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Cocalibrated G_2-manifolds with Ricci flat characteristic connection

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Abstract. Any 7-dimensional cocalibrated G_2-manifold admits a unique connection ∇ with skew symmetric torsion (see [8]). We study these manifolds under the additional condition that the ∇-Ricci tensor vanish. In particular we describe their geometry in case of a maximal number of ∇-parallel vector fields.

1 Introduction

Consider a triple (M^n, g, T) consisting of a Riemannian manifold (M^n, g) equipped with a 3-form T. We denote by ∇^g, Ric^g and Scal^g the Levi-Civita connection, the Riemannian Ricci tensor and the scalar curvature. The formula

$$\nabla_X Y := \nabla^g_X Y + \frac{1}{2}\,\mathrm{T}(X, Y, -)$$

defines a metric connection with torsion T. We will denote by Ric^∇ and Scal^∇ its Ricci tensor and scalar curvature respectively. If the Ricci tensor $\mathrm{Ric}^\nabla = 0$ vanishes, then T is a coclosed form, $\delta\mathrm{T} = 0$, and the Riemannian Ricci tensor is completely given by the 3-form T (see [8]),

$$\mathrm{Ric}^g(X, Y) = \frac{1}{4}\sum_{i,j=1}^{n} \mathrm{T}(X, e_i, e_j)\cdot\mathrm{T}(Y, e_i, e_j)\,, \quad \mathrm{Scal}^g = \frac{3}{2}\,\|\mathrm{T}\|^2\,.$$

In particular, the Ricci tensor Ric^g is non-negative, $\mathrm{Ric}^g(X, X) \geq 0$.

Let us introduce the 4-form σ_T depending on T,

$$\sigma_\mathrm{T} = \frac{1}{2}\sum_{i=1}^{n} (e_i \lrcorner \mathrm{T}) \wedge (e_i \lrcorner \mathrm{T})\,.$$

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If moreover there exists a ∇-parallel spinor field Ψ, then there is an algebraic link between $d\mathrm{T}$, $\nabla\mathrm{T}$ and σ_{T} (see [8]),

$$\big(X \lrcorner\, d\mathrm{T} + 2\,\nabla_X \mathrm{T}\big)\cdot\Psi = 0\,, \quad \big(3\,d\mathrm{T} - 2\sigma_{\mathrm{T}}\big)\cdot\Psi = 0\,.$$

The classification of flat metric connections with skew symmetric torsion has been investigated by Cartan and Schouten in 1926. Complete proofs are known since the beginning of the 70-ties. In [4] one finds a simple proof of this result. Therefore, we are interested in non-flat ($\mathcal{R}^\nabla \not\equiv 0$) and ∇-Ricci flat ($\mathrm{Ric}^\nabla \equiv 0$) metric connections with skew symmetric torsion $\mathrm{T} \not\equiv 0$.

In this paper we study the 7-dimensional case. Any cocalibrated G_2-manifold admits a unique connection ∇ with skew symmetric torsion and ∇-parallel spinor field Ψ. If this characteristic connection is Ricci flat, then we obtain a solution of the Strominger equations (see [8]),

$$\nabla\Psi = 0\,, \quad \mathrm{Ric}^\nabla = 0\,, \quad d*\mathrm{T} = 0\,.$$

If $\mathrm{T} = 0$, M^7 is a Riemannian manifold with holonomy G_2 and $\mathrm{Ric}^g = 0$ follows automatically. The case of $\mathrm{T} \not\equiv 0$ is different. The condition $\mathrm{Ric}^\nabla \equiv 0$ is not a consequence of the fact that the holonomy of ∇ is contained in G_2, it is a new condition for the cocalibrated G_2-structure. In this paper we investigate the geometry of the 7-manifolds under consideration. Moreover, we describe all these manifolds with a large number of ∇-parallel vector fields.

2 Examples of Ricci flat connections with skew symmetric torsion

Let us discuss some examples.

Example 1. Any Hermitian manifold admits a unique metric connection ∇ preserving the complex structure and with skew symmetric torsion (see [8]). In [10] the authors constructed on $(k-1)\,(S^2 \times S^4)\,\#\,k\,(S^3 \times S^3)$ a Hermitian structure with vanishing ∇-Ricci tensor, $\mathrm{Ric}^\nabla = 0$, for any $k \geq 1$. These examples are toric bundles over special Kähler 4-manifolds.

Example 2. There are 7-dimensional cocalibrated G_2-manifolds (M^7, g, ω^3) with characteristic torsion T such that

$$\nabla\mathrm{T} = 0\,, \quad d\mathrm{T} = 0\,, \quad \delta\mathrm{T} = 0\,, \quad \mathrm{Ric}^\nabla = 0\,, \quad \mathfrak{hol}(\nabla) \subset \mathfrak{u}(2) \subset \mathfrak{g}_2\,.$$

The regular G_2-manifolds of this type have been described in [7], Theorem 5.2 (the degenerate case $2a + c = 0$). M^7 is the product $X^4 \times S^3$, where X^4 is a Ricci-flat Kähler manifold and S^3 the round sphere.

Example 3. A suitable deformation of any Sasaki-Einstein manifold yields a metric connection with skew symmetric torsion and vanishing Ricci tensor, see [1].

Next we describe a similar method in order to construct 5-dimensional connections with skew symmetric torsion and vanishing Ricci tensor.

Theorem 1. *Let (Z^4, g, Ω^2) be a 4-dimensional Riemannian manifold equipped with a 2-form Ω^2 such that*

1. $d\Omega^2 = 0\,, \quad d*\Omega^2 = 0$ *and* $\Omega^2 \wedge \Omega^2 = 0$.
2. *The 2-dimensional distributions*
$$E^2 = \{X \in TZ^4 : X \lrcorner \Omega^2 = 0\}\,, \quad F^2 = \{X \in TZ^4 : X \perp E^2\}$$
are integrable.
3. *The 2-form is of the form $\Omega^2 = 2a\, f_1 \wedge f_2$, where a is constant and f_1, f_2 is an oriented orthonormal frame in F^2.*
4. *The Riemannian Ricci tensor of Z^4 has two non-negative eigenvalues of multiplicity two,*
$$\mathrm{Ric}^g = 4a^2\,\mathrm{Id} \text{ on } F^2, \quad \mathrm{Ric}^g = 0 \text{ on } E^2.$$
5. *Ω^2 is the curvature form of some $\mathbb{R}^1$- or S^1-connection η.*

Then the principal fibre bundle $\pi\colon N^5 \to Z^4$ defined by Ω^2 admits a Riemannian metric and the torsion form
$$\mathrm{T} = \pi^*(\Omega^2) \wedge \eta$$
yields a metric connection ∇ with the following properties:
$$||\mathrm{T}||^2 = 4a^2, \quad d\mathrm{T} = 0\,, \quad d*\mathrm{T} = 0\,, \quad \mathrm{Ric}^\nabla = 0\,, \quad \nabla\eta = 0\,.$$

Proof. Apply O'Neill's formulas and compute
$$\mathrm{Ric}^g(X,Y) - \frac{1}{4}\sum_{i,j=1}^{5} \mathrm{T}(X,e_i,e_j)\cdot\mathrm{T}(Y,e_i,e_j) = 0\,. \qquad \square$$

Example 4. Let $u = u(x,y)$ be a smooth function of two variables and consider the metric
$$g = e^u\,x\,(dx^2 + dy^2) + x\,dz^2 + \frac{1}{x}(dt + y\,dz)^2$$
defined on the set $Z^4 = \{(x,y,t,z) \in \mathbb{R}^4 : x > 0\}$. (Z^4, g) is a Kähler manifold and the Riemannian Ricci tensor has two eigenvalues, namely zero and
$$-\frac{u_{xx} + u_{yy}}{2xe^u},$$
both with multiplicity two (see [5], [11]). If the function u is a solution of the equation
$$-\frac{u_{xx} + u_{yy}}{2xe^u} = 4a^2,$$
Theorem 1 is applicable and we obtain a family of non-flat 5-dimensional examples. Remark that a compact Kähler manifold Z^4 of that type splits into $S^2 \times T^2$, see [6]. The corresponding connection ∇ on the Lie group $N^5 = S^3 \times T^2$ is flat, see [4].

3 Cocalibrated G_2-manifolds with vanishing characteristic Ricci tensor

Consider a cocalibrated G_2-manifold (M^7, g, ω^3),

$$d *\omega^3 = 0\,, \quad \| \omega^3\|^2 = 7\,,$$

and suppose that the G_2-structure ω^3 is not ∇^g-parallel (i.e. $d\omega^3 \not\equiv 0$). There exists a unique metric connection ∇ with skew symmetric torsion and preserving the G_2-structure ω^3. Its torsion form is given by the formula (see [8]),

$$\mathrm{T} = -*d\omega^3 + \mu\,\omega^3, \quad \mu = \frac{1}{6}\left(d\omega^3,\, *\omega^3\right).$$

The condition $\mathrm{Ric}^\nabla = 0$ becomes equivalent to $d\,\mathrm{T} = 0$ and $d*\mathrm{T} = 0$. Indeed, we have:

Theorem 2 ([8, Thm 5.4]). *The following conditions are equivalent:*

1. $\mathrm{Ric}^\nabla = 0\,.$
2. $d\,\mathrm{T} = 0$ *and* $d*\mathrm{T} = 0\,.$
3. $d\mu = 0$ *and* $d*d\omega^3 - \mu\, d\omega^3 = 0\,.$

Using the G_2-splitting of 3-forms, $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, we know that the characteristic torsion of a cocalibrated G_2-manifold belongs to $\mathrm{T} \in \Lambda^3_1 \oplus \Lambda^3_{27}$. In particular, we obtain

$$\mathrm{T} \wedge \omega^3 = 0\,.$$

Differentiating the latter equation and using $d\,\mathrm{T} = 0$ one gets

$$\left(*d\,\omega^3 - \mu\,\omega^3\right) \wedge \omega^3 = 0\,, \qquad \|\,d\,\omega^3\|^2 = 6\,\mu^2.$$

We compute the length of T,

$$\|\mathrm{T}\|^2 = \|d\omega^3\|^2 - 2\,\mu\left(*d\omega^3,\, \omega^3\right) + 7\,\|\omega^3\|^2 = 6\,\mu^2 - 12\,\mu^2 + 7\mu^2 = \mu^2.$$

Consequently, $\|\mathrm{T}\|^2$ is constant. Moreover, the Riemannian scalar curvature is constant, too,

$$\mathrm{Scal}^g = \frac{3}{2}\,\|\mathrm{T}\|^2 = \frac{3}{2}\,\mu^2.$$

Since $(\mathrm{T}, \omega^3) = \mu$, we decompose the torsion form into two parts according to the splitting of 3-forms,

$$\mathrm{T} = \mathrm{T}_1 + \mathrm{T}_{27}\,, \qquad \mathrm{T}_1 = \frac{1}{7}\,\mu\,\omega^3, \qquad \mathrm{T}_{27} = -*d\omega^3 + \frac{6}{7}\mu\omega^3.$$

Corollary 1 ([8, Remark 5.5]). *Let (M^7, g, ω^3) be a compact, cocalibrated G_2-manifold with $\mathrm{Ric}^\nabla = 0$ and $\mathrm{T} \neq 0$. Then the third cohomology group is non-trivial,*

$$H^3(M^7; \mathbb{R}) \;\neq 0\,.$$

Example 5. On the round sphere S^7 there exists a G_2-structure (not cocalibrated) such that $\mathcal{R}^\nabla = 0$ (see [4]). In particular, the Ricci tensor vanishes, $\mathrm{Ric}^\nabla = 0$. The characteristic torsion is coclosed, $\delta \mathrm{T} = 0$, but not closed, $d\,\mathrm{T} \neq 0$.

Remark 1. A cocalibrated G_2-manifold with $\mathrm{Ric}^\nabla = 0$ and $\mathrm{T} \not\equiv 0$ cannot be of pure type Λ^3_1 or Λ^3_{27}. Indeed, if

$$0 = \mathrm{T}_{27} = - *d\omega^3 + \frac{6}{7}\mu\omega^3$$

we differentiate,

$$0 = -d * d\,\omega^3 + \frac{6}{7}\,\mu\, d\,\omega^3$$

and combine the latter formula with equation (3) of Theorem 2. We conclude that $\mu = 0$, $d\omega^3 = 0$ and, finally, $\mathrm{T} = 0$. The second case, i.e. $\mathrm{T}_1 = 0$, implies immediately $\mu = 0$ and $\mathrm{T} = 0$.

There exists a canonical ∇-parallel spinor field Ψ_0 such that

$$\nabla\Psi_0 = 0\,, \qquad \omega^3 \cdot \Psi_0 = -\,7\,\Psi_0\,.$$

Since $\Lambda^3_{27} \cdot \Psi_0 = 0$ we obtain

$$\mathrm{T} \cdot \Psi_0 = \mathrm{T}_1 \cdot \Psi_0 = -\mu\,\Psi_0\ .$$

The integrability condition for a parallel spinor (see [8]) yields an algebraic restriction for the derivative $\nabla\mathrm{T}$, namely

$$\nabla_X\big(\mathrm{T}\cdot\Psi\big) = \big(\nabla_X\mathrm{T}\big)\cdot\Psi \ = \ 0\ , \quad \sigma_{\mathrm{T}}\cdot\Psi = 0\ , \quad \mathrm{T}^2\cdot\Psi = \|\,\mathrm{T}\,\|^2\,\Psi$$

for any vector $X \in TM^7$ and any ∇-parallel spinor field Ψ. In particular, the characteristic torsion T acts on the space of all ∇-parallel spinors. This condition is not so restrictive. For example, the space of 3-forms $\Sigma^3 \in \Lambda^3_{27}$ killing three spinors has dimension 14, the space killing four spinors has still dimension 9.

4 ∇-parallel vector fields

Via the Riemannian metric we identify vectors with 1-forms. Denote by $\mathcal{P}^\nabla$ the space of all ∇-parallel vector field (1-forms). Any ∇-parallel vector field θ is a Killing field and

$$2\,\nabla^g\theta = d\,\theta = \theta \lrcorner \mathrm{T}\ , \qquad \nabla^g_\theta\theta = 0\ .$$

holds. This formula together with $d\,\mathrm{T} = 0$ implies that T is preserved by the flow of θ,

$$\mathcal{L}_\theta\mathrm{T} = 0\,.$$

The Riemannian Ricci tensor on θ becomes

$$\mathrm{Ric}^g(\theta,\theta) = \frac{1}{2}\,\|\,d\,\theta\,\|^2\ .$$

The subgroup of G_2 preserving four vectors in $\mathbb{R}^7$ is trivial. The isotropy subgroups of two or three vectors in $\mathbb{R}^7$ coincide and this group is isomorphic to $SU(2) \subset G_2$. Finally, the isotropy subgroup of one vector is isomorphic to $SU(3) \subset G_2$ (see for example [7]). This algebraic observation proves immediately the following

Proposition 1. *If (M^7, g, ω^3) is not ∇-flat, then the possible dimensions of the space $\mathcal{P}^\nabla$ are* 0, 1, *or* 3.

4.1 The case of three ∇-parallel vector fields

We discuss the case that there are three orthonormal and ∇-parallel 1-forms θ_1, θ_2, θ_3. Then $\omega^3(\theta_1, \theta_2, -)$ is ∇-parallel, too. If it does not coincide with θ_3, then we have at least four ∇-parallel 1-forms, i.e. the G_2-connection ∇ is flat. Under our assumption $\mathcal{R}^\nabla \not\equiv 0$ we conclude that

$$\omega^3(\theta_1, \theta_2, -) = \theta_3 \, , \qquad \omega^3(\theta_1, \theta_2, \theta_3) = 1 \, .$$

The holonomy of the connection ∇ is contained in $\mathfrak{su}(2) \subset \mathfrak{g}_2$. Moreover, the spinors

$$\Psi_0 \, , \quad \Psi_1 := \theta_1 \cdot \Psi_0 \, , \qquad \Psi_2 := \theta_2 \cdot \Psi_0 \, , \qquad \Psi_3 := \theta_3 \cdot \Psi_0$$

are all ∇-parallel spinors. The torsion form T acts as a symmetric endomorphism on the space $\mathrm{Lin}(\Psi_0, \Psi_1, \Psi_2, \Psi_3)$ and $\mathrm{T} \cdot \Psi_0 = -\,\mu\,\Psi_0$. Consequently, T acts on the 3-dimensional space $\mathrm{Lin}(\Psi_1, \Psi_2, \Psi_3)$ and $\mathrm{T}^2 = \|\mathrm{T}\|^2 \cdot \mathrm{Id} = \mu^2 \cdot \mathrm{Id}$. We decompose the torsion form into

$$\mathrm{T} = \mathrm{T}_1 + \mathrm{T}_{27} = \frac{1}{7}\,\mu\,\omega^3 + \mathrm{T}_{27}$$

and we use the known action of ω^3 on spinors:

$$\omega^3 \cdot \Psi_0 = -7\Psi_0 \, , \qquad \omega^3 \cdot \Psi_i = \Psi_i \, , \; i = 1,2,3 \, , \qquad \mathrm{T}_{27} \cdot \Psi_0 = 0 \, .$$

Finally, $\mathrm{T}_{27} \in \Lambda^3_{27}$ preserves the space $\mathrm{Lin}(\Psi_1, \Psi_2, \Psi_3)$ and

$$\mathrm{T}_{27}^2 + \frac{2}{7}\,\mu\,\mathrm{T}_{27} = \frac{48}{49}\,\mu^2 .$$

Without loss of generality we may assume that Ψ_1, Ψ_2, Ψ_3 are eigenspinors of T_{27},

$$\mathrm{T}_{27} \cdot \Psi_i = m_i\,\Psi_i \, , \quad m_i^2 + \frac{2}{7}\,m_i\,\mu = \frac{48}{49}\,\mu^2, \qquad i = 1,2,3 \, .$$

We fix an orthonormal basis $e_1, \dots, e_7$ such that

$$\omega^3 = e_{127} + e_{135} - e_{146} - e_{236} - e_{245} + e_{347} + e_{567}$$

and $\theta_1 = e_1, \theta_2 = e_2, \theta_3 = e_7$. This is possible, since we already have $\omega^3(\theta_1, \theta_2, \theta_3) = 1$. Let

$$\mathrm{T}_{27} = \sum_{i<j<k} t_{ijk}\,e_{ijk}$$

be the 3-form T_{27} and introduce the following numbers:

$$a := t_{236} + t_{245} \, , \qquad b := t_{347} + t_{567} \, , \qquad c := t_{235} - t_{246} \, .$$

A purely algebraic computation yields the following

Lemma 1. *The space of all 3-forms $T_{27} \in \Lambda^3_{27}$ such that $T_{27} \cdot \Psi_i = m_i \Psi_i$, $i = 1, 2, 3$ is an affine space of dimension 9. A parameterization is given by*

$$\begin{aligned} T_{27} = & \Big(-\frac{m_1}{2} - b\Big) e_{127} - t_{156}\, e_{134} + \Big(\frac{m_1}{2} + t_{146} + a\Big) e_{135} \\ & - t_{145}\, e_{136} + t_{145}\, e_{145} + t_{146}\, e_{146} + t_{156}\, e_{156} - t_{256}\, e_{234} \\ & + t_{235}\, e_{235} + t_{236}\, e_{236} + t_{245}\, e_{245} + t_{246}\, e_{246} + t_{256}\, e_{256} + t_{347}\, e_{347} \\ & + t_{467}\, e_{357} - t_{457}\, e_{367} + t_{457}\, e_{457} + t_{467}\, e_{467} + t_{567}\, e_{567}\,. \end{aligned}$$

and

$$m_1 + 2a + 2b = m_2\,, \quad -2a + 2b = m_3\,, \quad c = 0\,.$$

Corollary 2. *For $X \perp \mathrm{Lin}(\theta_1, \theta_2, \theta_3)$ we have*

$$T(\theta_i, \theta_j, X) = 0\,, \quad T = (\theta_1 \lrcorner T) \wedge \theta_1 + (\theta_2 \lrcorner T) \wedge \theta_2 + (\theta_3 \lrcorner T) \wedge \theta_3\,.$$

We solve the linear system with respect to a and b:

$$a = -\frac{1}{4}\big(m_1 - m_2 + m_3\big)\,, \quad b = \frac{1}{4}\big(-\, m_1 + m_2 + m_3\big)\,.$$

In particular,

$$m_1 + 2b = \frac{1}{2}\big(m_1 + m_2 + m_3\big)\,.$$

We are interested in the value

$$T(\theta_1, \theta_2, \theta_3) = \frac{1}{7}\,\mu - \frac{m_1}{2} - b = \frac{1}{7}\,\mu - \frac{1}{4}\big(m_1 + m_2 + m_3\big)\,.$$

We have 8 possibilities, namely

$$m_i = \frac{6}{7}\,\mu \qquad \text{or} \qquad m_i = -\frac{8}{7}\,\mu\,.$$

Therefore

$$T(\theta_1, \theta_2, \theta_3) = 0\,, \quad \pm\frac{1}{2}\,\mu \quad \text{or} \quad \mu\,.$$

We summarize the result.

Theorem 3. *Let (M^7, g, ω^3) be a cocalibrated G_2-manifold and ∇ its characteristic connection. Suppose that $\mathrm{Ric}^\nabla = 0$, $\|T\|^2 = \mu^2 > 0$ and $\mathcal{R}^\nabla \not\equiv 0$. If $\theta_1, \theta_2, \theta_3$ are three orthonormal and ∇-parallel vector fields, then*

1. $\omega^3(\theta_1, \theta_2, \theta_3) = 1$.

2. $T(\theta_1, \theta_2, \theta_3)$ *is constant and has only four possible values:* $0, \pm\mu/2, \mu$.

3. $T(\theta_i, \theta_j, X) = 0$ *for* $X \perp \mathrm{Lin}(\theta_1, \theta_2, \theta_3)$.

In particular

$$\begin{aligned} \mathrm{T} &= (\theta_1 \lrcorner \mathrm{T}) \wedge \theta_1 + (\theta_2 \lrcorner \mathrm{T}) \wedge \theta_2 + (\theta_3 \lrcorner \mathrm{T}) \wedge \theta_3 \\ &= d\theta_1 \wedge \theta_1 + d\theta_2 \wedge \theta_2 + d\theta_3 \wedge \theta_3 \,. \end{aligned}$$

and

$$[\theta_1, \theta_2] = -\mathrm{T}(\theta_1, \theta_2, \theta_3)\, \theta_3$$

is proportional to θ_3*. The* 3*-dimensional space* $\mathrm{Lin}(\theta_1, \theta_2, \theta_3)$ *is closed with respect to the Lie bracket and is a Lie subalgebra of the Killing vector fields. This algebra is either commutative or isomorphic to* $\mathfrak{so}(3)$*.*

Remark 2. Since we do not assume that the torsion form T is ∇-parallel, it is not obvious by general arguments that $[\theta_1, \theta_2] = -\mathrm{T}(\theta_1, \theta_2)$ is again ∇-parallel.

We can classify the case of $\mathrm{T}(\theta_1, \theta_2, \theta_3) = \mu$ immediately. Indeed, we have then $\|\mathrm{T}\|^2 \geq \mu^2$. On the other hand, we know that $\|\mathrm{T}\|^2 = \mu^2$ holds. It follows that

$$\mathrm{T} = \mu\, \theta_1 \wedge \theta_2 \wedge \theta_3 \quad \text{and} \quad \nabla \mathrm{T} = 0\,.$$

Cocalibrated G_2-structures with characteristic holonomy $\mathfrak{su}(2)$ and a characteristic torsion of the given type have been classified at the end of our paper [7]. We apply this result and obtain

Theorem 4. *Let* (M^7, g, ω^3) *be a complete, cocalibrated* G_2*-manifold and* ∇ *its characteristic connection. Suppose that* $\mathrm{Ric}^\nabla = 0$*. If* $\theta_1, \theta_2, \theta_3$ *are three orthonormal and* ∇*-parallel vector fields and* $\mathrm{T}(\theta_1, \theta_2, \theta_3) = \mu$*, then the universal covering of* M^7 *is isometric to the product* $X^4 \times S^3$*, where* X^4 *is a complete anti-self dual and Ricci flat Riemannian manifold.*

If $\mathrm{T}(\theta_1, \theta_2, \theta_3) = 0$ the 3-dimensional abelian Lie group acts on M^7 locally free as a group of isometries and preserves the torsion form T. Moreover, we obtain the 2-forms $d\theta_i = \theta_i \lrcorner \mathrm{T}$ and

$$\mathcal{L}_{\theta_i}(\theta_j \lrcorner \mathrm{T}) = 0\,, \qquad \theta_i \lrcorner \theta_j \lrcorner \mathrm{T} = 0\,.$$

We will investigate the special case, where two of these 2-forms vanish, later.

Remark 3. We do not have any results in case of $|\mathrm{T}(\theta_1, \theta_2, \theta_3)| = \mu/2$.

4.2 Special ∇-parallel vector fields

There are special ∇-parallel vector fields (1-forms), namely

$$\mathcal{SP}^\nabla := \big\{\theta : \nabla^g \theta = 0 \text{ and } \theta \lrcorner \mathrm{T} = 0\big\} \subset \mathcal{P}^\nabla.$$

A consequence of the formula in Theorem 3 is the following

Corollary 3. *If* $\mathrm{T} \not\equiv 0$ *and* $\mathcal{R}^\nabla \not\equiv 0$*, then* $\dim(\mathcal{SP}^\nabla) \leq 2$*.*

Proposition 2. *If* $\theta \in \mathcal{SP}^\nabla$ *is special* ∇*-parallel, then*

$$\nabla^g_\theta\, \omega^3 = 0\,, \qquad d(\theta \lrcorner \omega^3) = \theta \lrcorner d\,\omega^3, \qquad \mathcal{L}_\theta(\theta \lrcorner \omega^3) = 0\,.$$

Proof. Since $\theta \lrcorner \mathrm{T} = 0$ we get

$$\nabla_\theta S = \nabla^g_\theta S + \frac{1}{2}\,\rho_*(\theta \lrcorner \mathrm{T})(S) = \nabla^g_\theta S$$

for any tensor S. Here ρ_* denotes action of $\mathfrak{so}(7)$ in the corresponding tensor representation. In particular,

$$\nabla^g_\theta\, \omega^3 = 0\,.$$

Since θ is ∇^g-parallel, we have $\nabla^g(\theta \lrcorner \omega^3) = \theta \lrcorner \nabla^g \omega^3$. Using an orthonormal frame with $\theta = e_7$ we compute the differential

$$d\,(\theta \lrcorner \omega^3) = \sum_{i=1}^{7} \nabla^g_{e_i}(\theta \lrcorner \omega^3) \wedge e_i = \sum_{i=1}^{6} (\theta \lrcorner \nabla^g_{e_i}\omega^3) \wedge e_i \ + \ 0 = \sum_{i=1}^{6} \theta \lrcorner (\nabla^g_{e_i}\omega^3 \wedge e_i)$$
$$= \sum_{i=1}^{6} \theta \lrcorner (\nabla^g_{e_i}\omega^3 \wedge e_i) \ + \ \theta \lrcorner (\nabla^g_\theta \omega^3 \wedge \theta) = \theta \lrcorner d\omega^3.$$

Finally, $\mathcal{L}_\theta(\theta \lrcorner \omega^3) = \theta \lrcorner d(\theta \lrcorner \omega^3) = \theta \lrcorner \theta \lrcorner d\omega^3 = 0$. □

Theorem 5. *Let (M^7, g, ω^3) be a compact, cocalibrated G_2-manifold and ∇ its characteristic connection. Suppose that $\mathrm{Ric}^\nabla = 0$, $\|\mathrm{T}\|^2 = \mu^2 > 0$ and $\mathcal{R}^\nabla \not\equiv 0$. Then the space of harmonic 1- forms coincides with $\mathcal{SP}^\nabla$,*

$$H^1(M^7; \mathbb{R}) = \big\{\theta : \Delta^g \theta = 0\big\} = \mathcal{SP}^\nabla.$$

In particular, the second Betti number is bounded, $b_2(M^7) \leq 2$.

Proof. The result follows directly from the Weitzenboeck formula for 1-forms and the link between Ric^g and the torsion form T,

$$0 = \int_{M^7} g(\Delta^g\theta, \theta) = \int_{M^7} \|\nabla^g\theta\|^2 + \int_{M^7} \mathrm{Ric}^g(\theta, \theta)$$
$$= \int_{M^7} \|\nabla^g\theta\|^2 + \frac{1}{2}\int_{M^7} \|\theta \lrcorner \mathrm{T}\|^2.$$ □

4.3 The case of two special ∇-parallel vector fields

Suppose that there exist two special ∇-parallel vector fields θ_1, θ_2,

$$\nabla^g\ \theta_1 = \nabla^g\theta_2 = 0\,, \qquad \theta_1 \lrcorner \mathrm{T} = \theta_2 \lrcorner \mathrm{T} = 0\,.$$

Then $\omega^3(\theta_1, \theta_2, -) = \theta_3$ is the third ∇-parallel (non-special) vector field and we have

$$\mathrm{T}(\theta_1, \theta_2, \theta_3) = 0\,, \qquad [\theta_1, \theta_2] = [\theta_1, \theta_3] = [\theta_2, \theta_3] = 0\,.$$

The conditions $\theta_1 \lrcorner \mathrm{T} = \theta_2 \lrcorner \mathrm{T} = 0$ restrict the algebraic type of the torsion form. In fact, Theorem 3 yields that the possible torsion forms depend on two parameters only. Indeed, there are two possibilities. The first case:

$$a = \frac{2}{7}\,\mu\,, \quad b = \frac{5}{7}\,\mu\,, \quad m_1 = -\frac{8}{7}\,\mu\,, \quad m_2 = m_3 = \frac{6}{7}\,\mu\,.$$

The second case:

$$a = \frac{2}{7}\,\mu\,, \quad b = -\,\frac{2}{7}\,\mu\,, \quad m_1 = \frac{6}{7}\,\mu\,, \quad m_2 = \frac{6}{7}\,\mu\,, \quad m_3 = -\frac{8}{7}\,\mu\,.$$

Introducing a new notation for the frame

$$f_1 := e_3\,, \quad f_2 := e_4\,, \quad f_3 := e_5\,, \quad f_4 := e_6\,, \quad f_5 := e_7$$

we obtain the following formula for the torsion form:

$$\begin{aligned} \mathrm{T} &= (t_{125} + \mu/7) f_{125} + t_{245}(f_{135} + f_{245}) \\ &\quad + t_{235}(-f_{145} + f_{235}) + (t_{345} + \mu/7) f_{345}\,, \\ b &= t_{125} + t_{345} = \frac{5}{7}\,\mu \quad \text{or} \quad -\,\frac{2}{7}\,\mu\,, \\ \mu^2 &= \|\mathrm{T}\|^2 = \Big(t_{125} + \frac{\mu}{7}\Big)^2 + \Big(t_{345} + \frac{\mu}{7}\Big)^2 + 2\,t_{245}^2 + 2\,t_{235}^2\,. \end{aligned}$$

If M^7 is complete, its universal covering splits into $N^5 \times \mathbb{R}^2$ and the torsion T as well as the form $\theta_3 = e_7 = f_5$ are forms on N^5. This follows form $\mathcal{L}_{\theta_i}\mathrm{T} = 0$, $\mathcal{L}_{\theta_i}\theta_3 = 0$ for $i = 1, 2$. We reduced the dimension. $(N^5, g, \nabla, \mathrm{T}, \theta_3)$ is a 5-dimensional Riemannian manifold equipped with a torsion form T as well as a metric connection ∇ such that

$$d * \mathrm{T} = 0\,, \quad d\,\mathrm{T} = 0\,, \quad ||\mathrm{T}||^2 = 0\,, \quad \mathrm{Ric}^{\nabla} = 0\,,$$

$$\mathcal{R}^{\nabla} \not\equiv 0\,, \quad \mathfrak{hol}(\nabla) \subset \mathfrak{su}(2) \subset \mathfrak{g}_2$$

hold. θ_3 is ∇-parallel on N^5,

$$\nabla\theta_3 = 0\,, \quad d\,\theta_3 = \theta_3 \lrcorner\, \mathrm{T}\,, \quad \mathrm{T} = \theta_3 \wedge d\,\theta_3\,, \quad 0 = d\,\mathrm{T} = d\,\theta_3 \wedge d\,\theta_3\,.$$

Consider the case of $b = -2\mu/7$. Then

$$t_{125} + \frac{\mu}{7} = -t_{345} - \frac{\mu}{7}$$

and we obtain

$$*\mathrm{T} = -\theta_3 \lrcorner\, \mathrm{T} = -d\theta_3\,, \quad *d\theta_3 = -\mathrm{T} = -d\theta_3 \wedge \theta_3\,.$$

We multiply the latter equation by $d\theta_3$:

$$\|d\theta_3\|^2 = d\theta_3 \wedge *d\theta_3 = -\theta_3 \wedge d\theta_3 \wedge d\theta_3 = 0\,.$$

Consequently, $b = -2\,\mu/7$ implies that the torsion form vanishes, $\mathrm{T} = 0$, i.e. the second case is impossible.

We observe that there are three ∇-parallel 2-forms on N^5, namely,

$$\Omega_i^2 := \theta_i \lrcorner\, \big(\omega^3 - \theta_1 \wedge \theta_2 \wedge \theta_3\big)\,.$$

Consequently, $\mathfrak{hol}(\nabla) \subset \mathfrak{su}(2)$. We can express these forms in our local frame,

$$\begin{aligned} \Omega^2_1 &= f_{13} - f_{24}\,, \\ \Omega^2_2 &= -\,f_{14} - f_{23}\,, \\ \Omega^2_3 &= f_{12} + f_{34}\,. \end{aligned}$$

Remark that

$$\big(\theta_3 \lrcorner \mathrm{T}\,,\, \Omega^2_1\big) = \big(\theta_3 \lrcorner \mathrm{T}\,,\, \Omega^2_2\big) = 0\,, \quad \big(\theta_3 \lrcorner \mathrm{T}\,,\, \Omega^2_3\big) = b + \frac{2}{7}\,\mu = \mu$$

holds.

