bmatrix}.$$

Then $\alpha_2 \neq 0$ and $\alpha_2' \neq 0$. Hence, we need only consider the cases:

(i) $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ and $0 < \alpha_2' \leq \alpha_1' < 1$,

(ii) $0 < \alpha_2 \leq \alpha_1 < 1$ and $0 < \alpha_2' \leq \alpha_1' < 1$,

(iii) $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ and $1 \leq \frac{1}{\alpha_2'} \leq \alpha_1'$.

Assume (i) holds. It follows that $R_1 = \begin{bmatrix} S_1 & 0 \\ 0 & \det S_1 \end{bmatrix}$ and $R_2 = \begin{bmatrix} S_2 & 0 \\ 0 & \det S_2 \end{bmatrix}$ for some $S_1, S_2 \in \mathsf{O}(2)$. Thus $K = \begin{bmatrix} \frac{1}{\alpha_1} & 0 \\ 0 & \frac{1}{\alpha_2} \end{bmatrix} S_1^{-1}$ and so

$$S_2 \begin{bmatrix} \frac{1}{\alpha_1} & 0 \\ 0 & \frac{1}{\alpha_2} \end{bmatrix} S_1^{-1} = \begin{bmatrix} \alpha_1' & 0 \\ 0 & \alpha_2' \end{bmatrix}.$$

By applying the mapping $A \mapsto AA^\top$, we get

$$S_2 \begin{bmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{\alpha_2^2} \end{bmatrix} S_2^\top = \begin{bmatrix} {\alpha_1'}^2 & 0 \\ 0 & {\alpha_2'}^2 \end{bmatrix}.$$

As $\frac{1}{\alpha_2} \geq \frac{1}{\alpha_1} \geq 0$ and $\alpha_1' \geq \alpha_2' \geq 0$, it follows that $\alpha_1^2 {\alpha_2'}^2 = 1$ and ${\alpha_1'}^2 \alpha_2^2 = 1$. Hence $\alpha_1' \geq 1$, a contradiction.

Similarly, if (ii) or (iii) hold, then we arrive at a contradiction. $\square$

Corollary 2. *Any four-input homogeneous system is equivalent to exactly one of the systems*

$$\Sigma_1^{(4,0)} : u_1 E_2 + u_2 E_3 + u_3 E_5 + u_4 E_6$$
$$\Sigma_{2,\boldsymbol{\alpha}}^{(4,0)} : u_1(E_4 - \alpha_1 E_1) + u_2(E_5 - \alpha_2 E_2) + u_3 E_3 + u_4 E_6$$

for some $\alpha_1, \alpha_2 \in \mathbb{R}$, where $0 = \alpha_2 \leq \alpha_1$ or $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ or $0 < \alpha_2 \leq \alpha_1 < 1$. Here $\boldsymbol{\alpha}$ parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

Remark 4. $\Sigma_1^{(4,0)}$ is controllable. $\Sigma_{2,\boldsymbol{\alpha}}^{(4,0)}$ is not controllable exactly when $\alpha_1 = \alpha_2 = 0$.

Lemma 3. *For any $M \in \mathbb{R}^{3\times 3}$ there exist $R_1, R_2 \in \mathsf{SO}(3)$ such that*

$$R_1 M R_2 = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)\,,$$

where $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$. Moreover, if $\operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)$ and $\operatorname{diag}(\alpha_1', \alpha_2', \alpha_3')$ are two such matrices and

$$R_1 \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3) R_2 = \operatorname{diag}(\alpha_1', \alpha_2', \alpha_3')$$

for some $R_1, R_2 \in \mathsf{SO}(3)$, then $\alpha_1 = \alpha_1'$, $\alpha_2 = \alpha_2'$, and $\alpha_3 = \alpha_3'$.

Theorem 3. *Any three-input homogeneous system is equivalent to exactly one of the systems*

$$\Sigma_{1,\beta}^{(3,0)} : u_1(E_1 + \beta E_4) + u_2 E_2 + u_3 E_6$$
$$\Sigma_{2,\boldsymbol{\alpha}}^{(3,0)} : u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5) + u_3(E_3 + \alpha_3 E_6)$$

for some $\alpha_1, \alpha_2, \alpha_3, \beta \in \mathbb{R}$, where $0 \leq \beta \leq 1$ and $0 = \alpha_3 \leq \alpha_2 \leq \alpha_1$ or $0 < |\alpha_3| \leq \alpha_2 < 1 \wedge \alpha_2 \leq \alpha_1$ or $\alpha_2 = 1 \leq \frac{1}{|\alpha_3|} \leq \alpha_1$. Here $\boldsymbol{\alpha}$ and β parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.

Remark 5. $\Sigma_{1,\beta}^{(3,0)}$ is controllable exactly when $\beta > 0$. $\Sigma_{2,\boldsymbol{\alpha}}^{(3,0)}$ is not controllable exactly when $\alpha_1 = \alpha_2 = \alpha_3 = 1$ or $\alpha_2 = 0$.

Proof. Let $\Sigma\colon \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ be a three-input system. (Here $M_1, M_2 \in \mathbb{R}^{3\times 3}$.) Clearly either $\max\{\operatorname{rank}(M_1), \operatorname{rank}(M_2)\} = 3$ or $\max\{\operatorname{rank}(M_1), \operatorname{rank}(M_2)\} = 2$. Suppose, $\operatorname{rank}(M_1) = 3$ or $\operatorname{rank}(M_2) = 3$. We may assume $\operatorname{rank}(M_1) = 3$. (If not, consider

$\Sigma\colon \zeta \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$.) Then there exists $K \in \mathsf{GL}(3,\mathbb{R})$ such that $M_1 K = I_3$. Thus Σ is equivalent to $\Sigma'\colon \begin{bmatrix} I_3 \\ M_2' \end{bmatrix}$. By lemma 3, there exist $R_2, K \in \mathsf{SO}(3)$ such that

$$R_2 M_2' K = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3)$$

for some $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$.

If $\alpha_3 = 0$ or $|\alpha_3| \leq \alpha_2 < 1$ or $1 = \alpha_2 \leq \frac{1}{|\alpha_3|} \leq \alpha_1$, then we are done. Suppose $1 < |\alpha_3| \leq \alpha_2 \leq \alpha_1$ or $0 < |\alpha_3| < 1 < \alpha_2 \leq \alpha_1$. If $\alpha_3 > 0$, then

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\frac{1}{\alpha_1} \\ 0 & \frac{1}{\alpha_2} & 0 \\ \frac{1}{\alpha_3} & 0 & 0 \end{bmatrix} = I_3$$

$$\text{and} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\frac{1}{\alpha_1} \\ 0 & \frac{1}{\alpha_2} & 0 \\ \frac{1}{\alpha_3} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\alpha_3} & 0 & 0 \\ 0 & \frac{1}{\alpha_2} & 0 \\ 0 & 0 & \frac{1}{\alpha_1} \end{bmatrix}.$$

If $\alpha_3 < 0$, then

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{\alpha_1} \\ 0 & \frac{1}{\alpha_2} & 0 \\ -\frac{1}{\alpha_3} & 0 & 0 \end{bmatrix} = I_3$$

$$\text{and} \qquad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{\alpha_1} \\ 0 & \frac{1}{\alpha_2} & 0 \\ -\frac{1}{\alpha_3} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\alpha_3} & 0 & 0 \\ 0 & \frac{1}{\alpha_2} & 0 \\ 0 & 0 & -\frac{1}{\alpha_1} \end{bmatrix}.$$

In both cases $0 < \frac{1}{\alpha_1} \leq \frac{1}{\alpha_2} \leq \frac{1}{|\alpha_3|}$. Thus Σ is equivalent to some system $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}'}$ with $0 < |\alpha_3'| \leq \alpha_2' < 1$ and $\alpha_2' \leq \alpha_1'$. Likewise, if $\frac{1}{|\alpha_3|} \geq \alpha_1 \geq \alpha_2 = 1$, then Σ is equivalent to some system $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}'}$ with $1 = \alpha_2' \leq \frac{1}{|\alpha_3'|} \leq \alpha_1'$.

On the other hand, suppose $\operatorname{rank}(M_1) = 2$ and $\operatorname{rank}(M_2) \in \{1, 2\}$. Again, a simple argument shows that Σ is equivalent to some system $\Sigma'\colon \begin{bmatrix} I_{2,0} \\ M_1' \end{bmatrix}$, where $I_{2,0} = \begin{bmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}$. If $\operatorname{rank}(M_1') = 1$, it is easy to show that Σ is equivalent to $\Sigma^{(3,0)}_{1,0}$. Assume that $\operatorname{rank}(M_1') = 2$. Then there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(3,\mathbb{R})$ such that

$$R_1 I_{2,0} K = I_{2,0} \qquad \text{and} \qquad R_2 M_1' K = \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

By the SVD there exist $S_1, S_2 \in \mathsf{O}(2)$ such that $S_2 \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} S_1 = \operatorname{diag}(\beta, 0)$ for some $\beta \geq 0$. Let

$$K' = \begin{bmatrix} S_1 & \mathbf{0} \\ \mathbf{0} & \det S_1 \end{bmatrix} \in \mathsf{SO}(3) \qquad \text{and} \qquad R_2' = \begin{bmatrix} S_2 & \mathbf{0} \\ \mathbf{0} & \det S_2 \end{bmatrix} \in \mathsf{SO}(3)\,.$$

Now

$$(K')^{-1} I_{2,0} K' = I_{2,0} \qquad \text{and} \qquad R_2' \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} K' = \begin{bmatrix} \beta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $\beta \leq 1$, then we are done (i.e., Σ is equivalent to $\Sigma^{(3,0)}_{1,\beta}$). Suppose that $\beta > 1$. Then

$$\zeta \cdot \langle E_1 + \beta E_4, E_2, E_6 \rangle = \left\langle \tfrac{1}{\beta} E_4 + E_1, E_5, E_3 \right\rangle .$$

It is a simple matter to show that there exists an automorphism ψ such that

$$\psi \cdot \left\langle \tfrac{1}{\beta} E_4 + E_1, E_5, E_3 \right\rangle = \left\langle E_1 + \tfrac{1}{\beta} E_4, E_2, E_6 \right\rangle .$$

Thus Σ is equivalent to $\Sigma^{(3,0)}_{1,\beta'}$ for some $0 \leq \beta' \leq 1$.

We now verify that none of these class representatives are equivalent. As the traces of $\Sigma^{(3,0)}_{1,\beta}$ and $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}}$, respectively, do not generate the same subalgebra (for any $\beta, \alpha_1, \alpha_2 \in \mathbb{R}$), they cannot be equivalent. Suppose two systems $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}}$ and $\Sigma^{(3,0)}_{2,\boldsymbol{\alpha}'}$, with $\alpha_1 \geq \alpha_2 \geq |\alpha_3| \geq 0$ and $\alpha_1' \geq \alpha_2' \geq |\alpha_3'| \geq 0$, are equivalent. We claim that $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$. Indeed, assume there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(3, \mathbb{R})$ such that $R_1 I_3 K = I_3$ and $R_2 \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) K = \mathrm{diag}(\alpha_1', \alpha_2', \alpha_3')$. Then, by lemma 3, it follows that $\boldsymbol{\alpha} = \boldsymbol{\alpha}'$. On the other hand, assume there exist $R_1, R_2 \in \mathsf{SO}(3)$ and $K \in \mathsf{GL}(3, \mathbb{R})$ such that $R_1 \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3) K = I_3$ and $R_2 I_3 K = \mathrm{diag}(\alpha_1', \alpha_2', \alpha_3')$. Then $\alpha_1^2 {\alpha_3'}^2 = 1$, $\alpha_2^2 {\alpha_2'}^2 = 1$ and $\alpha_3^2 {\alpha_1'}^2 = 1$. Clearly, $\alpha_3, \alpha_3' \neq 0$. Three possibilities remain, either

(i) $0 < |\alpha_3| \leq \alpha_2 < 1$ and $0 < |\alpha_3'| \leq \alpha_2' < 1$, or

(ii) $0 < |\alpha_3| \leq \alpha_2 < 1$ and $0 < |\alpha_3'| \leq \alpha_2' < 1 \ \wedge \ \alpha_2' \leq \alpha_1'$, or

(iii) $0 < |\alpha_3| \leq \alpha_2 < 1 \ \wedge \ \alpha_2 \leq \alpha_1$ and $0 < |\alpha_3'| \leq \alpha_2' < 1 \ \wedge \ \alpha_2' \leq \alpha_1'$.

Again (as in theorem 2), each case leads to a contradiction. □

Remark 6. There is only one six-dimensional affine subspace of $\mathfrak{so}(4)$, namely $\mathfrak{so}(4)$. Therefore any six-input system is equivalent to the system

$$\Sigma^{(6,0)} \colon u_1 E_1 + u_2 E_2 + u_3 E_3 + u_4 E_4 + u_5 E_5 + u_6 E_6 .$$

Clearly, this system is controllable.

5 Inhomogeneous systems

We now proceed to the classification of the inhomogeneous systems on $\mathsf{SO}(4)$. This classification is, in part, based on our classification of homogeneous systems. As before, we distinguish between the number ℓ of controls involved; this yields five types of systems. (Clearly there are no six-input inhomogeneous systems.) Suppose

$$\Sigma \colon A + u_1 B_1 + \cdots + u_\ell B_\ell$$

is an inhomogeneous system. Then the corresponding homogeneous system

$$\widetilde{\Sigma} \colon u_1 B_1 + \cdots + u_\ell B_\ell$$

is equivalent to exactly one homogeneous class representative Σ^0. Consequently, Σ is equivalent to a system Σ' with parametrization map $\Xi'(\mathbf{1}, u) = A' + \Xi^0(\mathbf{1}, u)$. Such an (arbitrary) system is then further simplified by applying automorphisms preserving the trace Γ^0 of Σ^0. Accordingly, for each homogeneous class representative Σ^0, representatives for the associated class of inhomogeneous systems are identified. We will, in addition, use vectors $\boldsymbol{\varepsilon} = (\varepsilon_i)$ to parametrize class representatives.

Again, it is convenient to write the condition of equivalence in matrix form. An ℓ-input inhomogeneous system specified by

$$\Sigma\colon \sum_{i=1}^{6} a^i E_i + u_1 \sum_{i=1}^{6} b_1^i E_i + \cdots + u_\ell \sum_{i=1}^{6} b_\ell^i E_i$$

will be written (in matrix form) as

$$\Sigma\colon \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} a^1 & b_1^1 & \dots & b_\ell^1 \\ \vdots & \vdots & & \vdots \\ a^6 & b_1^6 & \dots & b_\ell^6 \end{bmatrix}.$$

Here $M_1, M_2 \in \mathbb{R}^{3\times(\ell+1)}$. Two ℓ-input inhomogeneous systems $\Sigma\colon \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$ and $\Sigma'\colon \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}$ are equivalent if and only if there exist an automorphism $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$ and $K \in \mathsf{Aff}(\ell, \mathbb{R})$ such that

$$\psi \cdot \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} K = \begin{bmatrix} M_1' \\ M_2' \end{bmatrix}.$$

Here

$$\mathsf{Aff}(\ell, \mathbb{R}) = \left\{ \begin{bmatrix} 1 & \mathbf{0} \\ \boldsymbol{v} & N \end{bmatrix} : \boldsymbol{v} \in \mathbb{R}^{\ell\times 1},\ N \in \mathsf{GL}(\ell, \mathbb{R}) \right\}.$$

For an inhomogeneous system

$$\Sigma\colon A + u_1 B_1 + \cdots + u_\ell B_\ell\,,$$

with $A = \sum_{i=1}^{6} \varepsilon_i E_i$, it follows that $\sum_{i=1}^{6} \varepsilon_i^2 \neq 0$. We omit this condition in the statements of the theorems throughout this section. A proof sketch is provided for theorem 4 to elucidate the approach used in the classification of inhomogeneous systems. More details are provided in the proof of theorem 5. The proofs of theorems 6, 7, and 8 are similar and shall therefore be omitted.

Theorem 4. *Every single-input inhomogeneous system is equivalent to exactly one of the systems*

$$\Sigma_{\beta\boldsymbol{\varepsilon}}^{(1,1)}\colon A + u_1(E_1 + \beta E_4)$$

for some $0 \leq \beta \leq 1$, *where* *(i) if* $\beta = 0$ *then*

$$A = \varepsilon_2 E_2 + \varepsilon_4 E_4$$

with $\varepsilon_2, \varepsilon_4 \geq 0$, *and*

(ii) if $0 < \beta \leq 1$ *then*

$$A = \varepsilon_2 E_2 + \varepsilon_4 E_4 + \varepsilon_5 E_5$$

with $\varepsilon_2, \varepsilon_4, \varepsilon_5 \geq 0$ *and* $\big((\beta = 1 \wedge \varepsilon_4 = 0) \Rightarrow \varepsilon_2 \geq \varepsilon_5\big)$.

Here β *and* $\boldsymbol{\varepsilon}$ *parametrize a family of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 7. If $\beta = 0$, then $\Sigma^{(1,1)}_{\beta\boldsymbol{\varepsilon}}$ is not controllable. If $\beta > 0$, then $\Sigma^{(1,1)}_{\beta\boldsymbol{\varepsilon}}$ is not controllable exactly when $\varepsilon_2 = 0$ or $\varepsilon_5 = 0$ or $(\varepsilon_2 = \varepsilon_5 \wedge \varepsilon_4 = 0 \wedge \beta = 1)$.

Proof. Let $\Sigma\colon A + u_1 B_1$ be a single-input system. Then, by theorem 1, Σ is equivalent to a system

$$\widehat{\Sigma}\colon \sum_{i=2}^{6} \varepsilon_i E_i + u_1(E_1 + \beta E_4)$$

for some $0 \leq \beta \leq 1$. Suppose $\beta > 0$. Now

$$R_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad R_2 \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix} k = \begin{bmatrix} \beta \\ 0 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad R_1, R_2 \in \mathsf{SO}(3)$$

exactly when $k = \det S_1 = \det S_2$, $R_1 = \begin{bmatrix} \det S_1 & 0 \\ 0 & S_1 \end{bmatrix}$, $R_2 = \begin{bmatrix} \det S_2 & 0 \\ 0 & S_2 \end{bmatrix}$, and $S_1, S_2 \in \mathsf{O}(2)$. Accordingly, there exist $S_1, S_2 \in \mathsf{O}(2)$ such that

$$\begin{bmatrix} \det S_1 & 0 \\ 0 & S_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \varepsilon_2 & 0 \\ \varepsilon_3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \det S_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \varepsilon_2' & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{and} \qquad \begin{bmatrix} \det S_2 & 0 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} \varepsilon_4 & \beta \\ \varepsilon_5 & 0 \\ \varepsilon_6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \det S_1 \end{bmatrix} = \begin{bmatrix} \varepsilon_4' & \beta \\ \varepsilon_5' & 0 \\ 0 & 0 \end{bmatrix}$$

for some $\varepsilon_2', \varepsilon_4', \varepsilon_5' \geq 0$. Therefore Σ is equivalent to the system

$$\Sigma'\colon \varepsilon_2' E_2 + \varepsilon_4' E_4 + \varepsilon_5' E_5 + u_1(E_1 + \beta E_4)\,.$$

Moreover, if $\beta = 1$ and $\varepsilon_4' = 0$, then Σ can be shown to be equivalent to a system

$$\Sigma''\colon \varepsilon_2'' E_2 + \varepsilon_5'' E_5 + u_1(E_1 + E_4)$$

for some $\varepsilon_2'' \geq \varepsilon_5'' \geq 0$.

Likewise, if $\beta = 0$, it follows that Σ is equivalent to a system

$$\Sigma'\colon \varepsilon_2' E_2 + \varepsilon_4' E_4 + u_1 E_1$$

for some $\varepsilon_2', \varepsilon_4' \geq 0$. (Again, as in the homogeneous case, one verifies that all the systems obtained are distinct and non-equivalent.) □

Theorem 5. *Every two-input inhomogeneous system is equivalent to exactly one of the systems*

(1) $$\Sigma^{(2,1)}_{1,\boldsymbol{\varepsilon}}\colon \varepsilon_2 E_2 + \varepsilon_5 E_5 + u_1 E_1 + u_2 E_4 \quad \text{with} \quad \varepsilon_2 \geq \varepsilon_5 \geq 0$$

(2) $$\Sigma^{(2,1)}_{2,\boldsymbol{\alpha\varepsilon}}\colon A + u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5)$$

with $\alpha_1 \geq \alpha_2 = 0$ *or* $1 \leq \frac{1}{\alpha_2} \leq \alpha_1$ *or* $0 < \alpha_2 \leq \alpha_1 < 1$, *where*

(i) if $\alpha_1 = \alpha_2 = 0$ *then*
$$A = \varepsilon_3 E_3 + \varepsilon_4 E_4$$
with $\varepsilon_3, \varepsilon_4 \geq 0$, *and*

(ii) if $\alpha_1 = \alpha_2 > 0$ *then*
$$A = \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_6 E_6$$
with $\varepsilon_3 = 0 \Rightarrow \varepsilon_6 \geq 0$, $\varepsilon_6 \in \mathbb{R}$, $\varepsilon_3, \varepsilon_4 \geq 0$, *and*

(iii) if $\alpha_1 > \alpha_2 = 0$ *then*
$$A = \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_5 E_5 + \varepsilon_6 E_6$$
with $(\varepsilon_4 = 0 \vee \varepsilon_5 = 0) \Rightarrow \varepsilon_6 \geq 0$, $\varepsilon_6 \in \mathbb{R}$, $\varepsilon_3, \varepsilon_4, \varepsilon_5 \geq 0$, *and*

(iv) if $\alpha_1 > \alpha_2 > 0$ *then*
$$A = \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_5 E_5 + \varepsilon_6 E_6$$
with $(\varepsilon_3, \varepsilon_4 > 0) \vee (\varepsilon_3 > 0 \wedge \varepsilon_5 \geq 0) \vee (\varepsilon_4, \varepsilon_5 \geq 0) \vee (\varepsilon_5, \varepsilon_6 \geq 0)$, $\varepsilon_5, \varepsilon_6 \in \mathbb{R}$, $\varepsilon_3, \varepsilon_4 \geq 0$.

Here $\boldsymbol{\alpha}$ *and* $\boldsymbol{\varepsilon}$ *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 8. $\Sigma^{(2,1)}_{1,\boldsymbol{\varepsilon}}$ is controllable exactly when $\varepsilon_5 \neq 0$. $\Sigma^{(2,1)}_{2,\boldsymbol{\alpha\varepsilon}}$ is not controllable exactly when $\alpha_2 = 0 \wedge (\alpha_1 = 0 \vee \varepsilon_5 = \varepsilon_6 = 0)$ or $\alpha_1 = \alpha_2 = 1 \wedge \varepsilon_4 = 0 \wedge \varepsilon_3 = \varepsilon_6$.

Proof. Let $\Sigma\colon A + u_1 B_1 + u_2 B_2$ be a two-input system. Then, by theorem 2, Σ is equivalent either to
$$\widehat{\Sigma}_1\colon \sum_{i=1}^{6} \varepsilon_i E_i + u_1 E_1 + u_2 E_4$$
or
$$\widehat{\Sigma}_2\colon \sum_{i=3}^{6} \varepsilon_i E_i + u_1(E_1 + \alpha_1 E_4) + u_2(E_2 + \alpha_2 E_5)\,.$$
It is easy to show that $\widehat{\Sigma}_1$ is equivalent to $\Sigma^{(2,1)}_{1,\boldsymbol{\varepsilon}}$. Suppose Σ is equivalent to $\widehat{\Sigma}_2$ and $\alpha_1 > \alpha_2 > 0$. Now
$$R_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad R_2 \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix} N = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \\ 0 & 0 \end{bmatrix}$$

$R_1, R_2 \in \mathsf{SO}(3)$, and $N \in \mathsf{GL}(2,\mathbb{R})$ exactly when $N = S$, $R_1 = R_2 = \begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix}$, and $S = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$, $\sigma_1, \sigma_2 \in \{-1, 1\}$. Accordingly, (a tedious but straightforward computation shows that) there exists $\sigma_1, \sigma_2 \in \{-1, 1\}$ such that

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3' & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_1\sigma_2 \end{bmatrix} \begin{bmatrix} \varepsilon_4 & \alpha_1 & 0 \\ \varepsilon_5 & 0 & \alpha_2 \\ \varepsilon_6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \varepsilon_4' & \alpha_1 & 0 \\ \varepsilon_5' & 0 & \alpha_2 \\ \varepsilon_6' & 0 & 0 \end{bmatrix}$$

where $\varepsilon_3', \varepsilon_4' \geq 0$ and $(\varepsilon_3' = 0 \vee \varepsilon_4' = 0) \Rightarrow \varepsilon_5' \geq 0$ and $\varepsilon_3' = \varepsilon_4' = 0 \Rightarrow (\varepsilon_5', \varepsilon_6' \geq 0)$ and $\varepsilon_3' = \varepsilon_5' = 0 \Rightarrow \varepsilon_6' \geq 0$. These conditions are equivalent to those given in the theorem.

