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A New Variational Characterization Of Compact Conformally Flat 4-Manifolds

Faen Wu, Xinnuan Zhao

Abstract. In this paper, we give a new variational characterization of certain 4-manifolds. More precisely, let R and Ric denote the scalar curvature and Ricci curvature respectively of a Riemannian metric, we prove that if (M^4, g) is compact and locally conformally flat and g is the critical point of the functional

$$F(g) = \int_{M^4} (aR^2 + b|Ric|^2)\,\mathrm{d}v_g\,,$$

where

$$(a, b) \in \mathbb{R}^2 \setminus L_1 \cup L_2$$

$$L_1\colon 3a + b = 0\,; \quad L_2\colon 6a - b + 1 = 0\,,$$

then (M^4, g) is either scalar flat or a space form.

1 Introduction

Let (M^n, g) be an n-dimensional compact smooth manifold. Denote by $\mathcal{M}$ and $\mathcal{G}$ the space of smooth Riemannian metric and the diffeomorphism group of M respectively. We call a functional $F\colon \mathcal{M} \to R$ Riemannian if F is invariant under the action of $\mathcal{G}$, i. e. $F(\varphi^* g) = F(g)$ for $\varphi \in \mathcal{G}$ and $g \in \mathcal{M}$.

By letting $S_2(M)$ denote the bundle of symmetric $(0, 2)$ tensors on M^n, we say that F has a gradient ∇F at $g \in \mathcal{M}$ if for $h \in S_2(M)$

$$\frac{\mathrm{d}}{\mathrm{d}t} F(g + th)|_{t=0} = \int_M \langle h, \nabla F\rangle_g \,\mathrm{d}v_g$$

In [6], Gursky and Viaclovsky studied the functional

$$F(g) = \int_{M^3} \sigma_k(C_g)\,\mathrm{d}v_g$$

2010 MSC: 53C20, 53C25
Key words: conformally flat, 4-manifold, variational characterization

where $\sigma_k(C_g)$ is the k-th elementary symmetric function of the eigenvalues of the Schonten tensor $C_g = Ric - \frac{1}{2}\frac{R}{n-1}g$. They proved that

Theorem 1. *[6] Let M be a compact 3-manifold, then a metric g with $F_2(g) \geq 0$ is critical for $F_2|_{\mathcal{M}_1}$ if and only if g has constant sectional curvature,where $\mathcal{M}_1 = \{g \in \mathcal{M} \,|\, \mathrm{Vol}(g) = 1\}$*

This gives a new variational characterization of three-dimensional space forms. In [7], Hu and Li generalized the above result to the case $n \geq 5$. There are many deep on going results about the 4-manifolds. M. J. Gursky considered in [5] 4-manifolds with harmonic self-dual Weyl tensor and obtained a lower bound of the L^2 norm of the self-dual part of Weyl tensor. S.-Y. A. Chang, M. J. Gursky and P. Yang obtained in [3] some sufficient geometric conditions for a 4-manifold to have certain conformal class of metric and consequently to have finite fundamental group. C. LeBrun and B. Maskit [9] completely determined compact simply connected and oriented 4-manifolds up to homomorphism which admit scalar flat, anti-self-dual Riemannian metrics. There is a rich literature concerning results related to the variation of curvature functional [1], [4], [10], [11], [12].

Early in 1938, before the higher dimensional Gauss-Bonnet formula were discovered, C. Lanczos [8] studied the functional

$$\phi_{a,b,c}(g) = \int_{M^4} \left(a|Rie|^2 + b|Ric|^2 + cR^2\right) \mathrm{d}v_g$$

on 4-manifolds. He found that the functional $\phi_{1,-4,1}$ has a gradient which is identically zero. In fact this establishes that this integral is a differential invariant of the manifold M. It is even a topological invariant, namely $32\pi^2\chi(M)$, where $\chi(M)$ the Euler-Poincare characteristic of M, i. e.

$$32\pi^2\chi(M) = \int_{M^4} \left(|Rie|^2 - 4|Ric|^2 + R^2\right) \mathrm{d}v_g \tag{1}$$