Theorem 6. *The kernel of* T

$$E^2 := \big\{X \in TN^5 : X \lrcorner \mathrm{T} = 0\big\}$$

is a 2*-dimensional subbundle of* TN^5*. The tangent bundle splits into two subbundles of dimension* 2 *and* 3*, respectively,*

$$TN^5 = E^2 \oplus (E^2)^{\perp}.$$

θ_3 *belongs to* $(E^2)^{\perp}$ *and the torsion form is given by*

$$\mathrm{T} = \mu\, f^*_1 \wedge f^*_2 \wedge \theta_3\,,$$

where f^*_1, f^*_2, θ_3 *is an orthonormal basis in* $(E^2)^{\perp}$*. Both subbundles are involutive and* N^5 *splits locally (but the* 2*- und* 3*-dimensional leaves are not totally geodesic).*

Proof. We compute the determinant of the skew symmetric endomorphism $\theta_3 \lrcorner \mathrm{T}$ on the space of all vectors being orthogonal to θ_3,

$$\mathrm{Det}(\theta_3 \lrcorner \mathrm{T}) = \frac{1}{4}\Big(-b^2 - \frac{4}{7}\,b\,\mu + \frac{45}{49}\,\mu^2\Big)^2 = 0\,.$$

This proves that the dimension of E^2 equals two. Let $f^*_1, f^*_2, f^*_3, f^*_4, f^*_5 = \theta_3$ be an orthonormal frame such that

$$\mathrm{Lin}(f^*_1, f^*_2, f^*_5) = \big(E^2\big)^{\perp}, \quad \mathrm{Lin}(f^*_3, f^*_4) = E^2.$$

Since μ is constant and $d\,\mathrm{T} = d * \mathrm{T} = 0$ we have

$$d\big(f^*_1 \wedge f^*_2 \wedge f^*_5\big) = 0\,, \quad d\big(f^*_3 \wedge f^*_4\big) = 0\,.$$

We differentiate the equations $f^*_3 \wedge f^*_3 \wedge f^*_4 = 0$, $f^*_4 \wedge f^*_3 \wedge f^*_4 = 0$,

$$\begin{aligned} 0 &= df^*_3 \wedge (f^*_3 \wedge f^*_4) - f^*_3 \wedge d(f^*_3 \wedge f^*_4) = df^*_3 \wedge (f^*_3 \wedge f^*_4) \\ 0 &= df^*_4 \wedge (f^*_3 \wedge f^*_4) - f^*_4 \wedge d(f^*_3 \wedge f^*_4) = df^*_4 \wedge (f^*_3 \wedge f^*_4)\,. \end{aligned}$$

By the Frobenius Theorem, the bundle $(E^2)^{\perp}$ is involutive. Similarly we have

$$df^*_1 \wedge (f^*_1 \wedge f^*_2 \wedge f^*_5) = df^*_2 \wedge (f^*_1 \wedge f^*_2 \wedge f^*_5) = df^*_5 \wedge (f^*_1 \wedge f^*_2 \wedge f^*_5) = 0$$

and the bundle E^2 is involutive. □

This splitting is not ∇-parallel ($\nabla \mathrm{T} \neq 0$), but the flow of θ_3 preserves the splitting ($\mathcal{L}_{\theta_3}\mathrm{T} = 0$). The Ricci tensor preserves the splitting, too. Indeed, it depends only on T and we compute easily:

Theorem 7. *The Ricci tensor* Ric^g *preserves the splitting of the tangent bundle and*

$$\mathrm{Ric}^g_{|E^2} = 0\,, \qquad \mathrm{Ric}^g_{|(E^2)^\perp} = \frac{1}{2}\mu^2\,\mathrm{Id}\,.$$

In particular, the Ricci tensor of (N^5, g) *has constant eigenvalues, and these are* 0 *and* $\mu^2/2 > 0$.

The 2-form $d\theta_3$ is invariant under the flow of θ_3,

$$\mathcal{L}_{\theta_3}\big(d\theta_3\big) = 0 \quad \text{and} \quad d\theta_3 \wedge d\theta_3 = 0\,.$$

If the orbit space $Z^4 := N^5/\theta_3$ is smooth, its tangent bundle splits into two involutive 2-dimensional subbundles. $d\theta_3$ defines a 2-form on Z^4 satisfying all the conditions of Theorem 1. However, we have an additional condition for $(N^5, g, \nabla, \mathrm{T}, \theta_3)$, namely the holonomy of ∇ should be contained in $\mathfrak{su}(2) \subset \mathfrak{g}_2$ and the holonomy representation is in $\mathbb{C}^2 \subset \mathbb{R}^5$. This is equivalent to the condition that there are three ∇-parallel 2-forms $\Omega^2_1, \Omega^2_2, \Omega^2_3$. The 2-form Ω^2_3 plays a special role on N^5. Indeed, it projects down to a Kähler form on Z^4.

Proposition 3.

$$\nabla\,\Omega^2_3 = 0\,, \quad d\,\Omega^2_3 = 0\,, \quad \mathcal{L}_{\theta_3}\Omega^2_3 = 0\,.$$

In particular, if Z^4 *is smooth, then* $\Omega^2_3 \in \Lambda^2_+(Z^4)$ *defines a* ∇^g*-parallel, self-dual 2-form on* Z^4.

Proof. Using the frame $f_1, \ldots, f_5$ one easily computes the formula

$$\Omega^2_3 = \frac{1}{\mu}\left(\,*\mathrm{T} + d\,\theta_3\right) = \frac{1}{\mu}\left(\,*\mathrm{T} + \theta_3 \lrcorner\, \mathrm{T}\right)\,.$$

Since $d * \mathrm{T} = 0$ we obtain $d\,\Omega^2_3 = 0$. Moreover, $\mathcal{L}_{\theta_3}\mathrm{T} = 0$, and

$$\mathcal{L}_{\theta_3}\Omega^2_3 = \frac{1}{\mu}\,\mathcal{L}_{\theta_3}(d\theta_3) \;=\; \frac{1}{\mu}\big(\theta_3 \lrcorner\,(\theta_3 \lrcorner\, \mathrm{T})\big) = 0\,. \qquad \square$$

A similar algebraic computation yields the following formulas.

Proposition 4.

$$d\,\Omega^2_1 = \mu\,\Omega^2_2 \wedge \theta_3\,, \qquad d\,\Omega^2_2 = -\,\mu\,\Omega^2_1 \wedge \theta_3\,,$$

$$\mathcal{L}_{\theta_3}\Omega^2_1 = \mu\,\Omega^2_2\,, \qquad \mathcal{L}_{\theta_3}\Omega^2_2 = -\,\mu\,\Omega^2_1\,.$$

Proof. Since the 2-forms are ∇-parallel, we can compute the derivatives using the formula (see [2])

$$d\,\Omega^2 = \sum_{j=1}^{5}(f_j \lrcorner\, \Omega^2) \wedge (f_j \lrcorner\, \mathrm{T})\,. \qquad \square$$

Remark 4. In the frame $f_1^*, \ldots, f_5^*$ we have $\Omega^2_3 = f_1^* \wedge f_2^* + f_3^* \wedge f_4^*$, too. In particular, Ω^2_3 is completely defined by T and θ_3. If Z^4 is smooth and compact, then $Z^4 = S^2 \times T^2$, see [6], and the connection ∇ on $M^7 = N^5 \times \mathbb{R}^2 = S^3 \times T^2 \times \mathbb{R}^2$ becomes flat.

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Almost Abelian rings

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Abstract. A ring R is defined to be left almost Abelian if $ae = 0$ implies $aRe = 0$ for $a \in N(R)$ and $e \in E(R)$, where $E(R)$ and $N(R)$ stand respectively for the set of idempotents and the set of nilpotents of R. Some characterizations and properties of such rings are included. It follows that if R is a left almost Abelian ring, then R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is regular. Moreover it is proved that (1) R is an Abelian ring if and only if R is a left almost Abelian left idempotent reflexive ring. (2) R is strongly regular if and only if R is regular and left almost Abelian. (3) A left almost Abelian clean ring is an exchange ring. (4) For a left almost Abelian ring R, it is an exchange $(S, 2)$ ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R.

1 Introduction

Throughout this article, all rings are associative with identity, and all modules are unital. The symbols $J(R)$, $N(R)$, $U(R)$, $E(R)$ will stand respectively for the Jacobson radical, the set of all nilpotent elements, the set of all invertible elements, the set of all idempotent elements of a ring R. For any nonempty subset X of a ring R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the right annihilator of X and the left annihilator of X, respectively.

The ring R is called left almost Abelian if $ae = 0$ implies $aRe = 0$ for $a \in N(R)$ and $e \in E(R)$, and R is said to be semiabelian [4] if every idempotent of R is either left semicentral or right semicentral. The ring R is called Abelian [1] if every idempotent of R is central. Clearly, Abelian rings are semiabelian and left almost Abelian. Following [4], we know that there exists a semiabelian ring which is not Abelian.

The ring R is called π-regular [1] if for every $a \in R$ there exist $n \geq 1$ and $b \in R$ such that $a^n = a^n b a^n$, and in case of $n = 1$ the ring R is called von Neumann regular. So von Neumann regular rings are π-regular. A ring R is called strongly π-regular if for every $a \in R$ there exist $n \geq 1$ and $b \in R$ such that $a^n = a^{2n} b$,

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and in case of $n = 1$ the ring R is called strongly regular. So strongly regular rings are strongly π-regular. The case when the set $N(R)$ of nilpotent elements of a π-regular ring R is an ideal has been studied by many authors. For examples, in [1], it is shown that if R is an Abelian ring, then R is a π-regular ring if and only if $N(R)$ is an ideal of R and $R/N(R)$ is a strongly regular ring and in [4] it is shown that if R is a semiabelian ring, then R is a π-regular ring if and only if $N(R)$ is an ideal of R and $R/N(R)$ is a strongly regular ring. The goal of this paper is to study the properties of left almost Abelian rings, and to extend some known results on Abelian von Neumann regular rings, π-regular rings, and exchange rings. For instance we prove the following results: if R is a left almost Abelian ring, then R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is strongly regular.

2 Characterizations and Properties

It is easy to see that a ring R is Abelian if and only if $ae = 0$ implies $aRe = 0$ for each $a \in R$ and $e \in E(R)$. Motivated by this, we call a ring R left almost Abelian if $ae = 0$ implies $aRe = 0$ for each $a \in N(R)$ and $e \in E(R)$. Clearly, Abelian rings are left almost Abelian. The converse is not true in general. For example, if R is a reduced ring with $E(R) = \{0, 1\}$ then the 2×2 upper triangular matrix ring $UTM_2(R)$ is left almost Abelian but not Abelian.

According to [4], Abelian rings are semiabelian and the converse is not true in general. The following example implies that semiabelian rings need not be left almost Abelian.

Let R be a ring with $E(R) = \{0, 1\}$ and $N(R) \neq 0$. Then

$$E(UTM_2(R)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; a, b \in R \right\}.$$

Clearly, $\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$ is left semicentral and $\begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$ is right semicentral, so $UTM_2(R)$ is semiabelian, but not left almost Abelian. In fact, let $0 \neq a \in N(R)$. Then

$$\begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix} \in N(UTM_2(R)) \qquad \text{and} \qquad \begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 0,$$

but

$$\begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix} UTM_2(R) \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & aR \\ 0 & 0 \end{pmatrix} \neq 0.$$

Hence $UTM_2(R)$ is not left almost Abelian.

This example also implies that the upper triangular matrices rings over a left almost Abelian ring need not be left almost Abelian.

Proposition 1.

(1) *The subrings and direct products of left almost Abelian rings are left almost Abelian.*

(2) *Let R be a left almost Abelian ring and $e \in E(R)$. Then*

   (a) $(1 - e)Re \subseteq J(R)$.

(b) *If $ReR = R$, then $e = 1$.*

(c) *If M is a maximal left ideal of R and $e \notin M$, then $(1-e)R \subseteq M$.*

(d) *Let M be a maximal left ideal of R and $a \in R$. If $1 - ae \in M$, then $1 - ea \in M$.*

(e) *For any $x \in R$ and $n \geq 1$, $(exe)^n = ex^n e$.*

Proof. (1) is trivial.
(2) (a) For any $a \in R$, write $h = (1-e)a - (1-e)a(1-e)$. Then $h \in N(R)$ and $h(1-e) = 0$. Since R is a left almost Abelian ring, $(1-e)aeR(1-e) = hR(1-e) = 0$. Thus

$$(1-e)ReR(1-e) = \sum_{a\in R}(1-e)aeR(1-e) = 0$$

and so

$$\big((1-e)ReR\big)^2 = 0\,.$$

This implies $(1-e)Re \subseteq J(R)$.

(b) is an immediate consequence of (a).

(c) Since $e \notin M$, $Re + M = R$. By (a), $(1-e)Re \subseteq J(R) \subseteq M$, hence

$$(1-e)R = (1-e)Re + (1-e)M \subseteq M\,.$$

(d) Since $1 - ae \in M$, $e \notin M$. By (c), $(1-e)R \subseteq M$. Since $1 - ae = (1-a) + (a - ae)$, $1 - a \in M$, and $1 - ea = (1-a) + ((1-e)a)$ implies $1 - ea \in M$.

(e) Since

$$ex(1-e) \in N(R)\,, \quad ex(1-e)xe \in ((1-e)xe)Re\,,$$

i.e. $ex^2e = e(xe)^2$. Since

$$ex^2e = (exe)^2 + ex(1-e)xe\,, \quad ex^2e = (exe)^2.$$

By induction on n, we obtain $ex^ne = (exe)^n$. □

It is well known that a ring R is Abelian if and only if every idempotent of R is left semicentral and if and only if every idempotent of R is right semicentral. Hence we can construct a left almost Abelian ring which is not semiabelian.

Let R_1 and R_2 be left almost Abelian rings which are not Abelian. Take $e_1 \in R_1$ to be a right semicentral idempotent which is not central and $e_2 \in R_2$ to be a left semicentral idempotent which is not central, then the idempotent (e_1, e_2) is neither right nor left semicentral in $R_1 \oplus R_2$. Hence $R_1 \oplus R_2$ is not semiabelian, while by Proposition 1(1), $R_1 \oplus R_2$ is left almost Abelian.

A ring R is called directly finite if $xy = 1$ implies $yx = 1$ for $x, y \in R$, and R is called left *min-abelian* if for every

$$e \in ME_l(R) = \{e \in E(R) \mid Re \text{ is a minimal left ideal of } R\}\,,$$

e is left semicentral in R. It is well known that Abelian rings are directly finite and left min-abelian.

Corollary 1. *Let R be a left almost Abelian ring. Then*

(1) R is directly finite.

(2) R is left min-abelian.

Proof. (1) Let $ab = 1$, where $a, b \in R$. Set $e = ba$, then $e \in E(R), ae = a$ and $eb = b$. Since R is left almost Abelian, $(1-e)Re \subseteq J(R)$ by Proposition 1(2)(a). So we have $(1-e)a = (1-e)ae \in J(R)$. Therefore, $1-e = (1-e)ab \in J(R)$. This gives $1 = e = ba$, and R is directly finite.

(2) Let $e \in ME_l(R)$. If e is not left semicentral, then there exists $0 \neq a \in R$ such that $ae - eae \neq 0$. Let $h = ae - eae$. Then $eh = 0$, $he = h$ and $0 \neq h \in N(R)$. Since $hR(1-e) \subseteq (1-e)ReR(1-e)$, the equality $hR(1-e) = 0$ follows from the proof of Proposition 1(2)(a). Since $0 \neq Rh \subseteq Re$, $Rh = Re$. Hence $eR(1-e) = 0$, so also $eR = eRe$. Let $e = ch$ for some $c \in R$. Then $h = he = hee = hech = heceh = 0$ what contradicts to $h \neq 0$. Thus e is left semicentral and so R is a left min-abelian ring. □

The following example shows that the converse of Corollary 1 is not true in general.

Let F be a division ring and

$$R = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}.$$

For the idempotent $e = e_{11} + e_{33}$ we obtain that

$$eR(1-e)Re = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0,$$

and so R is not left almost Abelian. But by [19, Proposition 2.1] R is left quasi-duo, hence R is left min-abelian by [16, Theorem 1.2].

According to [13], an element e of a ring R is called op-idempotent if $e^2 = -e$. Clearly, an op-idempotent element may not be idempotent. For example, let $R = Z/3Z$. Then $\bar{2} \in R$ is op-idempotent, while it is not idempotent. In [3], Chen called an element $e \in R$ potent if there exists an integer $n \geq 2$ such that $e^n = e$. Clearly, idempotent is potent, while there exists a potent element which is not idempotent. For example, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(Z)$ is a potent element, while it is not idempotent. We denote by $E^o(R)$ and $PE(R)$ the set of all op-idempotent elements and the set of all potent elements of R, respectively. Write

$$P_l(R) = \left\{ k \in R \;\middle|\; {}_R Rk \text{ is projective} \right\}.$$

Clearly, $E(R) \subseteq P_l(R)$. Similarly, we can define $P_r(R)$. Recall that a ring R is left PP (i.e. principally left ideal of R is projective) if ${}_R Ra$ is projective for all $a \in R$. Evidently, R is a left PP ring if and only if $P_l(R) = R$. A ring R is called right GPP if for any $x \in R$, there exists $n \geq 1$ such that $x^n \in P_r(R)$.

Theorem 1. *The following conditions are equivalent for a ring R:*

(1) R is a left almost Abelian ring;

(2) $ae = 0$ implies $aRe = 0$ for each $a \in N(R)$ and $e \in E^o(R)$;

(3) $ae = 0$ implies $aRe = 0$ for each $a \in N(R)$ and $e \in PE(R)$;

(4) $ak = 0$ implies $aRk = 0$ for each $a \in N(R)$ and $k \in P_l(R)$.

Proof. (1) $\Longleftrightarrow$ (2), (3) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (1) are trivial.

(1) $\Longrightarrow$ (3) Let $e \in PE(R)$ and $a \in N(R)$ with $ae = 0$. Then there exists $n \geq 2$ such that $e^n = e$. Since $e^{n-1} \in E(R)$ and $ae^{n-1} = 0$, $aRe^{n-1} = 0$ by (1). Thus $aRe = aRe^n = aRe^{n-1}e = 0$.

(1) $\Longrightarrow$ (4) Assume that $a \in N(R)$ and $k \in P_l(R)$ are such that $ak = 0$. Since ${}_RRk$ is projective, there exists $e \in E(R)$ satisfying $l(k) = l(e)$. Hence $ae = 0$, and so $aRe = 0$ by (1). Since $k = ek$, $aRk = aRek = 0$. □

Corollary 2. *Let R be a left PP ring. Then the following conditions are equivalent:*

(1) R is a left almost Abelian ring;

(2) For each $a \in N(R)$ and $b \in R$, $ab = 0$ implies $aRb = 0$;

(3) For each $a \in N(R)$, $r(a)$ is an ideal of R.

A ring R is called left idempotent reflexive if $aRe = 0$ implies $eRa = 0$ for all $a \in R$ and $e \in E(R)$. Clearly, *Abelian* rings are left idempotent reflexive.

Theorem 2. *The following conditions are equivalent for a ring R:*

(1) R is an Abelian ring;

(2) R is an almost Abelian ring and left idempotent reflexive ring;

(3) R is a left idempotent reflexive ring and for any $a, b \in R$ and $e \in E(R)$ we have $eabe = eaebe$.

Proof. (1) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (3) By Proposition 1(2), $ea(1-e)be = 0$ for all $a, b \in R$. Hence $eabe = eaebe$.

(3) $\Longrightarrow$ (1) Let $e \in E(R)$. For any $a \in R$, write $h = ae - eae$. Then

$$hR(1-e) = (1-e)hR(1-e) = (1-e)h(1-e)R(1-e)$$

by (3), so $hR(1-e) = 0$ because $h(1-e) = 0$. Since R is a left idempotent reflexive ring, $(1-e)Rh = 0$, which implies $h = (1-e)h = 0$. Thus $ae = eae$ for all $a \in R$, showing that e is left semicentral. This implies that R is an Abelian ring. □

A ring R is called von Neumann regular if $a \in aRa$ for all $a \in R$ and R is said to be unit-regular if for any $a \in R$, $a = aua$ for some $u \in U(R)$. A ring R is called strongly regular if $a \in a^2R$ for all $a \in R$. Clearly, strongly regular $\Longrightarrow$ unit-regular $\Longrightarrow$ von Neumann regular. Since von Neumann regular rings are semiprime, it follows that von Neumann regular rings are left idempotent reflexive. And it is well known that R is strongly regular if and only if R is von Neumann regular and *Abelian*. In view of Theorem 2, we have the following corollary.

Corollary 3. *The following conditions are equivalent for a ring R:*

(1) *R is a strongly regular ring;*

(2) *R is an unit-regular ring and left almost Abelian ring;*

(3) *R is a von Neumann regular ring and left almost Abelian ring.*

Following [17], a ring R is called left NPP (nil left principally ideal of R is projective) if for any $a \in N(R)$, Ra is projective left R-module. A ring R is said to be reduced if $a^2 = 0$ implies $a = 0$ for each $a \in R$, or equivalently, $N(R) = 0$. Obviously, reduced rings are left NPP, semiprime and *Abelian*. The following theorem gives some new characterizations of reduced rings in terms of left almost Abelian rings and left NPP rings.

Theorem 3. *The following conditions are equivalent for a ring R:*

(1) *R is a reduced ring;*

(2) *R is a left NPP ring, semiprime ring and left almost Abelian ring;*

(3) *R is a left NPP ring, left idempotent reflexive ring and left almost Abelian ring.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1) By Theorem 2, R is an Abelian ring. Now let $a \in R$ such that $a^2 = 0$. Since R is left NPP, $l(a) = Re, e \in E(R)$. Hence $ea = 0$ and $a = ae$ because $a \in l(a)$. Thus $a = ae = ea = 0$. □

The following theorem is an immediate consequence of Proposition 1(1). We prove this directly.

Theorem 4. *If R is a subdirect product of a family of left almost Abelian rings $\{R_i : i \in I\}$, then R is left almost Abelian.*

Proof. Let $R_i = R/A_i$ where A_i be ideals of R with $\bigcap_{i\in I} A_i = 0$. Let $a \in N(R)$ and $e \in E(R)$ with $ae = 0$. Then $a_i = a + A_i \in N(R_i)$, $e_i = e + A_i \in E(R_i)$ and $(a + A_i)(e + A_i) = 0$ for any $i \in I$. Since each R_i is left almost Abelian, $a_iR_ie_i = 0$ for $i \in I$. This implies $aRe \subseteq A_i$ for all $i \in I$, so we have $aRe \subseteq \bigcap_{i\in I} A_i = 0$. Therefore R is left almost Abelian. □

Recall that a ring R has insertion-of-factors-property (IFP) if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$.

A ring R is called left WIFP (weakly IFP) if for any $a \in N(R)$ and $b \in R$, $ab = 0$ implies $aRb = 0$. By Corollary 2, we know that left PP left almost Abelian rings are left WIFP, and left WIFP rings are left almost Abelian.

Clearly, IFP rings are left WIFP.

Let $Z_2 = Z/2Z$. Then the 2×2 upper triangular matrix ring $R = \begin{pmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{pmatrix}$ is a left almost Abelian and left PP ring, so R is a left WIFP ring. Since R is not an Abelian ring, R is not an IFP ring. Thus there exists a left WIFP ring which is neither Abelian nor IFP.

It is well known that rings whose simple left R-modules are YJ-injective are always semiprime. But in general rings whose simple singular left R-modules are injective (hence also YJ-injective) need not be semiprime.

In [7], it is shown that if R is an IFP ring over which every simple singular left modules are YJ-injective, then R is a reduced weakly regular ring. We can generalize the result as follows.

Theorem 5. *If R is a left WIFP ring whose every simple singular left modules are YJ-injective, then R is a reduced weakly regular ring.*

Proof. First, we show that R is a reduced ring. Let $a^2 = 0$. Suppose that $a \neq 0$. Then there exists a maximal left ideal M containing $r(a)$ because $r(a) \neq R$ and $r(a)$ is a left ideal of R. If M is not essential left ideal of R, then $M = l(e)$ for some $e \in ME_l(R)$. Since $a \in r(a) \subseteq M = l(e)$, $ae = 0$. Hence $e \in r(a) \subseteq M = l(e)$, which is a contradiction. Therefore M must be an essential left ideal of R. Thus R/M is YJ-injective and so any R-homomorphism of Ra into R/M extends to one of R into R/M. Let $f : Ra \longrightarrow R/M$ be defined by $f(ra) = r + M$. Note that f is a well-defined R-homomorphism. Since R/M is YJ-injective, there exists $c \in R$ such that $1 + M = f(a) = ac + M$, but $ac \in r(a) \subseteq M$, which implies $1 \in M$, a contradiction. Hence $a = 0$ and so R is a reduced ring. Therefore R is an IFP ring. By [7, p. 2087–2096], R is also a weakly regular ring. □

Proposition 2. *Let R be a left almost Abelian ring and right GPP ring. Then for each $x \in R$, $x = u + a$, where $u \in P_r(R)$ and $a \in N(R)$.*

Proof. Since R is a right GPP ring, there exists $n \geq 1$ such that $x^n \in P_r(R)$. Clearly, there exists $e \in E(R)$ such that $x^n e = x^n$ and $r(x^n) = r(e)$. Since $xe = (xe)e$ and $r(xe) = r(e)$, $xe \in P_r(R)$ and

$$\big(x(1-e)\big)^{n+1} = x\big((1-e)x(1-e)\big)^n = x(1-e)x^n(1-e)$$

by Proposition 1(2)(e). Hence $x(1-e) \in N(R)$. Let $u = xe$ and $a = x(1-e)$. Then $x = u + a$, $u \in P_r(R)$ and $a \in N(R)$. □

A ring R is called left SF if every simple left R-module is flat, and R is said to be right NFB (nilpotent free Baer ring) if for any $a \in N(R)$, and $b \in R$ with $ab = 0$, there exists $e \in E(R)$ such that $ae = 0$ and $eb = b$. Clearly, right NPP rings are right NFB.

Proposition 3. *Let R be a left SF ring. If R is a left almost Abelian right NFB ring, then R is a strongly regular ring.*

Proof. It is well known that reduced left SF rings are strongly regular. We claim that R is reduced. In fact, if $a^2 = 0$, then $Ra + r(aR) = R$. If not, then there exists maximal left ideal M of R containing $Ra + r(aR)$. Since R is a left SF ring, R/M is flat as a left R-module. Since $a \in Ra \subseteq M$, $a = ab$ for some $b \in M$. Since

R is a right NFB ring, there exists $e \in E(R)$ such that $ae = 0$ and $e(1-b) = 1-b$. Since R is a left almost Abelian ring, $aRe = 0$. Hence $aR(1-b) = aRe(1-b) = 0$, which implies $1-b \in r(aR) \subseteq M$. This is a contradiction. Hence $Ra + r(aR) = R$. Let $1 = ca + x$, where $c \in R$ and $x \in r(aR)$. Therefore, $a = aca + ax = aca$. Since $a(1-ca) = 0$ and $1-ca \in E(R)$, $aR(1-ca) = 0$. Hence $ac(1-ca) = 0$, this gives $ac = acca$ and $a = aca = accaa = 0$. □

Corollary 4. *The following conditions are equivalent for a ring R:*

(1) R is a strongly regular ring;

(2) R is a left SF ring, left almost Abelian ring and right NFB ring;

(3) R is a left SF ring, left almost Abelian ring and right NPP ring;

(4) R is a left SF ring, left almost Abelian ring and right PP ring.

Let R be a ring and M a bimodule over R. The trivial extension of R and M is $R \propto M = \{(a,x) | a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by $(a,x)(b,y) = (ab, ay + xb)$. Clearly $R \propto M$ is a ring and $0 \propto M = \{(0,x) | x \in M\}$ is a nonzero nilpotent ideal of $R \propto M$.

Let R be a ring, M a bimodule over R. Write

$$T(R,M) = \left\{ \begin{pmatrix} c & x \\ 0 & c \end{pmatrix} \;\middle|\; c \in R, x \in M \right\},$$

then $T(R,M)$ is a ring and $T(R,M) \cong R \propto M$.

Let R be a ring and $R[x]$ denote the ring of polynomials over R. Clearly, $R[x]/(x^2) \cong R \propto R$.

A right R-module M is called normal if $me = 0$ implies $mRe = 0$ for each $m \in M$ and $e \in E(R)$. Clearly, every right module over an Abelian ring is normal.

Proposition 4. *Let M be a (R,R)-bimodule. Then $T(R,M)$ is a left almost Abelian ring if and only if R is a left almost Abelian ring and M is a right normal R-module.*

Proof. Assume that $T(R,M)$ is a left almost Abelian ring. Then R is a left almost Abelian ring by Proposition 1. Let $m \in M$ and $e \in E(R)$ satisfy $me = 0$. Then

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} = 0\,.$$

Since $T(R,M)$ is left almost Abelian,

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} = 0$$

for each $r \in R$. Therefore $mre = 0$ for each $r \in R$, that is, $mRe = 0$, and M is a right normal R-module.

Conversely, assume that R is left almost Abelian and M is a right normal R-module. Let

$$A = \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} \in N\big(T(R,M)\big)$$

and

$$E = \begin{pmatrix} e & y \\ 0 & e \end{pmatrix} \in E\big(T(R,M)\big)$$

satisfy $AE = 0$. Then $a \in N(R)$, $e \in E(R)$ and we have the following equations:

$$ey + ye = y\,, \tag{1}$$

$$ae = 0\,, \tag{2}$$

$$ay + xe = 0\,. \tag{3}$$

Since R is almost Abelian, $aRe = 0$ by (2). Hence, by (1), we have

$$ay = aey + aye = 0\,. \tag{4}$$

Thus (3) implies

$$xe = (ay + xe) - ay = 0\,. \tag{5}$$

Since M is right normal R-module, $xRe = 0$.

Now, for each $B = \begin{pmatrix} b & z \\ 0 & b \end{pmatrix} \in E(T(R,M))$, we have

$$ABE = \begin{pmatrix} abe & aby + aze + xbe \\ 0 & abe \end{pmatrix}\,. \tag{6}$$

Since $abe, aze, abye \in aRe$, $abe = abye = aze = 0$. Similarly $aby = abey + abye$ implies $aby = 0$ and $xbe \in xRe$ implies $xbe = 0$.

Thus $ABE = 0$, and this gives $AT(R,M)E = 0$. Hence $T(R,M)$ is a left almost Abelian ring. □

Corollary 5. *Let M be an (R,R)-bimodule. Then $R \propto M$ is a left almost Abelian ring if and only if R is a left almost Abelian ring and M is a right normal R-module.*

Let R be a left almost Abelian ring and I an ideal of R. If $I \subseteq N(R)$, then I is right normal as right R-module. Hence by Proposition 4 and Corollary 5, we have the following corollary.

Corollary 6. *Let I be an ideal of R and $I \subseteq N(R)$. Then the following conditions are equivalent:*

(1) R is a left almost Abelian ring;

(2) $T(R,I)$ is a left almost Abelian ring;

(3) $R \propto I$ is a left almost Abelian ring.

It is well known that a ring R is Abelian if and only if for each $e, g \in E(R)$, $ge = 0$ implies $gRe = 0$. Hence, a ring R is Abelian if and only if every right R-module is normal and if and only if R_R is normal. Thus, by Proposition 4, we have the following corollary.

Corollary 7. *Let R be a ring. Then the following conditions are equivalent:*

(1) R is an Abelian ring;

(2) $T(R, R)$ is a left almost Abelian ring;

(3) $R \propto R$ is a left almost Abelian ring;

(4) $R[x]/(x^2)$ is a left almost Abelian ring.

3 Almost Abelian π-regular rings

For convenience, we list the following notions which appeared in the first section of this paper. Let R be a ring and $a \in R$. Then a is called π-regular, if there exist $n \geq 1$ and $b \in R$ such that $a^n = a^n b a^n$. If $n = 1$, a is called von Neumann regular. Further a is said to be strongly π-regular, if $a^n = a^{n+1}b$, and if $n = 1$, a is called strongly regular. A ring R is called von Neumann regular, strongly regular, π-regular and strongly π-regular, if every element of R is von Neumann regular, strongly regular, π-regular and strongly π-regular, respectively. For convenience, we list some known facts which are necessary for the study of π-regularity of rings.

Lemma 1. *[11, Theorem 23.2] The following conditions are equivalent for a ring R.*

(1) R is strongly π-regular.

(2) Every prime factor ring of R is strongly π-regular.

(3) $R/P(R)$ is strongly π-regular.

Proposition 5. *Let R be a left almost Abelian ring and $x \in R$. Then:*

(1) If x is von Neumann regular, then x is strongly regular.

(2) If x is π-regular, then there exists an $e \in E(R)$ such that ex is von Neumann regular and $(1-e)x \in N(R)$.

(3) R is π-regular if and only if R is strongly π-regular.

Proof. (1) Let $x = xyx$ for some $y \in R$. Write $e = yx$. Then $e^2 = e \in R$ and $x = xe$. By Proposition 1(2),

$$e = eee = eyxe = eyexe = eyex = ey^2x^2$$

so, we have $x = xe = xy^2x^2$. Similarly, we can show that $x = x^2y^2x$. Therefore x is strongly regular.

(2) By hypothesis, there exists a positive integer n such that x^n is regular. By (1), x^n is strongly regular. By [10], $x^n = x^n u x^n$ and $x^n u = u x^n$ for some $u \in U(R)$. Let $e = x^n u$. Then $e \in E(R)$, $x^n = ex^n$ and $x^n = ev$, where $v = u^{-1}$. Since

$$(ex)(x^{n-1}u)(ex) = ex^n uex = evuex = ex\,,$$

ex is von Neumann regular. On the other hand, by Proposition 1(2),

$$((1-e)x)^n(1-e) = (1-e)x^n(1-e) = (1-e)ev(1-e) = 0\,,$$

so, we have $((1-e)x)^{n+1} = 0$. Hence $(1-e)x \in N(R)$.

(3) follows from (1). □

The module ${}_RM$ has the finite exchange property if for every module ${}_RA$ and any two decompositions $A = M' \oplus N = \oplus_{i\in I} A_i$ with $M' \cong M$ and I finite set, there exist submodules $A_i' \subseteq A_i$ such that $A = M' \oplus (\oplus_{i\in I} A_i')$.

Warfield [15] called a ring R an exchange ring if ${}_RR$ has the finite exchange property and showed that this definition is left-right symmetric. Nicholson [9] showed that R is an exchange ring if and only if idempotents can be lifted modulo every left (equivalently, right) ideal of R.

Theorem 6. *Let R be a left almost Abelian exchange ring. Then R/P is a local ring for every prime ideal of R.*

Proof. According to [14, Theorem 1], an exchange ring with only two idempotents is a local ring. Since R is an exchange ring, idempotents can be lifted modulo P. For any idempotent element g of R/P, there exists idempotent e of R such that $e + P = g$. Since R is a left almost Abelian, $eR(1-e)Re = 0$ by Proposition 1(2). Hence $gR/P(\bar{1}-g)R/Pg = 0$. Since R/P is a prime ring, $g = 0$ or $g = \bar{1}$, therefore R/P only has two idempotents. Since R/P is an exchange ring, R/P is a local ring. □

Corollary 8. *Let R be a left almost Abelian exchange ring. Then R/P is a division ring for every left (resp., right) primitive ideal of R.*

It is easy to show that if R is an exchange ring with $J(R) = 0$, then R is reduced if and only if R is left almost Abelian. Combining this fact with Theorem 3 and [8, Theorem 4.6], we have the following lemma.

Lemma 2. *If R is an exchange ring, then the following conditions are equivalent.*

(1) $R/J(R)$ is reduced.

(2) $R/J(R)$ is Abelian.

(3) $R/J(R)$ is left almost Abelian.

(4) R is quasi-duo.

(5) R is left quasi-duo.

Theorem 7. *Let R be an exchange ring, then the following conditions are equivalent.*

(1) $N(R) \subseteq J(R)$.