On the other hand, suppose Σ is equivalent to $\widehat{\Sigma}_2$ and $\alpha_1 = \alpha_2 > 0$. Then

$$R_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad R_2 \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \\ 0 & 0 \end{bmatrix} N = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \\ 0 & 0 \end{bmatrix}$$

$R_1, R_2 \in \mathsf{SO}(3)$, and $N \in \mathsf{GL}(2,\mathbb{R})$ exactly when $N = S^\top$, $R_1 = R_2 = \begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix}$, and $S \in \mathsf{O}(2)$. Therefore there exists $S \in \mathsf{O}(2)$ such that

$$\begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & S^\top \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_3' & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} S & 0 \\ 0 & \det S \end{bmatrix} \begin{bmatrix} \varepsilon_4 & \alpha_1 & 0 \\ \varepsilon_5 & 0 & \alpha_1 \\ \varepsilon_6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & S^\top \end{bmatrix} = \begin{bmatrix} \varepsilon_4' & \alpha_1 & 0 \\ 0 & 0 & \alpha_1 \\ \varepsilon_6' & 0 & 0 \end{bmatrix}$$

where $\varepsilon_3', \varepsilon_4' \geq 0$ and $\varepsilon_3' = 0 \Rightarrow \varepsilon_6' \geq 0$.

The (families of) equivalence representatives 2(i) and 2(iii) are obtained similarly. (Again, as in the homogeneous case, one verifies that all the systems obtained are distinct and non-equivalent.) □

Theorem 6. *Every three-input inhomogeneous system is equivalent to exactly one of the systems*

(1) $\Sigma^{(3,1)}_{1,\beta\boldsymbol{\varepsilon}} : A + u_1(E_1 + \beta E_4) + u_2 E_2 + u_3 E_6$ *with* $0 \leq \beta \leq 1$, *where*

(i) if $\beta = 0$ *then*

$$A = \varepsilon_3 E_3 + \varepsilon_4 E_4 \quad \textit{with} \quad \varepsilon_3, \varepsilon_4 \geq 0\,,$$

(ii) if $0 < \beta \leq 1$ *then*

$$A = \varepsilon_3 E_3 + \varepsilon_4 E_4 + \varepsilon_5 E_5$$

with $\big((\varepsilon_4 = 0 \wedge \beta = 1) \Rightarrow \varepsilon_3 \geq \varepsilon_5\big)$, $\varepsilon_3, \varepsilon_4, \varepsilon_5 \geq 0$.

(2) $\Sigma^{(3,1)}_{2,\boldsymbol{\alpha\varepsilon}} \colon A + u_1(E_1+\alpha_1 E_4) + u_2(E_2+\alpha_2 E_5) + u_3(E_3+\alpha_3 E_6)$
with $0 = \alpha_3 \le \alpha_2 \le \alpha_1$ *or* $0 < |\alpha_3| \le \alpha_2 < 1 \wedge \alpha_2 \le \alpha_1$ *or* $\alpha_2 = 1 \le \frac{1}{|\alpha_3|} \le \alpha_1$, *where*

(i) *if* $\alpha_1 = \alpha_2 = |\alpha_3|$ *then*

$$A = \varepsilon_4 E_4 \quad \textit{with} \quad \varepsilon_4 \ge 0\,,$$

(ii) *if* $\alpha_1 > \alpha_2 = |\alpha_3|$ *then*

$$A = \varepsilon_4 E_4 + \varepsilon_5 E_5 \quad \textit{with} \quad \varepsilon_4, \varepsilon_5 \ge 0\,,$$

(iii) *if* $\alpha_1 = \alpha_2 > |\alpha_3|$ *then*

$$A = \varepsilon_4 E_4 + \varepsilon_6 E_6 \quad \textit{with} \quad \varepsilon_4, \varepsilon_6 \ge 0\,,$$

(iv) *if* $\alpha_1 > \alpha_2 > |\alpha_3|$ *then*

$$A = \varepsilon_4 E_4 + \varepsilon_5 E_5 + \varepsilon_6 E_6 \quad \textit{with} \quad \varepsilon_6 \in \mathbb{R}\,,\ \varepsilon_4, \varepsilon_5 \ge 0\,.$$

Here $\boldsymbol{\alpha}$, β *and* $\boldsymbol{\varepsilon}$ *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 9. $\Sigma^{(3,1)}_{1,\beta\boldsymbol{\varepsilon}}$ is controllable exactly when $\beta \neq 0$ or $\varepsilon_4 \neq 0$. $\Sigma^{(3,1)}_{2,\boldsymbol{\alpha\varepsilon}}$ is not controllable exactly when $\alpha_2 = 0$ and $(\alpha_1 = 0 \vee \varepsilon_5 = 0)$.

Theorem 7. *Every four-input inhomogeneous system is equivalent to exactly one of the systems*

(1) $\Sigma^{(4,1)}_{1,\boldsymbol{\varepsilon}} \colon \varepsilon_1 E_1 + \varepsilon_4 E_4 + u_1 E_2 + u_2 E_3 + u_3 E_5 + u_4 E_6 \quad \textit{with} \quad \varepsilon_1 \ge \varepsilon_4 \ge 0$

(2) $\Sigma^{(4,1)}_{2,\boldsymbol{\alpha\varepsilon}} \colon A + u_1(E_4 - \alpha_1 E_1) + u_2(E_5 - \alpha_2 E_2) + u_3 E_3 + u_4 E_6$
with $\alpha_1 \ge \alpha_2 = 0$ *or* $1 \le \frac{1}{\alpha_2} \le \alpha_1$ *or* $0 < \alpha_2 \le \alpha_1 < 1$, *where*

(i) *if* $\alpha_1 = \alpha_2$ *then*

$$A = \varepsilon_1 E_1 \quad \textit{with} \quad \varepsilon_1 \ge 0\,,$$

(ii) *if* $\alpha_1 > \alpha_2$ *then*

$$A = \varepsilon_1 E_1 + \varepsilon_2 E_2 \quad \textit{with} \quad \varepsilon_1, \varepsilon_2 \ge 0\,.$$

Here $\boldsymbol{\alpha}$ *and* $\boldsymbol{\varepsilon}$ *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 10. Every four-input inhomogeneous system is controllable.

Theorem 8. *Every five-input inhomogeneous system is equivalent to exactly one of the systems*

$$\Sigma^{(5,1)}_{\beta\boldsymbol{\varepsilon}} \colon \varepsilon_1 E_1 + u_1(E_4 - \beta E_1) + u_2 E_2 + u_3 E_3 + u_4 E_5 + u_5 E_6$$

with $0 \le \beta \le 1$, $\varepsilon_1 \ge 0$. *Here* β *and* $\boldsymbol{\varepsilon}$ *parametrize families of class representatives, each different value corresponding to a distinct non-equivalent representative.*

Remark 11. Every five-input inhomogeneous system is controllable.

6 Conclusion

We have classified all left-invariant control affine systems on the orthogonal group $\mathsf{SO}(4)$ (cf. [1]). Specifically, we have shown that any system is equivalent to exactly one of a list of equivalence representatives. In addition, we have identified exactly which of the representative systems are controllable.

As a simple by-product of the classification of homogeneous systems, we recover a classification of subalgebras of $\mathfrak{so}(4)$. (Two subalgebras $\mathfrak{a}_1, \mathfrak{a}_2 \subset \mathfrak{so}(4)$ are equivalent if there exists $\psi \in \mathsf{Aut}(\mathfrak{so}(4))$ such that $\psi \cdot \mathfrak{a}_1 = \mathfrak{a}_2$). Any (non-trivial) subalgebra of $\mathfrak{so}(4)$ is equivalent to exactly one of the following subalgebras

$$\begin{aligned}
\mathfrak{a}^{(1)}_\alpha &= \langle E_1 + \alpha E_4 \rangle && = \varsigma \cdot \langle (\mathbf{E}_1, \alpha \mathbf{E}_1) \rangle \\
\mathfrak{a}^{(2)} &= \langle E_1, E_4 \rangle && = \varsigma \cdot \langle \mathbf{E}_1 \rangle \oplus \langle \mathbf{E}_1 \rangle \\
\mathfrak{a}^{(3)}_1 &= \langle E_1, E_2, E_3 \rangle && = \varsigma \cdot \mathfrak{so}(3) \oplus \{\mathbf{0}\} \\
\mathfrak{a}^{(3)}_2 &= \langle E_1 + E_4, E_2 + E_5, E_4 + E_6 \rangle && = \varsigma \cdot \{(A, A) : A \in \mathfrak{so}(3)\} \\
\mathfrak{a}^{(4)} &= \langle E_1, E_2, E_3, E_4 \rangle && = \varsigma \cdot \mathfrak{so}(3) \oplus \langle \mathbf{E}_1 \rangle \, .
\end{aligned}$$

Here $0 \leq \alpha \leq 1$ parametrizes a family of nonequivalent class representatives. (Only $\mathfrak{a}^{(3)}_1$ is an ideal.)

The classification of (controllable) systems should prove useful in the study of certain classes of invariant optimal control problems on $\mathsf{SO}(4)$. Generally, an (affine quadratic) invariant optimal control problem is given by the specification of

(1) a left-invariant control system $\Sigma = (\mathsf{G}, \Xi)$

(2) an affine quadratic cost function $\chi : \mathbb{R}^\ell \to \mathbb{R}$, $u \mapsto \mathcal{Q}(u - \mu)$ (here $\mathcal{Q}$ is assumed positive definite and $\mu \in \mathbb{R}^\ell$)

(3) boundary data (g_0, g_1, T), consisting of an initial state $g_0 \in \mathsf{G}$, a target state $g_1 \in \mathsf{G}$, and a (usually fixed) terminal time $T > 0$.

Explicitly, we want to *minimize* the functional

$$\mathcal{J} = \int_0^T \chi(u(t))\, \mathrm{d}t$$

over the trajectories of Σ subject to the boundary conditions. The equivalence of such problems has been considered in [8], [10]; this is called *cost equivalence*. It establishes a one-to-one correspondence between the associated optimal trajectories (resp. associated extremal curves) of equivalent problems. For two cost equivalent problems, the underlying left-invariant control systems must be equivalent. Hence (once a classification of systems has been found), only the transformations leaving each system invariant need be considered when investigating cost equivalence.

Some specific (invariant) optimal control problems on $\mathsf{SO}(4)$ have been studied by diverse authors in several contexts. For instance, D'Alessandro studied a particular (time) optimal control problem associated with a homogeneous three-input control affine system in the context of quantum control ([15]), whereas Puta et al. considered a particular optimal control problem for a homogeneous four-input control affine system in the broad context of motion control ([4]). Recently, Holderbaum et al. made attempts to compare different trajectories of some particular control systems on $\mathsf{SE}(3), \mathsf{SO}(1,3)$ and $\mathsf{SO}(4)$, in the context of rigid body dynamics ([5], [23]). Various variational problems associated with $\mathsf{SO}(4)$ (and its Lie algebra), like the Kowalewki's top or the integrable Suslin problem, have also been treated (see, e.g., [18], [22]). With a classification of controllable systems at hand a more unified approach to control problems on $\mathsf{SO}(4)$ may be feasible. This is a topic for future research.

Invariant optimal control problems naturally give rise to Hamilton-Poisson systems, via the Pontryagin Maximum Principle. Moreover, if two invariant optimal control problems are cost equivalent, then the associated Hamilton-Poisson systems are linearly equivalent ([8], [10]). In the context of Hamiltonian systems, Raţiu et al. studied the stability of equilibria for the $\mathfrak{so}(4)$ free rigid body ([13]). Furthermore, integrability (and explicit integration) of certain Euler equations on $\mathfrak{so}(4)$ and their physical applications were considered in [14], whereas (general) integrable quadratic Hamiltonians on $\mathfrak{so}(4)$ were also studied in [27].

Appendix: Classification of systems on SO(4) in matrix form

In the following tables, the homogeneous systems correspond to $A = 0$.

Single-input		
	$\Xi^0(\mathbf{1}, u)$	A
$\Sigma^{(1,1)}_{\beta\boldsymbol{\varepsilon}}$	$\begin{bmatrix}1\\0\\0\\\beta\\0\\0\end{bmatrix}$	$\underset{\beta=0}{\begin{bmatrix}0\\\varepsilon_2\\0\\\varepsilon_4\\0\\0\end{bmatrix}}$ $\underset{0<\beta\leq 1}{\begin{bmatrix}0\\\varepsilon_2\\0\\\varepsilon_4\\\varepsilon_5\\0\end{bmatrix}}$

Two-input		
	$\Xi^0(\mathbf{1}, u)$	A
$\Sigma^{(2,1)}_{1,\boldsymbol{\varepsilon}}$	$\begin{bmatrix}1 & 0\\0 & 0\\0 & 0\\0 & 1\\0 & 0\\0 & 0\end{bmatrix}$	$\begin{bmatrix}0\\\varepsilon_2\\0\\0\\\varepsilon_5\\0\end{bmatrix}$
$\Sigma^{(2,1)}_{2,\boldsymbol{\alpha\varepsilon}}$	$\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\\\alpha_1 & 0\\0 & \alpha_2\\0 & 0\end{bmatrix}$	$\underset{\alpha_1=\alpha_2=0}{\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\0\\0\end{bmatrix}}$ $\underset{\alpha_1=\alpha_2>0}{\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\0\\\varepsilon_6\end{bmatrix}}$ $\underset{\alpha_1>\alpha_2=0}{\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}}$ $\underset{\alpha_1>\alpha_2>0}{\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}}$

Three-input		
	$\Xi^0(\mathbf{1},u)$	A
$\Sigma^{(3,1)}_{1,\boldsymbol{\alpha\varepsilon}}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\\beta&0&0\\0&0&0\\0&0&1\end{bmatrix}$	$\underset{\beta=0}{\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\0\\0\end{bmatrix}}$ $\underset{0<\beta\leq1}{\begin{bmatrix}0\\0\\\varepsilon_3\\\varepsilon_4\\\varepsilon_5\\0\end{bmatrix}}$
$\Sigma^{(3,1)}_{2,\boldsymbol{\alpha\varepsilon}}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\\alpha_1&0&0\\0&\alpha_2&0\\0&0&\alpha_3\end{bmatrix}$	$\underset{\alpha_1=\alpha_2=\lvert\alpha_3\rvert}{\begin{bmatrix}0\\0\\0\\\varepsilon_4\\0\\0\end{bmatrix}}$ $\underset{\alpha_1>\alpha_2=\lvert\alpha_3\rvert}{\begin{bmatrix}0\\0\\0\\\varepsilon_4\\\varepsilon_5\\0\end{bmatrix}}$ $\underset{\alpha_1=\alpha_2>\lvert\alpha_3\rvert}{\begin{bmatrix}0\\0\\0\\\varepsilon_4\\0\\\varepsilon_6\end{bmatrix}}$ $\underset{\alpha_1>\alpha_2>\lvert\alpha_3\rvert}{\begin{bmatrix}0\\0\\0\\\varepsilon_4\\\varepsilon_5\\\varepsilon_6\end{bmatrix}}$

Four-input		
	$\Xi^0(\mathbf{1},u)$	A
$\Sigma^{(4,1)}_{1,\boldsymbol{\varepsilon}}$	$\begin{bmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$	$\begin{bmatrix}\varepsilon_1\\0\\0\\\varepsilon_4\\0\\0\end{bmatrix}$
$\Sigma^{(4,1)}_{2,\boldsymbol{\alpha\varepsilon}}$	$\begin{bmatrix}-\alpha_1&0&0&0\\0&-\alpha_2&0&0\\0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$	$\underset{\alpha_1=\alpha_2}{\begin{bmatrix}\varepsilon_1\\0\\0\\0\\0\\0\end{bmatrix}}$ $\underset{\alpha_1>\alpha_2}{\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\0\\0\\0\\0\end{bmatrix}}$

Five-input		
	$\Xi^0(\mathbf{1},u)$	A
$\Sigma^{(5,1)}_{\beta\boldsymbol{\varepsilon}}$	$\begin{bmatrix}-\beta&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\1&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$	$\begin{bmatrix}\varepsilon_1\\0\\0\\0\\0\\0\end{bmatrix}$

Six-input	
	$\Xi^0(\mathbf{1}, u)$
$\Sigma^{(6,0)}_{\beta\boldsymbol{\varepsilon}}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$\Sigma \colon A + \Xi^0(\mathbf{1}, u) \quad \Xi^0(\mathbf{1}, u) = u_1 B_1 + \cdots + u_\ell B_\ell \longleftrightarrow \begin{bmatrix} B_1 & \cdots & B_\ell \end{bmatrix}$$

References

[1] R.M. Adams, R. Biggs, C.C. Remsing: On the equivalence of control systems on the orthogonal group $\mathsf{SO}(4)$. *Recent Researches in Automatic Control, Systems Science and Communications* (2012) 54–59.

[2] R.M. Adams, R. Biggs, C.C. Remsing: Equivalence of control systems on the Euclidean group $\mathsf{SE}(2)$. *Control Cybernet.* 41 (2012) 513–524.

[3] A.A. Agrachev, Y.L. Sachkov: *Control Theory from the Geometric Viewpoint*. Springer (2004).

[4] A. Aron, I. Moş, A. Csaky, M. Puta: An optimal control problem on the Lie group $\mathsf{SO}(4)$. *Int. J. Geom. Methods Mod. Phys.* 5 (2008) 319–327.

[5] J.D. Biggs, W. Holderbaum: Singularities of optimal control problems on some six dimensional Lie groups. *IEEE Trans. Automat. Control* 52 (2007) 1027–1038.

[6] R. Biggs, C.C. Remsing: A category of control systems. *An. Şt. Univ. Ovidius Constanţa* 20 (2012) 355–368.

[7] R. Biggs, C.C. Remsing: A note on the affine subspaces of three-dimensional Lie algebras. *Bul. Acad. Ştiinţe Repub. Mold. Mat.* (3) (2012) 45–52.

[8] R. Biggs, C.C. Remsing: On the equivalence of cost-extended control systems on Lie groups. *Recent Researches in Automatic Control, Systems Science and Communications* (2012) 60–65.

[9] R. Biggs, C.C. Remsing: Control affine systems on semisimple three-dimensional Lie groups. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat.* 59 (2013) 399–414.

[10] R. Biggs, C.C. Remsing: Cost-extended control systems on Lie groups. To appear in Mediterr. J. Math. DOI: 10.1007/s00009-013-0355-0.

[11] R. Biggs, C.C. Remsing: Equivalence of control systems on the pseudo-orthogonal group $\mathsf{SO}(2,1)_0$. preprint.

[12] R. Biggs, C.C. Remsing: On the equivalence of control systems on Lie groups. preprint.

[13] P. Birtea, I. Caşu, T.S. Raţiu, M. Turhan: Stability of equilibria for the $\mathfrak{so}(4)$ free rigid body. *J. Nonlinear Sci.* 22 (2012) 187–212.

[14] O.I. Bogoyavlensky: Integrable Euler equations on $\mathsf{SO}(4)$ and their physical applications. *Commun. Math. Phys.* 93 (1984) 417–436.

[15] D. D'Alessandro: The optimal control problem on $\mathsf{SO}(4)$ and its applications to quantum control. *IEEE Trans. Automat. Control* 47 (2002) 87–92.

[16] A. Isidori: *Nonlinear Control Systems (Third Edition)*. Springer (1995).

[17] B. Jakubczyk: Equivalence and invariants of nonlinear control systems. *Nonlinear Controllability and Optimal Control* (1990) 177–218.

[18] B. Jovanović: Non-holonomic geodesic flows on Lie groups and the integrable Suslov problem on $\mathsf{SO}(4)$. *J. Phys. A: Math. Gen.* 31 (1998) 1415–1422.

[19] V. Jurdjevic: *Geometric Control Theory*. Cambridge University Press (1997).

[20] V. Jurdjevic, H.J. Sussmann: Control systems on Lie groups. *J. Diff. Equations* 12 (1972) 313–329.

[21] A. W. Knapp: *Lie Groups Beyond an Introduction (Second Edition)*. Birkhäuser (2005).

[22] I.V. Komarov, V.B. Kuznetsov: Kowalewski's top on the Lie algebras $\mathfrak{o}(4), \mathfrak{e}(3)$ and $\mathfrak{o}(3,1)$. *J. Phys. A: Math. Gen.* 23 (1990) 841–846.

[23] C. Linton, W. Holderbaum, J. Biggs: Rigid body trajectories in different 6D spaces. *ISRN Math. Phys.* (2012).

[24] H. Nijmeijer, A. van der Schaft: *Nonlinear Dynamical Control Systems*. Springer (1996).

[25] W. Respondek, I.A. Tall: Feedback equivalence of nonlinear control systems: a survey on formal approach. *Chaos in Automatic Control* (2006) 137–262.

[26] Y.L. Sachkov: Control theory on Lie groups. *J. Math. Sci.* 156 (2009) 381–439.

[27] V.V. Sokolov, T. Wolf: Integrable quadratic classical Hamiltonians on $\mathfrak{so}(4)$ and $\mathfrak{so}(1,3)$. *J. Phys. A: Math. Gen.* 39 (2006) 1915–1926.

[28] H.J. Sussmann: Lie brackets, real analyticity and geometric control. In: R.W. Brockett, R.S. Millman, H.J. Sussmann: *Differential Geometric Control Theory*. Birkhäuser (1983) 1–116.

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On some issues concerning polynomial cycles

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Abstract. We consider two issues concerning polynomial cycles. Namely, for a discrete valuation domain R of positive characteristic (for $N \geq 1$) or for any Dedekind domain R of positive characteristic (but only for $N \geq 2$), we give a closed formula for a set $\mathcal{CYCL}(R, N)$ of all possible cycle-lengths for polynomial mappings in R^N. Then we give a new property of sets $\mathcal{CYCL}(R, 1)$, which refutes a kind of conjecture posed by W. Narkiewicz.

1 Introduction

For a commutative ring R with unity and $\Phi = (\Phi_1, \dots, \Phi_N)$, where $\Phi_i \in R[X_1, \dots, X_N]$, we define a *cycle* for Φ as a k-tuple $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{k-1}$ of different elements of R^N such that

$$\Phi(\bar{x}_0) = \bar{x}_1,\ \Phi(\bar{x}_1) = \bar{x}_2\,,\ \dots\ ,\ \Phi(\bar{x}_{k-1}) = \bar{x}_0\,.$$

The number k is called the *length* of this cycle.

Let $\mathcal{CYCL}(R, N)$ be the set of all possible cycle-lengths for polynomial mappings in N variables with coefficients from R (we clearly assume that the elements of the considered cycles lie in R^N). For a material on various aspects of polynomial mappings and arithmetic of dynamical systems, see [1] and [4].

In Section 2 we examine $\mathcal{CYCL}(R, N)$ for a discrete valuation domain R with maximal ideal P. We assume that the residue field R/P has p^f elements (if R/P is infinite, then $\mathcal{CYCL}(R, N) = \mathbf{N}$). It is known (see Fact 1 in Section 2) that any element $k \in \mathcal{CYCL}(R, N)$ is of the form $k = a \cdot p^\alpha$, where all possible a were completely determined by the author. Thus, in order to know $\mathcal{CYCL}(R, N)$ it suffices for a given 'possible' a (as explained before) to find all α such that $a \cdot p^\alpha \in \mathcal{CYCL}(R, N)$. It is known that for a finite ramification index e the numbers α are bounded from above by some explicit function depending on e, p, f, N. In Theorem 1 we give a bound from below (for a given 'possible' a) for the biggest α

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such that $a \cdot p^\alpha \in \mathcal{CYCL}(R,N)$. Namely, we receive

$$\alpha \geq \left\lfloor \log_p\left(\frac{\log_p e}{2fN}\right)\right\rfloor .$$

We see that for fixed f, p, N the right-hand side of the last inequality grows to ∞ (when $e \to \infty$). Note that for a discrete valuation domain R the set $\mathcal{CYCL}(R,N)$ does not depend solely on p, e, f, N. Sometimes some subtler properties of R should be taken into account.

As a consequence of Theorem 1, in Theorem 2 we determine the sets $\mathcal{CYCL}(S,N)$ for some Dedekind domains S of positive characteristic and some N.

In Section 3 we consider properties of $\mathcal{A} := \mathcal{CYCL}(R,1)$ for a domain R. Any such $\mathcal{A}$ satisfies the following three 'obvious' properties:

(i) $1, 2 \in \mathcal{A}$;

(ii) $\mathcal{A}$ is closed under taking divisors;

(iii) for any prime p from $p \in \mathcal{A}$ it follows that $[1,p] \subseteq \mathcal{A}$.

Since there were no other obvious properties of $\mathcal{A}$, in mid-nineties W. Narkiewicz conjectured that for $\mathcal{A} \subseteq \mathbf{N}$ satisfying (i), (ii), (iii) there exists a domain R such that $\mathcal{A} = \mathcal{CYCL}(R,1)$. In Section 3 we give a negative answer to this question.

I think that it would be interesting to give a sensible conjecture concerning sets $\mathcal{CYCL}(R,N)$ for $N \geq 2$. In particular it is not clear whether the above property (iii) holds in this case.

2 Finding $\mathcal{CYCL}(R,N)$ for some rings of positive characteristic

Let R be a discrete valuation domain of any characteristic, and P is the unique maximal ideal of R. We assume that the field R/P is finite and has p^f elements (for prime p). Let π be a generator of the principal ideal P, and let v be the norm of R, normalized so that $v(\pi) = 1/p$. We denote by w the corresponding exponent, defined by

$$w(x) = -\frac{\log v(x)}{\log p} \quad \text{for } x \neq 0\,, \quad \text{and} \quad w(0) = \infty\,.$$

We put $e := w(p)$. Thus e is the ramification index of R. We extend w to R^N by putting $w(x_1, \ldots, x_N) = \min\{w(x_1), \ldots, w(x_N)\}$.