(2) $R/J(R)$ is a left almost Abelian ring.

If $J(R)$ is also nil, then the above conditions are equivalent to any of the following.

(3) $N(R)$ is a left ideal of R.

(4) $N(R)$ is a right ideal of R.

(5) R is an NI ring (i.e. the set of all nilpotent elements forms an ideal of R).

Proof. (1) $\Longrightarrow$ (2) Because R is an exchange ring there exists $e \in E(R)$ such that $e + J(R) = i$ for any $i \in E(R/J(R))$. On the other hand, for any $a \in R$, $ae - eae \in N(R)$, so, we have $ae - eae \in J(R)$ by (1). This shows that i is left semicentral in $R/J(R)$, hence $R/J(R)$ is left almost Abelian.

(2) $\Longrightarrow$ (1) By Lemma 2, $R/J(R)$ is reduced, therefore $N(R/J(R)) = 0$, so, we have $N(R) \subseteq J(R)$.

Now we assume that $J(R)$ is nil, then $J(R) \subseteq N(R)$.

By (1), $N(R) = J(R)$ is an ideal, so R is an NI ring. Thus (1) $\Longrightarrow$ (5).

(5) $\Longrightarrow$ (4) $\Longrightarrow$ (1) and (5) $\Longrightarrow$ (3) $\Longrightarrow$ (1) are trivial. □

It is known that π-regular rings are exchange and the Jacobson radical of π-regular ring is nil. Hence Theorem 7 implies that for a π-regular ring R, R is an NI ring if and only if $R/J(R)$ is a left almost Abelian ring.

The following corollary generalizes [1, Theorem 2].

Corollary 9. *Let R be a left almost Abelian π-regular ring. Then $N(R) = J(R)$, so R is an NI ring.*

Proof. It is an immediate consequence of Theorem 7 and Proposition 1(2)(b). □

In terms of Corollary 9, we have the following theorem, which generalizes [1, Theorem 3].

Theorem 8. *Let R be a left almost Abelian ring. Then R is π-regular if and only $N(R)$ is an ideal of R and $R/N(R)$ is von Neumann regular. In this case R is strongly π-regular.*

Proof. ($\Longrightarrow$) Suppose that R is π-regular. By Corollary 9, R is an NI ring and $N(R) = J(R)$. Therefore $R/N(R)$ is a reduced π-regular ring, so, $R/N(R)$ is strongly regular.

($\Longleftarrow$) Assume that $N(R)$ is an ideal of R and $\bar{R} = R/N(R)$ is a von Neumann regular ring. Then $R/N(R)$ is strongly regular because $R/N(R)$ is a reduced ring. To prove that R is π-regular, it is sufficient to prove (Lemma 1) that R/P is strongly π-regular for every prime ideal P of R. If $x \in R$, then $\bar{x} = x + J(R) \in \bar{R}$ is unit regular. So we have $\bar{x} = \bar{e}\bar{u} = \bar{u}\bar{e}$ with $e \in E(R)$ and $u \in U(R)$ because idempotents and units of $\bar{R}$ can be lifted modulo $N(R)$. Hence

$$x = eu + a = ue + b\,, \quad \text{where } a, b \in N(R)\,,$$

which implies

$$ex = e(u + a) \quad \text{and} \quad xe = (u + b)e\,,$$

and

$$(1-e)x = x - ex = (1-e)a \in N(R)\,,$$
$$x(1-e) = x - xe = b(1-e) \in N(R)\,.$$

So there exists a positive integer n such that $[(1-e)x]^n = [x(1-e)]^n = 0$. If $e \in P$, then $x^n \in P$ and $\hat{x} = x+P \in N(R/P)$, so $\hat{x}$ is strongly π-regular in R/P. If $e \notin P$, then since R is left almost Abelian, $eR(1-e)Re = 0 \subseteq P$ and $1-e \in P$, which gives $\hat{e} = \hat{1}$ in R/P. This implies $\hat{x} = \hat{e}\hat{x} = e\widehat{(u+a)} = \widehat{u+a}$ in R/P. Hence $\hat{x}$ is a unit and so it is a strongly $\pi-$regular element in R/P, and the proof is completed. □

Corollary 10. *Suppose R is left almost Abelian π-regular and let P be a prime ideal of R, then:*

(1) Every element of R/P is either nilpotent or unit.

(2) If $N(R) \subseteq P$, then R/P is a division ring.

(3) If P is left or right primitive ideal of R, then R/P is a division ring.

Hence R is strongly π-regular with $J(R) = N(R)$.

Corollary 11. *Let R be a left almost Abelian π-regular ring. If R is indecomposable, then R is local and $N(R) = J(R)$.*

Proof. By Theorem 8, $N(R) = J(R)$. Let $x \in R$. If $x \notin J(R)$, then $x \notin N(R)$. Since R is π-regular, there exists $n \geq 1$ and $y \in R$ such that $x^n = x^n y x^n$. Set $e = yx^n$. Then $e^2 = e$ and $x^n = x^n e$. Since R is indecomposable, either $e = 0$ or $e = 1$. Since $x \notin N(R)$, $e \neq 0$. Hence $e = 1$, that is $yx^n = 1$. By Corollary 1, R is directly finite, and x is invertible. This shows that R is a local ring. □

In [8, Theorem 4.6], it is proved that for a ring R, if $R/J(R)$ is an exchange ring, then R is left quasi-duo if and only if $R/J(R)$ is Abelian.

Theorem 9. *Let R be a left almost Abelian exchange ring. Then R is a left and right quasi-duo ring.*

Proof. Since R is a left almost Abelian exchange ring, $R/J(R)$ is Abelian exchange by the proof of Corollary 9. By Lemma 2, $R/J(R)$ is reduced, and by [8, Theorem 4.6], R is left and right quasi-duo. □

Combining Theorem 9 with Lemma 2 and [8, Corollary 4.7], we have the following corollary.

Corollary 12. *Let R be a left almost Abelian π-regular ring, then $R/J(R)$ is a duo ring and R is a quasi-duo ring.*

Proposition 6. *Let R be a π-regular ring such that $N(R)$ form a one-sided ideal of R. Then R is quasi-duo.*

Proof. We claim that $R/J(R)$ is reduced. To see this, let $x \in R$ be such that $x^2 \in J(R)$. Since $J(R)$ is nil, $(x^2)^m = 0$ for some $m \geq 1$. Therefore $x \in N(R)$. Since $N(R)$ is a one-sided ideal of R, $N(R) \subseteq J(R)$ and so, we have $x \in J(R)$. Having shown that $R/J(R)$ is reduced, R is quasi-duo by [8, Theorem 4.6]. □

Recall that a ring R is semi-π-regular if $R/J(R)$ is π-regular and idempotents can be lifted modulo $J(R)$. Combining Theorem 8 with Theorem 2, we have the following corollary.

Corollary 13. *Let R be a left almost Abelian semi-π-regular ring, then $R/J(R)$ is a strongly regular ring.*

We end this section with the following example which gives a non-Abelian left almost Abelian π-regular ring.

Let F be a division ring and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Clearly, R is a left almost Abelian π-regular ring. But R is not Abelian.

4 Applications

Following [9], a ring R is called clean if every element of R is a sum of a unit and an idempotent. Clean rings are always exchange rings, and the converse is true if R is Abelian.

Proposition 7. *Let R be a left almost Abelian ring. Then R is clean if and only if R is exchange.*

Proof. One direction is trivial.

For the other direction, let R be an exchange ring, then $R/J(R)$ is exchange and idempotents can be lifted modulo $J(R)$. By Proposition 1 (2)(b), $R/J(R)$ is Abelian. Therefore $R/J(R)$ is clean by [9], so, by [2, Proposition 7], R is a clean ring. □

In [5], it is shown that if R is a unit regular ring in which 2 is invertible, then every element in R is a sum of two units. The ring R is called an $(S, 2)$ ring [6] if every element in R is a sum of at least two units of R. In [1, Theorem 6] it is proved that if R is an Abelian π-regular ring, then R is an $(S, 2)$ ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R. We can generalize this result to left almost Abelian rings, however, we need the following lemma.

Lemma 3.

(1) R is an $(S, 2)$ ring if and only if $R/J(R)$ is an $(S, 2)$ ring.

(2) $\mathbb{Z}/2\mathbb{Z}$ is a homomorphic image of R if and only if $\mathbb{Z}/2\mathbb{Z}$ is a homomorphic image of $R/J(R)$.

Theorem 10. *Let R be a left almost Abelian π-regular ring. Then R is an $(S, 2)$ ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R.*

Proof. Since R is a left almost Abelian π-regular ring, $R/J(R)$ is strongly regular by Theorem 8 and Corollary 10. Hence $R/J(R)$ is Abelian π-regular. By [1, Theorem 6], $R/J(R)$ is an $(S, 2)$ ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R/J(R)$. Then Lemma 3 finishes the proof. □

In light of Theorem 10, we have the following corollaries:

Corollary 14. *Let R be a left almost Abelian π-regular ring such that $2 = 1 + 1 \in U(R)$. Then R is an $(S, 2)$ ring.*

Corollary 15. *Let R be a left almost Abelian π-regular ring. Then R is an $(S, 2)$ ring if and only if for some $d \in U(R)$, $1 + d \in U(R)$.*

Recall that a ring R is said to have stable range 1 [12] if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right invertible. Clearly, R has stable range 1 if and only if $R/J(R)$ has stable range 1. In [19, Theorem 6], it is showed that exchange rings with all idempotents central have stable range 1. We now generalize this result as follows.

Theorem 11. *Left almost Abelian exchange rings have stable range* 1*.*

Proof. Let R be a left almost Abelian exchange ring. Then $R/J(R)$ is exchange with all idempotents central, so, by [19, Theorem 6], $R/J(R)$ has stable range 1. Therefore R has stable range 1. □

In [18], the ring R is said to satisfy the unit 1-stable condition if for any $a, b, c \in R$ with $ab + c = 1$, there exists $u \in U(R)$ such that $au + c \in U(R)$. It is easy to prove that R satisfies the unit 1-stable condition if and only if $R/J(R)$ satisfies the unit 1-stable condition.

Proposition 8. *Let R be a left almost Abelian exchange ring, then the following conditions are equivalent:*

(1) R is an $(S, 2)$ ring.

(2) R satisfies the unit 1*-stable condition.*

(3) Every factor ring R_1 of R is an $(S, 2)$ ring.

(4) $\mathbb{Z}_2$ is not a homomorphic image of R.

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Eigenvalue relationships between Laplacians of constant mean curvature hypersurfaces in $\mathbb{S}^{n+1}$

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Abstract. For compact hypersurfaces with constant mean curvature in the unit sphere, we give a comparison theorem between eigenvalues of the stability operator and that of the Hodge Laplacian on 1-forms. Furthermore, we also establish a comparison theorem between eigenvalues of the stability operator and that of the rough Laplacian.

1 Introduction

Let M be an n-dimensional compact hypersurface with constant mean curvature in the unit sphere $\mathbb{S}^{n+1}(1)$. We let h_{ij} denote the components of the second fundamental form, S stand for the norm square of the second fundamental form, H be the mean curvature of M, respectively. A Schrödinger operator

$$J = -\Delta - (S + n)\,,$$

where Δ denotes the Laplace-Beltrami operator, is called a Jacobi operator. Since the spectral behavior is directly related to the instability of both minimal hypersurfaces and hypersurfaces with constant mean curvature in $\mathbb{S}^{n+1}(1)$ (for example, see [2], [10]), many mathematicians studied the first and the second eigenvalues of such Jacobi operator. The first eigenvalue of J on hypersurfaces in $\mathbb{S}^{n+1}(1)$ was studied by Simons [10] and Wu [11]. Ei Soufi and Ilias [6] studied the second eigenvalue of the Jacobi operator above. In 1993, Alencar, do Carmo and Colares [1] studied the stability of hypersurfaces with constant scalar curvature in $\mathbb{S}^{n+1}(1)$. Similarly to the case of both minimal hypersurfaces and hypersurfaces with constant mean curvature in $\mathbb{S}^{n+1}(1)$, we have a notion of Jacobi operator corresponding to compact hypersurfaces with constant scalar curvsture. For the first eigenvalue and the second eigenvalue of such Jacobi operator, the readers who are interested in it see [4], [8].

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Recently, Savo [9] considered compact minimal hypersurfaces of the unit sphere and proved a comparison theorem between the spectrum of the stability operator J and that of the Hodge Laplacian on 1-forms. In this paper, we consider hypersurfaces of the unit sphere with constant mean curvature. Now we state our result as follows:

Theorem 1. *Let $x\colon M^n \to \mathbb{S}^{n+1}(1)$ be an n-dimensional compact hypersurface with constant mean curvature H. We denote the norm square of the second fundamental form by S. Then*

$$\lambda_\alpha^J \le -2(n-1) + \lambda_{m(\alpha)}^{\Delta_1} + n|H| \max_M \sqrt{S}\,, \tag{1}$$

where λ_α^J is the α-th eigenvalue of J, $\lambda_{m(\alpha)}^{\Delta_1}$ is the $m(\alpha)$-th eigenvalue of the Hodge Laplacian Δ_1 with respect to 1-form. Here $m(\alpha) = \binom{n+2}{2}(\alpha-1)+1$.

In particular, Savo [9] has proved that for compact minimal hypersurfaces of the unit sphere, it holds that

$$\lambda_\alpha^J \le -2(n-1) + \lambda_{m(\alpha)}^{\Delta_1}\,. \tag{2}$$

Hence, the Theorem 1 above extends Theorem 1 in [9]. On the other hand, for eigenvalues of the stability operator J and the rough Laplacian, we have the following result:

Theorem 2. *Let $x\colon M^n \to \mathbb{S}^{n+1}(1)$ be an n-dimensional compact hypersurface with constant mean curvature. We have*

$$\lambda_\alpha^J \le -(n-1) + \lambda_{m(\alpha)}^{D^*D}\,, \tag{3}$$

where λ_α^J is the α-th eigenvalue of J, $\lambda_{m(\alpha)}^{D^*D}$ is the $m(\alpha)$-th eigenvalue of the rough Laplacian D^*D with respect to 1-form. Here $m(\alpha) = \binom{n+2}{2}(\alpha-1)+1$.

2 Proof of Theorems

Let $x\colon M^n \to \mathbb{S}^{n+1}(1)$ be an n-dimensional compact hypersurface with constant mean curvature. We adopt the following index convention:

$$1 \le i,j,k,l \le n\,, \quad 1 \le A,B \le n+2\,.$$

Choosing a local orthonormal frame $\{e_1,\dots,e_n,e_{n+1}\}$ and the dual coframe $\{\omega_1,\dots,\omega_n,\omega_{n+1}\}$ such that when restricted on M, $\{e_1,\dots,e_n\}$ forms a local orthonormal frame on M. Hence, $\omega_{n+1}=0$ on M and the following structure equations (see [5]):

$$\begin{aligned} dx &= \omega_i e_i\,,\\ de_i &= \omega_{ij}e_j + h_{ij}\omega_j e_{n+1} - \omega_i x\,,\\ de_{n+1} &= -h_{ij}\omega_j e_i\,, \end{aligned}$$

where h_{ij} denote the components of the second fundamental form of x, in which we used the summation convention on repeated indices. We will take this convention in the later part without any confusion. The Gauss equations (see [5], [7]) are

$$\begin{aligned} R_{ijkl} &= (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk})\,, \\ R_{ij} &= R_{ikjk} = (n-1)\delta_{ij} + nHh_{ij} - h_{ik}h_{jk}\,, \\ R &= n(n-1) + n^2H^2 - S\,, \end{aligned} \tag{4}$$

where R stands for the scalar curvature and $S = \sum_{ij} h_{ij}^2$ is the norm square of the second fundamental form, $H = \frac{1}{n}h_{ii}$ is the mean curvature of x. The Codazzi equations are given by

$$h_{ijk} = h_{ikj}\,, \quad \text{for } i,j,k = 1,\ldots,n\,.$$

Let f be a smooth function on M. The first and the second covariant derivatives of f are defined by

$$df = f_i\omega_i\,,$$

$$f_{ij}\omega_j = df_i + f_j\omega_{ji}\,.$$

Let a be a fixed vector in $\mathbb{R}^{n+2}$. Define

$$f^a = \langle a, x\rangle, \quad g^a = \langle a, e_{n+1}\rangle\,.$$

Then we have the following lemma:

Lemma 1. *(see [3]) Under the conceptions above, we have*

$$\begin{aligned} f_i^a &= \langle a, e_i\rangle\,, \quad g_i^a = -h_{ij}f_j^a, \\ f_{ij}^a &= h_{ij}g^a - f^a\delta_{ij}\,, \\ g_{ij}^a &= h_{ij}f^a - h_{ik}h_{jk}g^a - h_{ijk}f_k^a\,. \end{aligned}$$

Define the Hodge Laplacian Δ_p by

$$\Delta_p = d\delta + \delta d\colon A^p(M) \to A^p(M)$$

where $\delta = (-1)^{n(p+1)} * d*\colon A^p(M) \to A^{p-1}(M)$. For any $\psi \in A^p(M)$, one has

$$\Delta_p\,\psi = D^*D(\psi) - \mathrm{Ric}(\psi)\,,$$

where D^*D denotes the rough Laplacian which is given by

$$D^*D(\psi) = \sum_i (D_{e_i}D_{e_i} - D_{D_{e_i}e_i})\psi\,.$$

In particular, when $\xi = \xi_i\omega_i \in A^1(M)$, $D^*D(\xi) = \xi_{j,ii}\omega_j$, where the second covariant derivatives of ξ is defined by

$$\xi_{i,jk}\omega_k = d\xi_{i,j} + \xi_{k,j}\omega_{ki} + \xi_{i,k}\omega_{kj}\,.$$

In particular, for $f \in C^\infty(M)$, we have $\Delta_0 f = f_{ii} = \Delta f$. By (4), one gets

$$\mathrm{Ric}(\xi) = \xi_i R_{ij} e_j = (n-1)\xi + nH h_{ij}\xi_i e_j - h_{ik}h_{jk}\xi_i e_j$$

and hence,

$$D^*D(\xi) = \Delta_1\xi + (n-1)\xi + nHh_{ij}\xi_i e_j - h_{ik}h_{jk}\xi_i e_j. \tag{5}$$

Lemma 2. *Let a be a fixed vector in $\mathbb{R}^{n+2}$ and $a^\top$ denote the orthogonal projection onto M. Then*

$$\Delta_1 a^\top = -nHh_{ij}f^a_j e_i - nf^a_i e_i\,, \tag{6}$$

$$D^*D(a^\top) = -f^a_i e_i - h_{ik}h_{jk}f^a_i e_j\,. \tag{7}$$

Proof. By a direct calculation, one has from Lemma 1

$$\begin{aligned}\Delta_1 a^\top &= \Delta_1(\langle a^\top, e_i\rangle\omega_i) = \Delta_1(df^a) = d(\Delta f^a)\\ &= nHdg^a - ndf^a = -nHh_{ij}f^a_j e_i - nf^a_i e_i\,.\end{aligned}$$

Hence (6) is proved. Substituting ξ in (5) by $a^\top$ and using (6), we obtain (7). □

Lemma 3. *Let ξ be a vector field on M and a, b be two independent fixed vectors in $\mathbb{R}^{n+2}$. Then we have*

$$\begin{aligned}\Delta\big(\langle a, e_{n+1}\rangle\langle b^\top, \xi\rangle\big) &= \big((n-2)\xi_j + \langle e_j, \Delta_1\xi\rangle - 2h_{ik}h_{jk}\xi_i + nHh_{ij}\xi_i - S\xi_j\big)f^b_j g^a\\ &\quad - 2h_{ij}h_{ik}\xi_k f^a_j g^b - 2h_{ij}\xi_{k,i}f^a_j f^b_k - 2\xi_{ii}f^b g^a + 2h_{ij}\xi_i f^a_j f^b\\ &\quad + nH\xi_j f^a f^b_j + 2h_{ij}\xi_{i,j}g^b g^a.\end{aligned}$$

Proof. Given a point $p \in M$, let $\{e_i\}_{i=1}^n$ be an orthonormal frame which is geodesic at p. Then $\Delta f = e_ie_i(f)$ and we have from (5), Lemma 1 and Lemma 2,

$$\begin{aligned}\Delta\langle b^\top, \xi\rangle &= \langle D^*D(b^\top), \xi\rangle + 2f^b_{ij}\xi_{i,j} + \langle b^\top, D^*D(\xi)\rangle\\ &= -\xi_j f^b_j - h_{ik}h_{jk}\xi_i f^b_j + 2(h_{ij}g^b - f^b\delta_{ij})\xi_{i,j}\\ &\quad + \langle e_j, \Delta_1\xi\rangle f^b_j + (n-1)\xi_j f^b_j + nHh_{ij}\xi_i f^b_j - h_{ik}h_{jk}\xi_i f^b_j\\ &= \big((n-2)\xi_j + \langle e_j, \Delta_1\xi\rangle - 2h_{ik}h_{jk}\xi_i + nHh_{ij}\xi_i\big)f^b_j\\ &\quad + 2h_{ij}\xi_{i,j}g^b - 2\xi_{i,i}f^b\,,\end{aligned}$$

$$\begin{aligned}\big\langle\nabla\langle a, e_{n+1}\rangle, \nabla\langle b^\top, \xi\rangle\big\rangle &= g^a_i\big(\langle D_{e_i}b^\top, \xi\rangle + \langle b^\top, D_{e_i}\xi\rangle\big)\\ &= g^a_i(f^b_{ij}\xi_j + f^b_j\xi_{j,i})\\ &= -h_{ij}h_{ik}\xi_k f^a_j g^b + h_{ij}\xi_i f^a_j f^b - h_{ij}\xi_{k,i}f^a_j f^b_k\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta\big(\langle a,e_{n+1}\rangle\langle b^\top,\xi\rangle\big) &= \langle a,e_{n+1}\rangle\Delta\langle b^\top,\xi\rangle+\langle b^\top,\xi\rangle\Delta\langle a,e_{n+1}\rangle\\ &\quad+2\big\langle\nabla\langle a,e_{n+1}\rangle,\nabla\langle b^\top,\xi\rangle\big\rangle\\ &= g^a\Big(\big((n-2)\xi_j+\langle e_j,\Delta_1\xi\rangle-2h_{ik}h_{jk}\xi_i+nHh_{ij}\xi_i\big)f_j^b\\ &\quad+2h_{ij}\xi_{i,j}g^b-2\xi_{i,i}f^b\Big)+(nHf^a-Sg^a)\xi_jf_j^b\\ &\quad+2(-h_{ij}h_{ik}\xi_kf_j^ag^b+h_{ij}\xi_if_j^af^b-h_{ij}\xi_{k,i}f_j^af_k^b)\\ &= \big((n-2)\xi_j+\langle e_j,\Delta_1\xi\rangle-2h_{ik}h_{jk}\xi_i+nHh_{ij}\xi_i-S\xi_j\big)f_j^bg^a\\ &\quad-2h_{ij}h_{ik}\xi_kf_j^ag^b-2h_{ij}\xi_{k,i}f_j^af_k^b-2\xi_{i,i}f^bg^a+2h_{ij}\xi_if_j^af^b\\ &\quad+nH\xi_jf^af_j^b+2h_{ij}\xi_{i,j}g^bg^a.\end{aligned}$$

We conclude the proof of Lemma 3. □

Now we are in a position to prove Theorem 1.

Proof. (of Theorem 1) Let $\{E_A\}_{A=1}^{n+2}$ be a fixed orthonormal basis of $\mathbb{R}^{n+2}$. Define

$$X_{AB}^\top=\langle E_A,e_{n+1}\rangle E_B^\top-\langle E_B,e_{n+1}\rangle E_A^\top$$

and

$$u_{AB}=\langle X_{AB}^\top,\xi\rangle=-u_{BA}.$$

Let

$$f^A=\langle E_A,x\rangle,\quad g^A=\langle E_A,e_{n+1}\rangle\,.$$

Then from Lemma 3, we have

$$\begin{aligned}\Delta u_{AB} &= \Delta\big(\langle E_A,e_{n+1}\rangle\langle E_B^\top,\xi\rangle\big)-\Delta\big(\langle E_B,e_{n+1}\rangle\langle E_A^\top,\xi\rangle\big)\\ &= \big((n-2)\xi_j+\langle e_j,\Delta_1\xi\rangle-2h_{ik}h_{jk}\xi_i+nHh_{ij}\xi_i-S\xi_j\big)(f_j^Bg^A-f_j^Ag^B)\\ &\quad-2h_{ij}h_{ik}\xi_k(f_j^Ag^B-f_j^Bg^A)-2h_{ij}\xi_{k,i}(f_j^Af_k^B-f_j^Bf_k^A)\\ &\quad-2\xi_{ii}(f^Bg^A-f^Ag^B)+2h_{ij}\xi_i(f_j^Af^B-f_j^Bf^A)\\ &\quad+nH\xi_j(f^Af_j^B-f^Bf_j^A)\\ &= (n-2-S)u_{AB}+v_{AB}\,,\end{aligned}$$

where

$$\begin{aligned}v_{AB} &= \big(\langle e_j,\Delta_1\xi\rangle-2h_{ik}h_{jk}\xi_i+nHh_{ij}\xi_i\big)(f_j^Bg^A-f_j^Ag^B)\\ &\quad-2h_{ij}h_{ik}\xi_k(f_j^Ag^B-f_j^Bg^A)-2h_{ij}\xi_{k,i}(f_j^Af_k^B-f_j^Bf_k^A)\\ &\quad-2\xi_{i,i}(f^Bg^A-f^Ag^B)+2h_{ij}\xi_i(f_j^Af^B-f_j^Bf^A)\\ &\quad+nH\xi_j(f^Af_j^B-f^Bf_j^A)\,.\end{aligned}$$

Let λ_α^J be the α-th eigenvalue of J and φ_α be the orthonormal eigenfunction corresponding to λ_α^J, that is,

$$J\varphi_\alpha = \lambda_\alpha^J \varphi_\alpha, \quad \int_M \varphi_\alpha \varphi_\beta = \delta_{\alpha\beta}. \tag{8}$$

Denote by $V_m^{\Delta_1}$ the direct sum of the first m eigenspaces of Δ_1 such that the following orthogonality relations

$$\int_M \langle X_{AB}^\top, \xi\rangle \varphi_1 = \cdots = \int_M \langle X_{AB}^\top, \xi\rangle \varphi_{\alpha-1} = 0 \tag{9}$$

hold for all A, B. Note that $X_{AB}^\top$ is skew symmetric. Hence, we know that (9) has $\binom{n+2}{2}(\alpha-1)$ homogenous linear equations in $\xi \in V_m^{\Delta_1}$. If we let

$$m(\alpha) := \binom{n+2}{2}(\alpha-1)+1\,,$$

then we can find a non-zero vector field $\xi \in V_{m(\alpha)}^{\Delta_1}$ such that the function u_{AB} is orthogonal to the first $\alpha-1$ eigenfunctions of J for all A, B. By the Rayleigh-Ritz principle, we have

$$\begin{aligned} \lambda_\alpha^J \int_M u_{AB}^2 &\le \int_M u_{AB} J\, u_{AB} \\ &= -\int_M u_{AB}\Delta\, u_{AB} - \int_M (S+n)u_{AB}^2 \\ &= -\int_M \Big(2(n-1)u_{AB}^2 + u_{AB}v_{AB}\Big). \end{aligned} \tag{10}$$

It follows from $u_{AB} = \xi_l(f_l^B g^A - f_l^A g^B)$ that

$$\sum_{A,B} u_{AB}^2 = \xi_l \xi_k \sum_{A,B} (f_l^B g^A - f_l^A g^B)(f_k^B g^A - f_k^A g^B) = 2|\xi|^2, \tag{11}$$

$$\begin{aligned} \sum_{A,B} u_{AB}v_{AB} = \xi_l \Big\{ &(\langle e_j, \Delta_1 \xi\rangle - 2h_{ik}h_{jk}\xi_i + nHh_{ij}\xi_i) \\ &\times \sum_{A,B} (f_j^B g^A - f_j^A g^B)(f_l^B g^A - f_l^A g^B) \\ &- 2h_{ij}h_{ik}\xi_k \sum_{A,B} (f_j^A g^B - f_j^B g^A)(f_l^B g^A - f_l^A g^B) \\ &- 2h_{ij}\xi_{k,i} \sum_{A,B} (f_j^A f_k^B - f_j^B f_k^A)(f_l^B g^A - f_l^A g^B) \\ &- 2\xi_{i,i} \sum_{A,B} (f^B g^A - f^A g^B)(f_l^B g^A - f_l^A g^B) \end{aligned} \tag{12}$$

$$
\begin{aligned}
&+2h_{ij}\xi_i\sum_{A,B}(f_j^A f^B - f_j^B f^A)(f_l^B g^A - f_l^A g^B)\\
&+nH\xi_j\sum_{A,B}(f^A f_j^B - f^B f_j^A)(f_l^B g^A - f_l^A g^B)\Big\}\\
=&\,\xi_l\Big\{2(\langle e_j,\Delta_1\xi\rangle - 2h_{ik}h_{jk}\xi_i + nHh_{ij}\xi_i)\delta_{jl} + 4h_{ij}h_{ik}\xi_k\delta_{jl}\Big\}\\
=&\,2\langle\xi,\Delta_1\xi\rangle + 2nHh_{ij}\xi_i\xi_j,
\end{aligned}
$$

where we used

$$\sum_{A,B}\langle E_A, X\rangle\langle Y, E_B\rangle = \langle X, Y\rangle$$

for any X, Y. Applying (11) and (12) to (10) yields

$$
\begin{aligned}
\lambda_\alpha^J\int_M|\xi|^2 \le& -\int_M\Big(2(n-1)|\xi|^2 + \langle\xi,\Delta_1\xi\rangle + nHh_{ij}\xi_i\xi_j\Big)\\
\le& -2(n-1)\int_M|\xi|^2 + \lambda_{m(\alpha)}^{\Delta_1}\int_M|\xi|^2 + n|H|\max_M\sqrt{S}\int_M|\xi|^2
\end{aligned}
\tag{13}
$$

which shows that

$$\lambda_\alpha^J \le -2(n-1) + \lambda_{m(\alpha)}^{\Delta_1} + n|H|\max_M\sqrt{S}.$$

We complete the proof of Theorem 1. □

Proof. (of Theorem 2) From (5), we have

$$\big\langle\xi, D^*D(\xi)\big\rangle = \langle\xi,\Delta_1\xi\rangle + (n-1)|\xi|^2 + nHh_{ij}\xi_i\xi_j - h_{ik}h_{jk}\xi_i\xi_j. \tag{14}$$

Putting (14) into (13), one gets

$$
\begin{aligned}
\lambda_\alpha^J\int_M|\xi|^2 \le& -\int_M\Big(2(n-1)|\xi|^2 + \langle\xi,\Delta_1\xi\rangle + nHh_{ij}\xi_i\xi_j\Big)\\
=& -\int_M\Big((n-1)|\xi|^2 + \langle\xi, D^*D(\xi)\rangle + h_{ik}h_{jk}\xi_i\xi_j\Big)\\
\le& -\int_M\Big((n-1)|\xi|^2 + \langle\xi, D^*D(\xi)\rangle\Big)\\
\le& -(n-1)\int_M|\xi|^2 + \lambda_{m(\alpha)}^{D^*D}\int_M|\xi|^2,
\end{aligned}
$$

which gives

$$\lambda_\alpha^J \le -(n-1) + \lambda_{m(\alpha)}^{D^*D}.$$

Thus, the proof of Theorem 2 is completed. □

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Diophantine Approximation and special Liouville numbers

Johannes Schleischitz

Abstract. This paper introduces some methods to determine the simultaneous approximation constants of a class of well approximable numbers $\zeta_1, \zeta_2, \dots, \zeta_k$. The approach relies on results on the connection between the set of all s-adic expansions ($s \geq 2$) of $\zeta_1, \zeta_2, \dots, \zeta_k$ and their associated approximation constants. As an application, explicit construction of real numbers $\zeta_1, \zeta_2, \dots, \zeta_k$ with prescribed approximation properties are deduced and illustrated by Matlab plots.

1 Introduction

1.1 Basic facts and notations

This paper deals with the one parameter simultaneous approximation problem

$$\begin{aligned} |x| &\leq Q^{1+\theta} \\ |\zeta_1 x - y_1| &\leq Q^{-\frac{1}{k}+\theta} \\ &\vdots \\ |\zeta_k x - y_k| &\leq Q^{-\frac{1}{k}+\theta}, \end{aligned} \tag{1}$$

where $\zeta_1, \zeta_2, \dots, \zeta_k$ are real numbers which we will assume to be linearly independent together with 1 and $x, y_1, y_2, \dots, y_k$ are integers to be determined in dependence of the parameter $Q > 1$ in order to minimize θ. To be more precise, we define the function $\psi_j(Q)$ for $1 \leq j \leq k+1$ by setting $\psi_j(Q)$ the minimum over all $\theta \in \mathbb{R}$ such that there are j linearly independent vectors $(x, y_1, y_2, \dots, y_k) \in \mathbb{Z}^{k+1}$ that satisfy the system (1). In the sequel we will restrict to approximation vectors with $x > 0$, which clearly is no loss of generality as $(x, y_1, \dots, y_k) \mapsto (-x, -y_1, \dots, -y_k)$ does not affect approximation constants. Another equivalent way to view the functions ψ_j is to consider the lattice $\Lambda = \{(x, \zeta_1 x - y_1, \dots, \zeta_k x - y_k) : x, y_1, \dots, y_k \in \mathbb{Z}\}$

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and the convex body (in fact the parallelepiped) $K(Q)$ defined as the set of points $(z_1, z_2, \ldots, z_{k+1}) \in \mathbb{R}^{k+1}$ with

$$|z_1| \leq Q \tag{2}$$

$$|z_i| \leq Q^{-\frac{1}{k}}, \qquad 2 \leq i \leq k+1\,, \tag{3}$$

and to define $\lambda_j(Q)$ as the j-th *successive minimum* of Λ with respect to $K(Q)$. This j-th minimum is defined as the infimum over all $\lambda > 0$ for which the $\mathbb{R}$-span of $\lambda K(Q) \cap \Lambda$ has dimension at least j, or equivalently $\lambda K(Q)$ contains j linearly independent points of Λ.

With respect to these successive minima λ_j, the functions $\psi_j(Q)$ can also be determined by

$$Q^{\psi_j(Q)} = \lambda_j(Q)\,.$$

One has the inequalities

$$-1 \leq \psi_1(Q) \leq \psi_2(Q) \leq \cdots \leq \psi_{k+1}(Q) \leq \frac{1}{k} \tag{4}$$

as we will show later, and Dirichlet's Theorem states

$$\psi_1(Q) < 0 \qquad \text{for all} \quad Q > 1\,. \tag{5}$$

Minkowski's second convex body theorem yields for any convex body K with volume $V(K)$ and any lattice Λ

$$\frac{2^{k+1}}{(k+1)!}\frac{\det(\Lambda)}{V(K)} \leq \lambda_1\lambda_2\cdots\lambda_{k+1} \leq 2^{k+1}\frac{\det(\Lambda)}{V(K)}\,,$$

see [1], so that in our special case, as $V(K(Q)) = 1$ for every Q, we have

$$c_1(\Lambda) \leq \lambda_1(Q)\lambda_2(Q)\cdots\lambda_{k+1}(Q) \leq c_2(\Lambda)$$

uniformly in the parameter Q. With $q := \log(Q)$ and taking logarithms, this yields

$$q\left|\sum_{i=1}^{k+1}\psi_i(Q)\right| \leq C(\Lambda)\,, \tag{6}$$

with some constant $C(\Lambda)$ not depending on Q.