A polynomial cycle $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{k-1}$ is called a *(polynomial) $\star$-cycle* if

$$w(\bar{x}_i - \bar{x}_j) \geq 1 \quad \text{for all } i, j\,.$$

Let $\mathcal{CYCL}\star(R,N)$ be the set of all possible lengths of $\star$-cycles for polynomial mappings in N variables with coefficients from R.

In the fact below we collect some properties of $\mathcal{CYCL}(R,N)$ already proved by the author (see [2], [3]).

Fact 1. *Let $R, p, f, \ldots$ be as above. Then*

(i) a number k lies in $\mathcal{CYCL}(R,N)$ if and only if $k = ab$, where $a \leq p^{fN}$ and b is the length of a suitable $\star$-cycle in R^N.

(ii) If $\widehat{R}$ is the completion of R with respect to the norm v, then

$$\mathcal{CYCL}(R,N) = \mathcal{CYCL}(\widehat{R},N) \quad \text{and} \quad \mathcal{CYCL}\star(R,N) = \mathcal{CYCL}\star(\widehat{R},N)$$

(note that for $\widehat{R}$ the numbers p, e, f are the same as for R).

(iii) Let m be a positive integer not divisible by p. Then there is a $\star$-cycle of length m in R^N if and only if there are $r > 0$ and positive integers $a_1, \ldots, a_r$ with $a_1 + \cdots + a_r \le N$ such that m divides $[p^{fa_1} - 1, \ldots, p^{fa_r} - 1]$.

(iv) Let S be a Dedekind domain, and let $\mathcal{P}(S)$ denote the family of all nonzero prime ideals of S. If $N \ge 2$, then

$$\mathcal{CYCL}(S,N) = \bigcap_{\mathfrak{p}\in\mathcal{P}(S)} \mathcal{CYCL}(S_{\mathfrak{p}},N) = \bigcap_{\mathfrak{p}\in\mathcal{P}(S)} \mathcal{CYCL}(\widehat{S_{\mathfrak{p}}},N),$$

where $\widehat{S_{\mathfrak{p}}}$ is the completion of $S_{\mathfrak{p}}$ with respect to the obvious valuation.

In particular, to find $\mathcal{CYCL}(R,N)$ it suffices to know p^f and, for each m dividing $[p^{fa_1} - 1, \ldots, p^{fa_r} - 1]$ (for some $a_1, \ldots, a_r$ satisfying $a_1 + \cdots + a_r \le N$), to know for which $n \ge 0$ the number $m \cdot p^n$ lies in $\mathcal{CYCL}\star(R,N)$.

In this section we will prove that for any 'possible' m, as explained in Fact 1(iii), and for any n if the ramification index is sufficiently large, then $m{\cdot}p^n \in \mathcal{CYCL}\star(R,N)$. This, in turn, gives a closed formula for $\mathcal{CYCL}(S,N)$ for a Dedekind domain S of positive characteristic and $N \ge 2$. The fact that for any prime p and any $n \ge 0$ in $F_p[[X]]$ there are cycles of length p^n was established in the thesis of Zieve [5], who quoted an example due to Poonen.

Theorem 1. *Let R be as in this section. Let m be a divisor of $[p^{fa_1}-1, \ldots, p^{fa_r}-1]$ for some $a_1, \ldots, a_r$ satisfying $a_1 + \cdots + a_r \le N$. If $e \ge p^{2fNp^n}$, then $m \cdot p^n \in \mathcal{CYCL}\star(R,N)$.*

Proof. Owing to Fact 1, we may assume that $n \ge 1$ and R is complete. It suffices to take $m = [p^{fa_1} - 1, \ldots, p^{fa_r} - 1]$. Suppose that for $e \ge p^{2fNp^n}$ we have a $\star$-cycle of length $(p^{fa_1} - 1) \cdot p^n$ for a map $\Phi_1 \colon R^{a_1} \to R^{a_1}$. For $i \ge 2$, by Fact 1(iii), in R^{a_i} there is a $\star$-cycle of length $p^{fa_i} - 1$ for some mapping Φ_i of R^{a_i}. We see that $\Phi = (\Phi_1, \ldots, \Phi_r) \colon R^{a_1+\cdots+a_r} \longrightarrow R^{a_1+\cdots+a_r}$ constructed in the natural way has a $\star$-cycle of length $[(p^{fa_1} - 1) \cdot p^n, p^{fa_2} - 1, \ldots, p^{fa_r} - 1] = m \cdot p^n$.

So, it suffices to prove for any $M \le N$ that $(p^{fM} - 1) \cdot p^n \in \mathcal{CYCL}\star(R,M)$.

Put $q = p^{fM}$, and let ξ be a primitive root of unity of order $q-1$. By the usual Hensel's lemma (here we use the completeness of R) we have that the minimal over R polynomial of ξ is of degree M. Thus $R^M \sim R[\xi]$ as modules over R. Using this canonical isomorphism, we obtain that to any polynomial $F(X) \in R[\xi][X]$ there corresponds a polynomial mapping $\Phi \colon R^M \to R^M$ with coefficients from R. One can see that $R[\xi]$ is a complete discrete valuation domain with maximal ideal $\pi R[\xi]$, and the corresponding residue field has p^{fM} elements. We thus have a notion of $\star$-cycles in $R[\xi]$, and to a $\star$-cycle in $R[\xi]$ there corresponds a $\star$-cycle in R^M.

Thus it suffices to find a $\star$-cycle in $R[\xi]$ of length $(q-1)p^n$.

Take $F(X) = \pi + \xi X + \gamma X^q + X^d$, where $d = q^2$ and $q = p^{fM}$. We remember that $\binom{0}{0} = 1$.

Lemma 1. *For any $T \geq 0$ the T-th iteration of F satisfies*

$$
\begin{aligned}
F^T(0) &\equiv \sum_{t=1}^{T}\sum_{r=0}^{T-t}\xi^r\binom{T-t}{r}\pi^{d^{T-t-r}} + \gamma\sum_{t=1}^{T-1}\sum_{r=0}^{T-t-1}\xi^r\binom{T-t}{r}\pi^{d^{T-t-r-1}q}\\
&\equiv \sum_{t=0}^{T-1}\xi^{-t}\pi^{d^t}A_T(t) + \gamma\sum_{t=0}^{T-2}\xi^{-(t+1)}\pi^{d^t q}A_T(t+1) \mod (p\pi, \gamma^{q+1}),
\end{aligned}
$$

where $A_T(t) = \sum_{k=0}^{T-1}\binom{k}{t}\xi^k$. Moreover,

$$
A_T(t) + A_T(t+1) = \xi^{-1}\Big(A_T(t+1) + \xi^T\binom{T}{t+1}\Big).
$$

Proof. We use direct induction. One only has to remember that $\xi^q = \xi^d = \xi$ and $(x+y)^p \equiv x^p + y^p \mod (p)$. □

Lemma 2. *(i) If $q > 2$ and $T = (q-1)p^r$, then for $j = 0, 1, \ldots, p^r - 1$ we have $w(A_T(j)) \geq e$, and $A_T(p^r)$ is invertible.*

(ii) If $q = 2$, then $A_T(t) = \binom{T}{t+1}$.

Proof. (i) Since $\xi \neq 1$, we have

$$
A_T(0) = 1 + \xi + \cdots + \xi^{T-1} = 0.
$$

Using $w(\xi - 1) = 0$, simple properties of binomial coefficients and

$$
A_T(t) + A_T(t+1) = \xi^{-1}\left(A_T(t+1) + \xi^T\binom{T}{t+1}\right)
$$

we obtain the assertion.

(ii) In this case we have $\xi = 1$, and therefore the assertion follows from Lemma 1. □

Assume that $q > 2$. Put $\gamma = \pi^{d^{p^n-1}(d-q)}z$. In view of $(q+1)d^{p^n-1}(d-q) > d^{p^n}$ and $e \geq p^{2fNp^n} \geq d^{p^n}$ we obtain by Lemma 1 that

$$
F^T(0) \equiv \sum_{t=0}^{T-1}\xi^{-t}\pi^{d^t}A_T(t) + \pi^{d^{p^n-1}(d-q)}z\sum_{t=0}^{T-2}\xi^{-(t+1)}\pi^{d^t q}A_T(t+1) \mod (\pi^{d^{p^n}+1}R[\xi, z]).
$$

In particular, taking $T = (q-1)p^n$ we get, using Lemma 2,

$$
F^{(q-1)p^n}(0) = \pi^{d^{p^n}}\xi^{-p^n}A_{(q-1)p^n}(p^n)\big(1 + z + \pi h(z)\big),
$$

for some polynomial $h \in R[\xi][X]$. Thus $F^{(q-1)p^n}(0) = 0$ if and only if

$$
1 + z + \pi h(z) = 0.
$$

The existence of (a unique) $z \in R[\xi]$ satisfying $F^{(q-1)p^n}(0) = 0$ follows from the Hensel's lemma. Fix such z.

Now it is sufficient to show that the smallest $j > 0$ satisfying $F^j(0) = 0$ is $j = (q-1)p^n$.

If $F^j(0) \equiv 0 \mod (\pi^2)$, then, by Lemma 1, $A_j(0) \equiv 0 \mod (\pi)$ and $\xi^j \equiv 1 \mod (\pi)$, $q-1 \mid j$ follow. From the simple properties of cycles it follows that it suffices to show that $F^{(q-1)p^{n-1}}(0) \neq 0$. But, Lemma 1 gives

$$F^{(q-1)p^{n-1}}(0) \equiv \xi^{-p^{n-1}} A_{(q-1)p^{n-1}}(p^{n-1})\pi^{d^{p^{n-1}}} \mod (\pi^{d^{p^{n-1}}+1}),$$

and, by Lemma 2, we are done.

Assume that $q = 2$. Put $\gamma = \pi^{d^{p^n-2}(d-q)}z$. In view of $(q+1)d^{p^n-2}(d-q) > d^{p^n-1}$ and $e \geq p^{2fNp^n} \geq d^{p^n}$ we obtain by Lemma 1 that

$$F^T(0) = \sum_{t=0}^{T-1} \pi^{d^t} A_T(t) + \pi^{d^{p^n-2}(d-q)} z \sum_{t=0}^{T-2} \pi^{d^t q} A_T(t+1) \mod (\pi^{d^{p^n-1}+1} R[z]).$$

In particular, taking $T = p^n$ we get, using Lemma 2,

$$F^{p^n}(0) = \pi^{d^{p^n-1}} A_{p^n}(p^n-1)\big(1 + z + \pi h(z)\big),$$

for some polynomial $h \in R[X]$. Thus $F^{p^n}(0) = 0$ if and only if $1 + z + \pi h(z) = 0$. The existence of $z \in R$ satisfying $F^{p^n}(0) = 0$ follows from the Hensel's lemma. Fix such z.

Now it suffices to show that the smallest $j > 0$ satisfying $F^j(0) = 0$ is $j = p^n$.

From the simple properties of cycles it follows that it suffices to show that $F^{p^{n-1}}(0) \neq 0$. But, Lemma 1 gives

$$F^{p^{n-1}}(0) \equiv A_{p^{n-1}}(p^{n-1}-1)\pi^{d^{p^{n-1}-1}} \mod (\pi^{d^{p^{n-1}-1}+1}),$$

and, by Lemma 2, we are done.

This finishes the proof of the theorem. $\square$

Theorem 2. *(i) Let S be a Dedekind domain of characteristic $p > 0$. Let $\mathcal{F}(S)$ be the set of all natural f such that there is a nonzero prime ideal $\mathfrak{p}$ of S of norm p^f. Let $\mathcal{A}(f,N)$ consists of all numbers of the form $a \cdot b \cdot p^n$, where $a \leq p^{fN}$, $n \geq 0$ and $b \mid [p^{fa_1}-1, \ldots, p^{fa_r}-1]$ for some $a_1, \ldots, a_r$ satisfying $a_1 + \cdots + a_r \leq N$.*

If $N \geq 2$, then

$$\mathcal{CYCL}(S,N) = \bigcap_{f \in \mathcal{F}(S)} \mathcal{A}(f,N).$$

(ii) Let S be a discrete valuation domain of characteristic $p > 0$ such that the residue field has p^f elements. Then

$$\mathcal{CYCL}(S,1) = \{a \cdot b \cdot p^n : \ a \leq p^f,\ b \mid p^f - 1,\ n \geq 0\}.$$

Proof. Since $e = \infty$, the assertion follows from Theorem 1 and Fact 1. $\square$

Remark 1. (i) If in Theorem 2(i) $\mathcal{F}(S)$ is empty, then $\mathcal{CYCL}(S,N) = \mathbf{N}$. The similar happens to $\mathcal{CYCL}(S,1)$ in Theorem 2(ii) if $f=\infty$.

(ii) Note that $\mathcal{A}(f,N) \subseteq \mathcal{A}(fk,N)$ for any natural k. Hence, if all elements from $\mathcal{F}(S)$ are multiplicities of one element from $\mathcal{F}(S)$, then the formula in Theorem 2(i) may be significantly simplified.

Corollary 1. *We have*

$$\mathcal{CYCL}(F_p[X],2) = \{abp^n : \ a \le p^2, \ b \mid p^2-1, \ n \ge 0\}$$

and

$$\mathcal{CYCL}(F_p[X],3) = \{abp^n : \ a \le p^3, \ n \ge 0 \ \text{ and } \ b \mid p^2-1 \text{ or } p^3-1\}.$$

On the other hand

$$\mathcal{CYCL}(F_p[X],1) = \mathcal{CYCL}(F_p[X,Y],1) = \mathcal{CYCL}(F_p,1) = \{1,2,\ldots,p\}.$$

Proof. Taking into account Remark 1(ii) by Theorem 2 we obtain the first part. The second part follows from $\mathcal{CYCL}(A[X],1) = \mathcal{CYCL}(A,1)$ for any domain A. $\square$

3 A property of $\mathcal{CYCL}(R,1)$

For a domain R with unity, the set $\mathcal{A} = \mathcal{CYCL}(R) := \mathcal{CYCL}(R,1)$ satisfies

(i) $1,2 \in \mathcal{A}$;

(ii) $\mathcal{A}$ is closed under taking divisors;

(iii) for a prime p, $p \in \mathcal{A}$ implies that $\{1,2,\ldots,p\} \subseteq \mathcal{A}$ (the last property follows from the Lagrange interpolation formula).

W. Narkiewicz asked in mid-nineties, whether for a subset $\mathcal{A}$ of naturals, satisfying the above properties (i), (ii), (iii), there is a domain R with $\mathcal{CYCL}(R) = \mathcal{A}$.

In this section we emphasize another property of $\mathcal{CYCL}(R)$, and thus give a negative answer to the mentioned question.

Theorem 3. *For a domain R with unity, let $\mathcal{A} = \mathcal{CYCL}(R)$. Then for a prime number p we have that $p^2 \in \mathcal{A}$ implies $\{2r : r = 1,2,\ldots,p\} \subseteq \mathcal{A}$.*

Proof. Let a tuple $a_0, a_1, \ldots, a_{p^2-1}$ be a cycle for $f(X) \in R[X]$. Then $0 = b_0$, $1 = b_1$, $b_2, \ldots, b_{p^2-1}$, with $b_i = (a_i - a_0)/(a_1 - a_0) \in R$, is a cycle for

$$g(X) = (a_1-a_0)^{-1}\Big(f\big((a_1-a_0)X + a_0\big) - a_0\Big) \in R[X].$$

So assume that $a_0 = 0$, $a_1 = 1$.

One proves that if $(j-i,p) = 1$, then $a_j - a_i$ is invertible. Put $d = a_p$. If $(j-i,p^2) = p$, then $a_j - a_i \sim d$. Fix $2 \le r \le p$. We are going to show that $a_0, a_1, \ldots, a_{r-1}, a_p, a_{p+1}, \ldots, a_{p+r-1}$ is a cycle (of length $2r$) for a suitable polynomial $f(X)$ from $R[X]$. Namely let us take as $f(X)$ the unique polynomial of degree $\leq 2r-1$ with coefficients from the field of fractions of R satisfying

$$\begin{gathered} f(a_0)=a_1\,,\quad f(a_1)=a_2\,,\quad \dots\quad ,\quad f(a_{r-1})=a_p\,,\\ f(a_p)=a_{p+1}\,,\quad \dots\quad ,\quad f(a_{p+r-1})=a_0. \end{gathered} \tag{1}$$

Put $f(X)=c_0+c_1X+\cdots+c_{2r-1}X^{2r-1}$. Then (1) is equivalent to a system of linear equations with $c_0,\dots,c_{2r-1}$ to be found. From linear algebra we then get a formula for c_i.

Namely, putting $b_0=a_0,\ \dots\ ,\ b_{r-1}=a_{r-1},\ b_r=a_p,\ b_{r+1}=a_{p+1},\ \dots\ ,\ b_{2r-1}=a_{p+r-1}$, we have $c_i=\Delta_i/\Delta$, where $\Delta=\prod_{0\leq i<j\leq 2r-1}(b_j-b_i)$ and Δ_i is the determinant of the matrix

$$\begin{pmatrix} 1 & b_0 & \dots & b_0^{i-1} & b_1 & b_0^{i+1} & \dots & b_0^{2r-1}\\ 1 & b_1 & \dots & b_1^{i-1} & b_2 & b_1^{i+1} & \dots & b_1^{2r-1}\\ \multicolumn{8}{c}{\dotfill}\\ 1 & b_{r-1} & \dots & b_{r-1}^{i-1} & b_r & b_{r-1}^{i+1} & \dots & b_{r-1}^{2r-1}\\ \multicolumn{8}{c}{\dotfill}\\ 1 & b_{2r-1} & \dots & b_{2r-1}^{i-1} & b_0 & b_{2r-1}^{i+1} & \dots & b_{2r-1}^{2r-1} \end{pmatrix}.$$

We easily see that d divides all the terms in the differences of $r+1$-th and first rows, $r+2$-th and second rows,..., $2r$-th and r-th rows. Thus $d^r\mid\Delta_i$. From the properties of the differences a_j-a_i we get $\Delta\sim d^r$. Thus $c_i=\Delta_i/\Delta\in R$. □

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References

[1] W. Narkiewicz: *Polynomial Mappings, Lecture Notes in Mathematics, vol. 1600.* Springer-Verlag, Berlin (1995).

[2] T. Pezda: Cycles of polynomial mappings in several variables over rings of integers in finite extensions of the rationals. *Acta Arith.* 108 (2) (2003) 127–146.

[3] T. Pezda: Cycles of polynomial mappings in several variables over rings of integers in finite extensions of the rationals, II. *Monatsh. Math.* 145 (2005) 321–331.

[4] J.H. Silverman: *The Arithmetic of Dynamical Systems,* Graduate Texts in Mathematics, No. 241. Springer-Verlag (2007).

[5] M. Zieve: *Cycles of Polynomial Mappings,* PhD thesis. University of California at Berkeley (1996).

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A Reproducing Kernel and Toeplitz Operators in the Quantum Plane

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Abstract. We define and analyze Toeplitz operators whose symbols are the elements of the complex quantum plane, a non-commutative, infinite dimensional algebra. In particular, the symbols do not come from an algebra of functions. The process of forming operators from non-commuting symbols can be considered as a second quantization. To do this we construct a reproducing kernel associated with the quantum plane. We also discuss the commutation relations of creation and annihilation operators which are defined as Toeplitz operators. This paper extends results of the author for the finite dimensional case.

1 Introduction

Based on the formalism developed in [3], we have introduced and studied in a pair of papers (see [9], [10]) a reproducing kernel and its associated Toeplitz operators which have symbols in a non-commutative algebra which is a finite dimensional truncated version of the complex quantum plane called a paragrassmann algebra. We extend those results now to the case of the complex quantum plane, which is an infinite dimensional, non-commutative algebra. Creation and annihilation operators are defined as certain Toeplitz operators, and their commutation relations are discussed.

This is much like a quantization scheme according to a common intuition of what those words should mean: "operators instead of functions". However, here one must modify this catch phrase to say "operators instead of elements in a non-commutative algebra". This is so because here the symbols are not elements in an algebra isomorphic to an algebra of functions, since the latter is commutative. So, as we remarked in [10], the quantization scheme discussed here is more akin to what in physics is known as a *second quantization*, where one goes from one quantum

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theory to another quantum theory, rather than a *first quantization*, where one goes from a classical theory to a quantum theory.

The paper is organized as follows: The next section introduces the basic definitions and properties. Section 3 is about the reproducing kernel while in Section 4 we define and study Toeplitz operators, including the creation and annihilation operators. Section 5 is about the commutation relations of the creation and annihilation operators. The concluding remarks in Section 6 give some brief indications for possible future research.

2 Definitions and such

We study here the *complex quantum plane* defined as the algebra

$$\mathbb{C}Q_q(\theta,\overline{\theta}) := \mathbb{C}\{\theta,\overline{\theta}\}/\langle\theta\overline{\theta} - q\overline{\theta}\theta\rangle$$

where $\mathbb{C}\{\theta,\overline{\theta}\}$ is the free algebra over $\mathbb{C}$ on the two generators θ and $\overline{\theta}$ while $\langle\theta\overline{\theta} - q\overline{\theta}\theta\rangle$ is the two sided ideal generated by the element $\theta\overline{\theta} - q\overline{\theta}\theta$ for some $q \in \mathbb{C} \setminus \{0\}$. This is a non-commutative algebra provided that $q \neq 1$. It has a vector space basis $AW := \{\theta^j\overline{\theta}^k \mid j,k \in \mathbb{N}\}$, known as the *anti-Wick basis*, and so is infinite dimensional. In Ref. [3] the authors call this the anti-normal ordering, which is synonymous with anti-Wick ordering. This agrees with the definition of quantum plane in [5] (putting the field $k = \mathbb{C}$ there) and with the quantum q-plane in [6], except for notation. However, we will not be studying co-actions of quantum groups on this quantum space as is often done, but rather how its elements serve as the symbols for Toeplitz operators.

Moreover, we also define a *conjugation* (also called a $*$*-operation)* in $\mathbb{C}Q_q(\theta,\overline{\theta})$ on the basis AW by putting

$$(\theta^j\overline{\theta}^k)^* := \theta^k\overline{\theta}^j \tag{1}$$

and then by extending *anti-linearly* to linear combinations with coefficients in $\mathbb{C}$. This is easily shown to be an involution, i.e., $f^{**} = f$ for all $f \in \mathbb{C}Q_q(\theta,\overline{\theta})$. This conjugation makes θ and $\overline{\theta}$ into a pair of variables, each being the conjugate of the other. We will see that this $*$-operation relates well with the operation of taking the adjoint of a Toeplitz operator. However, we are not saying (nor do we need) that this $*$-operation converts $\mathbb{C}Q_q(\theta,\overline{\theta})$ into a $*$-algebra, that is $(fg)^* = g^*f^*$ for all $f,g \in \mathbb{C}Q_q(\theta,\overline{\theta})$. We do note without giving proof that this is a $*$-algebra if and only if $q \in \mathbb{R} \setminus \{0\}$.

We let $w = \{w_j \mid j \in \mathbb{N}\}$ be a sequence of strictly positive real numbers, that is, $w_j > 0$. These will be referred to as weights. We use these weights to define an inner product on $\mathbb{C}Q_q(\theta,\overline{\theta})$ as the sesquilinear extension (anti-linear in the first entry, linear in the second) of

$$\langle\theta^a\overline{\theta}^b, \theta^c\overline{\theta}^d\rangle_w := w_{a+d}\,\delta_{a+d,b+c} = w_{a+d}\,\delta_{a-b,c-d} \quad \text{for all } a,b,c,d \in \mathbb{N}, \tag{2}$$

with δ being the Kronecker delta. Notice that the condition $a - b = c - d$ is necessary and sufficient for the inner product in (2) to be non-zero. Clearly, given a pair $a, b \in \mathbb{N}$ there are infinitely many pairs $c, d \in \mathbb{N}$ such that $c - d = a - b$ and also satisfying $(c, d) \neq (a, b)$. Therefore AW is not even an orthogonal basis, let alone an orthonormal basis.

We wish to note, although without giving the relatively straightforward proof, that there is this compatibility between the inner product (2) and the conjugation (1), namely: $\langle f, g\rangle_w^* = \langle f^*, g^*\rangle_w$ for all $f, g \in \mathbb{C}Q_q(\theta, \overline{\theta})$, where the $*$-operation on the left side is complex conjugation in $\mathbb{C}$. We note that we also have the identity $\langle f, g\rangle_w^* = \langle g, f\rangle_w$.

The definition (2) is partly motivated by the inner product introduced in [3] and studied in [9] and [10]. There one has the paragrassmann algebra defined by

$$PG_{l,q}(\theta, \overline{\theta}) = \mathbb{C}(\theta, \overline{\theta})/\langle \theta\overline{\theta} - q\overline{\theta}\theta, \theta^l, \overline{\theta}^l\rangle$$

with $l \geq 2$ an integer. This is a quotient (as an algebra) of $\mathbb{C}Q_q(\theta, \overline{\theta})$ by the nilpotency relations $\theta^l = 0$ and $\overline{\theta}^l = 0$. In that case, using the notation in [9], the inner product used there satisfies

$$\langle \theta^a\overline{\theta}^b, \theta^c\overline{\theta}^d\rangle_w = \langle \theta^{a+d}, \theta^{b+c}\rangle = w_{a+d}\,\delta_{a+d,b+c}\chi_l(a+d). \tag{3}$$

Here χ_l is the characteristic function for the set of integers $\{0, 1, \ldots, l-1\}$. Its presence is due to the nilpotency relations. Equation (3) was not the actual definition of this inner product, although it could have been. Instead the definition of this inner product was given in terms of a Berezin type integral, thereby presenting $PG_{l,q}(\theta, \overline{\theta})$ as something quite analogous to a classical L^2 space. It seems to be impossible to write (2) as a Berezin type integral, since now there are no 'top classes' in the theory. However, it might be useful to express (2) as some sort of generalized L^2 inner product.