Another important property of the joint behaviour of the functions ψ_j is that for any given $1 \leq s \leq k$ there are arbitrarily large values $Q = Q(s)$ such that

$$\psi_s(Q) = \psi_{s+1}(Q) \tag{7}$$

provided that $1, \zeta_1, \zeta_2, \ldots, \zeta_k$ are linearly independent over $\mathbb{Q}$, see Theorem 1.1 in [5]. To quantify the behaviour of $\psi_j(Q)$ Summerer and Schmidt introduced the quantities

$$\underline{\psi}_i := \liminf_{Q\to\infty}\psi_i(Q)\,, \qquad \overline{\psi}_i := \limsup_{Q\to\infty}\psi_i(Q)\,,$$

and gave the estimates

$$\underline{\psi}_j \geq \frac{j-k-1}{kj}\,, \qquad 1 \leq j \leq k+1 \tag{8}$$

$$\overline{\psi}_j \geq \frac{j-k}{k(j+1)}\,, \qquad 1 \leq j \leq k\,, \tag{9}$$

where (9) requires $1, \zeta_1, \zeta_2, \ldots, \zeta_k$ to be linearly independent over $\mathbb{Q}$ again. Each of these bounds will be shown to be best possible in Corollary 3. Moreover, (7) implies

$$\underline{\psi}_{i+1} \leq \overline{\psi}_i\,, \qquad 1 \leq i \leq k\,. \tag{10}$$

In order to study the dynamical behaviour of the functions $\psi_j(Q)$ it will be convenient to work with functions

$$L_j(q) = q\psi_j(Q)$$

as these functions are piecewise linear with slopes among $\{-1, \frac{1}{k}\}$. Therefore we have (4). Definition (6) is equivalent to

$$\left|\sum_{i=1}^{k+1} L_i(q)\right| \leq C(\Lambda)\,. \tag{11}$$

We also introduce the classical approximation constants $\omega_j, \widehat{\omega}_j$ defined by Jarník, Bugeaud in addition to $\underline{\psi}_j, \overline{\psi}_j$. For fixed $\zeta_1, \zeta_2, \ldots, \zeta_k$ and for every $X > 0$ define the functions $\omega_j(X)$ as the supremum over all real numbers ν (in fact the maximum) such that the system

$$|x| \leq X\,, \qquad |\zeta_i x - y_i| \leq X^{-\nu}, \qquad 1 \leq i \leq k\,, \tag{12}$$

has j linearly independent solutions $(x, y_1, \ldots, y_k) \in \mathbb{Z}^{k+1}$. The approximation constants $\omega_j, \widehat{\omega}_j$ are now defined as

$$\omega_j = \limsup_{X\to\infty} \omega_j(X)\,, \qquad \widehat{\omega}_j = \liminf_{X\to\infty} \omega_j(X)\,.$$

We will put $\omega := \omega_1, \widehat{\omega} := \widehat{\omega}_1$ and denote by $\Omega = (\omega, \omega_2, \ldots, \omega_{k+1}, \widehat{\omega}, \ldots, \widehat{\omega}_{k+1}) \in \mathbb{R}^{2k+2}$ the vector of classical approximation constants (relative to $\zeta_1, \zeta_2, \ldots, \zeta_k$). Very similarly to the proof of Theorem 1.4 in [5], which treats the special case $j = 1$, one obtains

$$(1+\omega_j)(1+\underline{\psi}_j) = (1+\widehat{\omega}_j)(1+\overline{\psi}_j) = \frac{k+1}{k}\,, \qquad 1 \leq j \leq k+1\,. \tag{13}$$

One just needs to replace "a solution" by "j linearly independent solutions" at any place it occurs in the proof. Combining (13) with (8), (9) for $1, \zeta_1, \zeta_2, \ldots, \zeta_k$ linearly independent over $\mathbb{Q}$ we obtain the bounds

$$\frac{1}{k} \leq \omega \leq \infty\,, \tag{14}$$

$$\frac{1}{k} \leq \omega_2 \leq 1\,, \tag{15}$$

$$0 \leq \omega_j \leq \frac{1}{j-1}\,, \qquad 3 \leq j \leq k+1 \tag{16}$$

for the constants ω_j as well as

$$\frac{1}{k} \leq \widehat{\omega} \leq 1\,, \tag{17}$$

$$0 \leq \widehat{\omega}_j \leq \frac{1}{j}\,, \qquad 2 \leq j \leq k\,, \tag{18}$$

$$0 \leq \widehat{\omega}_{k+1} \leq \frac{1}{k} \tag{19}$$

for the constants $\widehat{\omega}_j$. Each considered individually, these bounds again are best possible.

1.2 Outline of the results

In the present paper, we will put our focus on simultaneous approximation of numbers that allow good individual as well as simultaneous approximation. Liouville numbers, that is real numbers ζ for which the inequality

$$\left|\zeta - \frac{p}{q}\right| \leq \frac{1}{q^\eta}$$

has infinitely many rational solutions $\frac{p}{q}$ for arbitrarily large $\eta \in \mathbb{R}$, will be suitable examples since they all satisfy $\omega = \infty$, where $\omega = \omega_1$ is defined by (12) in the one-dimensional case.

In section 2, Propositions 1, 2, we establish a connection between the s-adic expansions ($s \geq 2$) of the components ζ_j of $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ and the approximation constants $\omega, \widehat{\omega}$. These results are then applied to the case where all ζ_j admit good approximations in one fixed base s independent of j. After these considerations for suitable arbitrary $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ we put our focus on Liouville numbers, using heavily the fact that $\omega = \infty$ in this case. Theorem 1 will allow to compute all classical approximation constants $\omega_j, \widehat{\omega}_j$ for a special type of Liouville numbers and the resulting Corollary 2 will lead us to the construction of vectors $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ with prescribed approximation constants $\omega_j, \widehat{\omega}_j$ that are subject to certain restrictions. As consequences of these results we will be able to give an explicit example of a vector $\zeta_1, \zeta_2, \ldots, \zeta_k$ that shows a conjecture by Wolfgang Schmidt concerning successive minima of a lattice to be true. A non-constructive proof was given by Moshchevitin in a non-constructive way. Moreover we will construct cases where all functions ψ_j simultaneously take all possible values of their spectrum for arbitrarily large Q.

Inspired by methods used to deal with Liouville numbers, we then generalize Theorem 1 to a wider class of vectors $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ for which $\omega < \infty$. This will be the subject of Theorems 2,3 and lead to many more explicit constructions of special cases of the Schmidt Conjecture.

In the last section we will first discuss the special case where $\underline{\psi}_{j+1} = \overline{\psi}_j$ for $1 \leq j \leq k$ and give a constructive existence proof for the degenerate case $\underline{\psi}_1 = -1$ in arbitrary dimension. Throughout the paper we will illustrate the derived results by Matlab plots of the functions L_j for the special cases we consider to visualize derived results. These plots shall also lead to some insight into the dynamical behaviour of these functions in general. One should mention at this point that the plots often seem curved although the functions are piecewise linear, which is due to the non sufficient digital resolution, i.e. by zooming in one can see that they are indeed piecewise linear.

2 Results for Approximation constants

2.1 Estimates for $\omega, \widehat{\omega}$

In the sequel let $s \geq 2$ be an integer and $\zeta_i \in (0,1)$ for $1 \leq i \leq k$. For each $1 \leq i \leq k$ the non vanishing digits of the s-adic expansions of such ζ_i and $1-\zeta_i$ define two sequences $(a_n^{i,(s)})_{n\geq 1}$ and $(a_n^{\prime i,(s)})_{n\geq 1}$ by

$$\zeta_i = \sum_{n\geq 1} \alpha_{n,i}^{(s)} s^{-a_n^{i,(s)}}, \qquad a_1^{i,(s)} < a_2^{i,(s)} < \ldots, \qquad 0 < \alpha_{n,i} \leq s-1 \tag{20}$$

$$1-\zeta_i = \sum_{n\geq 1} \beta_{n,i}^{\prime(s)} s^{-a_n^{\prime i,(s)}}, \qquad a_1^{\prime(s)} < a_2^{\prime(s)} < \ldots, \qquad 0 < \beta_{n,i}^{\prime} \leq s-1\,. \tag{21}$$

We call the sequence $a_{n,i}^{\prime(s)}$ the *dual expansion* of ζ_i in base s. Set $(b_n^{(s)})_{n\geq 1}$ the monotonically ordered sequence of all $(a_{n,i}^{i,(s)})_{n\geq 1}$ and similarly $(b_n^{\prime(s)})_{n\geq 1}$ the monotonically ordered sequence of all $(a_n^{\prime i,(s)})_{n\geq 1}$. The following Theorem expresses the simultaneous approximation constant ω of $\zeta_1, \zeta_2, \ldots, \zeta_k$ in terms of the s-adic presentations of ζ_i $(s = 2,3,4,\ldots)$ by using these two ordered sequences. The proof is introductory to the rest of the work and for this purpose quite detailed.

Proposition 1. *We have*

$$\omega \geq \max\left\{ \limsup_{\|(s,n)\|_\infty\to\infty} \frac{b_{n+1}^{(s)} - b_n^{(s)} - 1}{b_n^{(s)}},\ \limsup_{\|(s,n)\|_\infty\to\infty} \frac{b_{n+1}^{\prime(s)} - b_n^{\prime(s)} - 1}{b_n^{\prime(s)}} \right\}, \tag{22}$$

and

$$\omega \leq \max\left\{ \limsup_{\|(s,n)\|_\infty\to\infty} \frac{b_{n+1}^{(s)} - b_n^{(s)}}{b_n^{(s)}},\ \limsup_{\|(s,n)\|_\infty\to\infty} \frac{b_{n+1}^{\prime(s)} - b_n^{\prime(s)}}{b_n^{\prime(s)}} \right\}. \tag{23}$$

where $\|(A,B)\|_\infty := \max\{|A|,|B|\}$ *(or any other norm since they are all equivalent in* $\mathbb{R}^2$*).*

Proof. We first prove (22). By definition of $(b_n^{(s)})_{n\geq 1}$ as the mixed ordered sequence of the sequences $(a_n)_{n\geq 1}$, all numbers $\zeta_1, \zeta_2, \ldots, \zeta_k$ will have zeros at the positions $b_n^{(s)}+1, b_n^{(s)}+2, \ldots, b_{n+1}^{(s)}-1$ behind the comma in base s for any $s \geq 2$. Since multiplication of ζ_j by $s^{b_n^{(s)}}$ only shifts the comma $b_n^{(s)}$ positions to the right, this means, for any $1 \leq j \leq k$ all $s^{b_n^{(s)}}\zeta_j$ start with $b_{n+1} - b_n - 1$ zeroes in base s behind the comma. For this reason any pair (s,n) satisfies

$$\|s^{b_n(s)}\zeta_j\| = \left|s^{b_n^{(s)}}\zeta_j - \lfloor s^{b_n^{(s)}}\zeta_j\rfloor\right| \leq s^{-(b_{n+1}^{(s)} - b_n^{(s)} - 1)}$$

for any $1 \leq j \leq k$, where $\|\cdot\|$ denotes the smallest distance of a real number to an integer. Analogously, for all $1-\zeta_j$ and all pairs (s,n) we have

$$\|s^{b_n^{\prime(s)}}\zeta_j\| = \|s^{b_n^{\prime(s)}}(1-\zeta_j)\| \leq |\lceil s^{b_n^{\prime(s)}}\zeta_j\rceil - s^{b_n^{\prime(s)}}\zeta_j| \leq s^{-(b_{n+1}^{\prime(s)}-b_n^{\prime(s)}-1)}.$$

We conclude that for any pair (s,n)

$$\max_{1\leq j\leq k}\|x\zeta_j\| \leq \max\left\{s^{-(b_{n+1}^{(s)}-b_n^{(s)}-1)}, s^{-(b_{n+1}^{\prime(s)}-b_n^{\prime(s)}-1)}\right\}, \tag{24}$$

with

$$x = s^{b_n^{(s)}} \quad \text{or} \quad x = s^{b_n^{\prime(s)}}.$$

Surely, $s^{b_n} \to \infty$ or $s^{b_n'} \to \infty$ is equivalent to $\|(s,n)\|_\infty \to \infty$, and we claim that (22) follows directly from the definition of the approximation constant ω. To see this we take a sequence of pairs (n,s) with $\|(s,n)\|_\infty \to \infty$, for which $\frac{b_{n+1}^{(s)}-b_n^{(s)}-1}{b_n^{(s)}}$ or $\frac{b_{n+1}^{\prime(s)}-b_n^{\prime(s)}-1}{b_n^{\prime(s)}}$ tend to the lim sup-values on the right hand side of (22). Putting $X_{\sigma(n,s)} := x_{\sigma(n,s)} := s^{b_n^{(s)}}$ or $X'_{\sigma(n,s)} := x_{\sigma(n,s)} := s^{b_n^{\prime(s)}}$, where σ is an arbitrary bijection $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, we obtain a sequence of X-values and x-values that leads via (24) to an approximation constant ω in (12) at least as large as both lim sup-values.

To prove (23), we first show the following assertion: It suffices to prove, that for any sufficiently large real parameter X there is a $s_0 = s_0(X)$, such that

$$\frac{b_2^{(s_0)}-b_1^{(s_0)}}{b_1^{(s_0)}} \geq \nu \quad \text{or} \quad \frac{b_2^{\prime(s_0)}-b_1^{\prime(s_0)}}{b_1^{\prime(s_0)}} \geq \nu, \tag{25}$$

where $\nu = \nu(s)$ is the largest exponent for which

$$\max_{1\leq j\leq k}\|\zeta_j s\| = s^{-\nu} \tag{26}$$

holds for all $s \leq X$.

For any sequence $(X_i)_{i\geq 1}$ let $(\nu_i)_{i\geq 1}$ be the largest exponent, for which (26) holds with ν_i in place of ν for all $s \leq X_i$. The existence of $s_0 = s_0(X)$ with (25) for any X implies the existence of a sequence $(\beta_i)_{i\geq 1}$ with $\beta_i \geq \nu_i$ for all i with β_i of the shape $\frac{b_2^{(s_0)}-b_1^{(s_0)}}{b_1^{(s_0)}}$ hence of the shape of the expressions involved in (23) in the case $n=1$. By definition of ω we may choose the sequence $(X_i)_{i\geq 1}$ such that $\lim_{i\to\infty}\nu_i = \limsup_{i\to\infty}\nu_i = \omega$. Furthermore we can assume without loss of generality that $(X_i)_{i\geq 1}$ satisfies $s_i = X_i$ for any i, as the exponent ν in the definition of ω in (12) for a fixed x decreases with growing X. Combining all these observations we get $\limsup_{i\to\infty}\beta_i \geq \omega$ where β_i fits in the lim sup term of (23) if we set $(s_i,n_i) = (s_i,1)$, where s_i plays the role of s_0 above, for $X = X_i$.

It remains to prove that for such sequences we have

$$\lim_{i\to\infty}\|(s_i,n_i)\|_\infty = \lim_{i\to\infty}\|(s_i,1)\|_\infty = \limsup_{i\to\infty} s_i = \infty.$$

This, however, is easy to see. As $s_i = X_i$ the definition of s_i guarantees that the number $s_i = s_i(X_i)$ minimizes $\max_{1\leq j\leq k}\|\zeta_j s_i\|$ among all $s_i \leq X_i$. On the other hand clearly $\liminf_{s\to\infty}\max_{1\leq j\leq k}\|\zeta_j\| = 0$ for any $\mathbb{Q}$-linearly independent $\zeta_1,\dots,\zeta_k$ and so by definition of $(s_i)_{i\geq 1}$ we also have $\lim_{i\to\infty}\max_{1\leq j\leq k}\|\zeta_j s_i\| = 0$. Consequently the sequence $(s_i)_{i\geq 1}$ cannot be bounded as only finitely many (strictly positive) values $\max_{1\leq j\leq k}\|\zeta_j s_i\|$ would appear, which proves

$$\limsup_{i\to\infty} s_i = \infty\,.$$

To complete the proof we have to find a value $s_0 = s_0(X)$ for which (25) holds. Note first, that for sufficiently large X and $s = s_0(X)$ we have $a_1^{j,(s)} = a_1^{\prime j,(s)} = 1$. Indeed for $s \geq \frac{1}{\min_i\|\zeta_i\|}$ and i_0 the index, for which the minimum is attained, we have $\{s\zeta_{i_0}\} \notin \{[0,\frac{1}{s}]\cup[\frac{s-1}{s},1)\}$, so the first digit after the comma in base s is neither 0 nor $(s-1)$. So we can assume X to be large enough to ensure $a_1^{j,(s)} = 1$ for all $1 \leq j \leq k$ and hence $b_1^{(s)} = 1$ as well. It is now easy to see that putting $s_0 := s$ is an appropriate choice, since (26) says that all $s\zeta_j$ respectively $s(1-\zeta_j)$ start with $\lfloor\nu\rfloor$ digits zero in base s behind the comma. This yields

$$\frac{a_2^{j,(s)} - a_1^{j,(s)}}{a_1^{j,(s)}} = a_2^{j,(s)} - 1 \geq \lfloor\nu\rfloor + 1 \geq \nu$$

for all $1 \leq j \leq k$, therefore

$$\frac{b_2^{(s)} - b_1^{(s)}}{b_1^{(s)}} = \frac{\min_j a_2^{j,(s)} - a_1^{j,(s)}}{a_1^{j,(s)}} \geq \nu\,,$$

respectively the same facts for $a'^{\cdot,(s)}_{\cdot}, b'^{(s)}_{\cdot}$. □

We easily deduce the following Corollary:

Corollary 1. *We have*

$$\omega \geq \max\left\{\sup_s \limsup_{n\geq 1}\frac{b_{n+1}^{(s)} - b_n^{(s)} - 1}{b_n^{(s)}}, \sup_s \limsup_{n\geq 1}\frac{b_{n+1}^{\prime(s)} - b_n^{\prime(s)} - 1}{b_n^{\prime(s)}}\right\}.$$

Similarly, we can give a lower bound for $\widehat{\omega}$ with respect to the s-adic representation of a real number.

Proposition 2. *For any $\zeta \in \mathbb{R}$ we have*

$$\widehat{\omega} \geq \max\left\{\sup_s \liminf_{n\geq 1}\max_{1\leq j\leq n}\frac{b_{j+1}^{(s)} - b_j^{(s)} - 1}{b_{n+1}^{(s)}}, \sup_s \liminf_{n\geq 1}\max_{1\leq j\leq n}\frac{b_{j+1}^{\prime(s)} - b_j^{\prime(s)} - 1}{b_{n+1}^{\prime(s)}}\right\}.$$

Proof. By definition of the supremum it is sufficient to prove

$$\widehat{\omega} \geq \mathscr{A}_s := \max\left\{\liminf_{n\geq 1}\max_{1\leq j\leq n}\frac{b_{j+1}^{(s)} - b_j^{(s)} - 1}{b_{n+1}^{(s)}}, \liminf_{n\geq 1}\max_{1\leq j\leq n}\frac{b_{j+1}^{\prime(s)} - b_j^{\prime(s)} - 1}{b_{n+1}^{\prime(s)}}\right\}$$

for any base s separately. So let s be fixed and put $b_n^{(s)}=b_n$. By definition of $\widehat{\omega}$ for arbitrary $\epsilon>0$ and sufficiently large $X=X(\epsilon)$ we have to find an approximation vector $(x,y_1,\ldots,y_k)\in\mathbb{Z}^{k+1}$ with $x\leq X$ and

$$\max_{1\leq j\leq k}|\zeta_j x-y_j|\leq X^{-\mathscr{A}_s+\epsilon}. \tag{27}$$

For $\epsilon>0$ and large X let n_0 be defined by $s^{b_{n_0}}\leq X<s^{b_{n_0+1}}$ respectively $s^{b'_{n_0}}\leq X<s^{b'_{n_0+1}}$. Put $x:=s^{b_j}$ respectively $x:=s^{b'_j}$ where j is the index, such that the inner maximum from the definition of $\mathscr{A}_s$ is attained for the given n_0. By definition of $(b_n)_{n\geq 1}$ as the mixed sequence the first $b_{j+1}-b_j-1$ positions behind the comma of each $\zeta_t s^{b_j}, 1\leq t\leq k$, respectively $(1-\zeta_t)s^{b_j}, 1\leq t\leq k$, are zeros in base s. We infer that putting $y_t:=\lfloor\zeta_t x\rfloor$ for all $1\leq t\leq k$ respectively $y_t:=\lceil\zeta_t x\rceil$ for all $1\leq t\leq k$ we have

$$\max_{1\leq t\leq k}\|\zeta_t x\|=\max_{1\leq t\leq k}|\zeta_t x-y_t|\leq s^{b_j-b_{j+1}+1}\leq X^{\frac{b_j-b_{j+1}+1}{b_{n_0+1}}} \tag{28}$$

$$\text{resp.}\quad \max_{1\leq t\leq k}\|\zeta_t x\|=\max_{1\leq t\leq k}|(1-\zeta_t)x-y_t|\leq s^{b'_j-b'_{j+1}+1}\leq X^{\frac{b'_j-b'_{j+1}+1}{b_{n'_0+1}}}. \tag{29}$$

For the left hand side inequalities compare the proof of Proposition 3, the right hand side inequalities follow from $X<s^{b_{n_0+1}}$ and $X<s^{b'_{n_0+1}}$ respectively. As (28), (29) holds for every large X, we may let n tend to ∞ to conclude that (27) has a solution for all sufficiently large X. Hence the exponent of X in (28) and (29) respectively is larger than $\mathscr{A}_s-\epsilon$. □

Now we turn to simultaneous approximation of vectors $(\zeta_1,\zeta_2,\ldots,\zeta_k)$ with good approximation in one fixed simultaneous base $s\geq 2$, and we skip the dual expansion. We want to use Corollary 1 and Proposition 2 to give estimates for the simultaneous approximation constants $\omega,\widehat{\omega}$. With respect to the notation above, meaning that the s-adic digits of ζ_i are given by $(a_{n,i}^{(s)})_{n\geq 1}$ as in (20) and the ordered mixed sequence by $(b_n^{(s)})_{n\geq 1}$, we get

Lemma 1. *For any $s\geq 2$ we have*

$$\min_i\liminf_{n\geq 1}\left(\frac{a_{n+1}^{i,(s)}}{a_n^{i,(s)}}\right)^{1/k}\leq\limsup_{n\geq 1}\frac{b_{n+1}^{(s)}}{b_n^{(s)}}\leq\min_i\limsup_{n\geq 1}\frac{a_{n+1}^{i,(s)}}{a_n^{i,(s)}}.$$

Proof. The right hand inequality is trivial. For the left hand inequality keep s fixed and put $C:=\min_i\liminf_{n\geq 1}\frac{a_{n+1}^{i,(s)}}{a_n^{i,(s)}}$ and choose n_0 large enough, such that for all i and all $n\geq n_0$ we have

$$\frac{a_{n+1}^i}{a_n^i}\geq C-\epsilon$$

(s has been dropped in the notation). For arbitrary $b_n, n\geq n_0$, there exist m,i_0 with $a_m^{i_0}=b_n$ by definition of $(b_n)_{n\geq 1}$. The interval $[a_m,(C-\epsilon)a_m]$ contains at most k numbers b_i, since it contains at most one element of every sequence $(a_n^i)_{n\geq 1}$ for $1 \leq i \leq k$. By the pigeon hole principle there are two numbers b_j, b_{j+1} in the interval $[a_m^{i_0}, a_{m+1}^{i_0}]$ whose quotient $\frac{b_{j+1}}{b_j}$ is at least $(C-\epsilon)^{1/k}$. The lemma follows with $\epsilon \to 0$. □

In combination with Corollary 1 we observe

$$\omega \geq \min_i \liminf_{n\geq 1} \left(\frac{a_{n+1}^{i,(s)}}{a_n^{i,(s)}} \right)^{1/k} - 1, \qquad \forall s \geq 2 .$$

Getting lower bounds for $\widehat{\omega}$ by just considering the sequences $a_{n+1}^{i,(s)}$ is more complicated and to some extent impossible as we will see in Corollary 3. In fact even if

$$\lim_{n\to\infty} \frac{a_{n+1}^{i,(s)}}{a_n^{i,(s)}} = \infty , \qquad 1 \leq i \leq k ,$$

we can have $\widehat{\omega} = 1/k$, which is the weakest lower bound for $\widehat{\omega}$ by (17). We only mention that if we construct sequences $a_{n+1}^{i,(s)}$ for which $\lim_{n\to\infty} \frac{b_{n+1}^{(s)}}{b_n^{(s)}} = \infty$, Proposition 2 yields

$$\widehat{\omega} \geq \liminf_{n\geq 1} \frac{b_{n+1}^{(s)} - b_n^{(s)}}{b_{n+1}^{(s)}} = 1 - \frac{1}{\liminf_{n\geq 1} \frac{b_{n+1}^{(s)}}{b_n^{(s)}}} = 1 ,$$

and consequently $\widehat{\omega} = 1$ in view of (17).

2.2 The case $\omega = \infty$

In the following theorem, we compute the classical approximation constants $\omega_j, \widehat{\omega}_j$ for a special type of Liouville numbers $\zeta_1, \zeta_2, \ldots, \zeta_k$, whose best approximation vectors $(x, y_1, y_2, \ldots, y_k)$ to $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ are easy to guess. The main arguments of the compilation will be carried out in the proofs of the following theorems.

Theorem 1. *Let k be a positive integer and for $1 \leq j \leq k$ let $\zeta_j = \sum_{n\geq 1} \frac{1}{q_{n,j}}$, where*

$$q_{1,1} < q_{1,2} < \cdots < q_{1,k} < q_{2,1} < q_{2,2} < \cdots < q_{2,k} < q_{3,1} < \cdots \tag{30}$$

are natural numbers, such that

$$q_{n,j} \mid q_{n,j+1} \quad \text{for} \quad 1 \leq j \leq k-1 \quad \text{and} \quad q_{n,k} \mid q_{n+1,1} \quad \text{for all} \quad n \geq 1 \tag{31}$$

and such that

$$\lim_{n\to\infty} \frac{\log(q_{n+1,1}) - \log(q_{n,k})}{\log(q_{n+1,k})} = \eta_1 , \tag{32}$$

$$\lim_{n\to\infty} \frac{\log(q_{n+1,i}) - \log(q_{n+1,i-1})}{\log(q_{n+1,k})} = \eta_i , \quad 2 \leq i \leq k , \tag{33}$$

$$\lim_{n\to\infty} \frac{\log(q_{n+1,1})}{\log(q_{n,k})} = \eta_{k+1} = \infty , \tag{34}$$

where $\eta = (\eta_1, \eta_2, \cdots, \eta_{k+1}) \in \mathbb{R}^k \times \overline{\mathbb{R}}$ *satisfy*

$$\eta_1 + \eta_2 + \cdots + \eta_k = 1 \tag{35}$$

$$\eta_{k+1} > \eta_k \geq \eta_{k-1} \geq \ldots \geq \eta_1 > 0 \tag{36}$$

$$\eta_{k+1} = \infty\,. \tag{37}$$

Then the classical approximation constants relative to the vector $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_k)$ *are given by*

$$\begin{aligned}
\omega_1 &= \eta_{k+1} = \infty =: \wp_1(\eta)\\
\omega_2 &= \max\left\{\frac{\eta_k}{\eta_k + \eta_{k-1} + \cdots + \eta_1}, \frac{\eta_{k-1}}{\eta_{k-1} + \eta_{k-2} + \cdots + \eta_1}, \ldots, \frac{\eta_1}{\eta_1}\right\} =: \wp_2(\eta)\\
\omega_3 &= \max\left\{\frac{\eta_{k-1}}{\eta_k + \eta_{k-1} + \cdots + \eta_1}, \frac{\eta_{k-2}}{\eta_{k-1} + \eta_{k-2} + \cdots + \eta_1}, \ldots, \frac{\eta_1}{\eta_2 + \eta_1}\right\} =: \wp_3(\eta)\\
&\vdots
\end{aligned}$$

$$\omega_{k+1} = \frac{\eta_1}{\eta_k + \eta_{k-1} + \cdots + \eta_1} =: \wp_{k+1}(\eta).$$

and

$$\begin{aligned}
\widehat{\omega}_1 &= \min\left\{\frac{\eta_k}{\eta_k + \eta_{k-1} + \cdots + \eta_1}, \frac{\eta_{k-1}}{\eta_{k-1} + \eta_{k-2} + \cdots + \eta_1}, \ldots, \frac{\eta_1}{\eta_1}\right\} =: \widehat{\wp}_1(\eta)\\
\widehat{\omega}_j &= 0, \qquad 2 \leq j \leq k+1.
\end{aligned}$$

Proof. We start with the constants ω_j and intend to prove the inequalities $\omega_j \geq \wp_j(\eta)$ and $\omega_j \leq \wp_j(\eta)$ separately for $1 \leq j \leq k+1$.
(1) $\omega_j \geq \wp_j(\eta)$:
Let $\wp_{j,l}$ be the l-th quotient of the maximum labelled $\wp_j(\eta)$. We give a detailed proof of $\omega_j \geq \wp_{j,1} = \eta_{k+2-j}$ and then mention how to generalize the proof to derive all the other inequalities $\omega_j \geq \wp_{j,l}$ for $l \neq 1$.

To prove $\omega_j \geq \wp_{j,1}$, we will construct j sequences of approximation vectors

$$\big(x^{(1,i)}, y_1^{(1,i)}, \ldots, y_k^{(1,i)}\big)_{i\geq 1}, \big(x^{(2,i)}, y_1^{(2,i)}, \ldots y_k^{(2,i)}\big)_{i\geq 1}, \ldots, \big(x^{(j,i)}, y_1^{(j,i)}, \ldots, y_k^{(j,i)}\big)_{i\geq 1}$$

which are linearly independent for each fixed $i \in \mathbb{N}$ and such that $\omega_j = \eta_{k+2-j}$ follows for $i \to \infty$. Indeed for p in a j-element subset of $\{1, 2, \ldots, k\}$ and any $\epsilon > 0$ we claim for i sufficiently large

$$\max_{1\leq t\leq k} -\frac{\log(|\zeta_t x^{(p,i)} - y_t^{(p,i)}|)}{\log(x^{(p,i)})} \geq \eta_{k+2-j} - \epsilon\,.$$

In analogy to the definition of $(b_n^s)_{n\geq 1}$ in subsection 2.1 let $(b_n)_{n\geq 1}$ be the combined sequence of the logarithms of the integers $q_{n,j}$ in increasing order, which means for any nonnegative integer M and $N \in \{1, 2, \ldots, k\}$ we have $b_{kM+N} = \log(q_{M,N})$. By (34) we have $\limsup \frac{b_{n+1}}{b_n} = \infty$ and thus by putting the first approximation vector $(q_{n,i}, \lfloor \zeta_1 q_{n,i} \rfloor, \ldots, \lfloor \zeta_k q_{n,i} \rfloor)$ with arbitrary i we may let n tend to infinity, to obtain $\omega = \infty$: indeed applying (31) we derive that all the remainder terms

$$\|\zeta_j q_{n,i}\| = \sum_{l: q_{n,l} > q_{n,i}} \frac{1}{q_{l,j}} q_{n,i} \leq 2 \frac{q_{n,i}}{q_{n,i+1}}$$

are small due to (32)–(34). In order to estimate ω_j for $j \geq 2$ we construct a sequence of parameters X and approximation vectors $(x, y_1, \ldots, y_k)$ with $x \leq X$ explicitly. For the fixed choice $X^{(n)} := q_{n,k}$ we will get $\omega_j \geq \wp_{j,1}$. To see this let $x^{(1,n)} := X(n)$ and $y_t^{(1,n)} := \lfloor x^{(1,n)} \zeta_t \rfloor$ for $1 \leq t \leq k$. Define the second approximation vector by taking $x^{(2,n)} = q_{n,k-1}$ and again $y_t^{(2,n)} := \lfloor x^{(2,n)} \zeta_t \rfloor$. By means of (32), (33), (34) and the definition of $\wp_2(\eta)$ we claim that for each $C < \eta_{k+2-2} = \eta_k$

$$\left| \zeta_t x^{(2,n)} - y_t^{(2,n)} \right| \leq (x^{(1,n)})^{-C} = (X^{(n)})^{-C} \tag{38}$$

holds for $n = n(C)$ large enough. This follows from

$$\left| \zeta_t x^{(2,n)} - y_t^{(2,n)} \right| = \left| q_{n,k-1} \zeta_t - \lfloor q_{n,k-1} \zeta_t \rfloor \right| = \sum_{i=n+1}^{\infty} \frac{q_{n,k-1}}{q_{i,t}}, \quad 1 \leq t \leq k-1,$$

$$\left| \zeta_k x^{(2,n)} - y_k^{(2,n)} \right| = \left| q_{n,k-1} \zeta_k - \lfloor q_{n,k-1} \zeta_k \rfloor \right| = \frac{q_{n,k-1}}{q_{n,k}} + \sum_{i=n+1}^{\infty} \frac{q_{n,k-1}}{q_{i,k}}$$

in view of the definition of ζ_t and our assumption (31). In every case all the values of $\zeta_t x^{(2,n)} - y_t^{(2,n)}$ for $1 \leq t \leq k$ are bounded by $\frac{q_{n,k-1}}{q_{n,k}}(1 + o(1))$. Using (32), (33), (34) this leads to (38).

Similarly, defining the j-th approximation vector for $2 \leq j \leq k$ by $x^{(j,n)} = q_{n,k+1-j}$ and for $j = k+1$ by $x^{(k+1,n)} = q_{n-1,k}$ and then putting $y_t^{(j,n)} := \lfloor x^{(j,n)} \zeta_t \rfloor$ yields the corresponding inequalities.

We now check that these vectors are linearly independent as required. To do this we prove that all the matrices $\boldsymbol{B}_n = \left(\boldsymbol{B}_n(i,j)\right)_{1 \leq i,j \leq k+1}$ obtained by writing the h-th approximation vector $(x^{(h,n)}, y_1^{(h,n)}, \ldots, y_k^{(h,n)})$ in the h-th row, i.e.

$$\boldsymbol{B}_n = \begin{pmatrix} q_{n,k} & q_{n,k} \sum\limits_{i=1}^{n} q_{i,1}^{-1} & q_{n,k} \sum\limits_{i=1}^{n} q_{n,2}^{-1} & \cdots & q_{n,k} \sum\limits_{i=1}^{n} q_{i,k}^{-1} \\ q_{n,k-1} & q_{n,k-1} \sum\limits_{i=1}^{n} q_{i,1}^{-1} & q_{n,k-1} \sum\limits_{i=1}^{n} q_{n,2}^{-1} & \cdots & q_{n,k-1} \sum\limits_{i=1}^{n-1} q_{i,k}^{-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{n,1} & q_{n,1} \sum\limits_{i=1}^{n} q_{i,1}^{-1} & q_{n,1} \sum\limits_{i=1}^{n-1} q_{n,k-1}^{-1} & \cdots & q_{n,1} \sum\limits_{i=1}^{n-1} q_{i,k}^{-1} \\ q_{n-1,k} & q_{n-1,k} \sum\limits_{j=1}^{n-1} q_{j-1,1}^{-1} & q_{n-1,k} \sum\limits_{j=1}^{n-1} q_{j-1,2}^{-1} & \cdots & q_{n-1,k} \sum\limits_{j=1}^{n-1} q_{j-1,k}^{-1} \end{pmatrix}$$

are nonsingular. Observe that if we subtract $\frac{\boldsymbol{B}_n(h,1)}{\boldsymbol{B}_n(h+1,1)}$ times the $(h+1)$-th row from the h-th row of the matrix $\boldsymbol{B}_n$, all entries in the new h-th line will be zero apart from a one in position $(h, k+2-h)$. Starting with this process at $h = 1$ and repeating it until $h = k$ we end up with the matrix

$$\boldsymbol{C}_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & 0 & \dots & 0 \\ q_{n-1,k} & q_{n-1,k} \sum_{j=1}^{n-1} q_{j-1,1}^{-1} & q_{n-1,k} \sum_{j=1}^{n-1} q_{j-1,2}^{-1} & \cdots & q_{n-1,k} \sum_{j=1}^{n-1} q_{j-1,k}^{-1} \end{pmatrix},$$

which is easily seen to have absolute value of the determinant equal to $q_{n-1,k} \neq 0$. Therefore also $\det(\boldsymbol{B}_n) = q_{n-1,k} \neq 0$, as required.

To obtain all the other inequalities $\omega_j \geq \wp_{j,i}$ for $2 \leq i$, where the upper bound of i depends on j, we proceed analogously. In the definition of $x^{(1,n)}$ we replace $q_{n,k}$ by $q_{n,k+1-i}$ and again for $1 \leq t \leq k$ we define $y_t^{(1,n)} = \lfloor x^{(1,n)} \zeta_t \rfloor$ for the first approximation vector. We define all the others by taking $x^{(2,t)} = q_{n,k+1-2}$, $x^{(3,n)} = q_{k+1-3,n}, \dots$ and again $y_t^{(i,n)} = \lfloor x^{(i,n)} \zeta_t \rfloor$ for $1 \leq t \leq k$ and $2 \leq i \leq k+1$. This construction yields the desired lower bounds (or 0 which is omitted in $\wp(\eta)$) as above again by (32), (33), (34).

$\omega_j \leq \wp_j(\eta)$:
We have to show that for $1 \leq j \leq k+1$ the approximation vectors

$$\left(x^{(j,n)}, y_1^{(j,n)}, \dots, y_k^{(j,n)}\right)$$

constructed in the first step of the proof are somehow best possible. We split the proof of this assertion in 3 steps. To simplify notation let $(c_n)_{n\geq 1} = (e^{b_n})_{n \geq 1}$ be the ordered mixed sequence $(q_{1,1}, q_{1,2}, \dots, q_{1,k}, q_{2,1}, \dots)$.

First step: For an arbitrary approximation vector $(x, y_1, \dots, y_k)$ let h be the index determined by $c_h \leq x < c_{h+1}$ and let g be the largest integer such that the index $g-1$ satisfies $c_{g-1} \mid x$. Since $x < c_{h+1}$ and consequently $c_{h+1} \nmid x$ we clearly have $g \leq h+1$. When $X \to \infty$ so does h and we claim that for $h \to \infty$

$$\max_{1 \leq t \leq k} |\zeta_t x - y_t| \geq \frac{c_{g-1}}{c_g} - o\left(\frac{c_{g-1}}{c_g}\right), \qquad g > h+1-k \tag{39}$$

$$\max_{1 \leq t \leq k} |\zeta_t x - y_t| \geq \frac{1}{c_{h+2-k}} - o\left(\frac{1}{c_{h+2-k}}\right), \qquad g \leq h+1-k\,. \tag{40}$$

Furthermore in the case $g \leq h+1-k$ (i.e. the assumption of (40)), the inequality $x < \frac{1}{2} c_{h+1} c_{h+1-k}^{-1}$ contradicts that

$$\max_{1 \leq t \leq k} |\zeta_t x - y_t| < \frac{1}{2} \frac{1}{c_{h+1-k}} - o\left(\frac{1}{c_{h+1-k}}\right) \tag{41}$$

holds for $h \to \infty$.

Second step: Let X be a real parameter from the definition of the approximation constants ω_j and $m = m(X)$ be the index such that $c_m \leq X < \frac{c_{m+1}}{4}$. Then for

$1 \leq j \leq k+1$ a set of j vectors $(x^{(i)}, y_1^{(i)}, \ldots, y_k^{(i)}), 1 \leq i \leq j$, satisfying the inequalities

$$x^{(i)} \leq X\,, \qquad 1 \leq i \leq j\,, \tag{42}$$

$$\max_{1\leq t\leq k} |x^{(i)}\zeta_t - y_t^{(i)}| \leq \frac{1}{2}\,, \qquad 1 \leq i \leq j\,, \tag{43}$$

can only be linearly independent if at least one $x^{(i)}$ is not divisible by c_{m+2-j}.

Third step: We intend to show by combining the first two steps and using (32)–(34), that for arbitrary X the choice of approximation vectors in the proof of $\omega_j \geq \wp_j(\eta)$ is somehow optimal, i.e. the approximation constants of this case cannot be improved.

Proof of first step: We first make the assumption $c_{h+1} \in (q_{n,k})_{n\geq 1}$, and will explain at the end how to extend this easily to the case where c_{h+1} belongs to another sequence. This assumption is equivalent to $c_{h+1} = q_{n_1,k}$ for some $n_1 \in \mathbb{N}$ and it follows that $c_h = q_{n_1,k-1}$. By (31) we have $c_l \mid x$ for all $l \leq g-1$ and $c_l \nmid x$ for all $l \geq g$, in particular $c_g \nmid x$. Recall $g \leq h+1$. To prove the assertions we now consider the corresponding cases separately:

Case 1: $c_g > q_{n_1-1,k}$. Note that since $q_{n_1-1,k} = c_{h+1-k}$ this is equivalent to $g \geq h+2-k$ or $c_g \geq c_{h+2-k}$. We can write $x = x_1 + x_2$ with $0 < x_1 < c_g$ and $c_g \mid x_2$, since by our definition of g we have $x_1 \neq 0$. Denote by $\overline{g}$ the congruence class of g in the residue system $\{1,2,\ldots,k\}$ mod k. Note, that $c_{\overline{g}}$ is the smallest value c_g with g in the residue class $\overline{g}$, or equivalently $\zeta_{\overline{g}} = \sum_{l\geq 0} c_{\overline{g}+kl}^{-1}$, which we will make use of. We claim that

$$\|x_1\zeta_{\overline{g}}\| \geq c_g^{-1}c_{g-1} - \sum_{l\geq 1} c_{g+lk}^{-1}c_{g-1} \geq c_g^{-1}c_{g-1} - 2c_{h+1}c_{h+2}^{-1} \tag{44}$$

$$\{x_2\zeta_{\overline{g}}\} = \|x_2\zeta_{\overline{g}}\| = \Big\|x_2 \sum_{l\geq 1} c_{g+kl}^{-1}\Big\| \leq 2c_{h+1}c_{h+2}^{-1}, \tag{45}$$

where $\|\cdot\|$ denotes the closest distance to an integer an $\{\cdot\}$ the fractional part of a real number. Inequality (44) relies on the fact that $0 < \frac{x_1}{c_g} < 1$ and $c_{g-1} \mid x_1$, which is seen to be true because $c_{g-1} \mid x$, $c_{g-1} \mid c_g$ and $c_g \mid x_2$ by definition, so putting these together we get $c_{g-1} \mid x - x_2$, but $x - x_2 = x_1$. Combination of these two facts and recalling that exactly $g+k, g+2k, \ldots$ are the indices greater than g belonging to the residue class $\overline{g}$ shows that $x_1\zeta_{\overline{g}}$ is of the form

$$x_1\zeta_{\overline{g}} = \frac{x_1}{c_g} + x_1\sum_{l\geq 1} c_{g+lk}^{-1} = \frac{Kc_{g-1}}{c_g} + x_1\sum_{l\geq 1} c_{g+lk}^{-1} \geq K\frac{c_{g-1}}{c_g} + x_1\sum_{l\geq 1} c_{g+lk}^{-1}$$

with $K \in \{1,2,\ldots,\frac{c_g}{c_{g-1}} - 1\}$ (note $c_{g-1} \mid c_g$). The assertion now follows by a combination of $x_1 < c_g \leq c_{h+1}$,

$$\sum_{l\geq 1} c_{g+lk}^{-1} < c_{g+k}^{-1}\Big(1 + \frac{1}{2} + \frac{1}{4} + \cdots\Big) = 2c_{g+k}^{-1}, \tag{46}$$

and $c_{g+k}^{-1} \leq c_{h+2}^{-1}$, which is true by the assumption of case 1.

Inequality (45) follows from the fact that for any $s \leq g$ by virtue of (31) we have $c_s \mid x_2$ which holds in particular for those c_s with s in the residue class $\overline{g}$. So all quantities $x_2c_s^{-1}$ with $s \leq g$ are integers. Thus the sum of quantities of order smaller than $x_2c_{g+k}^{-1}$ in $x_2\zeta_{\overline{g}} = \sum_{l\geq 0} x_2c_{g+kl}^{-1}$, i.e. $x_2\sum_{l\geq 1} c_{g+kl}^{-1}$, has the same fractional part as the entire sum. Now on the one hand we have $x_2 \leq x \leq c_{h+1}$, and on the other hand $c_{g+k} \geq c_{h+2}$ by the assumption of case 1. Together with (46) these assertions yield (45).

Summing (44) and (45) and noting that by (34) we have $c_{h+1}c_{h+2}^{-1} = o(c_g^{-1}c_{g-1})$, so that we can use the triangle inequality on the fractional parts, we further have

$$\|x\zeta_{\overline{g}}\| \geq c_g^{-1}c_{g-1} - 4c_{h+1}c_{h+2}^{-1}.$$

By (36), (37) and (34) the expression $c_m^{-1}c_{m-1}$ is monotonically decreasing in m and the error term $4c_{h+1}c_{h+2}^{-1}$ is obviously $o(c_hc_{h+1}^{-1})$ by (34). Hence for $h \to \infty$ we obtain

$$\frac{c_{g-1}}{c_g} - o\left(\frac{c_h}{c_{h+1}}\right) \leq \|\zeta_{\overline{g}}x\| \leq \max_{1\leq t\leq k} \|\zeta_t x\|.$$

This establishes (39) in this case as $\frac{c_h}{c_{h+1}} \leq \frac{c_{g-1}}{c_g}$ by (36) and (32)–(34).

If c_{h+1} belongs to another sequence $(q_{n,i})_{n\geq 1}$, which means $c_{h+1} = q_{n_1,i}$ with $i \neq 1$, we look at the case $c_g \geq q_{n_1-1,i}$ and again obtain

$$\|x_1\zeta_{\overline{g}}\| \geq c_g^{-1}c_{g-1} - \sum_{l\geq 1} c_{g+lk}^{-1}c_{g-1} \geq c_g^{-1}c_{g-1} - 2c_{h+1}c_{g+k}^{-1}$$
$$\{x_2\zeta_{\overline{g}}\} = \|x_2\zeta_{\overline{g}}\| = \Big\|x_2\sum_{l\geq 1} c_{g+kl}^{-1}\Big\| \leq 2c_{h+1}c_{g+k}^{-1},$$

as in the proof of the special case (without using $c_{h+2} \leq c_{g+k}$ as above from which we derived the weaker but sufficient conditions (44), (45)). However, by (32)–(34) we again have $c_{h+1}c_{g+k}^{-1} = o(c_g^{-1})$ as $h \to \infty$ (or equivalently $g \to \infty$ as $g \geq h+2-k$) and the rest of the argumentation is almost as above. Thus (39) holds in any case.

Case 2: $c_g \leq q_{n_1-1,k}$. In this case it is more convenient to work directly with the values $q_{.,.}$ instead of $c_.$. As in case 1 let $x = x_1 + x_2$ with $0 < x_1 < q_{n_1,1}$ and $q_{n_1,1} \mid x_2$. Again $q_{n_1-1,k} \mid q_{n_1,1}$ and the definition of g ensures $x_1 \neq 0$. Analogously to the proof of (44), (45) in case 1 we deduce

$$\|x_1\zeta_1\| \geq \frac{1}{q_{n_1,1}} - 2q_{n_1,k}q_{n_1+1,1}^{-1}$$
$$0 \leq x_2\zeta_1 \leq 2q_{n_1,k}q_{n_1+1,1}^{-1}.$$

Using again the triangle inequality and (37), we again deduce

$$\|x\zeta_1\| \geq \frac{1}{q_{n_1,1}} - 4\frac{q_{n_1,k}}{q_{n_1+1,1}} = \frac{1}{q_{n_1,1}} - 4\frac{c_{h+1}}{c_{h+2}}\,.$$

But by (32)–(34) again $\frac{c_{h+1}}{c_{h+2}} = o(c_{h+2-k}^{-1}) = o(c_{n_1,1}^{-1})$ for $h \to \infty$ so that finally

$$\frac{1}{q_{n_1,k}} - o(c_{h+2-k}^{-1}) \leq \|\zeta_1 x\| \leq \max_{1\leq t\leq k} \|\zeta_t x\|\,.$$

But $q_{n_1,1} = c_{h+2-k}$, so we have (40) in this case. If c_{h+1} belongs to another sequence $(q_{n,i})_{n\geq 1}$, $i \neq k$, we can apply very similar estimates with respect to $\zeta_{\overline{i+1}} = \zeta_{i+1}$ instead of ζ_1. So our assumption is no loss of generality in this case either. Thus (40) holds in any case.

We still have to prove that $x < \frac{1}{2}c_{h+1}c_{h+1-k}^{-1}$ contradicts (41). For simplicity we again discuss the case $c_{h+1} \in (q_{n,k})_{n\geq 1}$ first. Write $x = x_1 + x_2$ with $0 < x_1 < q_{n_1-1,k}$ and $q_{n_1-1,k} \mid x_2$. Note that again we have $x_1 \neq 0$ by the assumption $g \leq h+1-k$, so $c_g \leq c_{h+1-k} = q_{n_1-1,k}$, and the definition of g. Assume we have $x < \frac{1}{2}c_{h+1}c_{h+1-k}^{-1} = \frac{1}{2}q_{n_1,k}q_{n_1-1,k}^{-1}$. The fractional part of $x_2\zeta_k$ is $\sum_{l\leq 0} x_2 c_{h+1+lk}^{-1}$ as higher order summands are integers by definition of x_2. We split this expression in

$$\{x_2\zeta_k\} = \|x_2\zeta_k\| = \sum_{l\geq 0} x_2 c_{h+1+kl}^{-1} = x_2 c_{h+1}^{-1} + \sum_{l\geq 1} x_2 c_{h+1+kl}^{-1}$$

and using $x_2 \leq x$ we infer

$$\|x_2\zeta_k\| \leq \frac{1}{2}c_{h+1-k}^{-1} + \sum_{l\geq 1} c_{h+1}c_{h+1-k}^{-1}c_{h+1+kl}^{-1}\,, \tag{47}$$

which is obviously $\frac{1}{2}c_{h+1-k}^{-1} + o(c_{h+1-k}^{-1})$ as $h \to \infty$ by (34).

On the other hand, by definition of x_1 and $c_{h+1-k} = q_{n_1-1,k} \nmid x_1$ as $g \leq h+1-k$ by assumption and a very similar argument as in case 1 we have

$$\left\| x_1\left(\frac{1}{q_{1,k}} + \frac{1}{q_{2,k}} + \cdots + \frac{1}{q_{n_1-1,k}}\right)\right\| \geq \frac{1}{q_{n_1-1,k}}\,.$$

On the other hand by $x_1 < q_{n_1-1,k}$ the sum of the remainder terms of $\zeta_k x_1$, i.e. $x_1 \sum_{l\geq 0} \frac{1}{q_{n_1+l,k}}$, is bounded above by $2\frac{q_{n_1-1,k}}{q_{n_1,k}}$ with very similar estimates as in (46). So

$$\|x_1\zeta_k\| \geq \frac{1}{q_{n_1-1,k}} - 2\frac{q_{n_1-1,k}}{q_{n_1,k}} \tag{48}$$

by a very similar argument as in case 1. As $n_1 \to \infty$, we have

$$\frac{q_{n_1-1,k}}{q_{n_1,k}} = o\left(\frac{1}{q_{n_1-1,k}}\right) = c_{h+1-k}^{-1} - o(c_{h+1-k}^{-1})$$

(note that $h \to \infty$ if $n_1 \to \infty$) and thus by (48)

$$\|x_1\zeta_k\| \geq c_{h+1-k}^{-1} - o(c_{h+1-k}^{-1}) \tag{49}$$

Using triangular inequality on (47), (49) thus gives

$$\max_{1\leq t\leq k} \|\zeta_t x\| \geq \|\zeta_k x\| \geq \frac{1}{2}c_{h+1-k}^{-1} - o(c_{h+1-k}^{-1}).$$

Our last assertion is proved in this case and the assumption $c_h \in (q_{n,k})_{n\geq 1}$ can obviously be dropped again.

Proof of second step: Without loss of generality assume that $c_m \in (q_{n,k})_{n\geq 1}$, the proof for the other cases is essentially the same. This means $c_m = q_{m_1,k}$ for some $m_1 \in \mathbb{N}$ and consequently $c_{m+1} = q_{m_1+1,1}$. Suppose c_{m-j+2} divides $x^{(i)}$ for all $1 \leq i \leq j$. On the one hand, by our assumption the $(j-1)$ numbers $c_{m-j+2}, c_{m-j+3}, \ldots, c_m$ belong to the sequences $q_{n,k}, q_{n,k-1}, \ldots, q_{n,k-j+2}$. On the other hand, $c_u \mid c_{u+1}$ for all $u \geq 1$ combined with $c_{m-j+2} \mid x^{(i)}$ for all $1 \leq i \leq j$ implies that for all $s \leq m-j+2$ the number c_s divides $x^{(i)}$. From these two facts we conclude that for $g \notin \{k, k-1, \ldots, k+2-j\}$, i.e. $g \in G := \{1, 2, \ldots, k+1-j\}$, the partial sum $x^{(i)} \sum_{r=1}^{m_1} \frac{1}{q_{r,g}}$ of $\zeta_g x^{(i)}$ is an integer, since every summand $\frac{x^{(i)}}{q_{r,g}}$ is. As terms of order lower than m_1 in $\zeta_g x^{(i)}$ for $g \in G \setminus \{1\}$ obviously add up to a quantity smaller than $\frac{1}{2}$, for $1 \leq i \leq j$ and $g \in G \setminus \{1\}$ we have

$$\|\zeta_g x^{(i)}\| = \zeta_g x^{(i)} - \lfloor \zeta_g x^{(i)} \rfloor = \sum_{r=m_1+1}^{\infty} \frac{1}{q_{r,g}} < \frac{1}{2}\,, \tag{50}$$

and combined with (43) eventually

$$y_g^{(i)} = \lfloor \zeta_g x^{(i)} \rfloor = x^{(i)} \sum_{r=1}^{m_1} \frac{1}{q_{r,g}}\,. \tag{51}$$

In view of our assumption $X < \frac{c_{m+1}}{4} = \frac{q_{m_1+1,1}}{4}$ the results (50), (51) are also valid for $g = 1$. To sum up, for all $g \in G$ we have (51), which obviously yields

$$\frac{x^{(a)}}{x^{(b)}} = \frac{y_g^{(a)}}{y_g^{(b)}}, \qquad g \in G, \quad 1 \leq a, b \leq j\,.$$

Thus in the matrix, whose i-th row is the i-th approximation vector

$$(x^{(i)}, y_1^{(i)}, \ldots, y_k^{(i)}) \in \mathbb{Z}^{k+1}, \quad 1 \leq i \leq j\,,$$

the first $|G| = k - j + 3$ columns together have rank 1. The rank of the whole matrix therefore cannot exceed $1 + [(k+1) - (k-j+3)] = j - 1 < j$. This means the j rows are linearly dependent, a contradiction. So c_{m-j+2} cannot divide all the numbers $x^{(i)}$, as stated.

Proof of third step: We will prove for arbitrary j, that $\omega_j(X)$ is for $X \to \infty$ asymptotically bounded above by one of the fractions (depending on $\log(X)$) involved in the definition of $\wp_j(\eta)$, by which we mean that for any $\epsilon > 0$ and $X = X(\epsilon)$ large enough we have $\omega_j(X) < \wp_j(\eta) + \epsilon$. Since $\omega_j = \limsup_{X\to\infty} \omega_j(X)$, $\epsilon \to 0$ shows the required result.

So let X be arbitrary but fixed and let h be the index determined by $c_h \leq X < c_{h+1}$. We first prove that without loss of generality we may restrict to the case where X lies an interval of the shape $[c_h, \frac{c_{h+1}}{4})$.

This is the case because the logarithm to the base $X = \frac{c_{h+1}}{4}$ of

$$D_{\boldsymbol{x}} := \max_{1\leq t\leq k} |\zeta_t x - y_t|, \qquad \boldsymbol{x} := (x, y_1, \ldots, y_k)$$

for vectors $\boldsymbol{x}$ with $|x| \leq X = \frac{c_{h+1}}{4}$ is asymptotically the same as to the base c_{h+1}.

Indeed we have

$$\log_{c_{h+1}}(D_{\boldsymbol{x}}) = \frac{\log(D_{\boldsymbol{x}})}{\log(c_{h+1})}\,, \qquad \log_{\frac{c_{h+1}}{4}}(D_{\boldsymbol{x}}) = \frac{\log(D_{\boldsymbol{x}})}{\log(\frac{c_{h+1}}{4})}\,,$$

$$\lim_{h\to\infty} \frac{\log(c_{h+1})}{\log(\frac{c_{h+1}}{4})} = \lim_{c_h\to\infty} \frac{\log(c_{h+1})}{\log(\frac{c_{h+1}}{4})} = \lim_{c_h\to\infty} \frac{\log(c_{h+1})}{\log(c_{h+1})-\log(4)} = 1\,,$$

and hence

$$\lim_{h\to\infty} \frac{\log_{c_{h+1}}(D_{\boldsymbol{x}})}{\log_{\frac{c_{h+1}}{4}}(D_{\boldsymbol{x}})} = 1\,. \tag{52}$$

On the other hand, since we can restrict to $\boldsymbol{x}$ belonging to some ω_j with $j \geq 2$, all expressions $\log_{c_{h+1}}(D_{\boldsymbol{x}})$ are bounded above by $2\omega_2 \leq 2$ (see (15)) for h sufficiently large. Together with (52) and since this holds for every vector $\boldsymbol{x}$ for which $x \leq X$, the definition of the quantities ω_j immediately implies that they remain unaffected by this change of base.

So let $h = h(X)$ be the index determined by $c_h \leq X < \frac{c_{h+1}}{4}$. By the second step of the proof (putting $m = h$) at least one of the j linearly independent approximation vectors $(x, y_1, y_2, \ldots, y_k) \in \mathbb{Z}^{k+1}$ has to satisfy the condition $c_{h-j+2} \nmid x$. Consider one of the j approximation vectors with this property. This means if we let $g-1$ be the largest index with $c_{g-1} \mid x$ as in step 1, we have $g-1 \leq h-j+1$, i.e. $g \leq h-j+2$. Further let i be the index, for which $c_h = q_{N,i}$ belongs to the sequence $(q_{n,i})_{n\geq 1}$. At this point one should mention that we will repeatedly use step 1 in the following, neglecting the o-terms in the estimates (39), (40), (41) as they do not affect the asymptotic behaviour we aim to prove.

First we treat the case $c_{g-1} \geq q_{N-1,i}$ (case 1 step 1). Note, that $\frac{c_{m-1}}{c_m}$ is monotonically decreasing as m increases by (32)–(34) and (36), which we already used before. Thus by $g \leq h-j+2$ and (39) we have

$$\max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{c_{h-j+1}}{c_{h-j+2}} - o\left(\frac{c_{h-j+1}}{c_{h-j+2}}\right) \qquad \text{for} \quad h\to\infty\,. \tag{53}$$

So $X \geq c_h$ implies

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq -\frac{\log(\frac{c_{h-j+1}}{c_{h-j+2}})}{\log(c_h)} = \frac{\log(c_{h-j+2}) - \log(c_{h-j+1})}{\log(c_h)}.$$

It is now easy to see by (32)–(34) that for h in a fixed residue class $\overline{h}$ of the residue system $\{1, 2, \ldots, k\}$ mod k, the right hand side tends to one of the fractions (depending on $\overline{h}$) in the definition of $\omega_j(\eta)$ or to zero as $h\to\infty$ or equivalently $X\to\infty$. Clearly, each expression in $\wp_j(\eta)$ is induced by some $\overline{h}$ in that way as well. This shows, that indeed we have $\omega_j(X) < \wp_j + \epsilon$ for any $\epsilon > 0$ and $X = X(\epsilon)$ large enough.

In the remaining case $c_{g-1} < q_{N-1,i}$ (case 2 step 1) by (40) and as c_{h+1} in (40) corresponds to $q_{N,i}$, $\max_{1\leq t\leq k} |x\zeta_t - y_t|$ is essentially bounded below by $\frac{1}{q_{N-1,i+1}}$ (omitting the lower order terms and i in the residue system $\{1, 2, \ldots, k\}$ mod k).

We distinguish three cases now.

If we have $i \notin \{k-1,k\}$, approximation relative to base X is bad, as in this case we have $\lim_{N\to\infty} \frac{\log(q_{N-1,i+1})}{\log(q_{N,i})} = 0$ as a consequence of (34), so again by $X \geq c_h = q_{N,i}$, the expression

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq -\log_{c_h} \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{\log(q_{N-1,i+1})}{\log(c_h)} = \frac{\log(q_{N-1,i+1})}{\log(q_{N,i})}$$

tends to 0 as $X \to \infty$, and we are done again.

In the case $i = k$, or equivalently $q_{N,k} \leq X < q_{N+1,1}$, due to $c_{g-1} < q_{N-1,i} = q_{N-1,k}$ we know again by (40) that $\max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{1}{q_{N,1}}$ and so $X \geq q_{N,k}$ implies

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq -\log_{q_{N,k}} \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{\log(q_{N,1})}{\log(q_{N,k})}.$$

Hence by (34), for $h \to \infty$ we have the asyptotic

$$\frac{\log(q_{N,1})}{\log(q_{N,k})} \sim \frac{\log(q_{N,1}) - \log(q_{N-1,k})}{\log(q_{N,k})}.$$

The right hand side, however, converges to $\eta_1 = \wp_{k+1}(\eta)$ for $N \to \infty$ by (32). This shows that $\omega_j(X) \leq \wp_{k+1}(\eta) + \epsilon$ for any $\epsilon > 0$ and $X = X(\epsilon)$ large enough and together with $\wp_{k+1} \leq \wp_j(\eta)$ for $1 \leq j \leq k+1$ we get $\omega_j \leq \wp_j(\eta)$ as desired.

We divide the remaining case $i = k-1$, which means $q_{N,k-1} \leq X < q_{N,k}$, again into two cases. If $x < \frac{1}{2}\frac{q_{N,k}}{q_{N-1,k}}$, it follows that $\max_{1\leq t\leq k} |x\zeta_t - y_t|$ is essentially bounded below by $\frac{1}{2}\frac{1}{q_{N-1,k}}$ in view of (41). This gives the estimate

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{\log(\frac{1}{2}q_{N-1,k})}{\log(q_{N,k-1})} \leq \frac{\log(q_{N-1,k}) - \log(2)}{\log(q_{N,1})},$$

which tends to 0 for $X \to \infty$ by (34).

Otherwise $x \geq \frac{1}{2}\frac{q_{N,k}}{q_{N-1,k}}$, which clearly implies $\lim_{N\to\infty} \frac{\log(x)}{\log(q_{N,k})} = 1$ by (34). In particular for every $\epsilon > 0$ we have $\log(x) > (1+\epsilon)\log(q_{N,k})$ for $N = N(\epsilon)$ sufficiently large. Note $X \geq x$ and that $X \to \infty$ is equivalent to $N \to \infty$. Combination of these facts together with the fact that $\max_{1\leq t\leq k} |x\zeta_t - y_t|$ is essentially bounded below by $\frac{1}{q_{N,1}}$ by (40) yields the inequality

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{\log(q_{N,1} + o(\log(q_{N,1})))}{\log(q_{N,k}(1+\epsilon))}$$

for any $\epsilon > 0$ and $N = N(\epsilon)$ sufficiently large. However, the right hand side is of the form $\frac{\log(q_{N,1})}{\log(q_{N,k})} + o\Big(\frac{\log(q_{N,1})}{\log(q_{N,k})}\Big)$ as $N \to \infty$ and $\epsilon \to 0$, which tends to $\eta_1 = \wp_{k+1}(\eta)$ for $N \to \infty$ as in the case $i = k$ above and so it is no improvement either. This shows step three.

Now it only remains to determine the approximation constants $\widehat{\omega}_j$. However, for $j \geq 2$ they are easily seen to be zero as a consequence of $\omega = \infty$. Indeed in this case we have $\underline{\psi}_1 = -1$ by (13) and if for some $\epsilon > 0$ we had $\overline{\psi}_2 = \frac{1}{k} - \epsilon$, we would obtain

$$\sum_{j=1}^{k+1} \psi_j(q) = \psi_1(q) + \psi_2(q) + \sum_{j=3}^{k+1} \psi_j(q)$$
$$\leq (-1 + \frac{\epsilon}{3}) + (\frac{1}{k} - \epsilon + \frac{\epsilon}{3}) + (k-1)\frac{1}{k} < -\frac{\epsilon}{3}$$

for a sequence of arbitrary large values q, a contradiction to (6). So (13) again yields $\omega_j = 0$ for $2 \leq j \leq k+1$.

It remains to determine $\widehat{\omega}$. Let X be a real number of the form $X = \lfloor \frac{c_{h+1}}{4} - 1 \rfloor$, so that in particular we have $c_h \leq X < c_{h+1}$.

Putting $j = 1$ in (53) and noting $X = \frac{c_{h+1}}{4} - 1 \geq \frac{c_{h+1}}{5}$ in the case $g \geq h+2-k$ we obtain

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq -\log_{\frac{c_{h+1}}{5}} \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{\log(c_{h+1}) - \log(c_h)}{\log(c_{h+1}) - \log(5)}. \quad (54)$$

If we now fix a residue class $\overline{h}$ for the values of h in the residue system $\{1, 2, \ldots, k\}$ mod k, the right hand side of (54) tends to one of the fractions in the definition of $\widehat{\wp}(\eta)$ as $h \to \infty$. In fact, as $\overline{h}$ runs through the residue system $\{1, 2, \ldots, k\}$ mod k this induces a bijection between the residue classes $\overline{h}$ of the residue system $\{1, 2, \ldots, k\}$ mod k and the expressions of $\widehat{\omega}$.

In the case $g \leq h+1-k$ as $\max_{1\leq t\leq k} |x\zeta_t - y_t|$ is essentially bounded below by $\frac{1}{q_{N-1,i+1}}$ again, we have the upper estimate

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq -\log_{\frac{c_{h+1}}{5}} \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{\log(q_{N,h})}{\log(q_{N+1,h+1}) - \log(5)}.$$

The right hand side, however, is smaller than the corresponding value in (54) for every h, so the case $g \leq h+1-k$ does never give any improvement. Thus, by its definition, the quantity $\widehat{\omega}$ can be estimated above by the minimum of the expressions of $\widehat{\wp}(\eta)$, which simply is $\widehat{\wp}(\eta)$.

On the other hand, fixing a residue classes $\overline{h}$ for h in the residue system $\{1, 2, \ldots, k\}$ mod k again and putting $x := c_h$ and $y_t := \lfloor \zeta_t c_h \rfloor$ for $1 \leq t \leq k$, we obtain a bijection between the resulting values

$$\lim_{h\in\overline{h}, h\to\infty} -\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| = \lim_{h\in\overline{h}, X\to\infty} -\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t|$$

as $\overline{h}$ runs through $\{1, 2, \ldots, k\}$ and the expressions involved in the definition of $\widehat{\wp}_1(\eta)$. Hence $\widehat{\omega}$ is at least as large as the minimum of these expressions, which again is $\widehat{\omega}$. □

Note, that in the special case $k = 2$ we have $\eta_1 + \eta_2 = 1$, and the approximation constants of $\boldsymbol{\zeta}$ in Theorem 1 become

$$\omega = \infty\,, \qquad \omega_2 = 1\,, \qquad \omega_3 = \eta_1\,,$$
$$\widehat{\omega} = \eta_2\,, \qquad \widehat{\omega}_2 = 0\,, \qquad \widehat{\omega}_3 = 0\,.$$

Particularly, we see that $\widehat{\omega}+\omega_3=1$, which is easily seen by straightforward computation with repeated use of (13) to be equivalent to Jarník's identity in the form $\overline{\psi}_1+2\overline{\psi}_1\underline{\psi}_3+\underline{\psi}_3=0$, see Theorem 1.5 in [5].

So far we have not asked for the numbers $\zeta_1,\zeta_2,\ldots,\zeta_k$ to be $\mathbb{Q}$-linearly independent together with 1, which is the usual assumption. For this purpose, we apply Theorem 1 in the special case $\zeta_j=\sum_{n\geq 1}2^{-a_{n,j}}$ with suitable sequences $(a_{n,j})_{n\geq 1}$ for $1\leq j\leq k$.

Corollary 2. *For $1\leq j\leq k$ let $(a_{n,j})_{n\geq 1}$ be sequences with the properties*

$$a_{1,1}<a_{1,2}<\ldots<a_{1,k}<a_{2,1}<a_{2,2}<\ldots a_{2,k}<a_{3,1}<\ldots \tag{55}$$

and for $\eta\in\mathbb{R}^k\times\overline{\mathbb{R}}$ as in Theorem 1 put

$$\lim_{n\to\infty}\frac{a_{n+1,1}-a_{n,k}}{a_{n+1,k}}=\eta_1 \tag{56}$$

$$\lim_{n\to\infty}\frac{a_{n,i}-a_{n,i-1}}{a_{n,k}}=\eta_i,\quad 2\leq i\leq k. \tag{57}$$

$$\lim_{n\to\infty}\frac{a_{n+1,1}}{a_{n,k}}=\eta_{k+1}=\infty \tag{58}$$

and $\zeta_j=\sum_{n\geq 1}2^{-a_{n,j}}$ for $1\leq j\leq k$. Then the corresponding approximation constants are given as in Theorem 1.