Now another motivation for the inner product (2) is seen in the well known example of the Hilbert space

$$\mathcal{H} := L^2(\mathbb{C}, \pi^{-1}e^{-|z|^2}dx\,dy) \tag{4}$$

where the monomials $z^j\overline{z}^k$ form a basis (linearly independent set such that the closure of their algebraic span is the entire Hilbert space). Then using a result that goes back at least as far to Bargmann's paper [2] in the second equality, for $a, b, c, d \in \mathbb{N}$ this basis satisfies

$$\langle z^a\overline{z}^b, z^c\overline{z}^d\rangle_{\mathcal{H}} = \langle z^{a+d}, z^{b+c}\rangle_{\mathcal{H}} = (a+d)!\,\delta_{a+d,b+c},$$

where we are using here the standard L^2 inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ in $\mathcal{H}$. Hence we can think of w_j as some sort of deformation of $j!$, the usual factorial of $j \in \mathbb{N}$. Notice that an immediate consequence is that $\langle z^a, \overline{z}^d\rangle_{\mathcal{H}} = 0$ if either $a > 0$ or $d > 0$, while for $a = b = c = d = 0$ we have $\langle 1, 1\rangle_{\mathcal{H}} = 1$. In turn this implies for f holomorphic and g anti-holomorphic that

$$\langle f, g\rangle_{\mathcal{H}} = f(0)^*g(0).$$

In particular such a pair of f and g is orthogonal if and only if either $f(0) = 0$ or $g(0) = 0$. This example has an interesting consequence. Suppose that we take the weights in the quantum plane to be $w_j = j!$ for all $j \in \mathbb{N}$. Then the inner product on the quantum plane $\mathbb{C}Q_q(\theta, \overline{\theta})$ is positive definite since in this case

$\mathbb{C}Q_q(\theta,\overline{\theta})$ is unitarily isomorphic to a dense subspace D of the Hilbert space $\mathcal{H}$ for any $q \in \mathbb{C} \setminus \{0\}$. In fact the map $U\colon \mathbb{C}Q_q(\theta,\overline{\theta}) \to \mathcal{H}$ given on the basis AW by $U(\theta^i\overline{\theta}^j) := z^i\overline{z}^j$ is an isometry. Actually D is the commutative subalgebra $\mathbb{C}[z,\overline{z}]$ of complex polynomials in two commuting variables, and so the unitary isomorphism

$$U\colon \mathbb{C}Q_q(\theta,\overline{\theta}) \xrightarrow{\cong} D = \mathbb{C}[z,\overline{z}]$$

is not an algebra isomorphism for $q \neq 1$. Also the completion of $\mathbb{C}Q_q(\theta,\overline{\theta})$ with respect to the corresponding metric is unitarily isomorphic to the Hilbert space $\mathcal{H}$, again for any $q \in \mathbb{C} \setminus \{0\}$. And hence there are cases where the inner product defined by (2) is positive definite. Motivated in part by this example we call θ a *holomorphic variable* and $\overline{\theta}$ an *anti-holomorphic variable*. (Compare also with the usage of these terms in [9] and [10].)

However there are also cases for which the inner product (2) is not positive definite. To see how this can happen, we first note some elementary calculations:

$$\begin{aligned} \langle 1,1\rangle_w &= w_0, \\ \langle \theta\overline{\theta},1\rangle_w = \langle 1,\theta\overline{\theta}\rangle_w &= w_1, \\ \langle \theta\overline{\theta},\theta\overline{\theta}\rangle_w &= w_2. \end{aligned} \tag{5}$$

As an aside, we note that 1 is a normalized state (the 'ground state') if and only if $w_0 = 1$. Let $\alpha \in \mathbb{R}$ be a real number to be specified in more detail later. Then

$$\langle 1+\alpha\theta\overline{\theta}, 1+\alpha\theta\overline{\theta}\rangle_w = w_0 + 2\alpha w_1 + \alpha^2 w_2, \tag{6}$$

a quadratic polynomial in α which has distinct real roots if and only if its discriminant is positive, that is, $w_1^2 - w_0w_2 > 0$. Picking weights that satisfy this condition we see that the inner product in (6) will be zero for two distinct values of $\alpha \in \mathbb{R}$ and negative for values strictly between those two values. (Recall that $w_2 > 0$.) In short, the inner product will not be positive definite in such a case. This example also shows that $w_1^2 - w_0w_2 < 0$ is a necessary condition for the inner product to be positive definite.

The remarks in the previous paragraphs show that the situation for the quantum plane is rather different from the finite dimensional theory, where the inner product is never positive definite, but always non-degenerate, as shown in [9]. We now wish to establish a necessary and sufficient condition on the weights w_k so that the inner product $\langle\cdot,\cdot\rangle_w$ defined in (2) is non-degenerate. Here it is:

Theorem 1. *The inner product* (2) *is non-degenerate on* $\mathbb{C}Q_q(\theta,\overline{\theta})$ *if and only if for every integer* $R \geq 1$ *and every* $n \in \mathbb{Z}$ *we have that*

$$\left\{ W_{R,s} \in \mathbb{C}^R \;\middle|\; s \geq |n| \right\}^{\perp} = 0,$$

where $W_{R,s} = (w_{r+s-|n|})_{|n|\leq r\leq |n|+R-1}$ *is a vector in* $\mathbb{C}^R$ *for every* $s \geq |n|$.

Proof. To facilitate this argument we define a partition of the basis AW so that elements in disjoint subsets of the partition are orthogonal with respect to the inner product (2). So for each integer $n \in \mathbb{Z}$ we define

$$P_n := \left\{ \theta^a\overline{\theta}^b \;\middle|\; a \geq 0,\ b \geq 0,\ a-b=n \right\}.$$

Then we have $P_n \perp P_m$ for all $n, m \in \mathbb{Z}$ satisfying $n \neq m$ as well as

$$AW = \bigcup_{n \in \mathbb{Z}} P_n,$$

a disjoint union. So we have an algebraic orthogonal decomposition

$$\mathbb{C}Q_q(\theta, \overline{\theta}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{P}_n,$$

where $\mathcal{P}_n := \mathrm{span}_{\mathbb{C}} P_n$. (Here we let $\mathrm{span}_{\mathbb{C}}\, S$ denote the operation of forming the algebraic subspace over $\mathbb{C}$ generated by the indicated set S. So, we are taking here only *finite* linear combinations of elements in S.) It follows that the inner product (2) is non-degenerate if and only if it is non-degenerate on each of the summands $\mathcal{P}_n$.

It will be convenient for us to define the *max-degree* of each basis element in AW by

$$\mathrm{maxdeg}(\theta^a \overline{\theta}^b) := \max(a, b) \geq 0\,.$$

Then P_n contains exactly one element of max-degree $|n| + k$ for $k = 0, 1, 2, \dots$ (and no other elements). For example, for the integers $n \leq 0$ we have

$$P_n = \big\{ \overline{\theta}^{(-n)}, \theta \overline{\theta}^{(-n+1)}, \dots, \theta^k \overline{\theta}^{(-n+k)}, \dots \big\}.$$

A similar expression holds for $n > 0$. We denote the unique element of P_n of max-degree r by ε_r for each integer $r \geq |n|$. The reader can check that for $n \geq 0$ we have $\varepsilon_r = \theta^r \overline{\theta}^{r-n}$, while for $n < 0$ we have $\varepsilon_r = \theta^{r+n} \overline{\theta}^r$, where $r \geq |n|$ in both cases. We omit n from the notation ε_r.

Taking the pair of elements $\varepsilon_r, \varepsilon_s \in P_n$ for a given $n \in \mathbb{Z}$ and $r, s \geq |n|$ and then computing their inner product gives (as the reader can check) that

$$\langle \varepsilon_r, \varepsilon_s \rangle_w = w_{r+s-|n|}.$$

In the example (5) given earlier the two elements 1 and $\theta\overline{\theta}$ lie in $\mathcal{P}_0$ and satisfy $\mathrm{maxdeg}\, 1 = 0$ and $\mathrm{maxdeg}\, \theta\overline{\theta} = 1$. So $\varepsilon_0 = 1$ and $\varepsilon_1 = \theta\overline{\theta}$ in $\mathcal{P}_0$.

Suppose that $n \in \mathbb{Z}$ is given. We then consider the inner product (2) restricted to $\mathcal{P}_n$. Take an arbitrary element $f \in \mathcal{P}_n$ with $f \neq 0$. We write

$$f = \sum_{r \geq |n|} a_r \varepsilon_r,$$

where each $a_r \in \mathbb{C}$, but only finitely many are non-zero. But at least one of these coefficients a_r is non-zero, since $f \neq 0$. The inner product is non-degenerate on $\mathcal{P}_n$ if and only there exists $g \in \mathcal{P}_n$ (depending on f, of course) such that $\langle g, f \rangle_w \neq 0$. We expand g as

$$g = \sum_{s \geq |n|} b_s \varepsilon_s$$

for complex coefficients b_s only finitely many of which are non-zero. Then we evaluate

$$\langle g, f\rangle_w = \sum_{r\geq|n|,\, s\geq|n|} a_r b_s^* \langle\varepsilon_s, \varepsilon_r\rangle_w = \sum_{s\geq|n|} b_s^* \Big(\sum_{r\geq|n|} a_r w_{r+s-|n|} \Big). \tag{7}$$

For example, if $w_k = 1$ (or any other constant value) for all $k \geq 0$, then taking f above such that $\sum_r a_r = 0$ but some $a_r \neq 0$ gives us an element $f \neq 0$ satisfying $\langle g, f\rangle_w = 0$ for all g. So in this particular case the inner product is degenerate.

Notice that the expression in parentheses on the right in (7) is given to us, while the coefficients b_s are ours to choose as we please *provided that* only finitely many of them are non-zero. So we define

$$v_s(f) := \sum_{r\geq|n|} a_r w_{r+s-|n|} \in \mathbb{C} \tag{8}$$

for every $s \geq |n|$. (Recall that n is a given integer so we do not include it in the notation $v_s(f)$. The sum is well defined since only finitely many of the a_r's are non-zero.) If just one of these numbers is non-zero, say $v_{s_0}(f) \neq 0$, then we can put $b_s = 0$ for all $s \neq s_0$ and $b_{s_0} = 1$. And therefore (7) is non-zero. And such a choice indeed has only finitely many (namely, one) of the b_s's different from zero. The element g corresponding to this choice of the b_s's is $g = \varepsilon_{s_0}$, which satisfies $\langle g, f\rangle_w \neq 0$. Therefore in this case $\{f\}^\perp \neq \mathcal{P}_n$. (Recall that we have restricted the inner product to $\mathcal{P}_n$.)

So if the inner product is degenerate on $\mathcal{P}_n$ (which means that $\{h\}^\perp = \mathcal{P}_n$ for some $0 \neq h \in \mathcal{P}_n$), then there must exist some $f \neq 0$ (actually, $f = h$ works) such that $v_s(f) = 0$ for all $s \geq |n|$. Conversely, if there exists some $f \neq 0$ such that $v_s(f) = 0$ for all $s \geq |n|$, then for every g we have $\langle g, f\rangle_w = 0$ by (7) and so the inner product is degenerate on $\mathcal{P}_n$. We now re-write the definition (8) for $v_s(f)$ as

$$v_s(f) = \sum_{|n|\leq r\leq|n|+R-1} a_r w_{r+s-|n|} \in \mathbb{C} \tag{9}$$

for some integer $R \geq 1$. Notice that the existence of R is given to us implicitly as part of the information about f, since only finitely many of the a_r's are non-zero. R is not unique, but that is not important for this argument.

So we can consider $A_R(f) := (a_r^*)_{|n|\leq r\leq|n|+R-1}$ as a vector in $\mathbb{C}^R$. Similarly, $W_{R,s} := (w_{r+s-|n|})_{|n|\leq r\leq|n|+R-1}$ is considered as a vector in $\mathbb{C}^R$. Recall that n is fixed since we are working in P_n. However, $s \geq |n|$ is arbitrary. We will now use the standard Hermitian inner product $\langle\cdot,\cdot\rangle_{\mathbb{C}^R}$ on $\mathbb{C}^R$. Then equation (9) is the same as

$$v_s(f) = \langle A_R(f), W_{R,s}\rangle_{\mathbb{C}^R}.$$

Now $\{W_{R,s}\}_{s\geq|n|}$ is an infinite sequence of vectors in the finite dimensional vector space $\mathbb{C}^R$. Since $f = \sum_{r\geq n} a_r\varepsilon_r$ is an arbitrary non-zero element in $\mathcal{P}_n$ with

$$|n| \leq \max(\{r \mid a_r \neq 0\}) \leq |n| + R - 1,$$

it follows that $A_R(f)$ is an arbitrary non-zero vector in $\mathbb{C}^R$. Therefore the following statements are equivalent provided that $n \in \mathbb{Z}$ is given:

- The inner product is degenerate on $\mathcal{P}_n$.
- For some $f \in \mathcal{P}_n$ with $f \neq 0$, we have $v_s(f) = 0$ for all $s \geq |n|$.
- For some sequence $\{a_r \mid r \geq |n|\}$, not identically zero but with only finitely many terms not equal to zero, we have $v_s = 0$ for all $s \geq |n|$, where we define $v_s := \sum_{r \geq |n|} a_r w_{r+s-|n|}$ for $s \geq |n|$.
- There exist some integer $R \geq 1$ and some vector $A \in \mathbb{C}^R \setminus \{0\}$ such that for all $s \geq |n|$ we have $\langle A, W_{R,s} \rangle_{\mathbb{C}^R} = 0$.
- There exists some integer $R \geq 1$ so that $\{W_{R,s} \in \mathbb{C}^R \mid s \geq |n|\}^\perp \neq 0$.

Equivalently, the inner product is non-degenerate on $\mathcal{P}_n$ if and only if for every integer $R \geq 1$ we have

$$\left\{W_{R,s} \in \mathbb{C}^R \;\middle|\; s \geq |n|\right\}^\perp = 0\,.$$

We have already established that the inner product (2) is non-degenerate on $\mathbb{C}Q_q(\theta, \overline{\theta})$ if and only if it is non-degenerate on $\mathcal{P}_n$ for every integer $n \in \mathbb{Z}$. And so this finishes the proof. $\square$

Remark 1. This result gives an algebraic necessary and sufficient condition on the weights w_k for their associated inner product to be non-degenerate. While it looks clumsy, it should be amenable to verification in applications. Intuitively, the condition that an infinite sequence in a finite dimensional vector space spans the vector space seems to be a generic condition. And so countably many such conditions should also be generic. Theorem 1 contrasts with the result for the paragrassmann algebra in [9], where we proved that the inner product (3) is non-degenerate for all positive weights.

Inside the subalgebra

$$\mathrm{Pre}(\theta) := \mathrm{span}_{\mathbb{C}}\{\theta^j \mid j \in \mathbb{N}\} \cong \mathbb{C}[\theta] \subset \mathbb{C}Q_q(\theta, \overline{\theta})$$

generated by all powers of the holomorphic variable θ, we have as a particular case of the definition (2) that

$$\langle \theta^j, \theta^k \rangle_w = \delta_{j,k} w_j$$

for all $j, k \in \mathbb{N}$. So the inner product restricted to the 'holomorphic' subspace $\mathrm{Pre}(\theta)$ is positive definite. This means that $\mathrm{Pre}(\theta)$ is a pre-Hilbert space. Moreover, an orthonormal basis of $\mathrm{Pre}(\theta)$ is given by

$$\phi_j(\theta) := \frac{1}{w_j^{1/2}}\, \theta^j \qquad \text{for } j \in \mathbb{N}.$$

Similar comments hold for the anti-holomorphic subalgebra $\mathrm{Pre}(\overline{\theta})$ defined in a completely analogous way:

$$\mathrm{Pre}(\overline{\theta}) := \mathrm{span}_{\mathbb{C}}\{\overline{\theta}^j \mid j \in \mathbb{N}\} \cong \mathbb{C}[\overline{\theta}] \subset \mathbb{C}Q_q(\theta, \overline{\theta})$$

We let

$$\mathcal{B}(\theta) = \mathcal{B} := \mathrm{comp}_{\mathbb{C}}\ \mathrm{Pre}(\theta)$$

denote the *holomorphic space* (or the *Segal-Bargmann space*) of the quantum plane. By the operation $\mathrm{comp}_{\mathbb{C}}$ we mean the completion of the indicated pre-Hilbert space. The set $\{\phi_j(\theta) \mid j \in \mathbb{N}\}$ is also an orthonormal basis for $\mathcal{B}(\theta)$. Unlike the finite dimensional case studied in [3], [9] and [10], the Segal-Bargmann space $\mathcal{B}(\theta)$ here is not necessarily an algebra. However, it does contain the dense subspace $\mathrm{Pre}(\theta) \cong \mathbb{C}[\theta]$ which is an algebra, namely the algebra of polynomials in θ. But the multiplication map for $\mathbb{C}[\theta]$ is not necessarily continuous in the topology induced by the norm associated to the inner product (2) and, if that is the case, then it is not extendible by continuity to $\mathcal{B}(\theta)$.

Analogously, we define the *anti-holomorphic space* (or the *anti-Segal-Bargmann space*) of the quantum plane by

$$\mathcal{B}(\overline{\theta}) := \mathrm{comp}_{\mathbb{C}}\ \mathrm{Pre}(\overline{\theta}).$$

These two spaces $\mathcal{B}(\theta)$ and $\mathcal{B}(\overline{\theta})$ should be understood as 'almost' disjoint. Their 'intersection' is the one dimensional subspace spanned by $1 = \theta^0 = \overline{\theta}^0$.

3 Reproducing kernel

As a first step towards the definition of Toeplitz operators, we shall find a reproducing kernel for the Segal-Bargmann space. First of all we will need to define a functional calculus for the Segal-Bargmann space. As is well-known, there always is a functional calculus for polynomials $f \in \mathbb{C}[x]$ associated to any element in any associative algebra. Here we write

$$f = \sum_{j=0}^{m} a_j x^j \in \mathbb{C}[x]$$

with coefficients $a_j \in \mathbb{C}$ and then use the standard definition

$$f(\theta) := \sum_{j=0}^{m} a_j \theta^j.$$

But there are some elements in $\mathcal{B}(\theta)$ that are not so representable, since they are infinite sums of elements in the orthogonal basis $\{\theta^j\}$. However, any element $u \in \mathcal{B}(\theta)$ can be expanded as an infinite sum with respect to the orthonormal basis $\{\phi_j(\theta)\}$ giving

$$u = \sum_{j=0}^{\infty} a_j \phi_j(\theta) = \sum_{j=0}^{\infty} a_j w_j^{-1/2} \theta^j$$

with $a_j \in \mathbb{C}$ and $\sum_j |a_j|^2 < \infty$. Equivalently, for all $u \in \mathcal{B}(\theta)$ we have

$$u = \sum_{j=0}^{\infty} f_j \theta^j$$

with $f_j \in \mathbb{C}$ and $\sum_j |f_j|^2 w_j < \infty$. So associated to any sequence of positive real numbers $w = \{w_j \mid j \geq 0\}$ we define a weighted little l^2 space:

$$l^2(w) := \big\{ f = \{f_j \mid j \in \mathbb{N}\} \; \big| \; \sum_j |f_j|^2 w_j < \infty \big\}.$$

Then the full *functional calculus of* θ is the linear mapping

$$\Phi \colon l^2(w) \to \mathcal{B}(\theta)$$

defined by

$$\Phi(f) = \Phi\big(\{f_j\}\big) := \sum_{j=0}^{\infty} f_j \theta^j.$$

So Φ is a unitary isomorphism of Hilbert spaces. We also use the more suggestive notation $f(\theta) := \Phi(f)$ for all $f \in l^2(w)$.

Now the reproducing kernel $K(\theta, \eta)$ is supposed to satisfy the *reproducing kernel formula*, namely

$$f(\theta) = \langle K(\theta, \eta), f(\eta) \rangle_w \in \mathcal{B}(\theta) \tag{10}$$

for all $f \in l^2(w)$ and where $\eta \in \mathcal{B}(\eta)$ is another 'independent copy' of a holomorphic variable. The intuitive idea behind the inner product in (10) is that it should only take η into consideration while letting θ have a free ride as a 'passenger'. The usual structure of reproducing kernel functions in spaces of holomorphic functions suggests that we should have

$$K(\theta, \eta) \in \mathcal{B}(\overline{\theta}) \otimes \mathcal{B}(\eta),$$

the standard tensor product of Hilbert spaces. This expresses the intuition that $K(\theta, \eta)$ should be anti-holomorphic in θ and holomorphic in η. So we want to define an inner product $\langle L(\theta, \eta), f(\eta) \rangle_w$ for all $L(\theta, \eta) \in \mathcal{B}(\overline{\theta}) \otimes \mathcal{B}(\eta)$ and all $f \in l^2(w)$. Actually, we will start off this discussion by suppressing the Hilbert space structures and simply considering $f(\eta) = \sum_k f_k \eta^k$, a formal infinite sum, and

$$L(\theta, \eta) = \sum_{ij} \lambda_{ij} \, \overline{\theta}^i \otimes \eta^j,$$

another formal infinite sum (that is, no convergence requirements). We now make the following formal calculation in order to motivate a definition:

$$\begin{aligned}
\langle L(\theta, \eta), f(\eta) \rangle_w &= \sum_{ijk} \lambda_{ij}^* f_k \langle \overline{\theta}^i \otimes \eta^j, \eta^k \rangle_w \\
&= \sum_{ijk} \lambda_{ij}^* f_k \langle \eta^j, \eta^k \rangle_w \theta^i \\
&= \sum_{ijk} \lambda_{ij}^* f_k \delta_{j,k} w_j \theta^i \\
&= \sum_i \Big(\sum_j \lambda_{ij}^* f_j w_j \Big) \theta^i.
\end{aligned} \tag{11}$$

The inner sum in (11) over $j \geq 0$ is an infinite sum of complex numbers for every $i \geq 0$ and so will not be considered as a formal infinite sum. But to consider it as an absolutely convergent series, say, we will have to impose conditions on the coefficients λ_{ij} and f_k of the above formal expressions. (The weights w_j are considered as given.) After all the inner sums in (11) have been well defined we are left with a formal expression, namely a formal power series in the variable θ. This can be used as such. Or, if one prefers, some more conditions can be imposed so that this series converges in some topological vector space, which could be $\mathcal{B}(\theta)$ with one of its many topological structures (norm topology, weak topology, etc.).

For example, we can use Hölder's inequality to get the estimate

$$\sum_j |\lambda_{ij}^* f_j w_j| \leq \Big(\sum_j |\lambda_{ij}|^p w_j\Big)^{1/p} \Big(\sum_j |f_j|^{p'} w_j\Big)^{1/p'} \tag{12}$$

for any $1 < p < \infty$, where p' is the usual dual index of p. Consequently, if there exists some $1 < p < \infty$ such that the first sum on the right side of (12) is finite for every $i \geq 0$ and such that the second sum is finite, then we have that the formula (11) defines the inner product $\langle L(\theta,\eta), f(\eta)\rangle_w$ as a formal power series in θ.

We next consider the canonical orthogonal basis of $l^2(w)$ given by

$$\varepsilon_j = (0, \dots, 0, 1, 0, \dots)$$

(all zeros with one single occurrence of 1 in entry $j \in \mathbb{N}$). Then we have

$$\varepsilon_j(\theta) = \Phi(\varepsilon_j) = \theta^j.$$

So a necessary condition for (10) to hold is that

$$\theta^j = \langle K(\theta,\eta), \eta^j\rangle_w \tag{13}$$

for all $j \in \mathbb{N}$. We look for a solution $K(\theta,\eta) = \sum_{kl} a_{kl}\overline{\theta}^k \otimes \eta^l$, a formal series, for unknown coefficients $a_{kl} \in \mathbb{C}$. So we use our formal definition (11) to get

$$\langle K(\theta,\eta), \eta^j\rangle_w = \sum_k a_{kj}^*\, w_j \theta^k,$$

a formal power series in θ. So (13) holds if and only if

$$\theta^j = \sum_k a_{kj}^* w_j\, \theta^k \tag{14}$$

for all $j \in \mathbb{N}$. Of course, the left side of (14) is a *finite* series. Clearly, (14) is satisfied if and only if $a_{jk} = \delta_{j,k}/w_j$.

Putting this into the formula for the reproducing kernel gives us

$$\begin{aligned} K(\theta,\eta) &= \sum_{kl} a_{kl}\overline{\theta}^k \otimes \eta^l = \sum_{kl} \frac{\delta_{k,l}}{w_k}\overline{\theta}^k \otimes \eta^l = \sum_k \frac{1}{w_k}\overline{\theta}^k \otimes \eta^k \\ &= \sum_k \phi_k(\overline{\theta}) \otimes \phi_k(\eta). \end{aligned} \tag{15}$$

But this series is not convergent in the norm topology of the Hilbert space $\mathcal{B}(\overline{\theta}) \otimes \mathcal{B}(\eta)$, since the terms satisfy

$$\left\| \phi_k(\overline{\theta}) \otimes \phi_k(\eta) \right\| = 1.$$

However, there is another topology on $\mathcal{B}(\overline{\theta}) \otimes \mathcal{B}(\eta)$ for which this series is convergent. This other topology corresponds to the strong operator topology (see [7]) in the space $\mathrm{End}(\mathrm{Pre}(\theta))$ of bounded linear operators mapping $\mathcal{B}(\eta)$ to $\mathcal{B}(\theta)$. Without going into a lot of technical details, let us simply note that the formula (15) induces a unitary isomorphism $S \colon \mathcal{B}(\eta) \to \mathcal{B}(\theta)$ given in Dirac notation by

$$S = \sum_k |\phi_k(\theta)\rangle\langle\phi_k(\eta)|$$

which is an infinite sum of rank one operators, each of which has operator norm 1, and so is not convergent in the operator norm topology.