Proof. Clearly, if we put $q_{n,j}=2^{a_{n,j}}$, all conditions of Theorem 1 are satisfied. □

Now one can easily prove that there are uncountably many vectors $\boldsymbol{\zeta}\in\mathbb{R}^k$, such that additionally $1,\zeta_1,\zeta_2,\ldots,\zeta_k$ are linearly independent over $\mathbb{Q}$. The arguments of the proof of the following Proposition 3 are suitable to prove the existence of vectors $(\zeta_1,\zeta_2,\ldots,\zeta_k)$ for which $1,\zeta_1,\zeta_2,\ldots,\zeta_k$ are linearly independent subject to certain approximation properties if this existence was established without the linear independence condition.

Proposition 3. *One can choose sequences $(a_{n,j})_{n\geq 1}$ in Corollary 2 such that*

$$\{1,\zeta_1,\ldots,\zeta_k\}$$

is linearly independent over $\mathbb{Q}$.

Proof. Note that in the case $k=1$ (32)–(34) simply yield $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\infty$ and it follows from Liouville's Theorem that the corresponding number of the form $\zeta=\sum_{n\geq 1}2^{-a_n}$ is transcendental, in particular $\{1,\zeta\}$ is linearly independent over $\mathbb{Q}$. In the case $k\geq 2$ consider the numbers $a_{1,j}=j$ for $1\leq j\leq k$ and define the sequences $(a_{n,j})_{n\geq 1}$ by the recurrence relations

$$a_{n+1,1} = \left\lfloor \frac{1}{\eta_1} n \cdot a_{n,k}(\eta_1) \right\rfloor \tag{59}$$

$$a_{n+1,2} = \left\lfloor \frac{1}{\eta_1} n \cdot a_{n,k}(\eta_1+\eta_2) \right\rfloor \tag{60}$$

$$\vdots$$

$$a_{n+1,k} = \left\lfloor \frac{1}{\eta_1} n \cdot a_{n,k}(\eta_1+\eta_2+\cdots+\eta_k) \right\rfloor. \tag{61}$$

One checks that (59)–(61) imply (32)–(34). Now we prove that we can change (59)–(61) slightly such that $\{1,\zeta_1,\ldots,\zeta_k\}$ is linearly independent over $\mathbb{Q}$.

Let $(b_n)_{n\geq 1}$ be the ordered combined set of all $a_{n,j}$ defined as above. Note that we can obviously "disturb" the system (59)–(61) a little by adding one to the elements of the form b_a where $a\in A$ with A an arbitrary subset of $\mathbb{N}$, without violating (59)–(61). Noting that (59)–(61) imply $\lim_{n\to\infty}\frac{a_{n+1,1}}{a_{n,1}} = \infty$, by the considerations of the case $k=1$ we know $\{1,\zeta_1\}$ is a $\mathbb{Q}$-linearly independent set, where ζ_1 is generated by the sequence $(a_{n,1})_{n\geq 1}$ defined above, ie $\zeta_1=\sum_{n\geq 1} a_{n,1}^{-1}$. If we now consider the set of all sequences $\mathscr{A}_2 = \{(a'_{n,2})_{n\geq 1}\}$ which arise from $(a_{n,2})_{n\geq 1}$ as above by adding 1 to b_a with $a\in A_2$ for an arbitrary subset A_2 of $\mathbb{N}$, we see that $\mathscr{A}_2$ is uncountable. So there must be a number ζ'_2 generated by an $a'_{n,2}\in\mathscr{A}_2$ with the property that $\{1,\zeta_1,\zeta'_2\}$ are linearly independent over $\mathbb{Q}$, since the $\mathbb{Q}$-span of $\{1,\zeta_1\}$ is only countable. Now we proceed analoguously with sets $\mathscr{A}_j$ for $3\leq j\leq k$ and finally get a $\mathbb{Q}$-linearly independent set $\{1,\zeta_1,\zeta'_2,\ldots,\zeta'_k\}$. As mentioned above, the set $\{\zeta_1,\zeta'_2,\ldots\zeta'_k\}$ fulfills all the requirements. □

Remark 1. Since algebraic numbers have countable cardinality one can readily generalize the proof above to show that we can even ask $\boldsymbol{\zeta}$ to be algebraically independent.

We now give some applications of the above theorem. Note that (4), (5) and (8) imply that for all $\epsilon>0$ and sufficiently large $Q=Q(\epsilon)>0$ we have the bounds

$$-1 \leq \psi_1(Q) \leq 0$$

$$\frac{j-k-1}{kj} - \epsilon \leq \psi_j(Q) \leq \frac{1}{k}, \qquad 2\leq j\leq k+1.$$

In the first Corollary we construct $\zeta_1,\zeta_2,\ldots,\zeta_k$ for which each $\psi_j(Q)$ takes each of the values inside of the corresponding intervals $I_1:=(-1,0), I_j:=\left(\frac{j-k-1}{kj},\frac{1}{k}\right)$ for arbitrarily large Q simultaneously for all $1\leq j\leq k+1$. So roughly speaking in this case all ψ_j take their possible range of values for arbitrarily large (Q,∞). In particular the bounds in (8) are best possible.

Corollary 3. *There exist $\zeta_1, \zeta_2, \dots, \zeta_k$ for which the set $\{1, \zeta_1, \zeta_2, \dots, \zeta_k\}$ is $\mathbb{Q}$ linearly independent and such that*

$$\begin{aligned} \omega_j &= \frac{1}{j-1}\,, && 1 \le j \le k+1\,, \\ \widehat{\omega} &= \frac{1}{k}\,, && \\ \widehat{\omega}_j &= 0\,, && 2 \le j \le k+1\,. \end{aligned}$$

Proof. Note that by means of proposition 3 for every $\eta \in \mathbb{R}^{k+1}$ subject to the restrictions of Theorem 1 we can construct $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \dots, \zeta_k)$ together with 1 linearly independent over $\mathbb{Q}$ such that Theorem 1 holds. Putting $\eta_2 = \eta_3 = \cdots = \eta_{k+1} = \frac{1}{k}$ in Theorem 1 immediately gives all the stated equalities for this $\boldsymbol{\zeta}$. □

Now we want to give our first explicit construction of special cases of Schmidt's conjecture, which was proved by Moshchevitin in a nonconstructive way in [3]. The conjecture states, that for each integer pair (k, i) with $k \ge 2, 1 \le i \le k-1$ there exists a vector $\boldsymbol{\zeta} \in \mathbb{R}^k$ with $\{1, \boldsymbol{\zeta}\}$ linearly independent over $\mathbb{Q}$ such that $\lim_{q\to\infty} \lambda_i(q) = 0$ and $\lim_{q\to\infty} \lambda_{i+2} = \infty$. Note that we cannot have $\lim_{q\to\infty} \lambda_i(q) = 0$, $\lim_{q\to\infty} \lambda_{i+1}(q) = \infty$ for any i because of the assumption of linear independence because of (7), see also the introduction in [3]. We now give a generalisation of this fact in the special case $i = 1$ for arbitrary $k \ge 2$.

Corollary 4. *Let $k \ge 2$ and $3 \le r \le k+1$ be integers. Then there exists $\boldsymbol{\zeta} \in \mathbb{R}^k$ with $\{1, \boldsymbol{\zeta}\}$ linearly independent over $\mathbb{Q}$ such that*

$$\begin{aligned} &\overline{\psi}_1 < 0\,, && \\ &\underline{\psi}_j < 0 < \overline{\psi}_j\,, && 2 \le j \le r-1\,, \\ &\underline{\psi}_j > 0\,, && r \le j \le k+1\,. \end{aligned}$$

The case $r = 3$ clearly implies Schmidt's conjecture for $i = r-2 = 1$.

Proof. We may assume $k \ge 3$, because for $k = 2$ the Corollary only states that $\overline{\psi}_1 < 0$ and $\underline{\psi}_3 > 0$ is possible, which only requires $\overline{\psi}_1 < 0$ by (6) and for any choice of $\eta = (\eta_1, \eta_2) \neq (1/2, 1/2)$ the construction of Theorem 1 gives an example. We apply Theorem 1 with η defined by

$$\eta_1 = \frac{\alpha^{k-1}}{1 + \alpha + \cdots + \alpha^{k-1}}, \tag{62}$$

$$\frac{\eta_j}{\eta_{j+1}} = \alpha, \qquad 1 \le j \le k-1. \tag{63}$$

The parameter $\alpha \in \{0, 1\}$ will be chosen later in dependence of (r, k). First note that by (13) our inequalities translate to

$$\widehat{\omega} > \frac{1}{k}\,, \tag{64}$$

$$\widehat{\omega}_j < \frac{1}{k} < \omega_j\,, \qquad 2 \le j \le r-1\,, \tag{65}$$

$$\omega_j < \frac{1}{k}\,, \qquad r \le j \le k+1\,. \tag{66}$$

Now note, that the left hand side of (65) trivially holds by Theorem 1. With (62), (63) the quantity $\widehat{\wp}(\eta)$ of Theorem 1 becomes

$$\widehat{\omega} = \min\left\{1, \frac{1}{1+\alpha}, \dots, \frac{1}{1+\alpha+\cdots+\alpha^{k-1}}\right\} = \frac{1}{1+\alpha+\cdots+\alpha^{k-1}}.$$

Moreover the assumption $\alpha < 1$ implies that (64) holds. To obtain the remaining inequalities it is obviously sufficient to prove $\omega_r < \frac{1}{k} < \omega_{r-1}$ for some α, which by (62), (63) and Theorem 1 is equivalent to

$$\frac{\alpha^{r-2}}{1+\alpha+\cdots+\alpha^{r-2}} < \frac{1}{k} < \frac{\alpha^{r-3}}{1+\alpha+\cdots+\alpha^{r-3}}, \tag{67}$$

since the last term in $\wp_j(\eta)$ is easily seen to be the largest in $\wp_j$ in our special case of constant ratios. For $r = 3$ this reduces to $\frac{\alpha}{\alpha+1} < \frac{1}{k}$, which is obviously true if we choose any $\alpha \in (0, \frac{1}{k})$, so we can assume $r \ge 4$. Defining the functions

$$\Phi_u\colon \alpha \longmapsto \frac{\alpha^u}{1+\alpha+\cdots+\alpha^u}$$

shows that (67) in the cases left to consider is equivalent to $\phi_{u+1}(\alpha) < \frac{1}{k} < \phi_u(\alpha)$ for $1 \le u \le k-2$. It is easy to check that all these ϕ_u are continuous, $\phi_u(\alpha) > \phi_{u+1}(\alpha)$ and $\phi_u(0) = 0, \phi_u(1) = \frac{1}{u+1} > \frac{1}{k}$. Further more from

$$\begin{aligned}\Phi_u'(\alpha) &= \frac{u\alpha^{u-1}(1+\alpha+\cdots+\alpha^u) - \alpha^u(1+2\alpha+\cdots+u\alpha^{u-1})}{(1+\alpha+\cdots+\alpha^u)^2}\\ &= \frac{\alpha^{2u-2}+2\alpha^{2u-3}+\cdots+(u-1)\alpha^u+u\alpha^{u-1}}{(1+\alpha+\cdots+\alpha^u)^2} > 0\end{aligned}$$

we deduce that they are monotonically increasing in α. Combination of these properties implies that for fixed u there exists some $t \in (0,1)$ such that $\phi_u(t) = \frac{1}{k}$ by intermediate value theorem. It further follows from these considerations on the one hand $\phi_u(\alpha) > \frac{1}{k}$ for $\alpha > t$, and on the other hand the existence of an interval $\alpha \in (t_0, t_1)$ with $t_0 < t < t_1$ such that $\phi_{u+1}(\alpha) < \frac{1}{k}$. Thus, for all $\alpha \in (t, t_1)$ we have $\phi_{u+1}(\alpha) < \frac{1}{k} < \phi_u(\alpha)$. □

2.3 The case $\omega < \infty$

We now aim to give similar results for vectors $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \dots, \zeta_k)$ whose components ζ_j have one-dimensional approximation constant $\omega < \infty$. Hence in particular the simultaneous approximation constant ω is finite too, as by definition it cannot exceed the minimum of the one-dimensional constants. As in Theorem 1 each ζ_j will be the sum of the reciprocals of integers $q_{n,j}$ that satisfy (30), (31), but the conditions (32)–(37) will be altered in ways to obtain a symmetric situation in all ζ_j, which will be more convenient for the purposes of section 3. We start with proving the following

Theorem 2. *For* $1 \leq j \leq k$ *let* $\zeta_j = \sum_{n\geq 1} \frac{1}{q_{n,j}}$ *where* $(q_{n,j})_{n\geq 1}$ *are sequences of integers for which* (30), (31) *are satisfied and that for*

$$(b_n)_{n\geq 1} = \big(\log(q_{1,1}), \log(q_{2,1}), \ldots, \log(q_{k,1}), \log(q_{1,2}), \ldots\big)$$

the inequality

$$\liminf_{n\to\infty} \frac{b_{n+1}}{b_n} > 2$$

is satisfied. Then the first $(k-1)$ *approximation constants relative to* $\zeta_1, \zeta_2, \ldots, \zeta_k$ *are given by*

$$\omega = \limsup_{n\to\infty} \frac{b_{n+1} - b_n}{b_n}, \qquad \widehat{\omega} = \liminf_{n\to\infty} \frac{b_n - b_{n-1}}{b_n},$$

$$\omega_2 = \limsup_{n\to\infty} \frac{b_n - b_{n-1}}{b_n}, \qquad \widehat{\omega}_2 = \liminf_{n\to\infty} \frac{b_{n-1} - b_{n-2}}{b_n},$$

$$\omega_3 = \limsup_{n\to\infty} \frac{b_{n-1} - b_{n-2}}{b_n}, \qquad \widehat{\omega}_3 = \liminf_{n\to\infty} \frac{b_{n-2} - b_{n-3}}{b_n},$$

$$\vdots \qquad\qquad \vdots$$

$$\omega_{k-1} = \limsup_{n\to\infty} \frac{b_{n-k+3} - b_{n-k+2}}{b_n}, \qquad \widehat{\omega}_{k-1} = \liminf_{n\to\infty} \frac{b_{n-k+2} - b_{n-k+1}}{b_n}.$$

Further more we have the inequlities

$$\omega_k \geq \limsup_{n\to\infty} \frac{b_{n-k+2} - b_{n-k+1}}{b_n}, \qquad \widehat{\omega}_k \geq \liminf_{n\to\infty} \frac{b_{n-k+1} - b_{n-k}}{b_n},$$

$$\omega_{k+1} \geq \limsup_{n\to\infty} \frac{b_{n-k+1} - b_{n-k}}{b_n}, \qquad \widehat{\omega}_{k+1} \geq \liminf_{n\to\infty} \frac{b_{n-k} - b_{n-k-1}}{b_n}.$$

Proof. Denote the right hand side expressions by $\wp_j$ respectively $\widehat{\wp}_j$. We prove the inequalities $\omega_j \geq \wp_j$, $\widehat{\omega}_j \geq \widehat{\wp}_j$ for $1 \leq j \leq k+1$ and then $\omega_j \leq \wp_j$, $\widehat{\omega}_j \leq \widehat{\wp}_j$ for $1 \leq j \leq k-1$. This obviously yields the assertions of the Theorem.

Throughout the proof, for arbitrary X let the integer h be determined as the index for which $c_h \leq X < c_{h+1}$. Obviously $X \to \infty$ is equivalent to $h \to \infty$, which will often be used implicitly. We first prove $\omega_j \geq \wp_j$ for $1 \leq j \leq k+1$.

Assume j arbitrary but fixed. Observe that for every h if we put $x^{(i)} = c_{h+1-i}$, $y_t^{(i)} = \lfloor \zeta_t x^{(i)} \rfloor$ for $1 \leq t \leq k$ and $1 \leq i \leq j$, we obtain

$$\max_{1\leq i\leq j} \max_{1\leq t\leq k} |\zeta_t x^{(j)} - y_t^{(j)}| = |\zeta_{\overline{t_0}} x^{(j)} - y_{t_0}^{(j)}| = \frac{c_{h+1-j}}{c_{h+2-j}} + o\left(\frac{c_{h+1-j}}{c_{h+2-j}}\right) \qquad (68)$$

as $h \to \infty$ with $t_0 := h + 2 - j$, as explained in the proof of Theorem 1. Choose an integer sequence $(h_r)_{r\geq 1}$ of values for h in (68) such that with $(X_r)_{r\geq 1} := (x_r)_{r\geq 1} := (c_{h_r})_{r\geq 1}$ we have

$$\begin{aligned}\lim_{r\to\infty} -\log_{X_r} \frac{c_{h_r+1-j}}{c_{h_r+2-j}} &= \limsup_{h\to\infty} -\log_{c_h} \frac{c_{h+1-j}}{c_{h+2-j}} \\ &= \limsup_{h\to\infty} \frac{\log(c_{h+2-j}) - \log(c_{h+1-j})}{\log(c_h)}\,,\end{aligned} \tag{69}$$

which is possible by definition of the lim sup. Leaving out the linearly independence condition, the so constructed vectors $\big(x^{(i)}, y_1^{(i)}, \dots, y_k^{(i)}\big), 1 \leq i \leq j$, lead to the values $\wp_j$ as lower bounds for ω_j in view of (68). However, the missing linearly independence condition is now obtained exactly as in Theorem 1.

To prove $\widehat{\omega}_j \geq \widehat{\wp}_j$, consider any sequence $(h_r)_{r\geq 1}$ and the same approximation vectors as in the proof of $\omega_j \geq \wp_j$ but take logarithms to base $c_{h+1} > X$ instead of base c_h. This yields a lower estimate for $\omega_j(X)$ for $X \in [c_{h_r}, c_{h_{r+1}})$. As this is valid for any sequence $(h_r)_{r\geq 1}$ we obtain the lower bounds $\widehat{\wp}_j = \liminf_{X\to\infty} \omega_j(X)$ for $\widehat{\omega}_j$ by definition of lim inf, as we claimed. So far we have established the lower bounds $\wp_j$ (respectively $\widehat{\wp}_j$) for ω_j (respectively $\widehat{\omega}_j$) for $1 \leq j \leq k + 1$.

For the upper bounds note first that with basically the same arguments as in the proof of step 3 in Theorem 1 we can restrict to the case $c_h \leq X \leq \frac{1}{4}c_{h+1}$. Further step 1 and step 2 of $\wp_j(\eta) \leq \omega_j$ in the proof of Theorem 1 remain valid in the present situation. Indeed, the estimates (39), (40) are already valid under the assumtion $\frac{b_{n+1}}{b_n} > 2$, which is weaker than (34) used in Theorem 1. The proof of step 2 is analoguous.

Now for every fixed X we divide all approximation vectors $(x, y_1, \dots, y_k)$ with $x \leq X$ into two categories. Let g be the largest integer such that $c_{g-1} \mid x$ for an approximation vector $(x, y_1, \dots, y_k)$ as in step 1 of Theorem 1. The distinction of vectors with $g > h+1-k$, which we will call vectors of category 1, and $g \leq h+1-k$, which we will call vectors of category 2, now leads to 2 cases.

Case 1: If for fixed X with $c_h \leq X < \frac{1}{4}c_{h+1}$ we have $g > h + 1 - k$ for an approximation vector (i.e. it belongs to category 1), (39) implies that

$$\omega_j(X) \leq -\log_{c_h} \frac{c_{h+1-j}}{c_{h+2-j}} = \frac{\log(c_{h+2-j}) - \log(c_{h+1-j})}{\log(c_h)}, \qquad 1 \leq j \leq k + 1. \tag{70}$$

So we have that for every X the quantity $\wp_j$ is an upper bound for $\omega_j^1(X)$, by which we mean the supremum over all real numbers ν such that (12) has j linearly independent vector solutions all of which are of the first category. Hence if we define $\omega_j^1 = \limsup_{X\to\infty} \omega_j^1(X)$, we get

$$\omega_j^1 \leq \wp_j, \qquad 1 \leq j \leq k + 1. \tag{71}$$

In order to give a connection between the approximation constants $\widehat{\omega}_j$ and approximation vectors of category 1, we define $\widehat{\omega}_j^1 := \liminf_{X\to\infty} \omega_j^1(X)$. We start with an arbitrary sequence $(X_r)_{r\geq 1}$ with corresponding subsequence $(c_{h_r})_{r\geq 1}$ of $(c_h)_{h\geq 1}$ and define a sequence $(X'_r)_{r\geq 1}$ by putting $X'_r := \frac{1}{5}c_{h_r+1}$. In view of (39) and observing that the fractions $\frac{c_m}{c_{m+1}}$ are monotonically decreasing by our assumption $\frac{b_{n+1}}{b_n} > 2$, we get the upper estimate for the approximation constants $\omega_j^1(X_r')$

$$\omega_j^1(X_r') \leq -\log_{\frac{1}{5}X_{r+1}}\left(\frac{c_{h_r+2-j}}{c_{h_r+1-j}}\right) = \frac{\log(c_{h_r+2-j}) - \log(c_{h_r+1-j})}{\log(c_{h_r+1}) - \log(5)} \tag{72}$$

for $1 \leq j \leq k+1$. If we specify a sequence $(X_r)_{r\geq 1}$ for which the corresponding sequence $(h_r)_{r\geq 1}$ has the property

$$\begin{aligned} \lim_{r\to\infty} -\log_{h_r} \frac{c_{h_r+1-j}}{c_{h_r+2-j}} &= \liminf_{h\to\infty} -\log_{c_h} \frac{c_{h+1-j}}{c_{h+2-j}} \\ &= \liminf_{h\to\infty} \frac{\log(c_{h+2-j}) - \log(c_{h+1-j})}{\log(c_h)}, \end{aligned} \tag{73}$$

which is possible again by definition of lim inf, we put $n = h+1$ in the definition of $\widehat{\wp}_j$ so that the right hand side of (72) tends to $\widehat{\wp}_j$ as $r \to \infty$. Thus $\lim_{r\to\infty} \omega_j^1(X_{\underline{r}})$ exists and is bounded above by $\widehat{\wp}_j$. In particular

$$\widehat{\omega}_j^1 \leq \widehat{\wp}_j, \qquad 1 \leq j \leq k+1. \tag{74}$$

Case 2: In the other case $g \leq h+1-k$ (i.e. the vector belongs to category 2), consider first an arbitrary sequence $(X_r)_{r\geq 1}$ that tends monotonically to infinity and the corresponding subsequence $(c_{h_r})_{r\geq 1}$ of $(c_h)_{h\geq 1}$ determined by $c_{h_r} \leq X_r < c_{h_{r+1}}$. Recall that without loss of generality we can assume $c_{h_r} \leq X_r < \frac{1}{4}c_{h_r+1}$, as this does not affect the approximation constants. By (40) we have

$$\max_{1\leq t\leq k} |\zeta_t x - y_t| \geq \frac{1}{c_{h_r+2-k}} - o\left(\frac{1}{c_{h_r+2-k}}\right). \tag{75}$$

Furthermore define $\omega_j^2(X)$ as the supremum of all ν, such that (12) has j linearly independent vector solutions with at least one vector of category 2, for every fixed $X > 0$, and define $w_j^2 := \limsup_{X\to\infty} w_j^2(X)$. Taking logarithms to the bases $c_{h_r} < X_r$ for $r \to \infty$, (75) implies that for $1 \leq j \leq k+1$ the expression $\omega_j^2(X_r)$ is bounded above by

$$-\log_{X_r}\left(\frac{c_{h_r+2-k}}{c_{h_r+3-k}}\right) \leq -\log_{c_{h_r}}\left(\frac{c_{h_r+2-k}}{c_{h_r+3-k}}\right) = \frac{\log(c_{h_r+3-k}) - \log(c_{h_r+2-k})}{\log(c_{h_r})}. \tag{76}$$

Note that the bound in (76) is valid for any sequence $(X_r)_{r\geq 1}$ and the corresponding sequence $(c_{h_r})_{r\geq 1}$. Hence on the one hand we have

$$\mathscr{A} := \limsup_{h\to\infty} \frac{\log(c_{h+3-k}) - \log(c_{h+2-k})}{\log(c_h)} \geq \limsup_{X\to\infty} \omega_j^2(X) = \omega_j^2 \tag{77}$$

simply by the definition of lim sup. Observe $\mathscr{A} = \wp_{k-1}$ (where n in the definition of $\wp_{k-1}$ corresponds to h in the definition of $\mathscr{A}$) and hence

$$\omega_j^2 \leq \wp_{k-1}, \qquad 1 \leq j \leq k+1. \tag{78}$$

On the other hand, for any sequence $(X_r)_{r\geq 1}$ by (75) and our assumption $\frac{b_{n+1}}{b_n} = \frac{\log(c_{n+1})}{\log(c_n)} > 2$, for r sufficiently large and $1 \leq j \leq k-1$ we have

$$\begin{aligned}\omega_j^2(X_r) &\leq -\log_{X_r}\left(\frac{1}{c_{h_r+2-k}} - o\left(\frac{1}{c_{h_r+2-k}}\right)\right)\\ &< -\log_{X_r}\left(\frac{c_{h_r+2-k}}{c_{h_r+3-k}}\right) \leq \omega_{k-1}^1(X_r),\end{aligned} \tag{79}$$

where the right inequality is a consequence of the constructions $\omega_j \geq \wp_j$, $\widehat{\omega}_j \geq \widehat{\wp}_j$ in the case $j = k-1$ (see (68), (69)). Since this holds for any sequence $(X_r)_{r\geq 1}$, we have

$$\widehat{\omega}_j^2 := \liminf_{X\to\infty} \omega_j^2(X) \leq \liminf_{X\to\infty} \omega_{k-1}^1(X) = \widehat{\omega}_{k-1}^1 \leq \widehat{\omega}_j^1, \quad 1 \leq j \leq k-1. \tag{80}$$

We now combine the results above for the quantities $\omega_j^s, \widehat{\omega}_j^s$, $s \in \{1,2\}$ to derive the required upper bounds. As every approximation vector is either of category 1 or category 2 for fixed $X > 0$, the definitions of $\omega_j^s, s \in \{1,2\}$, imply $\omega_j(X) = \max\{\omega_j^1(X), \omega_j^2(X)\}$ for every $X > 0$. Observe that for any functions $f, g : \mathbb{R}^+ \mapsto \mathbb{R}$ we have

$$\max\left\{\limsup_{X\to\infty} f(X), \limsup_{X\to\infty} g(X)\right\} = \limsup_{X\to\infty} \max\left\{f(X), g(X)\right\}.$$

Applying this on $f(X) = \omega_j^1(X)$, $g(X) = \omega_j^2(X)$ implies that ω_j, which is by definition $\limsup_{X\to\infty} \omega_j(X)$, equals the maximum of $\omega_j^1 = \limsup_{X\to\infty} \omega_j^1(X)$ and $\omega_j^2 = \limsup_{X\to\infty} \omega_j^2(X)$. By (71), (78) for $1 \leq j \leq k-1$ this maximum is $\max\{\wp_j, \wp_{k-1}\} = \wp_j$, which proves the upper bounds for ω_j, $1 \leq j \leq k-1$.

It remains to check the upper estimates for the constants $\widehat{\omega}_j$. In view of $\omega_j(X) = \max\{\omega_j^1(X), \omega_j^2(X)\}$, (74) and (80), we obtain

$$\widehat{\omega}_j = \liminf_{X\to\infty} \max_{s=1,2}\{\omega_j^s(X)\} = \liminf_{X\to\infty} \omega_j^1(X) = \widehat{\omega}_j^1 \leq \widehat{\wp}_j$$

for $1 \leq j \leq k-1$. □

Remark 2. It is rather clear from the proof that Theorem 2 remains valid for $C = \infty$ too. We will need this later in Theorem 4.

Corollary 5. *Let the assumptions of Theorem 2 be satisfied and assume further the existence of the limit of the quotients* $\frac{b_{n+1}}{b_n}$, *i.e.*

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} =: C \geq 2\,.$$

Then the first $(k-1)$ *approximation constants are given by*

$$\omega = C - 1 \qquad \qquad \vdots$$

$$\omega_2 = \frac{C-1}{C} = \widehat{\omega} \qquad \qquad \omega_{k-1} = \frac{C-1}{C^{k-2}} = \widehat{\omega}_{k-2}$$

$$\omega_3 = \frac{C-1}{C^2} = \widehat{\omega}_2 \qquad \qquad \frac{C-1}{C^{k-1}} = \widehat{\omega}_{k-1}\,.$$

For the remaining approximation constants we have the inequalities

$$\omega_k \geq \frac{C-1}{C^{k-1}}$$
$$\omega_{k+1} \geq \widehat{\omega}_k \geq \frac{C-1}{C^k}$$
$$\widehat{\omega}_{k+1} \geq \frac{C-1}{C^{k+1}}.$$

Proof. For every $h \geq 1$ we have

$$\lim_{n\to\infty} \frac{b_{n+h}}{b_n} = \lim_{n\to\infty} \frac{b_{n+h}}{b_{n+h-1}} \frac{b_{n+h-1}}{b_{n+h-2}} \cdots \frac{b_{n+1}}{b_n} = C^h$$

and hence Theorem 2 yields the claimed result. □

Remark 3. The bounds for $\omega_k, \widehat{\omega}_k, \omega_{k+1}, \widehat{\omega}_{k+1}$ could be improved further to

$$\frac{C-1}{C^{k-1}} \leq \omega_k \leq \max\left\{\frac{C}{C^k-1}, \frac{C-1}{C^{k-1}}\right\}$$
$$\widehat{\omega}_k = \frac{C-1}{C^k-1}$$
$$\omega_{k+1} = \frac{1}{C^{k-1}}$$
$$\min\left\{\frac{1}{C^k-1}, \frac{C-1}{C^k}\right\} \leq \widehat{\omega}_{k+1} \leq \frac{1}{C^k-1}$$

by a rather long and technical proof that we will not present here. In particular in the case $C \geq \beta_k > 2$, where β_k is the largest real root of $P_k(x) = x^{k+1} - 2x^k - x + 1$, we have

$$\omega_k = \frac{C-1}{C^{k-1}}$$
$$\widehat{\omega}_k = \frac{C-1}{C^k-1}$$
$$\omega_{k+1} = \frac{1}{C^{k-1}}$$
$$\widehat{\omega}_{k+1} = \frac{1}{C^k-1}.$$

Let us call the assumptions of Theorem 2 without the growth condition of $\frac{b_{n+1}}{b_n}$ the *basic assumptions* of Theorem 2 in the sequel. We can generalize the idea of the proof of Theorem 2 to get

Theorem 3. *Given the basic assumptions of Theorem 2, we consider some fixed* $d \in \{1, 2, \ldots, k-1\}$ *and define* κ_d *to be the largest real root of* $P_d(x) := x^d - x^{d-1} - 1$. *Then if*

1. $\frac{b_{n+1}}{b_n} > \kappa_d$, *for all* $n \geq 1$,

2. the sequence $(d_n)_{n\geq 1} := (b_{n+1} - b_n)_{n\geq 1}$ *is monotonically increasing*

are satisfied, the first $(k-d)$ *approximation constants are given by*

$$\omega = \limsup_{n\to\infty} \frac{b_{n+1} - b_n}{b_n}, \qquad \widehat{\omega} = \liminf_{n\to\infty} \frac{b_n - b_{n-1}}{b_n},$$

$$\omega_2 = \limsup_{n\to\infty} \frac{b_n - b_{n-1}}{b_n}, \qquad \widehat{\omega}_2 = \liminf_{n\to\infty} \frac{b_{n-1} - b_{n-2}}{b_n},$$

$$\omega_3 = \limsup_{n\to\infty} \frac{b_{n-1} - b_{n-2}}{b_n}, \qquad \widehat{\omega}_3 = \liminf_{n\to\infty} \frac{b_{n-2} - b_{n-3}}{b_n},$$

$$\vdots \qquad\qquad \vdots$$

$$\omega_{k-d} = \limsup_{n\to\infty} \frac{b_{n-k+d+2} - b_{n-k+d+1}}{b_n}, \qquad \widehat{\omega}_{k-d} = \liminf_{n\to\infty} \frac{b_{n-k+d+1} - b_{n-k+d}}{b_n}.$$

Furthermore we have the inequalities

$$\omega_{k-d+1} \geq \limsup_{n\to\infty} \frac{b_{n-k+d+1} - b_{n-k+d}}{b_n}, \qquad \widehat{\omega}_{k-d+1} \geq \liminf_{n\to\infty} \frac{b_{n-k+d+1} - b_{n-k+d}}{b_n},$$

$$\vdots \qquad\qquad \vdots$$

$$\omega_{k+1} \geq \limsup_{n\to\infty} \frac{b_{n-k+1} - b_{n-k}}{b_n}, \qquad \widehat{\omega}_{k+1} \geq \liminf_{n\to\infty} \frac{b_{n-k} - b_{n-k-1}}{b_n}.$$

Proof. We proceed as in the proof of Theorem 2 using the fact that d_n is increasing in place of the equivalent fact that $\frac{c_m}{c_{m+1}}$ is monotonically decreasing, which we deduced from the stronger assumptions in Theorem 2 (where it was infered from the stronger assumptions in Theorem 2) up to equation (76). Instead of (76), by our weaker assumption $\frac{b_{n+1}}{b_n} > \kappa_d$ instead of $\frac{b_{n+1}}{b_n} > 2 > \kappa_d$, we obtain the weaker upper bound

$$\omega_j^2(X_r) \leq -\log_{X_r}\left(\frac{1}{c_{h_r+d+1-k}}\right) < -\log_{X_r}\left(\frac{c_{h_r+d+1-k}}{c_{h_r+d+2-k}}\right) \leq \omega_{k-d}^1(X_r),$$

which yields $\wp_{k-d}$ as an upper bound for ω_j instead of $\wp_{k-1}$. We proceed analogously again till (79), instead of which we obtain

$$\omega_j^2(X_r) \leq -\log_{X_r}\left(\frac{1}{c_{h_r+d+1-k}}\right) < -\log_{X_r}\left(\frac{c_{h_r+d+1-k}}{c_{h_r+d+2-k}}\right) \leq \omega_{k-d}^1(X_r).$$

This gives $\widehat{\wp}_{k-d}$ as an upper bound for $\widehat{\omega}_j$ instead of $\widehat{\wp}_{k-1}$. The remainder of the proof is essentially the same as in Theorem 2. □

Remark 4. We clearly have $\lim_{d\to\infty} \kappa_d = 1$. On the other hand $1 + \frac{1}{d} < \kappa_d < 2$ which can esily be derived using the well known monotonic convergence of $(1+\frac{1}{n})^n$ to the Euler number $e \approx 2.71$.