Nonetheless this infinite series of operators is convergent in the strong operator topology. It satisfies $S \colon \phi_k(\eta) \mapsto \phi_k(\theta)$ for the basis elements and therefore $S \colon f(\eta) \mapsto f(\theta)$ for $f \in l^2(w)$. This is quite tautological, since this is exactly what the mapping induced by the reproducing kernel, as given by the right side of equation (10), is supposed to do! And so it does. Intuitively, the expression in (15) expresses in this context the formula for the kernel of the Dirac delta as a 'smooth' object.

This section may seem like a lot of work to arrive at a result that appears to lack substance. However, the formula (15) will be used in the next section to define Toeplitz operators in a rather natural way. And these Toeplitz operators have some substantial, non-trivial properties. There may be other ways, still to be discovered, for defining these Toeplitz operators. But for the time being we seem to have found a reasonable approach.

Also, it is worth mentioning that the reproducing kernel K in (15) is not a function of two variables in the usual sense of those words. If it were, then $f(\theta)$ would be the 'value' of f at the 'point' θ. But $f(\theta)$ is an element in $\mathcal{B}(\theta)$ for all $f \in l^2(w)$. And θ itself is an element in the very same space $\mathcal{B}(\theta)$. So the sort of reproducing kernel as given in (15) is not included in the classical theory of reproducing kernel *functions* such as found in [1] and [8]. For example, the usual point-wise estimate, which follows immediately from the Cauchy-Schwarz inequality in the classical case, seems to have no good analogue here. Anyway, the Cauchy-Schwarz inequality does not apply to the general reproducing kernel formula in (10) nor to its special case (13).

But there are some properties of the reproducing kernel (15) that are analogous to standard properties of reproducing kernel functions. (See [1] and [8].) The correct interpretation of the following properties entails defining with some care notations which superficially appear obvious. We will not go into that analysis, but refer the reader to [9] where a similar analysis was made. We now present these properties:

1. Positive definite: $\sum_{n,m=1}^{N} \lambda_n^* \lambda_m K(\theta_n, \theta_m) \geq 0$ for $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$.

2. Complex symmetry: $K(\theta,\eta)^* = K(\eta,\theta)$.

3. Self reproducing: $K(\theta,\eta) = \langle K(\eta,\cdot), K(\theta,\cdot)\rangle_w$.

4. Positivity on the diagonal: $K(\theta,\theta) = \sum_k |\phi_k(\theta)\rangle\langle\phi_k(\theta)| = I_{\mathcal{B}(\theta)} \geq 0$.

Also, there is the question of constructing a space with a given $K(\theta,\eta)$ (satisfying properties 1 and 2) as its reproducing kernel. While this is a well known result in the theory of reproducing kernel *functions*, it appears that the analogous construction can not be made here since we are not dealing with functions.

4 Toeplitz Operators

Much of the above material about the reproducing kernel appears to be somewhat tautological in nature, though with a lot of technical details since here we are dealing with infinite dimensional spaces rather than the finite dimensional theory in [9]. But the real point of the reproducing kernel for us is that it can be extended in a 'natural' manner to the quantum plane and as such becomes one of the principle ingredients in defining a non-trivial theory of Toeplitz operators with symbols in the complex quantum plane, a non-commutative algebra for $q \neq 1$. As noted earlier in [10], passing from a symbol in a non-commutative algebra to its Toeplitz operator is an example of second quantization, since it is the quantization of a theory that is itself a non-commutative (i.e., quantum) theory to begin with. Nonetheless, the initial theory is still often referred to as the classical theory.

To start off this discussion we define the inner product of any finite sum or any infinite (formal) sum of the form

$$M(\theta,\eta) = \sum_{jk} m_{jk}\,\overline{\theta}^j \otimes \eta^k$$

with coefficients $m_{jk} \in \mathbb{C}$ for $j,k \geq 0$ and a basis element $\eta^a\overline{\eta}^b \in \mathbb{C}Q_q(\eta,\overline{\eta})$ in AW by

$$\begin{aligned}
\langle M(\theta,\eta)\,,\,\eta^a\overline{\eta}^b\rangle_w &:= \sum_j\Big(\sum_k m_{jk}^*\langle\eta^k,\eta^a\overline{\eta}^b\rangle_w\Big)\theta^j \\
&= \sum_j\Big(\sum_k m_{jk}^*\langle\eta^{k+b},\eta^a\rangle_w\Big)\theta^j \\
&= \sum_j\Big(\sum_k m_{jk}^*\delta_{k+b,a}w_a\Big)\theta^j \\
&= w_a\sum_j m_{j,a-b}^*\,\theta^j
\end{aligned} \tag{16}$$

provided that the sum on j converges in $\mathcal{B}(\theta)$, which is equivalent to

$$\sum_j w_j|m_{j,a-b}|^2 < \infty\,.$$

Or we could simply take (16) to be a formal series. Here we have introduced the convention that $m_{jk} = 0$ if $k < 0$. Then for any given arbitrary element

$$F = \sum_{ab} c_{ab} \eta^a \overline{\eta}^b \in \mathbb{C}Q_q(\eta, \overline{\eta})$$

(which is always a finite sum) such that for each pair (a, b) satisfying $c_{ab} \neq 0$ we have convergence in (16), we define

$$\langle M(\theta, \eta)\, , F \rangle_w := \sum_{ab} c_{ab} \langle M(\theta, \eta)\, ,\, \eta^a \overline{\eta}^b \rangle_w \, ,$$

which is also a finite sum. Notice that this inner product in general takes values in $\mathcal{B}(\theta)$ provided that we impose the convergence conditions, though in some specific cases the inner product could lie in some subspace of $\mathcal{B}(\theta)$.

Next we define the operator associated with the reproducing kernel K. This is the extension of the reproducing kernel to the quantum plane that we mentioned earlier.

Definition 1. The linear operator associated to the reproducing kernel of $\mathrm{Pre}(\theta)$, $P_K \colon \mathbb{C}Q_q(\theta, \overline{\theta}) \to \mathbb{C}Q_q(\theta, \overline{\theta})$, is defined for all $F(\theta, \overline{\theta}) \in \mathbb{C}Q_q(\theta, \overline{\theta})$ by

$$P_K F(\theta) := \langle K(\theta, \eta), F(\eta, \overline{\eta}) \rangle_w . \tag{17}$$

This definition comes down to a special case of the discussion in the previous paragraph. So we must show that the inner product in (17) is well defined. Also P_K is actually a symmetric projection as we prove next.

Theorem 2. *P_K is well defined and is a projection, that is, $P_K^2 = P_K$. Also, P_K is symmetric with respect to the inner product $\langle \cdot, \cdot \rangle_w$.*

Remark 2. Since this inner product is not necessarily non-degenerate, we do not always have that the adjoint of P_K exists. Nonetheless, it makes sense to speak of the symmetry of P_K. And in those cases when the inner product is non-degenerate, we do have $P_K^* = P_K$.

Proof. We write $F_{ab}(\theta, \overline{\theta}) := \theta^a \overline{\theta}^b$ for the elements in the basis AW. We extend the notation established above by setting $\theta^n = 0$ and $w_n = 1$ for all integers $n < 0$. As we noted earlier, this basis AW is not orthogonal.

Acting with P_K on the basis elements F_{ab} in AW we obtain

$$\begin{aligned}(P_K F_{ab})(\theta) &= \langle K(\theta, \eta), F_{ab}(\eta, \overline{\eta}) \rangle_w = \langle K(\theta, \eta), \eta^a \overline{\eta}^b \rangle_w \\ &= \sum_j \frac{1}{w_j} \langle \eta^j, \eta^a \overline{\eta}^b \rangle_w \, \theta^j = \sum_j \frac{1}{w_j} \delta_{j+b,a} w_a \, \theta^j = \frac{w_a}{w_{a-b}} \theta^{a-b} .\end{aligned} \tag{18}$$

This result corresponds in this case to the convergence in (16) for all a, b. In this particular case, the infinite series collapses to at most one non-zero term, and so we have convergence not only in $\mathcal{B}(\theta)$ but even to an element in its subspace $\mathrm{Pre}(\theta)$.

So by extending linearly to finite sums we see that the definition (17) makes sense. Moreover, (18) shows that $\operatorname{Ran} P_K \subset \operatorname{Pre}(\theta)$. In particular by putting $b = 0$ in (18) we find that $(P_K F_{a,0})(\theta) = F_{a,0}(\theta)$, which says $P_K \colon \theta^a \mapsto \theta^a$, that is, P_K acts as the identity on $\operatorname{Pre}(\theta)$. So, $P_K^2 = P_K$ and $\operatorname{Ran} P_K = \operatorname{Pre}(\theta)$ follow immediately.

For the symmetry of P_K we calculate various matrix elements for P_K with respect to the elements in the basis AW. First for P_K acting on the right entry we obtain:

$$\begin{aligned}\langle F_{ab}, P_K F_{cd}\rangle_w &= \Big\langle \theta^a\overline{\theta}^b, \frac{w_c}{w_{c-d}}\theta^{c-d}\Big\rangle_w = \frac{w_c}{w_{c-d}}\delta_{a,b+c-d} w_a H(c-d) \\ &= \frac{w_a w_c}{w_{c-d}}\delta_{a-b,c-d} H(c-d), \end{aligned} \tag{19}$$

where H is the (discrete) Heaviside function $H : \mathbb{Z} \to \{0,1\}$ defined by $H(n) := 1$ for $n \geq 0$ and $H(n) := 0$ for $n < 0$.

Next we calculate the matrix elements for P_K acting on the left entry:

$$\begin{aligned}\langle P_K F_{ab}, F_{cd}\rangle_w &= \Big\langle \frac{w_a}{w_{a-b}}\theta^{a-b}, \theta^c\overline{\theta}^d\Big\rangle_w = \frac{w_a}{w_{a-b}} H(a-b)\delta_{a-b+d,c} w_c \\ &= \frac{w_a w_c}{w_{a-b}} H(a-b)\delta_{a-b,c-d}. \end{aligned} \tag{20}$$

Since the matrix entries (19) and (20) with respect to the elements in the vector space basis AW of $\mathbb{C}Q_q(\theta,\overline{\theta})$ are equal, we can pass to finite linear combinations to get

$$\langle F, P_K G\rangle_w = \langle P_K F, G\rangle_w$$

for all $F, G \in \mathbb{C}Q_q(\theta,\overline{\theta})$, which is the desired symmetry of P_K. □

Because of the previous proof we can think of P_K as a mapping

$$P_K \colon \mathbb{C}Q_q(\theta,\overline{\theta}) \to \operatorname{Pre}(\theta) \subset \mathcal{B}(\theta).$$

For any $g \in \mathbb{C}Q_q(\theta,\overline{\theta})$ we define the linear map $M_g\colon \operatorname{Pre}(\theta) \to \mathbb{C}Q_q(\theta,\overline{\theta})$ to be multiplication by g on the right, that is

$$M_g \phi := \phi g$$

for all $\phi \in \operatorname{Pre}(\theta)$. It is straightforward to show that $\operatorname{Ran} M_g \subset \mathbb{C}Q_q(\theta,\overline{\theta})$.

Definition 2. We define the *Toeplitz operator* associated to a *symbol* $g \in \mathbb{C}Q_q(\theta,\overline{\theta})$ to be

$$T_g = P_K M_g,$$

that is, right multiplication by g followed by the projection operator associated to the reproducing kernel K. We also write

$$T_g\colon \operatorname{Pre}(\theta) \to \mathcal{B}(\theta)$$

with the domain of T_g defined by $\operatorname{Dom}(T_g) := \operatorname{Pre}(\theta) \subset \mathcal{B}(\theta)$ to indicate that T_g is a densely defined linear operator acting in (but not on) the Segal-Bargmann space $\mathcal{B}(\theta)$. An equivalent way to write this definition is

$$T_g f(\theta) = \big\langle K(\theta,\eta)\,,\ f(\eta)\, g(\eta,\overline{\eta})\big\rangle_w,$$

where $f \in \operatorname{Pre}(\theta)$.

Actually, we have that $\operatorname{Ran} T_g \subset \operatorname{Pre}(\theta)$, but we prefer to consider the codomain to be the larger space $\mathcal{B}(\theta)$ in order to be able to apply the theory of densely defined linear operators acting in a Hilbert space. For example, see [7]. The definition of T_g can be expressed as this composition:

$$\operatorname{Dom}(T_g) = \operatorname{Pre}(\theta) \xrightarrow{M_g} \mathbb{C}Q_q(\theta, \overline{\theta}) \xrightarrow{P_K} \operatorname{Pre}(\theta) \subset \mathcal{B}(\theta)$$

One of the first considerations here is to find necessary and sufficient conditions on g in order that T_g is bounded and so has a unique bounded extension to $\mathcal{B}(\theta)$. And when T_g is bounded, one would like some information, at best a formula but at least an estimate, about the operator norm of T_g. While bounded operators are important, we will also be interested in certain operators that are not bounded.

We have used the common way of defining Toeplitz operators: multiply by a symbol and then project back into the Hilbert space. However, we are making choices here that are somewhat arbitrary. For example, we could have used left multiplication instead of right multiplication. Also the choice of the Segal-Bargmann space is arbitrary too. We could just as well have chosen the anti-Segal-Bargmann space, which also has a reproducing kernel. And having chosen instead that space, we would again have two possible choices for the multiplication operator: left and right. In all, there are four different choices for the definition of Toeplitz operators, and we simply have opted for one of these. The other three choices lead to very similar theories and will not be discussed further.

Next we define the *Toeplitz mapping* $T \colon g \mapsto T_g$ giving us a linear function

$$T \colon \mathbb{C}Q_q(\theta, \overline{\theta}) \to \operatorname{End}(\operatorname{Pre}(\theta)),$$

where $\operatorname{End}(\operatorname{Pre}(\theta))$ is the complex vector space of all linear densely defined operators S acting in the Hilbert space $\mathcal{B}(\theta)$ with $\operatorname{Dom} S = \operatorname{Pre}(\theta)$ and leaving $\operatorname{Pre}(\theta)$ invariant, that is $S(\operatorname{Pre}(\theta)) \subset \operatorname{Pre}(\theta)$. Because of this last condition $\operatorname{End}(\operatorname{Pre}(\theta))$ is closed under composition and so is an algebra. We also call T the *Toeplitz quantization*.

One verifies that $T_1 = I_{\operatorname{Pre}(\theta)}$, the identity, as an immediate consequence of the fact that K is the reproducing kernel of $\operatorname{Pre}(\theta)$. However, even though T is a map from one algebra to another algebra, it is not an algebra morphism. The product on the domain space is determined by $q \in \mathbb{C} \setminus \{0\}$, while the operator T_g is defined using the inner product which depends on the weights w_k. Even when the weights are functions of q (and so are not independent quantities) it is not expected that T preserves products, given what happens with Toeplitz operators in other contexts. Here is a result which shows what is happening in a 'nice' case.

Theorem 3. *Suppose that we have symbols g_1 and g_2, but with $g_2 \in \operatorname{Pre}(\theta)$, that is g_2 'depends' only on θ. Then $T_{g_1} T_{g_2} = T_{g_2 g_1}$.*

Proof. The point is since $g_2 \in \operatorname{Pre}(\theta)$ we have that $T_{g_2} = P_K M_{g_2} = M_{g_2}$, because multiplication by g_2 leaves $\operatorname{Pre}(\theta)$ invariant. So we calculate

$$T_{g_1} T_{g_2} = P_K M_{g_1} P_K M_{g_2} = P_K M_{g_1} M_{g_2} = P_K M_{g_2 g_1} = T_{g_2 g_1},$$

where the second to last equality is left to the reader to check. □

Remark 3. In the standard theories of Toeplitz operators, the symbols are functions and so commute. So essentially the same argument in such cases (with the corresponding hypothesis!) gives $T_{g_1}T_{g_2} = T_{g_1g_2}$. The fact that the map T in this context reverses the order of multiplication in this special case is not important as such. The equation $P_K M_g = M_g$ is not true for all symbols g and this is what is behind the fact that T does not respect multiplication. In fact, Theorem 3 implies that $T_{\overline{\theta}}T_\theta = T_{\theta\overline{\theta}}$. In the next calculation we actually will use something ever so slightly stronger, namely $T_{\overline{\theta}}T_\theta = T_{\theta\overline{\theta}} \neq 0$, but this will become clear later on. So for $q \neq 1$ we have

$$T_{\overline{\theta}\theta} = T_{q^{-1}\theta\overline{\theta}} = q^{-1}T_{\theta\overline{\theta}} = q^{-1}T_{\overline{\theta}}T_\theta \neq T_{\overline{\theta}}T_\theta.$$

Later on we will also calculate $T_\theta T_{\overline{\theta}}$ and see that this is yet another operator also not equal, in general, to $T_{\overline{\theta}\theta}$.

Theorem 4. *The linear map* $T\colon \mathbb{C}Q_q(\theta,\overline{\theta}) \to \mathrm{End}(\mathrm{Pre}(\theta))$ *is a vector space monomorphism if and only if the inner product* (2) *is non-degenerate.*

Proof. We are looking for a necessary and sufficient for $\ker T = 0$. So we take $g \in \ker T$, which means that $T_g = 0$. In particular, this is equivalent to $T_g f_d = 0$ for all $d \geq 0$, where $f_d = \theta^d$, an orthogonal basis of $\mathrm{Pre}(\theta) = \mathrm{Dom}(T_g)$. We calculate

$$\begin{aligned} T_g f_d(\theta) &= \langle K(\theta,\eta)\,,\, f_d(\eta)\, g(\eta,\overline{\eta})\rangle_w = \sum_c \frac{1}{w_c}\langle \overline{\theta}^c \otimes \eta^c, \eta^d g(\eta,\overline{\eta})\rangle_w \\ &= \sum_c \frac{1}{w_c}\langle \eta^c\overline{\eta}^d, g(\eta,\overline{\eta})\rangle_w\, \theta^c. \end{aligned}$$

So, $T_g f_d(\theta) = 0$ for all $d \geq 0$ if and only if $\langle \eta^c\overline{\eta}^d\,, g(\eta,\overline{\eta})\rangle_w = 0$ for all $c, d \geq 0$ if and only if $g(\eta,\overline{\eta})$ is orthogonal to $\mathbb{C}Q_q(\theta,\overline{\theta})$. So $\ker T = \big(\mathbb{C}Q_q(\theta,\overline{\theta})\big)^\perp$ and the result follows. □

Remark 4. One way to interpret this theorem is that it tells us when the symbol of a Toeplitz operator is uniquely determined by the operator. In the finite dimensional theory presented in [9] the inner product is always non-degenerate and the corresponding result proved there is that the Toeplitz quantization is always a monomorphism. Moreover in the context of [9] the domain and codomain vector space of the Toeplitz quantization have the same *finite* dimension; therefore that Toeplitz quantization is automatically a vector space (but not algebra) isomorphism. Here one expects the situation to be more complicated due to the fact that the domain and codomain of T have infinite dimension. To be more precise one expects that T is not surjective, that is, there exist operators which are not Toeplitz. Moreover, in the current context Toeplitz operators are not necessarily bounded as we shall see momentarily.

We calculate next the Toeplitz operators for the basis elements $\theta^i\overline{\theta}^j$ of the symbol space $PG_{l,q}(\theta,\overline{\theta})$.

Theorem 5. *The action of the Toeplitz operator $T_{\theta^i \overline{\theta}^j}$ on the orthonormal basis elements $\phi_a(\theta) = w_a^{-1/2} \theta^a \in \mathrm{Pre}(\theta)$ with $a \geq 0$ is given by*

$$(T_{\theta^i \overline{\theta}^j} \phi_a)(\theta) = \frac{w_{i+a}}{(w_a \, w_{i+a-j})^{1/2}} \, \phi_{i+a-j}(\theta). \tag{21}$$

Proof. We evaluate as follows:

$$\begin{aligned}
(T_{\theta^i \overline{\theta}^j} \phi_a)(\theta) &= \langle K(\theta, \eta) \, , \, \phi_a(\eta) \, \eta^i \overline{\eta}^j \rangle_w \\
&= \Big\langle \sum_k \phi_k(\overline{\theta}) \otimes \phi_k(\eta) \, , \, w_a^{-1/2} \eta^a \eta^i \overline{\eta}^j \Big\rangle_w \\
&= w_a^{-1/2} \sum_k w_k^{-1/2} \langle \eta^{j+k} \, , \, \eta^{i+a} \rangle_w \, \phi_k(\theta) \\
&= w_a^{-1/2} \sum_k w_k^{-1/2} \delta_{j+k, i+a} \, w_{j+k} \, \phi_k(\theta) \\
&= \frac{w_{i+a}}{(w_a \, w_{i+a-j})^{1/2}} \, \phi_{i+a-j}(\theta).
\end{aligned}$$

Recall that $\theta^n = 0$ and $w_n = 1$ for $n < 0$. So we also have put $\phi_n(\theta) = 0$ for $n < 0$ in the above calculation. ☐

This result determines T_g for all symbols $g \in \mathbb{C}Q_q(\theta, \overline{\theta})$ by linearity. Also, this result exhibits $T_{\theta^i \overline{\theta}^j}$ as a weighted shift operator with the degree of the shift being $i - j$. Next to see when this operator is bounded or compact we apply some basic functional analysis to obtain immediately:

Corollary 1. *First, $T_{\theta^i \overline{\theta}^j}$ is a bounded operator if and only if*

$$||T_{\theta^i \overline{\theta}^j}||_{op} = \sup \Big\{ \frac{w_{i+a}}{(w_a \, w_{i+a-j})^{1/2}} \; \Big| \; a \geq 0 \Big\} < \infty,$$

where $|| \cdot ||_{op}$ denotes the operator norm. Secondly, $T_{\theta^i \overline{\theta}^j}$ is a compact operator if and only if

$$\lim_{a \to \infty} \frac{w_{i+a}}{(w_a \, w_{i+a-j})^{1/2}} = 0.$$

Knowing this, it is now easy to construct examples of Toeplitz operators which are not bounded provided that we are free to choose the weights w_k. Similarly, it is now straightforward to construct Toeplitz operators which are bounded, but not compact, given the same freedom. We also showed earlier that $T_1 = I_{\mathrm{Pre}(\theta)}$, which is bounded but not compact.

We next obtain a consequence which relates the adjoint of a Toeplitz operator with symbol g to the Toeplitz operator with the conjugate symbol g^*.

Theorem 6. *Let $g \in \mathbb{C}Q_q(\theta, \overline{\theta})$ be arbitrary. Then*

$$\langle T_g f_1, f_2 \rangle_w = \langle f_1, T_{g^*} f_2 \rangle_w \tag{22}$$

for all $f_1, f_2 \in \mathrm{Pre}(\theta)$.

Proof. It suffices to prove this for $g = \theta^i \overline{\theta}^j$ where $i, j \geq 0$ and for $f_1 = \phi_a$ and $f_2 = \phi_b$ where $a, b \geq 0$. So we compute each side of (22) for these choices. For the left side we get

$$\begin{aligned} \langle T_{\theta^i \overline{\theta}^j} \phi_a, \phi_b \rangle_w &= \frac{w_{i+a}}{(w_a\, w_{i+a-j})^{1/2}} \langle \phi_{i+a-j}, \phi_b \rangle_w \\ &= \frac{w_{i+a}}{(w_a\, w_{i+a-j})^{1/2}} \delta_{i+a-j,b}. \end{aligned} \tag{23}$$

Note that the Kronecker delta is enforcing the condition that $i + a - j = b \geq 0$. Next for the right side we have

$$\begin{aligned} \langle \phi_a, T_{(\theta^i \overline{\theta}^j)^*} \phi_b \rangle_w &= \langle \phi_a, T_{\theta^j \overline{\theta}^i} \phi_b \rangle_w \\ &= \frac{w_{j+b}}{(w_b\, w_{j+b-i})^{1/2}} \langle \phi_a, \phi_{j+b-i} \rangle_w \\ &= \frac{w_{j+b}}{(w_b\, w_{j+b-i})^{1/2}} \delta_{a,j+b-i}. \end{aligned} \tag{24}$$

This time the delta imposes the condition $j + b - i = a \geq 0$. So in each case we have the combined conditions $a, b \geq 0$ and $i + a = b + j$. Using these conditions one can see that the expressions in (23) and (24) are equal. □

Remark 5. This result holds even when the inner product is degenerate. However, even when the inner product is non-degenerate all it says about the adjoint of T_g is that $T_{g^*} \subset (T_g)^*$, that is, the adjoint of T_g is an extension of T_{g^*}. Of course, such details are typical of densely defined operators. We recall that the Toeplitz operators are densely defined operators, all of which have the same dense domain, namely $\mathrm{Pre}(\theta)$. Also, this relation $T_{g^*} \subset (T_g)^*$ shows a compatibility between our definition of the conjugation in $\mathbb{C}Q_q(\theta, \overline{\theta})$ and the adjoint of a Toeplitz operator.