Again, we easily deduce the following

Corollary 6. *Let the basic assumptions of Theorem 2 and condition 1 from Theorem 3 be satisfied. Let us further assume*

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = C \geq \kappa_d.$$

Then the first $(k-d)$ approximation constants are given by

$$\begin{aligned}
\omega &= C-1 & &\vdots \\
\omega_2 &= \frac{C-1}{C} = \widehat{\omega} & \omega_{k-d} &= \frac{C-1}{C^{k-d-1}} = \widehat{\omega}_{k-d-1} \\
\omega_3 &= \frac{C-1}{C^2} = \widehat{\omega}_2 & &\frac{C-1}{C^{k-d}} = \widehat{\omega}_{k-d}\,.
\end{aligned}$$

Further more we have the inequalities

$$\begin{aligned}
\omega_{k-d+1} &\geq \frac{C-1}{C^{k-d}} \\
\omega_{k-d+2} \geq \widehat{\omega}_{k-d+1} &\geq \frac{C-1}{C^{k-d+1}} \\
&\vdots \\
\omega_{k+1} \geq \widehat{\omega}_k &\geq \frac{C-1}{C^k} \\
\widehat{\omega}_{k+1} &\geq \frac{C-1}{C^{k+1}}.
\end{aligned}$$

Let us illustrate the results of Corollary 5 and Corollary 6 in the case $k=3$, $C=2$. In the first plot we put

$$\zeta_1' = 2^{-1} + 2^{-15}, \quad \zeta_2' = 2^{-3} + 2^{-31}, \quad \zeta_3' = 2^{-7}$$

which (for numerical purposes) are the initial terms of

$$\zeta_1 = \sum_{n\geq 0} 2^{-2^{3n+1}+1}, \quad \zeta_2 = \sum_{n\geq 0} 2^{-2^{3n+2}+1}, \quad \zeta_3 = \sum_{n\geq 1} 2^{-2^{3n}+1},$$

which clearly satisfy the conditions of Corollary 5 with $C=2$. Notice the special behaviour of $L_k = L_3$, $L_{k+1} = L_4$ in comparison to the first $(k-1) = 2$ functions which behave as predicted in Corollary 5.

The assumptions of Corollary 6 are weaker in the sense that either $C < 2$ or the quotients $\frac{b_{n+1}}{b_n}$ converge to $C = 2$ without being strictly larger than 2 for every sufficiently large n. To illustrate this latter case we may put

$$\zeta_1' = 2^{-2} + 2^{-9}, \quad \zeta_2' = 2^{-3} + 2^{-17}, \quad \zeta_3' = 2^{-5} + 2^{-33}.$$

which are the initial terms of

$$\zeta_1 = \sum_{n\geq 0} 2^{-a_{3n+1}}, \quad \zeta_2 = \sum_{n\geq 0} 2^{-a_{3n+2}}, \quad \zeta_3 = \sum_{n\geq 1} 2^{-a_{3n}}$$

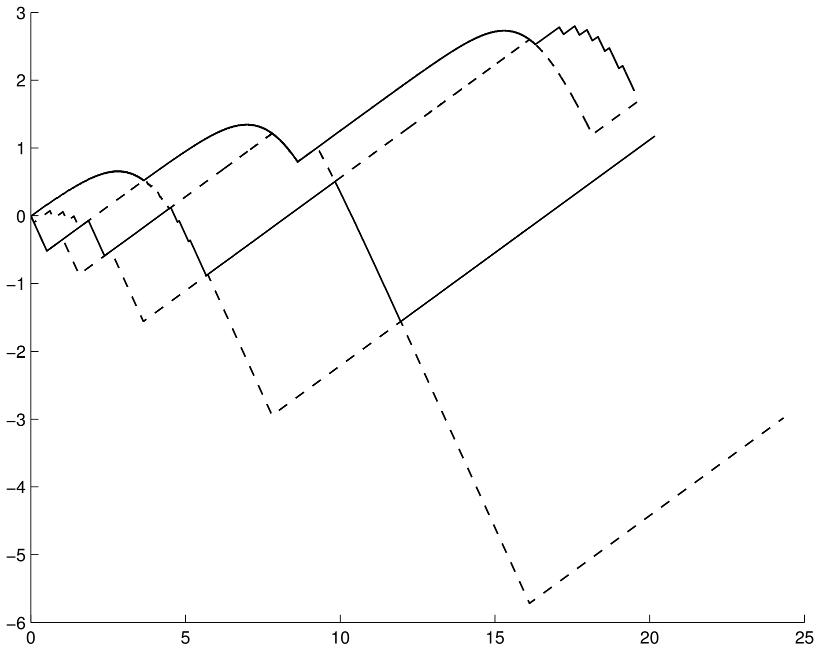

Figure 1: $k = 3$, $C = 2$; illustrates Corollary 5

with $a_1 = 2$ and $a_{n+1} = 2a_n - 1$ for all $n \geq 1$, which fulfills the conditions of Corollary 6 with $d = 2$. Indeed, we will see a different behaviour of L_2 compared to the previous picture. Only $L_{k-d} = L_1$ has the predicted shape. We can apply Corollary 6 to construct many more cases of Schmidt's conjecture explicitely. For simplicity of the proof we first deduce another easy Corollary from Corollary 6.

Corollary 7. *For $k \geq 2$ let $1 \leq d \leq k-1$ be an arbitrary integer. For any $C > \kappa_d$ there exists a sequence of positive integers $(b_n)_{n\geq 1}$ such that $\frac{b_{n+1}}{b_n} > C$ for all n and $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = C$. If $(a_{n,j})_{n\geq 1}$ for $1 \leq j \leq k$ are k sequences satisfying* (55) *such that $(b_n)_{n\geq 1}$ is their ordered mixed sequence, then for $\zeta_j = \sum_{n\geq 1} 2^{-a_{n,j}}$ the result of Corollary 6 is valid.*

Furthermore we can choose the sequence $(b_n)_{n\geq 1}$ such that $1, \zeta_1, \dots, \zeta_k$ are $\mathbb{Q}$-linearly independent.

Proof. The sequence $(b_n)_{n\geq 1}$ defined by $b_1 = S$ and $b_{n+1} = \lceil Cb_n \rceil$ with S sufficiently large that $b_2 > b_1$ clearly satisfies the stated properties. Putting $q_{n,j} = 2^{a_{n,j}}$ we see that the assumptions of Corollary 6 are satisfied, since it clearly makes no difference if we take the logarithm to base 2 instead of e as the quotients $\frac{b_{n+1}}{b_n}$ don't change. By a variation of $(b_n)_{n\geq 1}$ as in the proof of Corollary 3 we can guarantee the linear independence. □

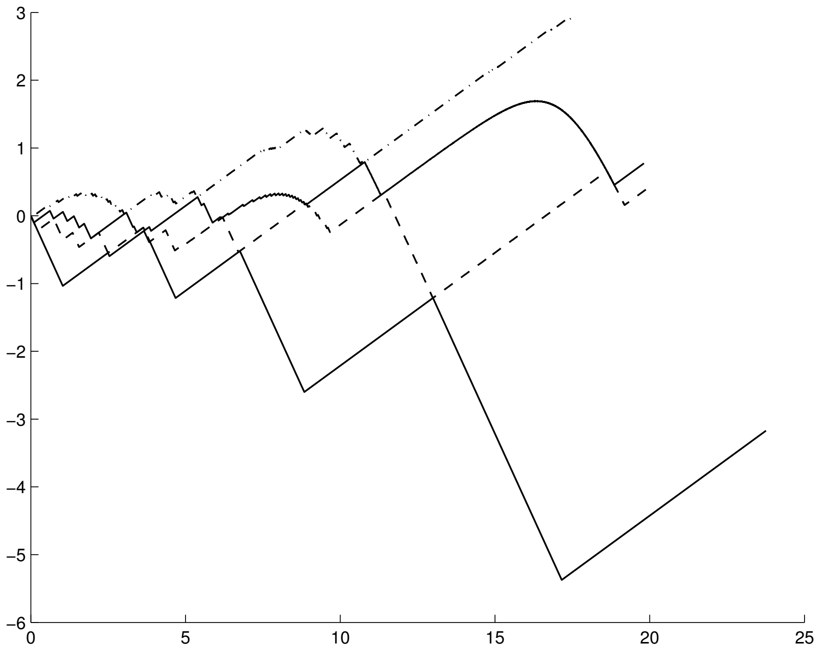

Figure 2: $k = 3$, $C = 2$; illustrates Corollary 6

Corollary 8. *With the notation of Corollary 7 there exists a constant $R(k)$ such that the following holds:*

- $R(k) > \frac{k}{\log(k)}$ *for k sufficiently large.*
- *For fixed $3 \leq T \leq R(k)$ there is some $C_0 = C_0(T)$, such that there exists a sequence $(b_n)_{n\geq 1}$ of positive integers satisfying $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = C_0$ such that the corresponding vector $(\zeta_1, \ldots, \zeta_k)$ constructed via Corollary 7 with $C = C_0$ has approximation constants that satisfy*

$$\overline{\psi}_{T-2} < 0 \quad \textit{and} \quad \underline{\psi}_T > 0. \tag{81}$$

A possible choice of $R(k)$ is $R(k) := k-1+(k-2)\frac{\log\left(k^{\frac{1}{k-2}}-1\right)}{\log(k)}$. This provides explicit examples *for Schmidt's Conjecture.*

Proof. Let k be an arbitrary but fixed integer. In view of (13) for a given $3 \leq T \leq R(k)$ we need to find $C_0 = C_0(T)$ such that a vector $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ that arises from Corollary 7 with $C = C_0$ satisfies $\widehat{\omega}_{T-2} > \frac{1}{k} > \omega_T$ to obtain (81). We will implicitly identify such a vector $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ with the resulting value C from the limit of the quotients $\frac{b_{n+1}}{b_n}$ in Corollary 7. This is well defined as the approximation constants we consider don't depend on the choice of the exact vector but depend only on C by Corollary 7. As was shown there the set of such vectors $(\zeta_1, \zeta_2, \ldots, \zeta_k)$

is nonempty for every $C > \kappa_d > 1$ and we can assume $(\zeta_1, \dots, \zeta_k)$ together with 1 to be $\mathbb{Q}$-linearly independent.

For any positive integer u define the function $\Psi_u(x) = \frac{x-1}{x^u}$. Each function $\Psi_.$ is easily seen to be continuous and Ψ_u increases on $[1, \frac{u}{u-1}]$ and decreases on $[\frac{u}{u-1}, \infty)$ with limit 0 as $x \to \infty$.

We use the notation of Theorem 3, in particular κ_d is the largest real root of $P_d(x) = x^d - x^{d-1} - 1$. We first prove that we can choose $C_0 = C_0(T)$ such that (81) holds for a given T that has the property

$$\Psi_{T-1}(\kappa_{k-T}) > \frac{1}{k} \tag{82}$$

with the constructions of Corollary 7 and the particular choice $C = C_0$.

Put $u = T-1$. If (82) is valid, the facts about the functions Ψ_u show that there is $x > \kappa_{k-T}$ such that $\Psi_{T-1}(x) = \frac{1}{k}$ and Ψ_{T-1} already decreases at x. Furthermore there is an interval $C_0 \in (x, x+\delta)$ such that $\Psi_{T-1}(C_0) < \frac{1}{k} < \Psi_{T-2}(C_0)$. Since $C_0 > x \geq \kappa_{k-T}$ we can apply Corollary 7 with $d := k-T, C := C_0$ and obtain

$$\omega_T = \Psi_{T-1}(C_0) < \frac{1}{k} < C_0\frac{1}{k} = C_0\Psi_{T-1}(C_0) = \Psi_{T-2}(C_0) < \widehat{\omega}_{T-2}\,,$$

as intended.

Now assume k is fixed, $1 \leq T \leq k+1$ and that for $C_0 := \kappa_{k-T}$

$$\frac{C_0 - 1}{C_0^{k-1} - C_0^{k-2}} > \frac{1}{k} \tag{83}$$

holds. By definition of $C_0 = \kappa_{k-T}$ we have $C_0^{T-1} = C_0^{k-1} - C_0^{k-2}$ and (83) further implies

$$\psi_{T-1}(C_0) = \frac{C_0 - 1}{C_0^{T-1}} = \frac{C_0 - 1}{C_0^{k-1} - C_0^{k-2}} > \frac{1}{k}$$

for such T, i.e. (82). Combining what we have shown so far, (81) can be obtained for T and $C_0(T) = \kappa_{k-T}$, provided (83) holds. We now show that (83) is true for $3 \leq T \leq R(k)$.

In view of $C_0 > 1$, inequality (83) is equivalent to $\kappa_{k-T} = C_0 \leq k^{\frac{1}{k-2}}$. Since P_{k-T}, whose largest root is $\kappa_{k-T} > 1$, increases on the interval $[1, \infty)$, this is equivalent to $P_{k-T}(k^{\frac{1}{k-2}}) \geq 0$, i.e.

$$k^{\frac{k-T}{k-2}} - k^{\frac{k-T-1}{k-2}} - 1 > 0\,.$$

Basic rearrangements show this is equivalent to

$$T \leq k - 1 + (k-2)\frac{\log\big(k^{\frac{1}{k-2}} - 1\big)}{\log(k)} =: R(k)\,.$$

To finish up the proof we are left to show that $R(k) > \frac{k}{\log(k)}$ holds for k sufficiently large. To see this we claim that for sufficiently large k we have

$$R(k) > \frac{k}{2}\left[1 + \frac{\log(k^{\frac{1}{k-2}} - 1)}{\log(k)}\right] > \frac{k}{2}\left[1 + \frac{\log(k^{\frac{1}{k}} - 1)}{\log(k)}\right] > \frac{k}{\log(k)}\,.$$

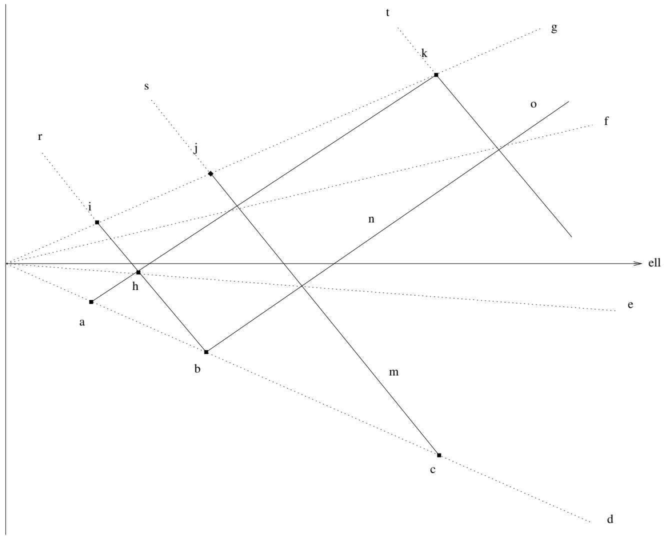

Figure 3: $\underline{\psi}_{j+1} = \overline{\psi}_j$, special case

For $k \geq 4$ only the right most inequality is non trivial. It is equivalent to $k(k^{\frac{1}{k}} - 1) > e^2$ or $k^{\frac{1}{k}} > \frac{e^2}{k} + 1$. However, this is easily seen to be true by $k^{\frac{1}{k}} > 1 + \frac{M}{k}$ for $k \geq k_0(M)$ for arbitrary M, which follows from the monotonic convergence of $\lim_{k\to\infty}(1 + \frac{M}{k})^k = e^M$ for every M. Putting $M > e^2$ arbitrary proves the claim. □

Remark 5. We have $\lim_{k\to\infty} \frac{R(k)}{k} = 0$, but the convergence is very slow, in particular we have seen it is slower than $\frac{1}{\log(k)}$.

3 The case $\underline{\psi}_{j+1} = \overline{\psi}_j$

In this section we want to treat the case of $\zeta_1, \zeta_2, \ldots, \zeta_k$ that define functions ψ_j with the property

$$\underline{\psi}_{j+1} = \overline{\psi}_j, \qquad 1 \leq j \leq k. \tag{84}$$

We will exclude the generic case where all approximation constants $\underline{\psi}_j, \overline{\psi}_j$ are zero, and call the cases where $\underline{\psi}_1 = -1$ the *degenerate cases*. Note, that the numbers Corollary 5 deals with lead to functions ψ_j that satisfy the equalities of (84) for $1 \leq j \leq k-2$, but not for $j \in \{k-1, k\}$ in the case $C < \infty$ (see the bounds given in the remark following Corollary 5). A *special case* of (84), for which an idealized picture is shown below, is of particular interest. If $(x^{(i)})_{i\geq 1}$ denotes the sequence of the first coordinates of approximation vectors $(x, y_1, \ldots, y_k)$, then we consider the special case where there is a sequence of $(x^{(i)})_{i\geq 1}$ with (ideally) constant value of $\frac{\log(x^{(i+1)})}{\log(x^{(i)})}$ such that

$$\max_{1\leq j\leq k} |\zeta_j x^{(i)} - y_j| \sim (x^{(i)})^{-\omega}.$$

This sequence causes the second minimum to attain the value ω_2, and so on. Thus in particular we have

$$\lim_{i\to\infty} \frac{\log(x^{(i+1)})}{\log(x^{(i)})} = \frac{\omega_j}{\omega_{j+1}}, \qquad 1 \leq j \leq k \tag{85}$$

$$\lim_{i\to\infty} -\frac{\log\bigl(\max_{1\leq t\leq k}|x^{(i)}\zeta_t - y_t^{(i)}|\bigr)}{\log(x^{(i)})} = \omega. \tag{86}$$

Let the equistence of a sequence $(x^{(i)})_{i\geq 1}$ such that (85), (86) holds and additionally for $i \geq i_0$ every $k+1$ consecutive approximation vectors $(x^{(j)}, y_1^{(j)}, \ldots, y_k^{(j)})$ belonging to $x^{(i)}, x^{(i+1)}, \ldots, x^{(i+k)}$ (i.e. $j \in \{i, i+1, \ldots, i+k\}$) are linearly independent be our definition of the *special case* mentioned above.

Roy shows, that numbers he defines as *extremal numbers* ζ in the introduction of [4] satisfy the property of the special case of (84) for $k = 2$ and $\zeta_1 = \zeta, \zeta_2 = \zeta^2$ and yield $\omega_j = \gamma^{j-1}$ for $1 \leq j \leq 3$ and $\widehat{\omega}_3 = \gamma^3$ with $\gamma := \frac{\sqrt{5}-1}{2}$. We are interested in other particular cases of the special case of (84).

It follows in general by (85), (86) that all the values $\underline{\psi}_., \overline{\psi}_.$ are determined by the value $\underline{\psi}_1$ (or equivalently ω). This holds for the degenerate case in particular. However, we will show in Theorem 4 that this phenomenon holds for *all* $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ in the degenerate case of (84). By virtue of Corollary 5 we can easily provide concrete examples for the degenerate case. Before we do so, for the sake of completeness we give a general result about the degenerate case of (84).

Proposition 4. *Assume the approximation functions arising from $\zeta_1, \ldots, \zeta_k$ satisfy $\underline{\psi}_1 = -1$ and (84). Then they already satisfy*

$$\underline{\psi}_1 = -1 \tag{87}$$

$$\overline{\psi}_1 = \frac{1-k}{2k} = \underline{\psi}_2 \tag{88}$$

$$\overline{\psi}_j = \frac{1}{k} = \underline{\psi}_{j+1}, \qquad 2 \leq j \leq k, \tag{89}$$

$$\overline{\psi}_{k+1} = \frac{1}{k}, \tag{90}$$

and hence in particular fall under the special case.

Proof. First note that in general if $\underline{\psi}_1 = -1$ we have $\overline{\psi}_j = \frac{1}{k}$ for $2 \leq j \leq k+1$ by means of (6), see the proof of Theorem 1. Consequently by (84) we have (89) and (90).

For (88) note first that $\frac{1-k}{2k}$ is always a lower bound as established in (8), (9). So by (10) it suffices to prove $\overline{\psi}_1 \leq \frac{1-k}{2k}$.

Suppose we had $\overline{\psi}_1 > \frac{1-k}{2k}$. This means for some sequence $(q_n)_{n\geq 1}$ tending to infinity we have $\psi_1(q_n) > V$ for some $V > \frac{1-k}{2k}$. Putting $V_0 := 2(V - \frac{1-k}{2k}) > 0$ and using $\psi_2(q_n) \geq \psi_1(q_n)$ we have

$$\lim_{n\to\infty} \sum_{j=1}^{k+1} \psi_j(q_n) > 2V + (k-1)\left[\frac{1}{k} - \frac{V_0}{2(k-1)}\right] = \frac{V_0}{2} > 0,$$

a contradiction to (6) since $\lim_{n\to\infty} q_n = \infty$. □

Theorem 4. *For any $k \geq 2$ there exist real numbers $\zeta_1, \zeta_2, \ldots, \zeta_k$ as in Corollary 5 with $C = \infty$ together with 1 linearly independent over $\mathbb{Q}$ that satisfy the degenerate case of (84) and hence (87)–(90) by Proposition 4.*

Proof. Using (13) we obtain the equivalent system

$$\omega = \infty \tag{91}$$

$$\widehat{\omega} = 1 = \omega_2 \tag{92}$$

$$\widehat{\omega}_j = 0 = \omega_{j+1}, \qquad 2 \leq j \leq k \tag{93}$$

$$\widehat{\omega}_{k+1} = 0\,. \tag{94}$$

In the case $k \geq 3$, we can just apply Corollary 5 with $C = \infty$ because then we obviously have

$$\lim_{C\to\infty} \frac{C-1}{C} = 1, \qquad \lim_{C\to\infty} \frac{C-1}{C^j} = 0, \quad j \geq 2,$$

which gives (91)–(94). However, in the case $k = 2$ and $C = \infty$ we can also apply Corollary 5 with a slightly more sophisticated argumentation. Of course we directly infer (91) and the left equation in (92) follows as for $k \geq 3$. From (91) we can immediately deduce $\widehat{\omega}_j = 0$ for $j = 2, 3$ as in the proof of Theorem 1, which is a rephrasing of the left hand side of (93) and (94). By the left equation in (92) and Jarník's identity $\omega_3 + \widehat{\omega} = 1$ (see comments between the end of proof of Theorem 1 and Corollary 2) we get $\omega_3 = 0$, i.e. the right hand side of (93). For the missing right hand equation of (92) note that $\omega_2 \geq \widehat{\omega}$ is always true by (10) and on the other hand $\omega_2 \leq 1$ by (15), so by the left hand equality in (92) we infer the right hand equality of (92). □

This allows to show that the bounds in (14)–(19) are best possible if considered independently by using only three types of vectors $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ depending on the dimension k. These types are:

- a set of together with 1 $\mathbb{Q}$-linearly independent *algebraic* numbers $\zeta_1, \zeta_2, \ldots, \zeta_k$ (leading to the generic case)
- $\zeta_1, \zeta_2, \ldots, \zeta_k$ as in Corollary 5 with $C = \infty$, for example $\zeta_j = \sum_{n\geq 1} \frac{1}{(nk+j)!}$ for $1 \leq j \leq k$
- $\zeta_1, \zeta_2, \ldots, \zeta_k$ as in Corollary 3

Corollary 9. *The bounds (14)–(19) are all (each for itself) optimal among $(\zeta_1, \ldots, \zeta_k)$ that are $\mathbb{Q}$-linearly independent together with 1.*

Proof. In Corollary 3 we have seen, that the upper bounds in (14), (15), (16) as well as the lower bounds in (17), (18), (19) cannot be improved.

In Theorem 4 we've just seen, that the left hand side of (16) and the right hand side of (17) are optimal.

However, all the other bounds are $1/k$ and it is well known that all constants $\omega_j, \widehat{\omega}_j$ are equal to $1/k$ in the generic case. To give concrete examples, an implication of Schmidt's subspace theorem says, that for all $\mathbb{Q}$-linearly independent algebraic numbers all approximation constants take the value $1/k$ (which follows already from $\omega = 1/k$ by (13) and (6)). So the lower bounds of (14), (15) such as the upper bounds of (18), (19) cannot be improved either, and the list is complete. □

Let $\zeta_1, \zeta_2, \dots, \zeta_k$ be real numbers that lead to a special case of (84), i.e. (85), (86) hold. It follows directly from (85), (86) that all the constants $\omega_j, \widehat{\omega}_j$ only depend on ω. It is easy to check that more precisely we have

$$\frac{(1+\omega)^{k+1}}{\omega} = \frac{(1+\widehat{\omega}_{k+1})^{k+1}}{\widehat{\omega}_{k+1}} \tag{95}$$

$$\omega_j = \omega^{1-\frac{j-1}{k+1}} \widehat{\omega}_{k+1}^{\frac{j-1}{k+1}}, \qquad 1 \le j \le k+1. \tag{96}$$

Using this we now prove a lower bound for $\widehat{\omega}$ in dependence of ω.

Proposition 5. *In the special case of (84) for $k \ge 2$ we have*

$$\frac{\omega}{\omega+1} < \widehat{\omega} \le 1.$$

Proof. The right hand side inequality is just (17).

Suppose for some $k \ge 2$ we had $\widehat{\omega} \le \frac{\omega}{\omega+1}$. Putting $j = 2$ in (96) (note $\omega_2 = \widehat{\omega}$ by definition) we have

$$\widehat{\omega}_{k+1} \le \left[\left(\frac{\omega}{\omega+1}\right)\omega^{-\frac{k}{k+1}}\right]^{k+1} = \frac{\omega}{(\omega+1)^{k+1}}. \tag{97}$$

Denote

$$f_k(x) := \frac{(x+1)^{k+1}}{x}, \qquad k \ge 1.$$

Differentiating shows that f_k decreases on $x \in (0, \frac{1}{k})$ and increases on $x \in (\frac{1}{k}, \infty)$, so its global minimum on $(0, \infty)$ is at $x = \frac{1}{k}$. Combining this with $\widehat{\omega}_{k+1} < \frac{1}{k}$, (95) and (97) we obtain

$$f_k(\omega) = f_k(\widehat{\omega}_{k+1}) \ge f_k\left(\frac{1}{f_k(\omega)}\right), \qquad k \ge 1.$$

Putting $z := \frac{1}{f_k(\omega)}$ this gives $\frac{1}{z} \ge f_k(z)$, which is false, as $\frac{1}{z}$ is an expression in the binomial expansion of $f_k(z)$. □

Remark 6. One can prove that for $k \ge 2$ we have $\lim_{\omega \to \infty} \omega + 1 - \frac{\omega}{\widehat{\omega}} = 0$.

Observe, that in Corollary 6 with arbitrary C we always have $\widehat{\omega} = \frac{\omega}{\omega+1}$. Proposition 5 shows that given ω the resulting special case of (84) leads to a larger value of $\widehat{\omega}$. It may be conjectured that among all $\zeta_1, \zeta_2, \dots, \zeta_k$ linearly independent together with 1 with prescribed $\omega = \omega_0$, the quantity $\widehat{\omega}$ is maximised for the special case of (84) with the value $\omega = \omega_0$.

Remark 7. Observe that the inequality $\frac{\widehat{\omega}^2}{1-\widehat{\omega}} \leq \omega$ always holds as established by Jarník, see Theorem 1 page 331 in [2]. So together with Proposition 5 in the special case of (84) we have

$$\frac{\widehat{\omega}^2}{1-\widehat{\omega}} \leq \omega \leq \frac{\widehat{\omega}}{1-\widehat{\omega}}\,.$$

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The conformal change of the metric of an almost Hermitian manifold applied to the antiholomorphic curvature tensor

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Abstract. By using the technique of decomposition of a Hermitian vector space under the action of a unitary group, Ganchev [2] obtained a tensor which he named *the Weyl component of the antiholomorphic curvature tensor*. We show that the same tensor can be obtained by direct application of the conformal change of the metric to the antiholomorphic curvature tensor. Also, we find some other conformally curvature tensors and examine some relations between them.

1 Introduction

Let (M, g, J) be an almost Hermitian manifold, $\dim M = 2n \geq 4$, with the complex structure J and the Hermitian metric g, i.e., $J^2 = -\operatorname{Id}$, $g(JX, JY) = g(X, Y)$ for all $X, Y \in T_p(M)$, where $T_p(M)$ is the tangent vector space of M at $p \in M$. Then the fundamental 2-form is $F(X, Y) = g(JX, Y) = -F(Y, X)$.

We consider the tensors having the standard symmetries of the curvature tensors of a Riemannian manifold. In [7] the linear space $\mathcal{R}(V)$ of such tensors over a $2n$-dimensional Hermitian vector space V was decomposed into irreducible components under the action of the unitary group. Furthermore, all conformally invariant subspaces of $\mathcal{R}(V)$ were found.

In [1] and [2] the holomorphic and antiholomorphic curvature tensors for an almost Hermitian manifold are introduced, and, using the same technique as in the paper [7], the *generalized Bochner curvature tensor* and the *Weyl component of the antiholomorphic curvature tensor* are obtained.

We ask the question: Is it possible to get these tensors in the classical way, i.e., by direct application of the conformal change of this metric?

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Key words: Almost Hermitian manifold, antiholomorphic curvature tensor, conformal change of metric

In [6], we examined conformally invariant tensors associated with holomorphic curvature tensor, and we found, among others, the generalized Bochner curvature tensor.

In the present paper, we deal with the antiholomorphic curvature tensor and, in Section 3, we find the Weyl component of the antiholomorphic curvature tensor. In Sections 2 and 4, we determine some other conformally invariant tensors. In Section 5 we find some relations between obtained conformally invariant tensors, and in Section 6 we discuss the case of Kähler manifold.

We denote by R the Riemannian curvature tensor. Then the first and the second Ricci tensors are defined as follows: $\rho(X,Y) = \sum_{i=1}^{2n} R(e_i, X, Y, e_i)$, and $\overset{*}{\rho}(X,Y) = \sum_{i=1}^{2n} R(e_i, X, JY, Je_i)$, where $\{e_i\}$ is an orthonormal basis of $T_p(M)$. Finally, the first and the second scalar curvatures are

$$\tau = \sum_{i=1}^{2n} \rho(e_i, e_i), \quad \overset{*}{\tau} = \sum_{i=1}^{2n} \overset{*}{\rho}(e_i, e_i)\,.$$

In general, the second Ricci tensor is not symmetric, but it satisfies the condition

$$\overset{*}{\rho}(JX, JY) = \overset{*}{\rho}(Y, X)\,. \tag{1}$$

To make some formulas clearer we use the following notations:

$$\begin{aligned}
\pi_1(X,Y,Z,W) &= g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\,,\\
\pi_2(X,Y,Z,W) &= F(X,W)F(Y,Z) - F(X,Z)F(Y,W) - 2F(X,Y)F(Z,W)\,,\\
\varphi(X,Y,Z,W) &= g(X,W)\rho(Y,Z) + g(Y,Z)\rho(X,W)\\
&\quad - g(X,Z)\rho(Y,W) - g(Y,W)\rho(X,Z),\\
(J\varphi)(X,Y,Z,W) &= \varphi(JX,JY,JZ,JW) = g(X,W)\rho(JY,JZ) + g(Y,Z)\rho(JX,JW)\\
&\quad - g(X,Z)\rho(JY,JW) - g(Y,W)\rho(JX,JZ)\,,\\
\psi(X,Y,Z,W) &= F(X,W)\big[\rho(Y,JZ) - \rho(JY,Z)\big] + F(Y,Z)\big[\rho(X,JW)\\
&\quad - \rho(JX,W)\big] - F(X,Z)\big[\rho(Y,JW) - \rho(JY,W)\big]\\
&\quad - F(Y,W)\big[\rho(X,JZ) - \rho(JX,Z)\big] - 2F(X,Y)\big[\rho(Z,JW)\\
&\quad - \rho(JZ,W)\big] - 2F(Z,W)\big[\rho(X,JY) - \rho(JX,Y)\big]\,,\\
\overset{*}{\varphi}(X,Y,Z,W) &= g(X,W)\big[\overset{*}{\rho}(Y,Z) + \overset{*}{\rho}(Z,Y)\big] + g(Y,Z)\big[\overset{*}{\rho}(X,W) + \overset{*}{\rho}(W,X)\big]\\
&\quad - g(X,Z)\big[\overset{*}{\rho}(Y,W) + \overset{*}{\rho}(W,Y)\big] - g(Y,W)\big[\overset{*}{\rho}(X,Z) + \overset{*}{\rho}(Z,X)\big]\,,\\
\overset{*}{\psi}(X,Y,Z,W) &= F(X,W)\big[\overset{*}{\rho}(Y,JZ) - \overset{*}{\rho}(JY,Z)\big] + F(Y,Z)\big[\overset{*}{\rho}(X,JW)\\
&\quad - \overset{*}{\rho}(JX,W)\big] - F(X,Z)\big[\overset{*}{\rho}(Y,JW) - \overset{*}{\rho}(JY,W)\big]\\
&\quad - F(Y,W)\big[\overset{*}{\rho}(X,JZ) - \overset{*}{\rho}(JX,Z)\big]\\
&\quad - 2F(X,Y)\big[\overset{*}{\rho}(Z,JW) - \overset{*}{\rho}(JZ,W)\big]\\
&\quad - 2F(Z,W)\big[\overset{*}{\rho}(X,JY) - \overset{*}{\rho}(JX,Y)\big]\,.
\end{aligned} \tag{2}$$

A $(0,4)$ tensor $T(X,Y,Z,W)$ is said to be generalized curvature tensor if it satisfies all algebraic properties of the Riemannian curvature tensor, i.e.,

$$T(X,Y,Z,W) = -T(Y,X,Z,W) = -T(X,Y,W,Z) = T(Z,W,X,Y)\,,$$
$$T(X,Y,Z,W) + T(Y,Z,X,W) + T(Z,X,Y,W) = 0\,.$$

All tensors in the relations (2) are generalized curvature tensors. Hence we can consider the corresponding first Ricci tensor and the second Ricci tensor of each of them, and denote them as $\rho(T)$ and $\overset{*}{\rho}(T)$, respectively. In particular, we have $\rho(R) = \rho$ and $\overset{*}{\rho}(R) = \overset{*}{\rho}$.