Corollary 2. *If $g \in \mathbb{C}Q_q(\theta, \overline{\theta})$ is a self-adjoint element (meaning $g^* = g$), then the Toeplitz operator T_g is a symmetric operator.*

Proof. By Theorem 6 and $g^* = g$ we have

$$\langle T_g f_1, f_2 \rangle_w = \langle f_1, T_g f_2 \rangle_w$$

for all $f_1, f_2 \in \mathrm{Pre}(\theta) = \mathrm{Dom}(T_g)$. And this is exactly what it means for a densely defined operator to be symmetric. (See [7].) □

Remark 6. If $g^* = g$, then it behooves us to study the self-adjoint extensions of the symmetric operator T_g. This remains an open problem.

Corollary 3. *Every Toeplitz operator T_g is closable. Moreover, its closure satisfies $\overline{T}_g = (T_g)^{**} \subset (T_{g^*})^*$.*

Proof. This follows rather directly from Theorem VIII.1b in [7]. We get from that reference that T_g is closable if and only if $\mathrm{Dom}(T_g)^*$ is a dense subspace. But this is so since $\mathrm{Dom}(T_g)^* \supset \mathrm{Dom}\, T_{g^*} = \mathrm{Pre}(\theta)$ and $\mathrm{Pre}(\theta)$ is dense. The equality $\overline{T}_g = (T_g)^{**}$ follows from the cited theorem. The inclusion $(T_g)^{**} \subset (T_{g^*})^*$ follows from Theorem 6. □

We now analyze various particular cases of (21). First for $i = j = 0$ we have

$$(T_1\phi_a)(\theta) = \frac{w_a}{(w_a\, w_a)^{1/2}}\, \phi_a(\theta) = \phi_a(\theta)$$

for all $a \geq 0$, so that $T_1 = I_{\mathrm{Pre}(\theta)}$, the identity map, as already noted above.

For the case $i = j$ of (21) we obtain for all $a \geq 0$ that

$$(T_{\theta^i\bar{\theta}^i}\phi_a)(\theta) = \frac{w_{i+a}}{(w_a\, w_{i+a-i})^{1/2}}\, \phi_{i+a-i}(\theta) = \frac{w_{i+a}}{(w_a\, w_a)^{1/2}}\, \phi_a(\theta) = \frac{w_{i+a}}{w_a}\, \phi_a(\theta).$$

Hence the basis $\phi_a(\theta)$ diagonalizes simultaneously the family of symmetric operators $T_{\theta^i\bar{\theta}^i}$ for $i \geq 0$. By Corollary 2 we see that $T_{\theta^i\bar{\theta}^i}$ is symmetric.

Next we consider (21) for the case $j = 0$ and get

$$(T_{\theta^i}\phi_a)(\theta) = \frac{w_{i+a}}{(w_a\, w_{i+a})^{1/2}}\, \phi_{i+a}(\theta) = \frac{w_{i+a}^{1/2}}{w_a^{1/2}}\, \phi_{i+a}(\theta)$$

or, equivalently, $T_{\theta^i} : \theta^a \mapsto \theta^{i+a}$ which itself can be written as $T_{\theta^i} = M_{\theta^i}$. Of course, this also follows from the definition $T_{\theta^i} = P_K M_{\theta^i} = M_{\theta^i}$, since M_{θ^i} leaves $\mathrm{Pre}(\theta)$ invariant and P_K acts as the identity on $\mathrm{Pre}(\theta)$. A subcase here is $T_\theta = M_\theta$, which merits the name *creation operator* since it increases by 1 the degree of the elements in $\mathrm{Pre}(\theta)$, which are exactly the polynomials in θ. Moreover, $T_{\theta^i} = (T_\theta)^i$ also is immediate. (Recall that T_θ leaves $\mathrm{Pre}(\theta)$ invariant, and so $(T_\theta)^i$ is defined.)

So, if T_θ is bounded (resp., compact), then T_{θ^i} is bounded (resp., compact) for all $i \geq 1$. In the Hilbert space introduced by Bargmann in [2], one has $w_a = a!$ and $\theta = z$, so that $T_{\theta^i} = T_{z^i}$ is not bounded for $i \geq 1$ in that space. One might expect that with w_a being some reasonable deformation of the factorial function the corresponding operators T_{θ^i} would also not be bounded. However, the boundedness of these operators depends completely on the choice of weights w_a, nothing else. So for some choices (such as, for example, w_a constant) these operators will be bounded.

Yet another interesting special case of (21) is when $i = 0$. Then we have

$$(T_{\bar{\theta}^j}\phi_a)(\theta) = \frac{w_a}{(w_a\, w_{a-j})^{1/2}}\, \phi_{a-j}(\theta) = \left(\frac{w_a}{w_{a-j}}\right)^{1/2} \phi_{a-j}(\theta)$$

or, in terms of the unnormalized monomials,

$$T_{\bar{\theta}^j} : \theta^a \mapsto \frac{w_a}{w_{a-j}}\, \theta^{a-j}$$

for all $a \geq 0$. In particular, for $j = 1$ we can see that

$$T_{\bar{\theta}} \colon \theta^a \mapsto \frac{w_a}{w_{a-1}}\, \theta^{a-1}$$

deserves to be called an *annihilation operator*, since it lowers the degree of any non-constant polynomial by 1 and sends constants to zero. A simple argument shows that $T_{\overline{\theta}^j} = (T_{\overline{\theta}})^j$. And similar to the above situation, we see that if $T_{\overline{\theta}}$ is bounded (resp., compact), then $T_{\overline{\theta}^j}$ is bounded (resp., compact) for all $j \geq 1$. Again, the space in [2] is an important example for which the operators $T_{\overline{\theta}^j}$ are not bounded. And again, the boundedness of these operators depends solely on the weights.

Using Theorem 3 in the first equality and two properties established above in the second equality, we see that

$$T_{\theta^i \overline{\theta}^j} = T_{\overline{\theta}^j} T_{\theta^i} = (T_{\overline{\theta}})^j (T_\theta)^i.$$

The last expression here is in anti-Wick order, which by definition means a product of creation and annihilation operators such that all of the creation operators are to the right of all of the annihilation operators. By linearity every Toeplitz operator T_g will then be a sum of terms, each of which is in anti-Wick order. Because of this property one says that the Toeplitz quantization is an *anti-Wick quantization*.

There is another way of viewing the annihilation operator $T_{\overline{\theta}}$. We note that in the case when $w_a = a!$ as in [2], we have that

$$T_{\overline{\theta}} \colon \theta^a \mapsto \frac{w_a}{w_{a-1}}\, \theta^{a-1} = \frac{a!}{(a-1)!}\, \theta^{a-1} = a\, \theta^{a-1},$$

which is the derivative operator from elementary calculus. So we can think of $T_{\overline{\theta}}$ in this more general context as a deformation of the classical derivative. We call it the *w-deformed derivative* and denote it by ∂_w. If we define the w-deformed integers to be $[n]_w := w_n / w_{n-1}$ for every integer $n \geq 1$ and $[0]_w := 0$, then we have

$$\partial_w = T_{\overline{\theta}} \colon \theta^a \mapsto [a]_w\, \theta^{a-1}.$$

The upshot of this paragraph is merely a change to another notation that is more compatible with notations used elsewhere in the literature, nothing else really.

Notice again that $T_{\overline{\theta}^j} T_{\theta^i} = T_{\theta^i \overline{\theta}^j}$ follows from Theorem 3. We now calculate $T_{\theta^i} T_{\overline{\theta}^j}$ using the individual formulas derived above for T_{θ^i} and $T_{\overline{\theta}^j}$. So,

$$\phi_a \xrightarrow{T_{\overline{\theta}^j}} \left(\frac{w_a}{w_{a-j}} \right)^{1/2} \phi_{a-j} \xrightarrow{T_{\theta^i}} \left(\frac{w_a}{w_{a-j}} \right)^{1/2} \left(\frac{w_{i+a-j}}{w_{a-j}} \right)^{1/2} \phi_{a-j+i}$$

which gives

$$T_{\theta^i} T_{\overline{\theta}^j} \phi_a = \frac{(w_a w_{i+a-j})^{1/2}}{w_{a-j}} \phi_{a-j+i}.$$

This is different from the formula (21) derived above for $T_{\theta^i \overline{\theta}^j}$. In particular, for the case $i = j = 1$ which we left unfinished earlier we have

$$T_\theta T_{\overline{\theta}} \phi_a = \frac{w_a}{w_{a-1}} \phi_a = [a]_w \phi_a.$$

For the sake of completeness we note that the operator $N_\theta := T_\theta T_{\overline{\theta}}$ is called the *w-deformed number operator*. On the other hand from equation (21) we have that

$$T_{\overline{\theta}} T_\theta \phi_a = T_{\theta \overline{\theta}} \phi_a = \frac{w_{a+1}}{w_a} \phi_a = [a+1]_w \phi_a.$$

5 Canonical Commutation Relations

This final section is a continuation of the two calculations just made at the end of the last section. First, we define the q-commutator of any two elements a and b in any (associative, say) algebra over $\mathbb{C}$ by

$$[a, b]_q := ab - qba,$$

where $q \in \mathbb{C} \setminus \{0\}$. This is the commutator which is appropriate for the study of q-deformations.

The Toeplitz quantization starts with the 'classical' space $\mathbb{C}Q_q(\theta, \overline{\theta})$ of symbols and from them produces operators acting in the 'quantum' Segal-Bargmann space $\mathcal{B}(\theta)$. The point here is that before the Toeplitz quantization we have the *homogeneous* q-commutation relation in $\mathbb{C}Q_q(\theta, \overline{\theta})$, namely

$$[\theta, \overline{\theta}]_q = \theta\overline{\theta} - q\overline{\theta}\theta = 0. \tag{25}$$

Speaking roughly without going into the rigorous details, in quantum theory we have creation operators and annihilations operators which come in pairs, say A for an annihilation operator and A^* for its corresponding creation operator. Then a typical commutation relation is something more or less like

$$[A, A^*] = I, \quad \text{the identity.}$$

This is called a *canonical commutation relation*. So in general in a quantum theory we expect *inhomogeneous* canonical commutation relations.

Now the Toeplitz quantization of the q-commutator $[\theta, \overline{\theta}]_q$ is

$$[T_\theta, T_{\overline{\theta}}]_q = T_\theta T_{\overline{\theta}} - qT_{\overline{\theta}} T_\theta.$$

But recall that T_θ is the creation operator and that $T_{\overline{\theta}}$ is the annihilation operator; so this q-commutator has the form $[A^*, A]_q$. And this is not the form of a canonical commutation relation. However, since it is homogeneous and $q \neq 0$ we can trivially rewrite (25) as

$$[\overline{\theta}, \theta]_{q^{-1}} = \overline{\theta}\theta - q^{-1}\theta\overline{\theta} = 0. \tag{26}$$

In fact we have an identification $\mathbb{C}Q_q(\theta, \overline{\theta}) \cong \mathbb{C}Q_{q^{-1}}(\overline{\theta}, \theta)$. What this means is that at the classical level we can not distinguish the q-deformed theory associated to the holomorphic (resp., anti-holomorphic) variable θ (resp., $\overline{\theta}$) from the q^{-1}-deformed theory associated to the holomorphic (resp., anti-holomorphic) variable $\overline{\theta}$ (resp., θ). (The previous sentence does not contain a typographical error. It makes perfect sense to consider $\overline{\theta}$ as a holomorphic variable whose associated anti-holomorphic variable is θ.) Another way of saying this is that as far as our theory is concerned only with the classical level we have no way to distinguish between q-deformations and q^{-1}-deformations nor between holomorphic and anti-holomorphic variables.

However, the quantizations of θ and $\overline{\theta}$ are distinguishable. In this sense Toeplitz quantization breaks a symmetry. And the choice of quantization determines exactly how the symmetry is broken. For example, if we define a Toeplitz quantization as in this paper, but using instead the anti-Segal-Bargmann space $\mathcal{B}(\overline{\theta})$ as the Hilbert space in which the quantized operators act, then θ quantizes to the annihilation operator while $\overline{\theta}$ quantizes to the creation operator, just the reverse of what we have obtained with the present Toeplitz quantization in the Segal-Bargmann space $\mathcal{B}(\theta)$. These comments indicate that naming a particular order in $\mathbb{C}Q_q(\theta,\overline{\theta})$ the anti-Wick ordering (that is, all creation operators to the right of all annihilation operators) is not justifiable in terms of mathematical structures of $\mathbb{C}Q_q(\theta,\overline{\theta})$ alone. We have simply decided to follow the nomenclature used in [3] as indicated earlier.

Now the Toeplitz quantization of the q^{-1}-commutator $[\overline{\theta},\theta]_{q^{-1}}$ is

$$[T_{\overline{\theta}}, T_\theta]_{q^{-1}} = T_{\overline{\theta}} T_\theta - q^{-1} T_\theta T_{\overline{\theta}}.$$

And this has the virtue of being of the form $[A, A^*]$. So we require this canonical commutation relation to hold:

$$[T_{\overline{\theta}}, T_\theta]_{q^{-1}} = T_{\overline{\theta}} T_\theta - q^{-1} T_\theta T_{\overline{\theta}} = I_{\mathrm{Pre}(\theta)}, \quad \text{the identity on } \mathrm{Pre}(\theta). \tag{27}$$

This gives us the recursion relation

$$[a+1]_w - q^{-1}[a]_w = 1$$

for all $a \geq 0$. But we already have $[0]_w = 0$. So the sequence $[a]_w$ is uniquely determined by q (or by q^{-1} if one wishes to consider this as the primary parameter). It is rather straightforward to find an explicit formula for $[a]_w$. The next definition is standard, though not universal. See [3] for a different, more symmetric definition.

Definition 3. Let $r \in \mathbb{C}$. For each integer $n \geq 0$ we define

$$[n]_r := 1 + r + r^2 + \cdots + r^{n-1} \quad \text{if } n \geq 1$$

and $[0]_r := 0$. This is called the r-deformation of n.

For example, $[1]_r = 1$ and $[2]_r = 1+r$. Taking $r = 1$ gives $[n]_r = n$ for every integer $n \geq 0$. This justifies saying that these are deformations of the integers and that r in the deformation parameter. If $r \neq 1$, then we have the alternative expression $[n]_r = \frac{1-r^n}{1-r}$, which often appears in the literature.

Proposition 1. *The unique solution of the recursion relation*

$$[a+1]_w - q^{-1}[a]_w = 1$$

for all integers $a \geq 0$ with $[0]_w = 0$ is $[a]_w = [a]_{q^{-1}}$.

Proof. The recursion relation for $[n]_r$ is $[n+1]_r - r[n]_r = 1$, as the reader can easily check. Taking $r = q^{-1}$ shows that the sequences $[a]_w$ and $[a]_{q^{-1}}$ satisfy the same recursion relation. But they both start out with $[0]_w = 0 = [0]_{q^{-1}}$, which ends the proof. □

Now it is a matter of going from the deformed integers $[a]_w = [a]_{q^{-1}}$ to the weights w_k. Now for every integer $a \geq 1$ we have

$$[a]_{q^{-1}} = [a]_w = \frac{w_a}{w_{a-1}} \tag{28}$$

by definition of $[a]_w$. It turns out that $[0]_w = 0$ carries no information about the weights. Then (28) gives a sequence of identities

$$w_1 = [1]_{q^{-1}} w_0, \qquad w_2 = [2]_{q^{-1}} w_1, \qquad w_3 = [3]_{q^{-1}} w_2,$$

and so on. The solution for $k \geq 1$ is clearly

$$w_k = [k]!_{q^{-1}} w_0,$$

where the q^{-1}-deformed factorial is defined by

$$[k]!_{q^{-1}} := [k]_{q^{-1}} [k-1]_{q^{-1}} \cdots [2]_{q^{-1}} [1]_{q^{-1}}$$

and where $w_0 > 0$ is arbitrary. In this way we have defined a unique sequence (up to a multiplicative positive constant) of weights $w_k = w_k(q)$, which are functions of the one parameter q such that

$$[T_{\overline{\theta}}, T_\theta]_{q^{-1}} = T_{\overline{\theta}} T_\theta - q^{-1} T_\theta T_{\overline{\theta}} = I_{\mathrm{Pre}(\theta)}.$$

In particular, $[T_{\overline{\theta}}, T_\theta]_{q^{-1}}$ is bounded. By putting the deformation parameter q equal to 1 and normalizing $1 \in \mathbb{C}Q_q(\theta, \overline{\theta})$ by putting $w_0 = 1$, we recover the weights $w_k = k!$ of the Hilbert space $\mathcal{H}$ defined in (4). Recall that the Segal-Bargmann space based on the phase space $\mathbb{C}$ in [2] is the closed subspace of $\mathcal{H}$ consisting of the holomorphic functions in $\mathcal{H}$.

If we wish to have some other operator instead of the identity on the 'right side' of the canonical commutation relation, the same method applies to give the corresponding weights.

6 Concluding Remarks

Since the Toeplitz operators introduced here are only densely defined, one has the standard problems in the analysis of such operators. For example, we know they are closable, but can we identify exactly what the closure is? And if a Toeplitz operator is symmetric, then we would like to know what its self-adjoint extensions are. In particular, we would like to know exactly what are the conditions for a Toeplitz operator to be essentially self-adjoint.

We have given necessary and sufficient conditions for the Toeplitz $T_{\theta^i \overline{\theta}^j}$ to be bounded or compact. But the full story remains to be told for T_g where g is an arbitrary symbol, though our results allow us to form sufficient conditions for boundedness and compactness by expanding T_g as a linear combination of $T_{\theta^i \overline{\theta}^j}$'s. We expect such conditions to be far from necessary.

Another possibility for further research is to define coherent states in this context, much as was done in [3] in a similar finite dimensional case. This would allow the introduction of a coherent state transform and a coherent state quantization. (Also see [4].) This would relate the material in this paper with yet another aspect of mathematical physics. Also it might be of interest to study in more detail the classical space $\mathbb{C}Q_q(\theta, \overline{\theta})$ from a physics point of view as a sort of non-commutative phase space.

Given the positive result in the finite dimensional case presented in [9] it seems reasonable to conjecture that $\mathbb{C}Q_q(\theta,\overline{\theta})$ also has its own reproducing kernel, at least in the case when its inner product is non-degenerate. We also leave this as a problem for another day.

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References

[1] N. Aronszajn: Theory of reproducing kernels. *Trans. Am. Math. Soc.* 108 (1950) 337–404.

[2] V. Bargmann: On a Hilbert space of analytic functions and its associated integral transform. I. *Commun. Pure Appl. Math.* 14 (1961) 187–214.

[3] M. El Baz, R. Fresneda, J-P. Gazeau, Y. Hassouni: Coherent state quantization of paragrassmann algebras. *J. Phys. A: Math. Theor.* 43 (2010). 385202 (15pp). Also see the Erratum for this article in arXiv:1004.4706v3

[4] J-P. Gazeau: *Coherent States in Quantum Physics*. Wiley-VCH (2009).

[5] C. Kassel: *Quantum Groups*. Springer (1995).

[6] M. Khalkhali: *Basic Noncommutative Geometry*. European Math. Soc. (2009).

[7] M. Reed, B. Simon: *Mathematical Methods of Modern Physics,* Vol. I, Functional Analysis. Academic Press (1972).

[8] S. Saitoh: *Theory of reproducing kernels and its applications,* Pitman Research Notes, Vol. 189. Longman Scientific & Technical, Essex (1988).

[9] S.B. Sontz: Paragrassmann Algebras as Quantum Spaces, Part I: Reproducing Kernels, Geometric Methods in Physics. *XXXI Workshop 2012. Trends in Mathematics* (2013) 47–63. arXiv:1204.1033v3

[10] S.B. Sontz: Paragrassmann Algebras as Quantum Spaces, Part II: Toeplitz Operators. *Journal of Operator Theory*. To appear. arXiv:1205.5493.

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Supplementary balance laws for Cattaneo heat propagation

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Abstract. In this work we determine for the Cattaneo heat propagation system all the supplementary balance laws (shortly SBL) of the same order (zero) as the system itself and extract the constitutive relations (expression for the internal energy) dictated by the Entropy Principle. The space of all supplementary balance laws (having the functional dimension 8) contains four original balance laws and their deformations depending on 4 functions of temperature $(\lambda^0(\vartheta), K^A(\vartheta), A = 1, 2, 3)$. The requirements of the II law of thermodynamics leads to the exclusion of three functional degrees $(K^A = 0, A = 1, 2, 3)$ and to further restriction to the form of internal energy. In its final formulation, entropy balance represents the deformation of the energy balance law by the functional parameter $\lambda^0(\vartheta)$.

1 Introduction

Systems of balance equations form the cornerstone of the Continuum Thermodynamics, [1], [2], [4], [5]. With each system of this type, there is associated the space of "supplementary balance laws" (see next Section) playing, for the systems of balance equations, the role similar to the role the conservation laws play for general systems of differential equations. In this work we determine explicitly all supplementary balance laws for the Cattaneo heat propagation system (CHP-system) (1) of the same order (zero) that the Cattaneo system itself. We will solve directly the Lagrange-Liu system of differential equations associated with the CHP model [7], [8], and, on our way, specify the constitutive relation – the form of internal energy as the function of temperature θ and heat flux q. If this condition is fulfilled, the total space of SBL (modulo trivial balance laws) is 8-dimensional. If this condition does not hold, there are no new SBL except trivial (see [6]). Then we show that the positivity condition for the production in the new balance laws place additional restriction to the form of internal energy and determine the unique SBL having nonnegative production – the entropy balance law.

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2 Supplementary balance laws of a balance system

Let one have a system of balance equations for the fields $y^i(t, x^A, A = 1, 2, 3)$

$$\partial_t F_i^0 + \partial_{x^A} F_i^A = \Pi_i\,, \quad i = 1, \ldots, m, \tag{1}$$

with the densities F_i^0, fluxes $F_i^A, A = 1, 2, 3$ and sources Π_i being functions of space-time point $(t, x^A, A = 1, 2, 3)$, fields y^i and their derivatives (by t, x^A) up to the order $k \geqq 0$. Number k is called the order of balance system (1). In continuum thermodynamics people mostly work with the balance system of order 0 (case of Rational Extended Thermodynamics) and 1.

Definition 1. A balance law (2) of order r (in the same sense as the system (1) is of order k)

$$\partial_t K^0 + \sum_{A=1}^{3} \partial_{x^A} K^A = Q \tag{2}$$

is called a *supplementary balance law (SBL) for the system* (1) if *every solution of the system* (1) *is, at the same time, solution of the balance equation* (2).

Examples of supplementary balance laws are: entropy balance, provided the Entropy Principle is admitted for system (1) (see [5], [10], [11]), Noether balance laws corresponding to the Lie groups of symmetry (see [7], [8]) and some linear combinations of the balance equations of original balance system with variable coefficients satisfying some condition ("gauge symmetries" of system (1), see [7], [8]).

As a rule, in classical physics one looks for entropy balance laws of the same order as the original balance systems. Higher order SBL are also of an interest for studying the balance system (1) – for example, a study of integrable systems leads to the hierarchy of conservation laws (often having form of conservation laws themselves) of higher order.

For a balance system (1) of order 0 (case of Rational extended Thermodynamics, see [5]), density and flux of a SBL (2) satisfy the system of equations

$$\lambda^i F^\mu_{i,y^j} = K^\mu_{,y^j}\,, \tag{3}$$

where summation over repeated indices is considered. Functions $\lambda^i(y^j)$ (main fields in terminology of [5]) are to be found from the conditions of solvability of this system. We call this system the LL-system referring to the Liu method of using Lagrange method for formulating dissipative inequality for a system (1), see [3], [5]. Source/production in the system (2) is then found as $Q = \sum_i \lambda^i \Pi_i$.

3 Cattaneo Heat propagation balance system

Consider the heat propagation model containing the temperature ϑ and heat flux q as the independent dynamical fields $y^0 = \vartheta$, $y^A = q^A$, $A = 1, 2, 3$.

Balance equations of this model have the form

$$\begin{cases} \partial_t(\rho\epsilon) + \operatorname{div}(q) = 0\,, \\ \partial_t(\tau q) + \nabla\Lambda(\vartheta) = -q\,. \end{cases} \tag{4}$$

The second equation in (4) can be rewritten in the conventional form

$$\partial_t(\tau q) + \lambda \cdot \nabla\vartheta = -q\,,$$

where $\lambda = \frac{\partial\Lambda}{\partial\vartheta}$. If the coefficient λ may depend on the density ρ, the equation is more complex.

Constitutive relations specify dependence of the internal energy ϵ on ϑ, q and possible dependence of coefficients τ, Λ on the temperature (including the requirement $\Lambda_{,\vartheta} \neq 0$). The simplest case is the linear relation $\epsilon = k\vartheta$, but for our purposes it is too restrictive, see [2, Sec.2.1].

Since ρ is not considered here as a dynamical variable, we merge it with the field ϵ and from now on and till the end it will be omitted. On the other hand, in this model the energy ϵ depends on temperature ϑ *and on the heat flux* q (see [2, Sec.2.1.2]) or, by change of variables, temperature $\vartheta = \vartheta(\epsilon, q)$ will be considered as the function of dynamical variables.

Cattaneo equation $q + \tau\partial_t(q) = -\lambda \cdot \nabla\vartheta$ has the form of the vectorial balance law and, as a result there is no need for the constitutive relations to depend on the derivatives of the basic fields. No derivatives appear in the constitutive relation, therefore, this is the RET model. In the second equation there is a nonzero production $\Pi^A = -q^A$. The model is homogeneous, there is no explicit dependence of any functions on t, x^A.