For the latter use, we note that

$$\begin{aligned}
\rho(\pi_1)(X,Y) &= (2n-1)g(X,Y), \quad \rho(\pi_2)(X,Y) = 3g(X,Y)\,,\\
\rho(\varphi)(X,Y) &= 2(n-1)\rho(X,Y) + \tau g(X,Y)\,,\\
\rho(J\varphi)(X,Y) &= 2(n-1)\rho(JX,JY) + \tau g(X,Y)\,,\\
\rho(\psi)(X,Y) &= -6[\,\rho(X,Y) + \rho(JX,JY)]\,,\\
\rho(\overset{*}{\varphi})(X,Y) &= 2(n-1)\big[\,\overset{*}{\rho}(X,Y) + \overset{*}{\rho}(Y,X)\big] + 2\overset{*}{\tau} g(X,Y)\,,\\
\rho(\overset{*}{\psi})(X,Y) &= -6\big[\,\overset{*}{\rho}(X,Y) + \overset{*}{\rho}(Y,X)\big]\,,\\
\overset{*}{\rho}(\pi_1)(X,Y) &= g(X,Y), \quad \overset{*}{\rho}(\pi_2)(X,Y) = (2n+1)g(X,Y)\,,\\
\overset{*}{\rho}(\varphi)(X,Y) &= \rho(X,Y) + \rho(JX,JY)\,,\\
\overset{*}{\rho}(J\varphi)(X,Y) &= \rho(X,Y) + \rho(JX,JY)\,,\\
\overset{*}{\rho}(\psi)(X,Y) &= -2(n-1)[\,\rho(X,Y) + \rho(JX,JY)] - 2\tau g(X,Y)\,,\\
\overset{*}{\rho}(\overset{*}{\varphi})(X,Y) &= 2[\,\overset{*}{\rho}(X,Y) + \overset{*}{\rho}(Y,X)]\,,\\
\overset{*}{\rho}(\overset{*}{\psi})(X,Y) &= -2(n-1)[\,\overset{*}{\rho}(X,Y) + \overset{*}{\rho}(Y,X)] - 2\overset{*}{\tau}(X,Y)\,.
\end{aligned} \tag{3}$$

2 Some conformally invariant tensors

For an almost Hermitian manifold (M,g,J), we consider the conformal change of metric $\bar{g} = e^{2f}g$, where f is a scalar function. If ∇ and $\bar{\nabla}$ are the Levi-Civita connections with respect to the metrics g and $\bar{g}$ respectively, we have

$$(\bar{\nabla} - \nabla)(X,Y) = \theta(X)(Y) + \theta(Y)(X) - g(X,Y)U\,,$$

for any vector fields $X, Y \in T(M)$, where $\theta = df$, and U is the vector field such that $g(U,X) = \theta(X)$.

From now on, all geometric objects in $(M,\bar{g},J)$ will be denoted by analogous letters as in (M,g,J), but with "bar".

It is well known (see e.g., [3]) that the Riemannian curvature tensors $\bar{R}$ and R are related as follows

$$\begin{aligned}
e^{-2f}\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) &+ g(X,W)\sigma(Y,Z) + g(Y,Z)\sigma(X,W)\\
&- g(X,Z)\sigma(Y,W) - g(Y,W)\sigma(X,Z)
\end{aligned} \tag{4}$$

where

$$\sigma(X,Y) = (\nabla_X\theta)(Y) - \theta(X)\theta(Y) + \frac{1}{2}g(X,Y)\theta(U)\,.$$

We note that $\sigma(X,Y)=\sigma(Y,X)$.

The relation (4) implies

$$\bar{\rho}(Y,Z)=\rho(Y,Z)+2(n-1)\sigma(Y,Z)+g(Y,Z)\sigma\,,$$

where $\sigma=\sum_{i=1}^{2n}\sigma(e_i,e_i)$. Therefore

$$\sigma=\frac{e^{2f}\bar{\tau}-\tau}{2(2n-1)}\,, \tag{5}$$

and

$$\begin{aligned}\sigma(Y,Z)=&\frac{1}{2(n-1)}\Big[\bar{\rho}(Y,Z)-\frac{\bar{\tau}}{2(2n-1)}\bar{g}(Y,Z)\Big]\\&-\frac{1}{2(n-1)}\Big[\rho(Y,Z)-\frac{\tau}{2(2n-1)}g(Y,Z)\Big]\,.\end{aligned} \tag{6}$$

Thus

$$\begin{aligned}\sigma(Y,Z)+\sigma(JY,JZ)=&\frac{1}{2(n-1)}\big[\bar{\rho}(Y,Z)+\bar{\rho}(JY,JZ)\big]-\frac{\bar{\tau}}{2(n-1)(2n-1)}\bar{g}(Y,Z)\\&-\frac{1}{2(n-1)}\big[\rho(Y,Z)+\rho(JY,JZ)\big]\\&+\frac{\tau}{2(n-1)(2n-1)}g(Y,Z)\,.\end{aligned} \tag{7}$$

On the other hand, the relation (4) implies

$$\begin{aligned}e^{-2f}\bar{R}(X,Y,JZ,JW)=&R(X,Y,JZ,JW)-F(X,W)\sigma(Y,JZ)\\&-F(Y,Z)\sigma(X,JW)+F(X,Z)\sigma(Y,JW)\\&+F(Y,W)\sigma(X,JZ)\,,\end{aligned}$$

wherefrom it follows

$$\sigma(Y,Z)+\sigma(JY,JZ)=\bar{\overset{*}{\rho}}(Y,Z)-\overset{*}{\rho}(Y,Z)\,, \tag{8}$$

or

$$\sigma(Y,Z)+\sigma(JY,JZ)=\frac{1}{2}\big[\bar{\overset{*}{\rho}}(Y,Z)+\bar{\overset{*}{\rho}}(Z,Y)\big]-\frac{1}{2}\big[\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)\big] \tag{9}$$

and

$$\sigma=\frac{1}{2}\left(e^{2f}\bar{\overset{*}{\tau}}-\overset{*}{\tau}\right)\,. \tag{10}$$

Comparing (7) and (9), as well as (5) and (10), we find

$$\begin{aligned}&\frac{1}{n-1}\big[\bar{\rho}(Y,Z)+\bar{\rho}(JY,JZ)\big]-\big[\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)\big]-\frac{\bar{\tau}}{(n-1)(2n-1)}\bar{g}(Y,Z)\\&=\frac{1}{n-1}\big[\rho(Y,Z)+\rho(JY,JZ)\big]-\big[\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)\big]-\frac{\tau}{(n-1)(2n-1)}g(Y,Z),\end{aligned} \tag{11}$$

and

$$e^{2f}\left(\bar{\overset{*}{\tau}}-\frac{\bar{\tau}}{2n-1}\right)=\overset{*}{\tau}-\frac{\tau}{2n-1}\,. \tag{12}$$

The relation (11) shows that the tensor

$$\begin{aligned}V(Y,Z)=&\frac{1}{n-1}[\,\rho(Y,Z)+\rho(JY,JZ)]-[\,\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)]\\&-\frac{\tau}{(n-1)(2n-1)}g(Y,Z)\end{aligned} \tag{13}$$

is conformally invariant, i.e., $\bar{V}(Y,Z)=V(Y,Z)$. Thus

$$e^{-2f}\bar{g}(X,W)\bar{V}(Y,Z)=g(X,W)V(Y,Z)$$

and therefore

$$\begin{aligned}&e^{-2f}\left\{\bar{g}(X,W)\bar{V}(Y,Z)+\bar{g}(Y,Z)\bar{V}(X,W)-\bar{g}(X,Z)\bar{V}(Y,W)-\bar{g}(Y,W)\bar{V}(X,Z)\right\}\\&\quad=g(X,W)V(Y,Z)+g(Y,Z)V(X,W)-g(X,Z)V(Y,W)-g(Y,W)V(X,Z)\,.\end{aligned}$$

This relation, in view of (13) and (2), can be rewritten in the form

$$\begin{aligned}&e^{-2f}\left[\frac{1}{n-1}(\bar{\varphi}+J\bar{\varphi})-\bar{\overset{*}{\varphi}}-\frac{2\bar{\tau}}{(n-1)(2n-1)}\bar{\pi}_1\right]\\&\qquad\qquad=\frac{1}{n-1}(\varphi+J\varphi)-\overset{*}{\varphi}-\frac{2\tau}{(n-1)(2n-1)}\pi_1\,.\end{aligned}$$

This means that the tensor

$$W_3=\frac{1}{n-1}(\varphi+J\varphi)-\overset{*}{\varphi}-\frac{2\tau}{(n-1)(2n-1)}\pi_1 \tag{14}$$

satisfies the condition

$$e^{-2f}\bar{W}_3=W_3\,. \tag{15}$$

Since all the tensors $\varphi, J\varphi, \overset{*}{\varphi}$ and π_1 are generalized curvature tensors, the tensor (14) is also a generalized curvature tensor.

In a similar way, we find that the tensors

$$\begin{gathered}W_4=\frac{1}{n-1}\psi-\overset{*}{\psi}+\frac{2\tau}{(n-1)(2n-1)}\pi_2\\W_5=\left(\frac{\tau}{2n-1}-\overset{*}{\tau}\right)\pi_1\qquad\text{and}\qquad W_6=\left(\frac{\tau}{2n-1}-\overset{*}{\tau}\right)\pi_2\end{gathered} \tag{16}$$

satisfy the conditions

$$e^{-2f}\bar{W}_4=W_4,\quad e^{-2f}\bar{W}_5=W_5\quad\text{and}\quad e^{-2f}\bar{W}_6=W_6 \tag{17}$$

respectively.

Thus, we can state the following theorem.

Theorem 1. *For an almost Hermitian manifold we have*

1. *the tensor* (13) *is conformally invariant;*
2. *the generalized curvature tensors* (16) *satisfy the conditions* (17) *respectively.*

3 The Weyl component of the antiholomorphic curvature tensor

A 2-plane $\alpha \in T_p(M)$ is said to be *holomorphic* if $J\alpha = \alpha$ and *antiholomorphic* if $J\alpha \perp \alpha$. The sectional curvatures with respect to such 2-planes are *holomorphic* and *antiholomorphic* respectively. The holomorphic sectional curvatures can be examined using the *holomorphic curvature tensor* (see e.g., [2], [4], [5]). In [6] we found some conformally invariant tensors associated with holomorphic curvature tensor.

To examine the antiholomorphic sectional curvatures, G. Ganchev [1], [2] introduced *antiholomorphic curvature tensor* which is

$$\begin{aligned}(AR)(X,Y,Z,W) = R(X,Y,Z,W)& \\ + \frac{1}{2n+2}\Big\{&F(X,W)\overset{*}{\rho}(Y,JZ) + F(Y,Z)\overset{*}{\rho}(X,JW) \\ &- F(X,Z)\overset{*}{\rho}(Y,JW) - F(Y,W)\overset{*}{\rho}(X,JZ) \\ &- 2F(X,Y)\overset{*}{\rho}(Z,JW) - 2F(Z,W)\overset{*}{\rho}(X,JY)\Big\} \\ + \frac{\overset{*}{\tau}}{(2n+2)(2n+1)}&\pi_2(X,Y,Z,W)\,.\end{aligned} \tag{18}$$

The corresponding Ricci tensor is

$$\begin{aligned}\rho(AR)(X,Y) &\equiv \sum_{i=1}^{2n}(AR)(e_i,X,Y,e_i) \\ &= \rho(X,Y) - \frac{3}{2n+2}\big[\overset{*}{\rho}(X,Y) + \overset{*}{\rho}(Y,X)\big] + \frac{3\overset{*}{\tau}}{(2n+2)(2n+1)}g(X,Y)\,,\end{aligned} \tag{19}$$

and therefore the corresponding scalar curvature is

$$\tau(AR) = \sum_{i=1}^{2n}\rho(AR)(e_i,e_i) = \tau - \frac{3\overset{*}{\tau}}{2n+1}\,, \tag{20}$$

while the second Ricci tensor vanishes.

In this section we apply the conformal change of the metric to the tensor (18). Let $A\bar{R}$ be the antiholomorphic curvature tensor with respect to the metric $\bar{g}$. Then, using (4), (8), and (10), we get

$$\begin{aligned}e^{-2f}(A\bar{R})(X,Y,Z,W) &= (AR)(X,Y,Z,W) + g(X,W)\sigma(Y,Z) \\ &+ g(Y,Z)\sigma(X,W) - g(X,Z)\sigma(Y,W) - g(Y,W)\sigma(X,Z) \\ &+ \frac{1}{2n+2}\Big\{F(X,W)[\sigma(Y,JZ) - \sigma(JY,Z)] \\ &\quad + F(Y,Z)[\sigma(X,JW) - \sigma(JX,W)] - F(X,Z)[\sigma(Y,JW) - \sigma(JY,W)] \\ &\quad - F(Y,W)[\sigma(X,JZ) - \sigma(JX,Z)] - 2F(X,Y)[\sigma(Z,JW) - \sigma(JZ,W)] \\ &\quad - 2F(Z,W)[\sigma(X,JY) - \sigma(JX,Y)]\Big\} \\ &+ \frac{\sigma}{(n-1)(2n+1)}\pi_2(X,Y,Z,W)\,.\end{aligned} \tag{21}$$

To determine $\sigma(X,Y)$, we put into (21) $X=W=e_i$, sum up and obtain

$$\begin{aligned}\rho(A\bar{R})(Y,Z)-\rho(AR)(Y,Z)=&\frac{2n^2-5}{n+1}\sigma(Y,Z)-\frac{3}{n+1}\sigma(JY,JZ)\\&+\frac{2n^2+3n+4}{(n+1)(2n+1)}g(Y,Z)\sigma\,.\end{aligned}\tag{22}$$

It follows from (22) that

$$e^{2f}\tau(A\bar{R})-\tau(AR)=\frac{8(n^2-1)}{2n+1}\sigma\,,$$

or, in view of (20),

$$\sigma=\frac{2n+1}{8(n^2-1)}\Big\{e^{2f}\Big(\bar{\tau}-\frac{3\bar{\overset{*}{\tau}}}{2n+1}\Big)-\Big(\tau-\frac{3\overset{*}{\tau}}{2n+1}\Big)\Big\}\,.\tag{23}$$

On the other hand, the relation (22) yields

$$\begin{aligned}4(n-1)(n^2-4)\sigma(Y,Z)=&(2n^2-5)[\,\rho(A\bar{R})(Y,Z)-\rho(AR)(Y,Z)]\\&+3[\,\rho(A\bar{R})(JY,JZ)-\rho(AR)(JY,JZ)]\\&-\frac{2(n^2-1)(2n^2+3n+4)}{(n+1)(2n+1)}g(Y,Z)\sigma\end{aligned}$$

wherefrom, using (19) and (23) and supposing $n>2$, we find

$$\begin{aligned}\sigma(Y,Z)=&\frac{1}{4(n-1)(n^2-4)}\big[(2n^2-5)\bar{\rho}(Y,Z)+3\bar{\rho}(JY,JZ)\big]\\&-\frac{3}{4(n^2-4)}\Big[\bar{\overset{*}{\rho}}(Y,Z)+\bar{\overset{*}{\rho}}(Z,Y)\Big]\\&-\frac{1}{16(n^2-1)(n^2-4)}\Big[(2n^2+3n+4)\bar{\tau}-9n\bar{\overset{*}{\tau}}\Big]\bar{g}(Y,Z)\\&-\Big\{\frac{1}{4(n-1)(n^2-4)}\big[(2n^2-5)\rho(Y,Z)+3\rho(JY,JZ)\big]\\&\quad-\frac{3}{4(n^2-4)}\big[\,\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)\big]\\&\quad-\frac{1}{16(n^2-1)(n^2-4)}\big[(2n^2+3n+4)\tau-9n\overset{*}{\tau}\big]g(Y,Z)\Big\}\,.\end{aligned}\tag{24}$$

Finally, substituting (23) and (24) into (21) and using the notations (2), we get

$$\begin{aligned}e^{-2f}\Big\{A\bar{R}+&\frac{1}{4(n^2-4)}\Big[-\frac{2n^2-5}{n-1}\bar{\varphi}-\frac{3}{n-1}J\bar{\varphi}-\bar{\psi}+3\Big(\bar{\overset{*}{\varphi}}+\frac{1}{n+1}\bar{\overset{*}{\psi}}\Big)\\&+\frac{(2n^2+3n+4)\bar{\tau}-9n\bar{\overset{*}{\tau}}}{2(n^2-1)}\bar{\pi}_1-\frac{3}{2(n^2-1)}\Big(n\bar{\tau}-\frac{7n-4}{2n+1}\bar{\overset{*}{\tau}}\Big)\bar{\pi}_2\Big]\Big\}\\=AR+&\frac{1}{4(n^2-4)}\Big[-\frac{2n^2-5}{n-1}\varphi-\frac{3}{n-1}J\varphi-\psi+3\Big(\overset{*}{\varphi}+\frac{1}{n+1}\overset{*}{\psi}\Big)\\&+\frac{(2n^2+3n+4)\tau-9n\overset{*}{\tau}}{2(n^2-1)}\pi_1-\frac{3}{2(n^2-1)}\Big(n\tau-\frac{7n-4}{2n+1}\overset{*}{\tau}\Big)\pi_2\Big]\,.\end{aligned}\tag{25}$$

Thus, putting

$$W_1 = AR + \frac{1}{4(n^2-4)}\Big[-\frac{2n^2-5}{n-1}\varphi - \frac{3}{n-1}J\varphi - \psi + 3\Big(\overset{*}{\varphi} + \frac{1}{n+1}\overset{*}{\psi}\Big) + \frac{(2n^2+3n+4)\tau - 9n\overset{*}{\tau}}{2(n^2-1)}\pi_1 - \frac{3}{2(n^2-1)}\Big(n\tau - \frac{7n-4}{2n+1}\overset{*}{\tau}\Big)\pi_2\Big] \tag{26}$$

we have

$$e^{-2f}\bar{W}_1 = W_1\,.$$

In [2] the tensor (26) is obtained applying the technique of a decomposition of the Hermitian vector space under an action of a unitary group, is denoted by $(AR)_W$ and is called the *Weyl component of the antiholomorphic curvature tensor*. It is here obtained as a result of a direct application of the conformal change of the metric to the antiholomorphic curvature tensor.

It can be proved, using the relation (3) that $\rho(W_1) = \overset{*}{\rho}(W_1) = 0$, and this explains the name "Weyl component..." for the tensor (26)

4 The second conformally invariant tensor associated with the antiholomorphic curvature tensor

Putting into (7) JZ instead of Z, we find

$$\sigma(Y,JZ) - \sigma(JY,Z) = \frac{1}{2(n-1)}\big[\bar{\rho}(Y,JZ) - \bar{\rho}(JY,Z)\big] + \frac{\bar{\tau}}{2(n-1)(2n-1)}\bar{F}(Y,Z) - \frac{1}{2(n-1)}\big[\rho(Y,JZ) - \rho(JY,Z)\big] - \frac{\tau}{2(n-1)(2n-1)}F(Y,Z)\,.$$

Substituting this relation as well as the relations (5) and (6) into (21), and using the notation (2), we obtain

$$e^{-2f}\Big\{A\bar{R} - \frac{1}{2(n-1)}\bar{\varphi} - \frac{1}{4(n^2-1)}\bar{\psi} + \frac{\bar{\tau}}{2(n-1)(2n-1)}\Big(\bar{\pi}_1 - \frac{3n}{(n+1)(2n+1)}\bar{\pi}_2\Big)\Big\} = AR - \frac{1}{2(n-1)}\varphi - \frac{1}{4(n^2-1)}\psi + \frac{\tau}{2(n-1)(2n-1)}\Big(\pi_1 - \frac{3n}{(n+1)(2n+1)}\pi_2\Big) \tag{27}$$

or

$$e^{-2f}\bar{W}_2 = W_2\,, \tag{28}$$

where

$$W_2 = AR - \frac{1}{2(n-1)}\varphi - \frac{1}{4(n^2-1)}\psi + \frac{\tau}{2(n-1)(2n-1)}\Big(\pi_1 - \frac{3n}{(n+1)(2n+1)}\pi_2\Big)\,. \tag{29}$$

We say that (29) is the *second conformally invariant* (in the sense of (28)) *curvature tensor* of an *almost Hermitian manifold associated with the antiholomorphic curvature tensor*.

In the case $n > 2$, the relation (27) can also be obtained from (25), using (11) and (12). Namely, we have, according to (11)

$$\begin{aligned}\bar{\overset{*}{\rho}}(Y,Z)+\bar{\overset{*}{\rho}}(Z,Y)=&\frac{1}{(n-1)}\big[\bar\rho(Y,Z)+\bar\rho(JY,JZ)\big]-\frac{\bar\tau}{(n-1)(2n-1)}\bar g(Y,Z)\\&+\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)-\frac{1}{(n-1)}\big[\rho(Y,Z)+\rho(JY,JZ)\big]\\&+\frac{\tau}{(n-1)(2n-1)}g(Y,Z)\end{aligned}.$$

Thus

$$\begin{aligned}e^{-2f}\bar g(X,W)\Big[\bar{\overset{*}{\rho}}(Y,Z)+\bar{\overset{*}{\rho}}(Z,Y)\Big]=e^{-2f}\Big\{&\frac{1}{(n-1)}\bar g(X,W)\big[\bar\rho(Y,Z)+\bar\rho(JY,JZ)\big]\\&-\frac{\bar\tau}{(n-1)(2n-1)}\bar g(X,W)\bar g(Y,Z)\Big\}\\&+g(X,W)\big[\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)\big]\\&-\frac{1}{(n-1)}g(X,W)\big[\rho(Y,Z)+\rho(JY,JZ)\big]\\&+\frac{\tau}{(n-1)(2n-1)}g(X,W)g(Y,Z)\,,\end{aligned}$$

such that

$$\begin{aligned}e^{-2f}\bar{\overset{*}{\varphi}}=e^{-2f}&\Big\{\frac{1}{n-1}(\bar\varphi+J\bar\varphi)-\frac{2\bar\tau}{(n-1)(2n-1)}\bar\pi_1\Big\}\\&+\overset{*}{\varphi}-\frac{1}{n-1}(\varphi+J\varphi)+\frac{2\tau}{(n-1)(2n-1)}\bar\pi_1.\end{aligned}$$

We get, in a similar way,

$$\begin{aligned}e^{-2f}\bar{\overset{*}{\psi}}=e^{-2f}&\Big\{\frac{1}{n-1}\bar\psi+\frac{2\bar\tau}{(n-1)(2n-1)}\bar\pi_2\Big\}\\&+\overset{*}{\psi}-\frac{1}{n-1}\psi-\frac{2\tau}{(n-1)(2n-1)}\pi_2\,.\end{aligned}$$

Therefore

$$\begin{aligned}3e^{-2f}\Big(\bar{\overset{*}{\varphi}}+\frac{\bar{\overset{*}{\psi}}}{n+1}\Big)=e^{-2f}&\Big\{\frac{3}{n-1}(\bar\varphi+J\bar\varphi)+\frac{3\bar\psi}{n^2-1}\\&-\frac{6\bar\tau}{(n-1)(2n-1)}\Big(\pi_1-\frac{1}{n+1}\pi_2\Big)\Big\}\\&+3\Big(\overset{*}{\varphi}+\frac{\overset{*}{\psi}}{n+1}\Big)-\frac{3}{n-1}(\varphi+J\varphi)-\frac{3\psi}{n^2-1}\\&+\frac{6\tau}{(n-1)(2n-1)}\Big(\pi_1-\frac{1}{n+1}\pi_2\Big)\,.\end{aligned}$$

Substituting this into (25), and using (12), we get just (27).

Thus we can state a theorem.

Theorem 2. *For $n > 2$ the relation* (27) *is a consequence of* (25), (11) *and* (12).

The first Ricci tensor of the tensor W_2 is

$$\rho(W_2)(Y,Z) = \frac{3}{2(n+1)}\Big\{\frac{1}{n-1}[\rho(Y,Z)+\rho(JY,JZ)] - [\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)] - \frac{1}{2n+1}\Big[\frac{3n}{(n-1)(2n-1)}\tau - \overset{*}{\tau}\Big]g(Y,Z)\Big\},$$

while for the second Ricci tensor we have $\overset{*}{\rho}(W_2) = 0$.

5 The relations between some conformally invariant tensors

5.1

According to (14) and (16), we have

$$\overset{*}{\varphi} = -W_3 + \frac{1}{n-1}(\varphi + J\varphi) - \frac{2\tau}{(n-1)(2n-1)}\pi_1\,,$$
$$\overset{*}{\psi} = -W_4 + \frac{1}{n-1}\psi + \frac{2\tau}{(n-1)(2n-1)}\pi_2\,.$$

Substituting this into (26), we get

$$\begin{aligned}4(n^2-4)W_1 = {} & 4(n^2-4)\Big[AR - \frac{\varphi}{2(n-1)} - \frac{\psi}{4(n^2-1)}\Big] - 3W_3 - \frac{3}{n+1}W_4 \\ & + \Big[\Big(\frac{2n^2+3n+4}{2(n^2-1)} - \frac{6}{(n-1)(2n-1)}\Big)\tau - \frac{9n}{2(n^2-1)}\overset{*}{\tau}\Big]\pi_1 \\ & + \Big[\Big(-\frac{3n}{2(n^2-1)} + \frac{6}{(n^2-1)(2n-1)}\Big)\tau + \frac{3(7n-4)}{2(n^2-1)(2n+1)}\overset{*}{\tau}\Big]\pi_2\,.\end{aligned}$$

But, in view of (29), we have

$$AR - \frac{1}{2(n-1)}\varphi - \frac{1}{4(n^2-1)}\psi = W_2 - \frac{\tau}{2(n-1)(2n-1)}\pi_1 + \frac{3n\tau}{2(n^2-1)(4n^2-1)}\pi_2,$$

because of which and using (16) the preceding relation can be rewritten in the form

$$\begin{aligned}W_1 \equiv (AR)_W = W_2 & - \frac{3}{4(n^2-4)}\Big(W_3 + \frac{1}{n+1}W_4\Big) \\ & + \frac{3}{8(n^2-1)(n^2-4)}\Big[3nW_5 - \frac{7n-4}{2n+1}W_6\Big]\,.\end{aligned} \tag{30}$$

Thus, we can state the following theorem.

Theorem 3. *The Weyl component of the antiholomorphic curvature tensor can be expressed as a linear combination of the tensors W_2, W_3, W_4, W_5 and W_6 such that* (30) *holds. Each of the tensors W is a generalized curvature tensor and each satisfies the condition of the type $e^{-2f}\bar{W} = W$.*

5.2

It is well known (see e.g., [3]) that the Weyl conformal curvature tensor for the Riemannian manifold (M, g), $\dim M = 2n$, is

$$\begin{aligned}C(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{2(n-1)}\big[g(X,W)\rho(Y,Z) \\ + g(Y,Z)\rho(X,W) - g(X,Z)\rho(Y,W) - g(Y,W)\rho(X,Z)\big] \\ + \frac{\tau}{2(n-1)(2n-1)}\,\pi_1(X,Y,Z,W)\end{aligned} \tag{31}$$

and that satisfies the condition

$$e^{-2f}\bar{C}(X,Y,Z,W) = C(X,Y,Z,W)\,.$$

Now, let us apply (18) to the tensor (31), instead to the tensor R. In such a way we obtain

$$\begin{aligned}(AC)(X,Y,Z,W) = C(X,Y,Z,W) + \frac{1}{2(n+1)}\big[F(X,W)\overset{*}{\rho}(C)(Y,JZ) \\ + F(Y,Z)\overset{*}{\rho}(C)(X,JW) - F(X,Z)\overset{*}{\rho}(C)(Y,JW) \\ - F(Y,W)\overset{*}{\rho}(C)(X,JZ) - 2F(X,Y)\overset{*}{\rho}(C)(Z,JW) \\ - 2F(Z,W)\overset{*}{\rho}(C)(X,JY)\big] \\ + \frac{\overset{*}{\tau}(C)}{(2n+2)(2n+1)}\pi_2(X,Y,Z,W)\,.\end{aligned} \tag{32}$$

The first Ricci tensor of the tensor (31) vanishes. But, for the second Ricci tensor we have

$$\begin{aligned}\overset{*}{\rho}(C)(Y,Z) &= \sum_{i=1}^{2n} C(e_i, Y, JZ, Je_i) \\ &= \overset{*}{\rho}(Y,Z) - \frac{1}{2(n-1)}[\,\rho(Y,Z) + \rho(JY,JZ)] \\ &\quad + \frac{\tau}{2(n-1)(2n-1)}g(Y,Z)\,,\end{aligned}$$

wherefrom it follows

$$\overset{*}{\tau}(C) = \overset{*}{\tau} - \frac{\tau}{2n-1}\,.$$

Substituting this and (31) into (32) and using (18), we get

$$A(C) = AR - \frac{1}{2(n-1)}\Big[\varphi + \frac{\psi}{2(n+1)} - \frac{\tau}{2n-1}\Big(\pi_1 - \frac{3n}{(n+1)(2n+1)}\pi_2\Big)\Big]\,,$$

such that, and in view of (29), we can state the following theorem.

Theorem 4. *The second conformally invariant curvature tensor, associated with the antiholomorphic curvature tensor, satisfies the relation $W_2 = A(C)$, while* (11) *can be expressed in the form*

$$\overset{*}{\rho}(\bar{C})(Y,Z) + \overset{*}{\rho}(\bar{C})(Z,Y) = \overset{*}{\rho}(C)(Y,Z) + \overset{*}{\rho}(C)(Z,Y)\,.$$

5.3

We note that, starting from (14), and using (3), we have

$$\begin{aligned}\overset{*}{\rho}(W_3)(Y,Z) &= \sum_{i=1}^{2n} W_3(e_i,Y,JZ,Je_i)\\ &= 2\Big\{\frac{1}{n-1}\big[\,\rho(Y,Z)+\rho(JY,JZ)\big]-\big[\,\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)\big]\\ &\quad -\frac{\tau}{(n-1)(2n-1)}g(Y,Z)\Big\}\\ &= 2V(Y,Z)\,,\end{aligned}$$

wherefrom it follows

$$\overset{*}{\tau}(W_3) = 4\Big(\frac{\tau}{2n-1}-\overset{*}{\tau}\Big)\,.$$

Substituting this into

$$\begin{aligned}A(W_3)(X,Y,Z,W) = W_3(X,Y,Z,W) &+ \frac{1}{2(n+1)}\big\{F(X,W)\overset{*}{\rho}(W_3)(Y,JZ)\\ &+F(Y,Z)\overset{*}{\rho}(W_3)(X,JW)-F(X,Z)\overset{*}{\rho}(W_3)(Y,JW)\\ &-F(Y,W)\overset{*}{\rho}(W_3)(Y,JZ)-2F(X,Y)\overset{*}{\rho}(W_3)(Z,JW)\\ &-2F(Z,W)\overset{*}{\rho}(W_3)(X,JY)\big\}\\ &+\frac{\overset{*}{\tau}(W_3)}{2(n+1)(2n+1)}\,\pi_2(X,Y,Z,W)\,,\end{aligned}$$

we get

$$A(W_3) = W_3 + \frac{1}{n+1}W_4 + \frac{2}{(n+1)(2n+1)}W_6\,. \qquad (33)$$

As for the tensor W_4, we have

$$\begin{aligned}\overset{*}{\rho}(W_4)(Y,Z) = -2(n+1)&\Big\{\frac{1}{n-1}[\,\rho(Y,Z)+\rho(JY,JZ)]-[\,\overset{*}{\rho}(Y,Z)+\overset{*}{\rho}(Z,Y)]\Big\}\\ &+2\Big[\frac{2\tau}{(n-1)(2n-1)}+\overset{*}{\tau}\Big]g(Y,Z)\,,\end{aligned}$$

wherefrom it follows

$$\overset{*}{\tau}(W_4) = -4(2n+1)\Big[\frac{\tau}{2n-1}-\overset{*}{\tau}\Big]\,.$$

Thus

$$A(W_4) = W_4 - \frac{1}{n-1}\psi + \overset{*}{\psi} - \frac{2\tau}{(n-1)(2n-1)}\pi_2 = 0\,.$$

In a similar way we find

$$A(W_5) = \Big(\frac{\tau}{2n-1}-\overset{*}{\tau}\Big)\Big(\pi_1-\frac{\pi_2}{2n+1}\Big)\,, \qquad A(W_6)=0\,.$$

Summing up the preceding results, we can state the following theorem.

Theorem 5. *Applying* (18) *to the tensors* W, *we get*

$$A(W_1) \equiv A(AR_W) = (AR)_W\,, \qquad A(W_2) = W_2\,,$$
$$A(W_3) = W_3 + \frac{1}{n+1}W_4 + \frac{2}{(n+1)(2n+1)}W_6\,, \qquad A(W_4) = 0\,,$$
$$A(W_5) = W_5 - \frac{1}{2n+1}W_6\,, \qquad A(W_6) = 0\,.$$

6 Kähler spaces

In the case of Kähler spaces, we have

$$\overset{*}{\rho} = \rho, \quad \overset{*}{\tau} = \tau, \quad \rho(JX, JY) = \rho(X, Y)\,, \tag{34}$$

and therefore, the conformally invariant tensor (13) has the form

$$V(Y,Z) = \frac{2(n-2)}{n-1}\rho(Y,Z) - \frac{\tau}{(n-1)(2n-1)}g(Y,Z)\,.$$

Also, putting

$$\begin{aligned}\psi_0(X,Y,Z,W) = &\, F(X,W)\rho(Y,JZ) + F(Y,Z)\rho(X,JW)\\ &- F(X,Z)\rho(Y,JW) - F(Y,W)\rho(X,JZ)\\ &- 2F(X,Y)\rho(Z,JW) - 2F(Z,W)\rho(X,JY)\,,\end{aligned}$$

we get

$$\psi = \overset{*}{\psi} = 2\psi_0\,. \tag{35}$$

Thus, for Kähler manifolds, the antiholomorphic curvature tensor is

$$AR = R + \frac{1}{2(n+1)}\psi_0 + \frac{\tau}{2(n+1)(2n+1)}\pi_2\,. \tag{36}$$

Using (34), (35) and (36), we find that, for Kähler manifolds, the relation (26) becomes

$$W_1 \equiv (AR)_W = R - \frac{1}{2(n+2)}(\varphi - \psi_0) + \frac{\tau}{4(n+1)(n+2)}(\pi_1 + \pi_2)\,,$$

or, explicitly,

$$\begin{aligned}(W_1)(X,Y,Z,W) \equiv &\,(AR)_W(X,Y,Z,W) = R(X,Y,Z,W)\\ &- \frac{1}{2(n+1)}\Big\{g(X,W)\rho(Y,Z) + g(Y,Z)\rho(X,W) - g(X,Z)\rho(Y,W)\\ &- g(Y,W)\rho(X,Z) - F(X,W)\rho(Y,JZ) - F(Y,Z)\rho(X,JW) + F(X,Z)\rho(Y,JW)\\ &+ F(Y,W)\rho(X,JZ) + 2F(X,Y)\rho(Z,JW) + 2F(Z,W)\rho(X,JY)\Big\}\\ &+ \frac{\tau}{4(n+1)(n+2)}[\pi_1(X,Y,Z,W) + \pi_2(X,Y,Z,W)]\,.\end{aligned}$$

But this is the Bochner curvature tensor of a Kähler manifold. Thus, we can state the following theorem.

Theorem 6. *For a Kähler manifold, the Weyl component of the antiholomorphic curvature tensor, $(AR)_W$, is Bochner curvature tensor.*

In a similar way we find that, in the case of Kähler manifolds,

$$W_2 = R - \frac{1}{2(n-1)}\varphi + \frac{n-2}{2(n^2-1)}\psi_0 + \tau\Big[\frac{\pi_1}{2(n-1)(2n-1)} + \frac{2n^2-6n+1}{2(n^2-1)(4n^2-1)}\pi_2\Big],$$

$$W_3 = -\frac{2(n-2)}{n-1}\varphi - \frac{2\tau}{(n-1)(2n-1)}\pi_1, \qquad W_5 = -\frac{2(n-1)}{2n-1}\pi_1,$$

$$W_4 = -\frac{2(n-2)}{n-1}\psi_0 + \frac{2\tau}{(n-1)(2n-1)}\pi_2, \qquad W_6 = -\frac{2(n-1)}{2n-1}\pi_2.$$

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