4 LL-system for supplementary balance laws of CHP-system

To study the LL-system for the supplementary balance laws we start with the $i \times \mu$ matrix of density/flux components

$$F_i^\mu = \begin{pmatrix} \epsilon & \tau q^1 & \tau q^2 & \tau q^3 \\ q^1 & \Lambda(\vartheta) & 0 & 0 \\ q^2 & 0 & \Lambda(\vartheta) & 0 \\ q^3 & 0 & 0 & \Lambda(\vartheta) \end{pmatrix}.$$

Assuming that coefficients τ and the function Λ are independent on the heat flux variables q^A we get the "vertical (i.e. by fields ϑ, q^A) differentials" of densities and flux components F_i^μ

$$d_v F_i^\mu = \begin{pmatrix} \epsilon_\vartheta\, d\vartheta + \epsilon_{q^A}\, dq^A & \tau_\vartheta q^1\, d\vartheta + \tau\, dq^1 & \tau_\vartheta q^2\, d\vartheta + \tau\, dq^2 & \tau_\vartheta q^2\, d\vartheta + \tau\, dq^3 \\ dq^1 & \Lambda_\vartheta\, d\vartheta & 0 & 0 \\ dq^2 & 0 & \Lambda_\vartheta\, d\vartheta & 0 \\ dq^3 & 0 & 0 & \Lambda_\vartheta\, d\vartheta \end{pmatrix}.$$

Let now

$$\partial_t K^0(x,y) + \partial_{x^A} K^A(x,y) = Q(x,y) \tag{5}$$

be a supplementary balance law for the Cattaneo balance system (4). It is easy to see that the LL-system has the form:
Subsystem of LL-system with $\mu = 0$ has the form:

$$\begin{cases} \lambda^0 \epsilon_\vartheta + \tau_\vartheta \lambda^A q^A = K^0_{,\vartheta} \\ \lambda^0 \epsilon_{q^A} + \lambda^A \tau = K^0_{,q^A} \end{cases}, \;\; A = 1,2,3\,. \tag{6}$$

For $\mu = A = 1,2,3$, using cyclic notations, we have LL-equations:

$$\begin{cases} \Lambda_\vartheta \lambda^A = K^A_{,\vartheta} \\ \lambda^0 = K^A_{,q^A} \\ 0 = K^A_{q^{A+1}} \\ 0 = K^A_{q^{A+2}} \end{cases}, \ A = 1,2,3\,. \tag{7}$$

Looking at systems (6), (7) we see that if we make the change of variables $\tilde{\vartheta} = \Lambda(\vartheta)$ then the system of equations (6), (7) takes the form (wherever is the derivative by ϑ we multiply this equation by $\Lambda_{,\vartheta}$)

$$\begin{cases} \lambda^0 \epsilon_{\tilde{\vartheta}} + \tau_{\tilde{\vartheta}} \lambda^A q^A = K^0_{,\tilde{\vartheta}} \\ \lambda^0 \epsilon_{q^A} + \lambda^A \tau = K^0_{q^A} \end{cases} ; \quad \begin{cases} K^A_{,\tilde{\vartheta}} = \lambda^A \\ K^A_{q^B} = \lambda^0 \delta^A_B \end{cases}, \ A, B = 1,2,3. \tag{8}$$

The second subsystem is equivalent to the relation

$$d_v K^A = \lambda^A d\tilde{\vartheta} + \lambda^0 dq^A.$$

These integrability conditions imply the expression $K^A = K^A(x^\mu, \tilde{\vartheta}, q^A)$ and

$$K^A_{q^A} = \lambda^0, \ A = 1,2,3 \ \Rightarrow \lambda^0 = \lambda^0(\tilde{\vartheta}).$$

Integrating equation $K^A_{q^A} = \lambda^0(\tilde{\vartheta})$ by q^A we get

$$K^A = \lambda^0(\tilde{\vartheta}) q^A + \tilde{K}^A(\tilde{\vartheta}) \tag{9}$$

with some functions $\tilde{K}^A(\tilde{\vartheta})$.

The first equation of each system now takes the form

$$\lambda^A = K^A_{\tilde{\vartheta}} = \lambda^0_{\tilde{\vartheta}} q^A + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta}). \tag{10}$$

Substituting these expressions for λ^A into the 0-th system

$$\begin{cases} \lambda^0 \epsilon_{\tilde{\vartheta}} + \tau_{\tilde{\vartheta}} \lambda^A q^A = K^0_{,\tilde{\vartheta}} \\ \lambda^0 \epsilon_{q^A} + \lambda^A \tau = K^0_{q^A} \end{cases}, \ A = 1,2,3,$$

we get

$$\begin{cases} K^0_{,\tilde{\vartheta}} = \lambda^0 \epsilon_{\tilde{\vartheta}} + \tau_{\tilde{\vartheta}} \big(\lambda^0_{\tilde{\vartheta}} \|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta}) q^A \big) \\ K^0_{q^A} = \lambda^0 \epsilon_{q^A} + \tau \big(\lambda^0_{\tilde{\vartheta}} q^A + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta}) \big) \end{cases}, \ A = 1,2,3, \tag{11}$$

where $\|q\|^2 = \sum_A q^{A\ 2}$.

Integrating A-th equation by q^A and comparing results for different A we obtain the following representation

$$K^0 = \lambda^0 \epsilon + \tau(\tilde{\vartheta}) \Big[\frac{1}{2} \lambda^0_{\tilde{\vartheta}} \|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta}) q^A \Big] + f(\tilde{\vartheta}) \tag{12}$$

for some function $f(\tilde{\vartheta}, x^\mu)$.

Calculate derivative by $\tilde{\vartheta}$ in the last formula for K^0 and subtract the first formula of the previous system. We get

$$0 = \lambda^0_{,\tilde{\vartheta}}\epsilon + \tau(\tilde{\vartheta})\Big[\Big(\frac{1}{2}\lambda^0_{\tilde{\vartheta}}\|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta})q^A\Big)\Big]_{,\tilde{\vartheta}} - \frac{1}{2}\tau_{,\tilde{\vartheta}}\lambda^0_{,\tilde{\vartheta}}\|q\|^2 + f_{,\tilde{\vartheta}}(\tilde{\vartheta}). \tag{13}$$

This is the compatibility condition for the system (6) for K^0. As such, it is realization of the general compatibility system (8).

Take $q^A = 0$ in the last equation, i.e. consider the case where *there is no heat flux*. Then the internal energy reduces to its *equilibrium value* $\epsilon^{\mathrm{eq}}(\tilde{\vartheta})$ and we get $f_{,\tilde{\vartheta}}(\tilde{\vartheta}) = -\lambda^0_{,\tilde{\vartheta}}\epsilon^{\mathrm{eq}}$. Integrating here we find

$$f(\tilde{\vartheta}) = f_0(x^\mu) - \int^{\tilde{\vartheta}} \lambda^0_{,\tilde{\vartheta}}(s)\epsilon^{\mathrm{eq}}(s)\, ds\,. \tag{14}$$

Substituting this value for f into the previous formula we get expressions for K^μ:

$$\begin{cases} K^0 = \lambda^0\epsilon - \displaystyle\int^{\tilde{\vartheta}} \lambda^0_{,\tilde{\vartheta}}\epsilon^{\mathrm{eq}}\, ds + \tau(\tilde{\vartheta})\Big[\frac{1}{2}\lambda^0_{\tilde{\vartheta}}\|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta})q^A\Big] + f_0 \\ K^A = \lambda^0(\tilde{\vartheta})q^A + \tilde{K}^A(\tilde{\vartheta}) \end{cases}, \quad A = 1, 2, 3. \tag{15}$$

In addition to this, from (13) and obtained expression for $f(\tilde{\vartheta})$, we get the *expression for internal energy*

$$\epsilon = \epsilon^{\mathrm{eq}}(\tilde{\vartheta}) + \frac{1}{2}\tau_{,\tilde{\vartheta}}\|q\|^2 - \frac{\tau(\tilde{\vartheta})}{\lambda^0_{\tilde{\vartheta}}(\tilde{\vartheta})}\Big[\frac{1}{2}\lambda^0_{,\tilde{\vartheta}\tilde{\vartheta}}\|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}\tilde{\vartheta}}(\tilde{\vartheta})q^A\Big]. \tag{16}$$

This form for internal energy presents the restriction to the constitutive relations in Cattaneo model placed on it by the entropy principle.

The zeroth main field λ^0 is an arbitrary function of $\tilde{\vartheta}$ while λ^A are given by the relations (15):

$$\lambda^A = (\lambda^0_{\tilde{\vartheta}}q^A + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta})). \tag{17}$$

Using this we find the source/production term for the SBL (5)

$$Q = \lambda^A\Pi_A = -\lambda^A q^A = -\big(\lambda^0_{\tilde{\vartheta}}\|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}}(\tilde{\vartheta})q^A\big)\,.$$

Now we combine obtained expressions for components of a secondary balance law. We have to take into account that the LL-system defines K^μ only $\mathrm{mod}\, C^\infty(X)$. This means first of all that all the functions may depend explicitly on x^μ. For energy ϵ, field $\Lambda(\vartheta)$ and the coefficient τ this dependence is determined by constitutive relations and is, therefore, fixed. Looking at (16) we see that the coefficients of terms linear and quadratic by q^A are also defined by the constitutive relation, i.e. in the representation

$$\begin{aligned} \epsilon &= \epsilon^{\mathrm{eq}}(\tilde{\vartheta}) + \mu(\tilde{\vartheta})\|q\|^2 + M_A(\tilde{\vartheta})q^A \\ &= \epsilon^{\mathrm{eq}}(\tilde{\vartheta}) + \frac{1}{2}\tau_{,\tilde{\vartheta}}\|q\|^2 - \frac{\tau(\tilde{\vartheta})}{\lambda^0_{\tilde{\vartheta}}(\tilde{\vartheta})}\Big[\frac{1}{2}\lambda^0_{,\tilde{\vartheta}\tilde{\vartheta}}\|q\|^2 + \tilde{K}^A_{,\tilde{\vartheta}\tilde{\vartheta}}(\tilde{\vartheta})q^A\Big], \end{aligned} \tag{18}$$

coefficients

$$\mu(\tilde{\vartheta},x)=\frac{1}{2}\tau_{,\tilde{\vartheta}}-\frac{1}{2}\frac{\tau(\tilde{\vartheta})}{\lambda^0_{\tilde{\vartheta}}(\tilde{\vartheta})}\lambda^0_{\tilde{\vartheta}\tilde{\vartheta}}\,,\quad M_A=-\frac{\tau(\tilde{\vartheta})}{\lambda^0_{\tilde{\vartheta}}(\tilde{\vartheta})}\tilde{K}^A_{,\tilde{\vartheta}\tilde{\vartheta}}(\tilde{\vartheta}) \tag{19}$$

are defined by the CR – by expression of internal energy as the quadratic function of the heat flux.

More than this, quantities $\frac{\lambda^0_{\tilde{\vartheta}\tilde{\vartheta}}}{\lambda^0_{\tilde{\vartheta}}}$ and $\frac{\tilde{K}^A_{,\tilde{\vartheta}\tilde{\vartheta}}(\tilde{\vartheta})}{\lambda^0_{\tilde{\vartheta}}}$ are also defined by the constitutive relations.

Rewriting the first relation (18) we get

$$\begin{aligned}\left(\ln(\lambda^0_{\tilde{\vartheta}})\right)_{,\tilde{\vartheta}}=\ln(\tau)_{,\tilde{\vartheta}}-2\frac{\mu(\tilde{\vartheta})}{\tau(\tilde{\vartheta})}&\Rightarrow \ln(\lambda^0_{\tilde{\vartheta}})=\ln(\tau)+b^0-2\int^{\tilde{\vartheta}}\frac{\mu}{\tau}(s)\,ds\\ &\Rightarrow \lambda^0_{\tilde{\vartheta}}=\alpha\tau e^{-2\int^{\tilde{\vartheta}}\frac{\mu}{\tau}(s)ds},\ \alpha=e^{b^0}>0\,.\end{aligned}$$

From this relation we find

$$\lambda^0(\tilde{\vartheta},x)=a^0+\alpha\hat{\lambda}^0=a^0+\alpha\int^{\tilde{\vartheta}}\left[\tau e^{-2\int^{u}\frac{\mu(s)}{\tau(s)}\,ds}\right]du \tag{20}$$

Here a^0 and α are constants (or, maybe, functions of x^μ).

Using obtained expression for $\lambda^0(\tilde{\vartheta},x)$ in the second formula (19) we get the expression for coefficients $\tilde{K}^A$ and, integrating twice by $\tilde{\vartheta}$, for the functions $K^A(\tilde{\vartheta})$

$$\begin{aligned}\tilde{K}^A_{,\tilde{\vartheta}\tilde{\vartheta}}&=-M_A\cdot\frac{\lambda^0_{\tilde{\vartheta}}(\tilde{\vartheta})}{\tau(\tilde{\vartheta})}=-M_A\alpha e^{-2\int^{\tilde{\vartheta}}\frac{\mu}{\tau}(s)ds}\\ \Rightarrow \tilde{K}^A&=k^A\tilde{\vartheta}+m^A+\alpha\cdot\hat{K}^A(\tilde{\vartheta})\\ &=k^A\tilde{\vartheta}+m^A-\alpha\int^{\tilde{\vartheta}}dw\int^{w}[M_A(u)e^{-2\int^{u}\frac{\mu}{\tau}(s)ds}]\,du\,.\end{aligned} \tag{21}$$

Functions $\hat{K}^A(\tilde{\vartheta})$ are defined by the second formula in the second line.

Thus, functions $\lambda^0_{\vartheta},\tilde{K}^A_{,\vartheta\vartheta}$ are defined by the constitutive relations while coefficients $\alpha>0,a^0,k^A,m^A$ are arbitrary functions of x^μ.

5 Supplementary balance laws for CHP-system

Combining obtained results, returning to the variable ϑ (and using repeatedly the relation $f_{,\tilde{\vartheta}}=\vartheta_{,\tilde{\vartheta}}f_{,\vartheta}=(\tilde{\vartheta}_{,\vartheta})^{-1}f_{,\vartheta}=\Lambda^{-1}_{,\vartheta}f_{,\vartheta}$) we get the general expressions for admissible densities/fluxes of the supplementary balance laws

$$
\begin{cases}
K^0 = \lambda^0\epsilon - \displaystyle\int^{\tilde\vartheta} \lambda^0_{,\tilde\vartheta}\epsilon^{\mathrm{eq}}\,ds + \tau(\tilde\vartheta)\Big[\frac{1}{2}\lambda^0_{\tilde\vartheta}\|q\|^2 + \tilde K^A_{,\tilde\vartheta}(\tilde\vartheta)q^A\Big] + f_0 \\
\quad = (a^0 + \alpha\hat\lambda^0)\epsilon - \alpha\displaystyle\int^{\vartheta}\hat\lambda^0_{,\vartheta}\epsilon^{\mathrm{eq}}\,ds \\
\qquad + \dfrac{\tau(\vartheta)}{\Lambda_{,\vartheta}}\Big[\dfrac{\alpha}{2}\hat\lambda^0_{\vartheta}\|q\|^2 + \big(\Lambda_{,\vartheta}k^A + \alpha\hat K^A_{,\vartheta}(\vartheta)\big)q^A\Big] + f_0, \\
K^A = \lambda^0(\tilde\vartheta)q^A + \tilde K^A(\tilde\vartheta) \\
\quad = (a^0 + \alpha\hat\lambda^0(\vartheta))q^A + k^A\Lambda(\vartheta) + m^A + \alpha\hat K^A(\vartheta)\,, \quad A = 1,2,3 \\
Q = -\lambda^A q^A = -\big(\lambda^0_{\tilde\vartheta}\|q\|^2 + \tilde K^A_{,\tilde\vartheta}(\tilde\vartheta)q^A\big) \\
\quad = -\Lambda^{-1}_{,\vartheta}\big(\lambda^0_{\vartheta}\|q\|^2 + \Lambda_{,\vartheta}k^A q^A + \alpha\hat K^A_{,\vartheta}(\vartheta)q^A\big) \\
\quad = -\Lambda^{-1}_{,\vartheta}\big(\alpha\hat\lambda^0_{\vartheta}\|q\|^2 + \Lambda_{,\vartheta}k^A q^A + \alpha\hat K^A_{,\vartheta}(\vartheta)q^A\big)\,.
\end{cases}
$$

Collecting previous results together we present obtained expressions for secondary balance laws first in short form and then in the form where original balance laws and the trivial balance laws are separated from the general form of SBL

$$
\begin{pmatrix} K^0 \\ K^1 \\ K^2 \\ K^3 \\ Q \end{pmatrix} =
\begin{pmatrix}
\lambda^0\epsilon - \int^{\vartheta}\lambda^0_{,\vartheta}\epsilon^{\mathrm{eq}}\,ds + \tau(\vartheta)\Lambda^{-1}_{\vartheta}[\frac12\lambda^0_{\vartheta}\|q\|^2 + \alpha\tilde K^A_{,\vartheta}(\vartheta)q^A] + f_0 \\
\lambda^0(\vartheta)q^1 + \tilde K^1(\vartheta) \\
\lambda^0(\vartheta)q^2 + \tilde K^2(\vartheta) \\
\lambda^0(\vartheta)q^3 + \tilde K^3(\vartheta) \\
-\Lambda^{-1}_{,\vartheta}(\lambda^0_{,\vartheta}\|q\|^2 + \tilde K^A_{,\vartheta}(\vartheta)q^A)
\end{pmatrix}
$$

$$
= a^0\begin{pmatrix} \epsilon \\ q^1 \\ q^2 \\ q^3 \\ 0 \end{pmatrix} + \sum_A k^A \begin{pmatrix} \tau(\vartheta)q^A \\ \delta^1_A\Lambda(\vartheta) \\ \delta^2_A\Lambda(\vartheta) \\ \delta^3_A\Lambda(\vartheta) \\ -q^A \end{pmatrix} + \begin{pmatrix} \alpha\tau\Lambda(\vartheta)^{-1}\hat K^A_{,\vartheta}(\vartheta)q^A \\ \hat K^1(\vartheta) \\ \hat K^2(\vartheta) \\ \hat K^3(\vartheta) \\ -\Lambda^{-1}_{,\vartheta}\hat K^A_{,\vartheta}(\vartheta)q^A \end{pmatrix} \tag{22}
$$

$$
+\,\alpha\begin{pmatrix}
\hat\lambda^0\epsilon - \int^{\vartheta}\hat\lambda^0_{,\vartheta}\epsilon^{\mathrm{eq}}\,ds + \tau(\vartheta)\Lambda^{-1}_{\vartheta}[\frac12\hat\lambda^0_{,\vartheta}\|q\|^2] \\
\hat\lambda^0(\vartheta)q^1 \\
\hat\lambda^0(\vartheta)q^2 \\
\hat\lambda^0(\vartheta)q^3 \\
-\Lambda^{-1}_{,\vartheta}\hat\lambda^0_{,\vartheta}\|q\|^2
\end{pmatrix} + \begin{pmatrix} f_0 \\ m^1 \\ m^2 \\ m^3 \\ 0 \end{pmatrix}.
$$

To get the second presentation of the SBL we use the decompositions (21) $\lambda^0 = \alpha\hat\lambda_0 + a^0$ and (20) $\tilde K^A(\tilde\vartheta) = k^A\tilde\vartheta + m^A - \hat K^A$.

Remark 1. Notice the duality between the tensor structure of the basic fields of Cattaneo system – one scalar field (temperature ϑ) and one vector field (heat flux q^A, $A = 1, 2, 3$) – and the structure of space $\mathcal{SBL}(C)$ of supplementary balance laws – elements of $\mathcal{SBL}(C)$ depend on one scalar function of temperature $\lambda^0(\vartheta)$ and one covector function of temperature $\hat{K}_A$.

Remark 2. It is easy to see that none of new SBL can be written as a linear combination of original balance equations with variable coefficients (Noether balance laws generated by vertical symmetries $v = v^k(y^i)\partial_{y^k}$, see [7], [8]). The easiest way to prove this is to compare the source terms of different balance equations.

Returning to the variable ϑ in the expression (16) and using the relation $\partial_{\tilde{\vartheta}} = \frac{1}{\Lambda(\vartheta)_{,\vartheta}}\partial_\vartheta$ we get the expression for the internal energy

$$\begin{aligned}\epsilon &= \epsilon^{\mathrm{eq}}(\vartheta) + \frac{\tau_{,\vartheta}}{2\Lambda_{,\vartheta}}\|q\|^2 - \frac{\tau(\vartheta)}{\lambda^0_{,\vartheta}}\left[\frac{1}{2}\left(\frac{\lambda^0_{,\vartheta}}{\Lambda_{,\vartheta}}\right)_{,\vartheta}\|q\|^2 + \left(\frac{\tilde{K}^A_{,\vartheta}}{\Lambda_{,\vartheta}}\right)_{,\vartheta} q^A\right] \\ &=^{\Lambda_{,\vartheta}=\kappa-\mathrm{const}} \epsilon^{\mathrm{eq}}(\vartheta) + \frac{\tau_{,\vartheta}}{2\kappa}\|q\|^2 - \frac{\tau(\vartheta)}{\kappa\lambda^0_{,\vartheta}}\left[\frac{1}{2}\lambda^0_{,\vartheta\vartheta}\|q\|^2 + \tilde{K}^A_{,\vartheta\vartheta}q^A\right].\end{aligned}$$

Notice that for $\lambda^0 = 0$, balance laws given by the 4th column in (22) (the one with coefficient α) vanish. The same is true for deformations of the Cattaneo equation (second column) defined by the third column when $\tilde{K}^A(\vartheta) = 0$.

The first and second balance laws in (22) are the balance laws of the original Cattaneo system. The last one is the *trivial balance law*. Third and fourth columns give the balance law

$$\begin{aligned}\partial_t\left[\hat{\lambda}^0\epsilon - \int^\vartheta \lambda^0_{,\vartheta}\epsilon^{\mathrm{eq}}\,ds + \tau(\vartheta)\Lambda_\vartheta^{-1}\left[\frac{1}{2}\lambda^0_\vartheta\|q\|^2 + \hat{K}^A_{,\vartheta}(\vartheta)q^A\right]\right] \\ + \partial_{x^A}\left[\hat{\lambda}^0(\vartheta)q^A + \hat{K}^A(\vartheta)\right] = -\Lambda^{-1}_{,\vartheta}\left(\hat{\lambda}^0_\vartheta\|q\|^2 + \hat{K}^A_{,\vartheta}(\vartheta)q^A\right).\end{aligned} \tag{23}$$

Source/production term in (23) equation has the form

$$\begin{aligned}-\Lambda^{-1}_{,\vartheta}\left(\hat{\lambda}^0_\vartheta\|q\|^2 + \hat{K}^A_{,\vartheta}(\vartheta)q^A\right) &= -\Lambda^{-1}_{,\vartheta}\hat{\lambda}^0_\vartheta\left(\|q\|^2 + \frac{\hat{K}^A_{,\vartheta}(\vartheta)}{\hat{\lambda}^0_\vartheta}q^A\right) \\ &= -\Lambda^{-1}_{,\vartheta}\hat{\lambda}^0_\vartheta\left[\sum_A\left(q^A + \frac{\hat{K}^A_{,\vartheta}(\vartheta)}{2\hat{\lambda}^0_\vartheta}\right)^2 - \sum_A\left(\frac{\hat{K}^A_{,\vartheta}(\vartheta)}{2\hat{\lambda}^0_\vartheta}\right)^2\right].\end{aligned}$$

By physical reasons, $\Lambda_{,\vartheta} > 0$. As (20) shows, $\lambda_{,\vartheta}$ may have any sign. We assume that this sign does not depend on ϑ.

For a fixed ϑ expression for the production in the balance law (23) may have constant sign *for all values of* q^A *if and only if* $\hat{K}^A_{,\vartheta}(\vartheta) = 0$, $A = 1, 2, 3$. Therefore this is possible only if the internal energy has the form

$$\begin{aligned}\epsilon &= \epsilon^{\mathrm{eq}}(\vartheta) + \left[\frac{\tau_{,\vartheta}}{2\Lambda_{,\vartheta}} - \frac{\tau(\vartheta)}{2\hat{\lambda}^0_{,\vartheta}}\left(\frac{\hat{\lambda}^0_{,\vartheta}}{\Lambda_{,\vartheta}}\right)_{,\vartheta}\right]\|q\|^2 \\ &=^{\tau-\mathrm{const},\ \Lambda_{,\vartheta}-\mathrm{const}} \epsilon^{\mathrm{eq}}(\vartheta) - \frac{\tau(\vartheta)}{2k\hat{\lambda}^0_{,\vartheta}}\hat{\lambda}^0_{,\vartheta\vartheta}\|q\|^2\end{aligned}$$

with some function $\hat{\lambda}^0(\vartheta)$. This being so, Cattaneo system has the entropy (supplementary balance) law

$$\partial_t\left[\hat{\lambda}^0\epsilon - \int^{\vartheta} \hat{\lambda}^0_{,\vartheta}\epsilon^{\mathrm{eq}}\,ds + \frac{1}{2}\tau(\vartheta)\Lambda^{-1}_{\vartheta}\hat{\lambda}^0_{,\vartheta}\|q\|^2\right] + \partial_{x^A}\left[\hat{\lambda}^0_{,\vartheta}q^A\right] \;=\; -\Lambda^{-1}_{,\vartheta}\hat{\lambda}^0_{,\vartheta}\|q\|^2$$

with the production term that may have constant sign – nonnegative, provided (we use the fact that $\hat{\lambda}^0_{,\vartheta} = \lambda^0_{,\vartheta}$*)*

$$\Lambda^{-1}_{,\vartheta}\lambda^0_{,\vartheta} \leqq 0\,. \tag{24}$$

This inequality (which is equivalent, if $\Lambda_{,\vartheta} \geqq 0$*, to the inequality* $\lambda^0_{,\vartheta} \leqq 0$*) is the II law of thermodynamics for Cattaneo heat propagation model.*

If we take $q = 0$ in obtained entropy balance we have to get the value of entropy at the equilibrium s^{eq}:

$$s^{\mathrm{eq}} = \hat{\lambda}^0\epsilon^{\mathrm{eq}} - \int^{\vartheta} \lambda^0_{,\vartheta}\epsilon^{\mathrm{eq}}\,ds = \int^{\vartheta} \hat{\lambda}^0\epsilon^{\mathrm{eq}}_{,\vartheta}\,d\vartheta.$$

From this it follows that *at a homogeneous state* $ds^{\mathrm{eq}} = \hat{\lambda}^0\,d\epsilon^{\mathrm{eq}}$. Comparing this with the Gibbs relation $d\epsilon^{\mathrm{eq}} = \vartheta ds^{\mathrm{eq}}$ we conclude that

$$\hat{\lambda}^0 = \frac{1}{\vartheta}. \tag{25}$$

Using (17) we also conclude that

$$\lambda^A = -\frac{q^A}{\vartheta^2}\,, \quad A = 1,2,3\,.$$

It follows from this that the condition (24) (II law) takes here the form well known from thermodynamics (see [2], [4], [5]):

$$\Lambda_{,\vartheta} \geqq 0\,.$$

Substituting (14) into (17) and calculating

$$\begin{aligned}
-\frac{\tau(\vartheta)}{2\hat{\lambda}^0_{,\vartheta}}\left(\frac{\hat{\lambda}^0_{,\vartheta}}{\Lambda_{,\vartheta}}\right)_{,\vartheta} &= \frac{\tau(\vartheta)\vartheta^2}{2}\left(\frac{-1}{\vartheta^2\Lambda_{,\vartheta}}\right)_{,\vartheta} = -\frac{\tau(\vartheta)\vartheta^2}{2}\,\frac{-(2\vartheta\Lambda_{,\vartheta} + \vartheta^2\Lambda_{,\vartheta\vartheta})}{\vartheta^4\Lambda^2_{,\vartheta}}\\
&= \frac{\tau(\vartheta)}{\vartheta\Lambda_{,\vartheta}} + \frac{\tau(\vartheta)\Lambda_{,\vartheta\vartheta}}{2(\Lambda_{,\vartheta})^2}
\end{aligned}$$

we get the expression for internal energy in the form

$$\begin{aligned}
\epsilon &= \epsilon^{\mathrm{eq}}(\vartheta) + \left[\frac{\tau_{,\vartheta}}{2\Lambda_{,\vartheta}} + \frac{\tau}{\vartheta\Lambda_{\vartheta}} + \frac{\tau\Lambda_{,\vartheta\vartheta}}{2(\Lambda_{,\vartheta})^2}\right]\|q\|^2\\
&=^{\tau-\mathrm{const},\ \Lambda_{,\vartheta}-\mathrm{const}}\ \epsilon^{\mathrm{eq}}(\vartheta) + \frac{\tau}{\vartheta\Lambda_{,\vartheta}}\|q\|^2.
\end{aligned} \tag{26}$$

For the entropy density we have

$$
\begin{aligned}
s &= s^{\mathrm{eq}} + \hat{\lambda}^0(\epsilon - \epsilon^{\mathrm{eq}}) + \frac{1}{2}\tau(\vartheta)\Lambda_{\vartheta}^{-1}\hat{\lambda}^0_{,\vartheta}\|q\|^2 = \\
&= s^{\mathrm{eq}} + \frac{1}{\vartheta}\left[\frac{\tau_{,\vartheta}}{2\Lambda_{,\vartheta}} + \frac{\tau}{\vartheta\Lambda_{\vartheta}} + \frac{\tau\Lambda_{,\vartheta\vartheta}}{2(\Lambda_{,\vartheta})^2}\right]\|q\|^2 - \frac{\tau(\vartheta)}{2\vartheta^2\Lambda_{\vartheta}}\|q\|^2 \\
&= s^{\mathrm{eq}} + \frac{1}{\vartheta}\left[\frac{\tau_{,\vartheta}}{2\Lambda_{,\vartheta}} + \frac{\tau}{2\vartheta\Lambda_{\vartheta}} + \frac{\tau\Lambda_{,\vartheta\vartheta}}{2(\Lambda_{,\vartheta})^2}\right]\|q\|^2 \\
&= s^{\mathrm{eq}} + \frac{\tau}{2\vartheta\Lambda_{,\vartheta}}\left[\frac{\tau_{,\vartheta}}{\tau} + \frac{1}{\vartheta} + \frac{\Lambda_{,\vartheta\vartheta}}{\Lambda_{,\vartheta}}\right]\|q\|^2 \\
&=^{\tau-\mathrm{const},\ \Lambda_{,\vartheta}-\mathrm{const}} s^{\mathrm{eq}} + \frac{\tau}{2\vartheta^2\Lambda_{,\vartheta}}\|q\|^2.
\end{aligned}
\tag{27}
$$

Correspondingly, the entropy balance law takes the form

$$
\partial_t\left(s^{\mathrm{eq}} + \frac{\tau}{2\vartheta\Lambda_{,\vartheta}}\left[\frac{\tau_{,\vartheta}}{\tau} + \frac{1}{\vartheta} + \frac{\Lambda_{,\vartheta\vartheta}}{\Lambda_{,\vartheta}}\right]\|q\|^2\right) + \partial_{x^A}\left(\frac{q^A}{\vartheta}\right) = \frac{1}{\Lambda_{,\vartheta}}\left\|\frac{q}{\vartheta}\right\|^2.
$$

Remark 3. If in the absence of the heat flow ($q = 0$) the "equilibrium state" is not homogeneous, more general constitutive relations with λ^0 different from (25) and more general form of energy and entropy balances satisfying the II law of Thermodynamics, are possible.

We collect obtained results in the following

Theorem 1. 1. *For the Cattaneo heat propagation balance system* (1) *compatible with the entropy principle and having a nontrivial supplementary balance law that is not a linear combination of the original balance laws with constant coefficients, the internal energy has the form* (7). *If* (7) *holds, all supplementary balance laws for Cattaneo balance system (including original equations and the trivial ones) are listed in* (6). *New supplementary balance laws depend on the 4 functions of temperature –* $\hat{\lambda}^0(\vartheta)$, $\tilde{K}^A(\vartheta)$, $A = 1, 2, 3$. *Corresponding main fields* λ^μ, $\mu = 0, 1, 2, 3$, *have the form* (17), (20).

2. *The additional balance law* (23) *given by the sum of third and fourth columns in* (22) has the nonnegative production term *if and only if the internal energy* ϵ *has the form* (26) *and, in addition, the condition* (24) *holds. Cattaneo systems satisfying these conditions depend on one arbitrary function of time* $\epsilon^{\mathrm{eq}}(\vartheta)$.

3. *The supplementary balance law having nonnegative production term (entropy) is unique modulo linear combination of original balance laws and the trivial balance laws.*

6 Conclusion

Description of the supplementary balance laws for Cattaneo heat propagation system given in this paper can probably be carried over for other systems of balance equations for the couples of fields: scalar + vector field.

One observes a kind of duality between the tensorial structure of dynamical fields (here ϑ, q) and the list of free functions of temperature $\lambda^0(\vartheta)$, $\tilde{K}^A(\vartheta)$, $A = 1, 2, 3$ entering the description of SBL.

It would be interesting to follow up if similar duality exists for the balance systems of more complex tensorial structure and for the systems of order 1 (recently the author completed the classification of SBL of order 0 and 1 for the Navier-Stokes fluid balance system, [9]).

In the case of Cattaneo heat propagation system, the II law of thermodynamics – existence of the SBL having the nonnegative production term – defines the entropy balance uniquely (modulo addition of trivial balance laws and the linear combination of the original balance laws). It would be interesting to look at other balance systems to determine the character of non-unicity of the SBL with the positive production – "abstract entropy balances" – to find the place of "physical entropy" in this list and to see if this "physical entropy balance" is "optimal" in some sense.

References

[1] P. Glensdorf, I. Prigogine: *Thermodynamical Theory of Structure, Stability and fluctuations*. Wiley, Brussels (1971).

[2] D. Jou, J. Casas-Vasquez, G. Lebon: *Extended Irreversible Thermodynamics*, 3rd ed. Springer (2001).

[3] I-Shish Liu: Method of Lagrange multipliers for exploatation of the entropy principle. *Arch. Rational Mech. Anal.* 46 (1972) 131–148.

[4] I. Muller: *Thermodynamics*. Pitman Adv. Publ. (1985).

[5] I. Muller, T. Ruggeri: *Rational Extended Thermodynamics*, 2nd ed. Springer (1998).

[6] P. Olver: *Applications of Lie Groups to Differential Equations*, 2nd ed. Springer-Verlag New York (2000).

[7] S. Preston: Geometrical Theory of Balance Systems and the Entropy Principle. *Proceedings of GCM7, Lancaster, UK, J. Phys.: Conf. Ser.* 62 (2007) 102–154. DOI: 10.1088/1742-6596/62/1/007

[8] S. Preston: Variational theory of balance systems. *Int. J. Geom. Methods of Mod. Phys.* 7 (2010). DOI: 10.1142/S0219887810004543

[9] S. Preston: Supplementary balance laws for the Navier-Stokes-Fourier Fluid. Manuscript, unpublished.

[10] T. Ruggeri: Galilean Invariance and Entropy Principle For Systems of Balance Laws. *Cont. Mech. Thermodyn.* 1 (1) (1989) 3–20.

[11] T. Ruggeri: The Entropy Principle: from Continuum Mechanics to Hyperbolic Systems of Balance Laws. *Entropy* 10 (2008) 319–333.

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Method of infinite ascent applied on $-(2^p \cdot A^6) + B^3 = C^2$

Susil Kumar Jena

Abstract. In this paper, the author shows a technique of generating an infinite number of coprime integral solutions for (A, B, C) of the Diophantine equation $-(2^p \cdot A^6) + B^3 = C^2$ for any positive integral values of p when $p \equiv 1 \pmod 6$ or $p \equiv 2 \pmod 6$. For doing this, we will be using a published result of this author in The Mathematics Student, a periodical of the Indian Mathematical Society.

1 Introduction

Many people, viz., Lebesgue [14], Ljunggren [15], Nagell [19], [20], Chao [8], Cohn [10], Mignotte & de Weger [18], Bugeaud, Mignotte & Siksek [7] have investigated on the solution of the Diophantine equation $x^2 + C = y^n$ with $x \geq 1$, $y \geq 1$, $n \geq 3$ and C is any integer, positive or negative for different values of $|C| \leq 100$. Le [13], Luca [16]; Arif & Muriefah [1] have considered a different form of the equation $x^2 + C = y^n$, when C is no longer a fixed integer but the power of one or two fixed primes.

For other related results concerning equation $x^2+C = y^n$ see [2], [3], [4], [5], [9], [11], [17], [21], [22], [23], [24]. For a survey relating equation $x^2 + C = y^n$ see [6]. Allowing C to be the product of some power of 2 and an integral sixth power, Theorem 3 and Theorem 4 give the main results of this paper. From a paper of Jena [12], we reproduce two useful Theorems relating to the Diophantine equation

$$mA^6 + nB^3 = C^2 \tag{1}$$

for any pair of integers (m, n) and the integral variables (A, B, C). Basing on these two theorems we obtain the main results of this paper.

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Key words: higher order Diophantine equations, method of infinite ascent, Diophantine equation $-(2^p \cdot A^6) + B^3 = C^2$

Theorem 1 (Jena [12]). *For any integer m, p and q,*

$$m(2pq)^6 + (mp^6 - q^2)(9mp^6 - q^2)^3 = (27m^2p^{12} - 18mp^6q^2 - q^4)^2. \tag{2}$$

Proof. The proof is got by expanding the terms of both the LHS and RHS of (2) and noting their equality. □

Theorem 2 (Jena [12]). *If (A_t, B_t, C_t) is a solution of the Diophantine equation $mA^6 + nB^3 = C^2$ with m, n, A, B and C as integers then $(A_{t+1}, B_{t+1}, C_{t+1})$ is also a solution of the same equation such that*

$$\begin{aligned}(A_{t+1}, B_{t+1}, C_{t+1})&\\ &= \left\{(2A_tC_t), -B_t(9mA_t^6 - C_t^2), (27m^2A_t^{12} - 18mA_t^6C_t^2 - C_t^4)\right\}\end{aligned} \tag{3}$$

and if mA_t, nB_t and C_t are pairwise coprime where nB_t is an odd integer and 3 is not a factor of C_t then mA_{t+1}, nB_{t+1} and C_{t+1} are also pairwise coprime where nB_{t+1} is an odd integer and 3 is not a factor of C_{t+1}; in addition to this, mA_{t+1} will be always an even integer and C_{t+1} always an odd integer.

Proof. We can get details of the proof in paper [12]. □

Now, let us proceed to the next section to note the principal results of this paper.

2 Results

In this paper, we prove that for any positive integer p, when $p \equiv 1 \pmod 6$ or $p \equiv 2 \pmod 6$ the Diophantine equation $-(2^p \cdot A^6) + B^3 = C^2$ has infinitely many coprime integral solutions for (A, B, C). This is equivalent to proving the statements of Theorem 3 and Theorem 4.

Theorem 3. *For any positive integer $q \geq 1$, the Diophantine equation*

$$-(2^{6q-5} \cdot A^6) + B^3 = C^2 \tag{4}$$

has infinitely many coprime integral solutions for (A, B, C).

Proof. We will prove Theorem 3 in three steps. Firstly, we have to establish that equation (4) has infinitely many coprime integral solutions for (A, B, C) when $q = 1$. Secondly, we will see how to use these coprime solutions of first step to find the initial coprime solutions for (A, B, C) of equation (4) for other values of $q > 1$. Next, we will show that the conditions of generating infinite number of coprime integral solutions, as proposed by Theorem 2, are applicable to (4) for each value of q.

Step I. Putting $q = 1$ in (4) we get

$$-(2^1 \cdot A^6) + B^3 = C^2. \tag{5}$$

We will denote the i^{th} solution for (A, B, C) of equation (4) when $q = j$ as $(A_i, B_i, C_i)_{q=j}$, where i and j take positive integral values. Now, we know that

$$-2 \cdot 1^6 + 3^3 = 5^2. \tag{6}$$

Using the result of (6), we get the starting solution for (A, B, C) of equation (4) as

$$(A_1, B_1, C_1)_{q=1} = (1, 3, 5). \tag{7}$$

Comparing (5) with (1) we get $m = -2$ and $n = 1$. The conditions of generating an infinite number of coprime integral solutions as proposed by Theorem 2 are applicable for equation (5), because the three terms mA_1, nB_1 and C_1 take values -2, 3 and 5 respectively, and are pairwise coprime; nB_1 is an odd integer and 3 is not a factor of C_1. Thus, Theorem 2 can be used repeatedly to generate an infinite number of coprime integral solutions for (A, B, C). Using (3) we have

$$\begin{aligned}(A_2, B_2, C_2)_{q=1} &= \big\{(2A_1C_1), -B_1(9mA_1^6 - C_1^2), \\ &\qquad (27m^2A_1^{12} - 18mA_1^6C_1^2 - C_1^4)\big\} \\ &= \big\{(2 \cdot 1 \cdot 5), -3 \cdot (9 \cdot (-2) \cdot 1^6 - 5^2), \\ &\qquad (27 \cdot (-2)^2 \cdot 1^{12} - 18 \cdot (-2) \cdot 1^6 \cdot 5^2 - 5^4)\big\} \\ &= (2^1 \cdot 5, 129, 383). \end{aligned} \tag{8}$$

Using equation (3), we calculate the k^{th} solution of (5) as

$$(A_k, B_k, C_k) = (2^{k-1} \cdot A'_k, B_k, C_k)$$

where the integer $k > 1$, $A_k = 2^{k-1}A'_k$ and all three terms A'_k, B_k and C_k are odd. By repeated use of equation (3) one can find any number of coprime integral solutions for (A, B, C) of equation (5).

Step II. The first solution for (A, B, C) of equation (5) is $(1, 3, 5)$. Using these values for (A, B, C) in (5) we have

$$\begin{aligned} -2 \cdot 1^6 + 3^3 &= 5^2. \\ \text{Or} \quad -2 \cdot 2^0 \cdot 1^6 + 3^3 &= 5^2. \end{aligned} \tag{9}$$

The second solution for (A, B, C) of equation (5) is $(2^1 \cdot 5, 129, 383)$. Using these values for (A, B, C) in (5) we get

$$\begin{aligned} -2 \cdot 2^6 \cdot 5^6 + 129^3 &= 383^2. \\ \text{Or} \quad -2^7 \cdot 5^6 + 129^3 &= 383^2. \end{aligned} \tag{10}$$

The k^{th} solution for (A, B, C) of equation (5) is $(2^{k-1} \cdot A'_k, B_k, C_k)$. Using these values for (A, B, C) in (5) we obtain

$$-(2^{6k-5} \cdot A'^6_k) + B_k^3 = C_k^2. \tag{11}$$

When $q = 1$, from (9) we get the starting solution for (A, B, C) of equation (4) as $(2^0 \cdot 1, 3, 5)$.
When $q = 2$, from (10) we get the starting solution for (A, B, C) of equation (4) as $(5, 129, 383)$.

When $q = k$, from (11) we get the starting solution for (A, B, C) of equation (4) as (A'_k, B_k, C_k).

Step III. In Step I, we have already proved the validity of the statement of Theorem 3 for $q = 1$. Putting $q = 2$ in (4) we get

$$-(2^7 \cdot A^6) + B^3 = C^2. \tag{12}$$

Now, for each integral value of $q > 1$, there is a starting solution for (A, B, C) for equation (4) as we showed in Step II. Since the values of B and C in these starting solutions are the same values which are generated by the subsequent solutions of equation (4), they should be coprime; B and C are odd integers; and 3 is not a factor of C. Hence, for any integer $q > 1$, the statement of Theorem 3 is valid, because the conditions of generating infinite number of coprime integral solutions as proposed by Theorem 2 are satisfied.

Thus, combining these three steps, we complete the proof of Theorem 3. □

Theorem 4. *For any positive integer $q \geq 1$, the Diophantine equation*

$$-(2^{6q-4} \cdot A^6) + B^3 = C^2 \tag{13}$$

has infinitely many coprime integral solutions for (A, B, C).

Proof. Since $-(2^2 \cdot 1^6) + 5^3 = 11^2$, we get the first coprime solution for (A, B, C) of the Diophantine equation (13) when $q = 1$ as

$$(A_1, B_1, C_1)_{q=1} = (1, 5, 11). \tag{14}$$

Using Theorem 2 we obtain

$$(A_2, B_2, C_2)_{q=1} = (2^1 \cdot 11, 785, -5497) = (2^1 \cdot 11, 785, 5497). \tag{15}$$

We can use (15) to get the first coprime solution for (A, B, C) of the Diophantine equation (13) when $q = 2$ as

$$(A_1, B_1, C_1)_{q=2} = (11, 785, 5497).$$

Steps similar to the proof of Theorem 3 should be followed in establishing the statement of Theorem 4. □

3 Conclusion

The proof of Theorem 3 and Theorem 4 establishes the infinitude characteristics of the Diophantine equation

$$-(2^p \cdot A^6) + B^3 = C^2$$

for any positive integral values of p when $p \equiv 1 \pmod 6$ or, $p \equiv 2 \pmod 6$. But, what about the status of this equation when $p \equiv 0$, 3, 4, or 5 (mod 6)? Well, we don't have the answer, because an initial starting coprime solution for (A, B, C) in each of these cases is not available with us. It needs further investigation.

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References

[1] S.A. Arif, F.S. Abu Muriefah: The Diophantine equation $x^2 + 3^m = y^n$. *Int. J. Math. Math. Sci.* 21 (3) (1998) 619–620.

[2] S.A. Arif, F.S. Abu Muriefah: On the Diophantine equation $x^2 + 2^k = y^n$ II. *Arab J. Math. Sci.* 7 (2) (2001) 67–71.

[3] F.S. Abu Muriefah, F. Luca, A. Togbé: On the Diophantine equation $x^2 + 5^a 13^b = y^n$. *Glasg. Math. J.* 50 (1) (2008) 175–181.

[4] A. Bérczes, I. Pink: On the Diophantine equation $x^2 + p^{2k} = y^n$. *Archiv der Mathematik* 91 (6) (2008) 505–517.

[5] A. Bérczes, I. Pink: On the Diophantine equation $x^2 + d^{2l+1} = y^n$. *Glasg. Math. J.* 54 (2) (2012) 415–428.

[6] Y. Bugeaud, F.S. Abu Muriefah: The Diophantine equation $x^2 + c = y^n$: a brief overview. *Rev. Colomb. Mat.* 40 (1) (2006) 31–37.

[7] Y. Bugeaud, M. Mignotte, S. Siksek: Classical and modular approaches to Exponential Diophantine equations II: The Lebesgue-Nagell equation. *Compositio Mathematica* 142 (2006) 31–62.

[8] K. Chao: On the Diophantine equation $x^2 = y^n + 1, xy \neq 0$. *Sci. Sinica* 14 (1965) 457–460.

[9] J.H.E. Cohn: The diophantine equation $x^2 + 2^k = y^n$. *Arch. Math. (Basel)* 59 (4) (1992) 341–344.

[10] J.H.E. Cohn: The Diophantine equation $x^2 + C = y^n$. *Acta Arith.* 65 (4) (1993) 367–381.

[11] E. Goins, F. Luca, A. Togbé: On the Diophantine equation $x^2 + 2^\alpha 5^\beta 13^\gamma = y^n$. *ANTS VIII Proceedings: A. J. van der Poorten and A. Stein (eds.), ANTS VIII, Lecture Notes in Computer Science 5011* (2008) 430–442.

[12] S.K. Jena: Method of Infinite Ascent applied on $mA^6 + nB^3 = C^2$. *Math. Student* 77 (2008) 239–246.

[13] M. Le: Diophantine equation $x^2 + 2^m = y^n$. *Chinese Sci. Bull.* 42 (18) (1997) 1515–1517.

[14] V.A. Lebesgue: Sur l'impossibilité en nombres entiers de l'équation $x^m = y^2 + 1$. *Nouv. Ann. Math.* 99 (1850) 178–181. (French)

[15] W. Ljunggren: Über einige Arcustangensgleichungen die auf interessante unbestimmte Gleichungen führen. *Ark. Mat.* 29A (13) (1943) 1–11. (German)

[16] F. Luca: On the equation $x^2 + 2^a \cdot 3^b = y^n$. *Int. J. Math. Math. Sci.* 29 (4) (2002) 239–244.

[17] F. Luca, A. Togbé: On the Diophantine equation $x^2 + 2^a 5^b = y^n$. *Int. J. Number Theory* 4 (6) (2008) 973–979.

[18] M. Mignotte, B.M.M. de Weger: On the Diophantine equations $x^2 + 74 = y^5$ and $x^2 + 86 = y^5$. *Glasgow Math. J.* 38 (1) (1996) 77–85.

[19] T. Nagell: Sur l'impossibilité de quelques équations à deux indéterminées. *Norsk. Mat. Forensings Sknifter* 13 (1923) 65–82. (French)

[20] T. Nagell: Contributions to the theory of Diophantine equations of the second degree with two unknowns. *Nova Acta Soc. Sci. Upsal. Ser (4)* 16 (2) (1955) 38–38.

[21] I. Pink, Zs. Rábai: On the Diophantine equation $x^2 + 5^k 17^l = y^n$. *Commun. Math.* 19 (1) (2011) 1–9.

[22] N. Saradha, A. Srinivasan: Solutions of some generalized Ramanujan-Nagell equations. *Indag. Math. (N.S.)* 17 (1) (2006) 103–114.

[23] N. Saradha, A. Srinivasan: Solutions of some generalized Ramanujan-Nagell equations via binary quadratic forms. *Publ. Math. Debrecen* 71 (3-4) (2007) 349–374.

[24] H. Zhu, M. Le: On some generalized Lebesgue-Nagell equations. *J. Number Theory* 131 (3) (2011) 458–469.

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Erratum

Communications in Mathematics No. 1, Vol. 20 (2012), page 20, proof of Proposition 1

The Editorial Office regrets that there has been a printing error in the paper "Locally variational invariant field equations and global currents: Chern-Simons theories" by M. Francaviglia, M. Palese, E. Winterroth, Comm. Math. No. 1, Vol. 20 (2012). On page 20, one line is missing at the end of the proof of Proposition 1. The correct version is:

… *As stated in the section above it is clear that, if* $\mathcal{L}_\Xi d_H \gamma_{ij} = 0$, *then* $d_H(\nu_i + \epsilon_i)$ *is global. Generators of such a global current lie in the kernel of the second variational derivative and are symmetries of the variationally trivial Lagrangian* $d_H \gamma_{ij}$. $\square$

The technical editors are responsible for this mistake which occured during the final preparation of the paper for publication. In the online version the mistake has already been corrected.

The Editorial Office apologizes to the authors and the readers.

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