Taking this Gauss-Bonnet formula into account, we naturally study the functional

$$F(g) = \int_{M^4} \left(aR_g^2 + b|Ric_g|^2\right) \mathrm{d}v_g \tag{2}$$

We obtain a new variational characterization of 4-manifolds as follow

Theorem 2. *Suppose that (M^4, g) is compact and locally conformally flat. If g is a critical point of the functional (2) with any pairs (a, b) in the real plane with two fixed lines deleted, that is*

$$(a, b) \in \mathbb{R}^2 \setminus L_1 \cup L_2\,; \quad L_1\colon 3a + b = 0\,; \quad L_2\colon 6a - b + 1 = 0\,,$$

then (M^4, g) is either scalar flat or a space form.

2 Preliminaries

Recently, the first author [13] studied the variation formulas of a metric by the moving frame method. He obtained the first and the second variation formulas for the Riemannian curvature tensor, Ricci tensor and scalar curvature of a metric in another formalism which should be equivalent to the classical ones. He also obtained some interesting applications of these formulas. We believe that these formulas are more convenient in the computations of calculus of variation, especially in the computations where the second variation of a metric is involved. We follow the notations as in [13]. Classical variational formulas of metric can be found in [2] and [12].

Suppose that

$$g(t) = \sum_{i=1}^{n} \theta_i^2(t)$$

is a variation of a given metric g. For the sake of simplicity, from now on we use Einstein summation convention; i. e., the repeated indices imply summation. The indices $i, j, k, \ldots$ are from 1 to n unless otherwise stated. Let $\theta_{ij}(t)$ and $\Omega_{ij}(t)$ are connection one-forms and curvature two-forms determined respectively by

$$\mathrm{d}\theta_i(t) = \theta_{ij}(t) \wedge \theta_j(t)$$

$$\Omega_{ij}(t) = \mathrm{d}\theta_{ij}(t) - \theta_{ik}(t) \wedge \theta_{kj}(t) = -\frac{1}{2} R_{ijkl}(t)\theta_k(t) \wedge \theta_l(t)$$

where d is the exterior differential operator on the manifold. These equations are known as the structural equation of the Levi-Civita connection of the metric. R_{ijkl} are the components of the $(0,4)$ type Riemannian curvature tensor. Assume that

$$\theta_i(t) = \theta_i + \omega_i t + o(t) \qquad R_{ijkl}(t) = R_{ijkl} + r_{ijkl} t + o(t)$$

where $\theta_i = \theta_i(t)|_{t=0}$, $\omega_i = \left.\frac{\mathrm{d}\theta_i(t)}{\mathrm{d}t}\right|_{t=0} = a_{ij}\theta_j$, $R_{ijkl} = R_{ijkl}(t)|_{t=0}$, $r_{ijkl} = \left.\frac{\mathrm{d}R_{ijkl}(t)}{\mathrm{d}t}\right|_{t=0}$.

By a crucial lemma proved in [13], there exists an isometry of $g(t)$, such that a_{ij} are symmetric. So we may always assume $a_{ij} = a_{ji}$ without loss of generality. With these preparation we have [13]

$$r_{ijkl} = -(a_{ik,jl} - a_{il,jk} + a_{jl,ik} - a_{jk,il} + R_{ijkm}a_{ml} + R_{ijml}a_{mk}) \tag{3}$$

where $a_{ij,kl}$ is defined by

$$a_{ij,kl}\theta_l = da_{ij,k} + a_{lj,k}\theta_{li} + a_{il,k}\theta_{lj} + a_{ij,l}\theta_{lk}$$

and $a_{ij,k}$ is defined by

$$a_{ij,k}\theta_k = da_{ij} + a_{kj}\theta_{ki} + a_{ik}\theta_{kj} \,,$$

$$\theta_{ij} = \theta_{ij}(t)|_{t=0}$$

$a_{ij,k}$ and $a_{ij,kl}$ are the first and the second covariant derivatives of a_{ij} with respect to the initial metric g.

Defined the Ricci curvature

$$R_{ij}(t) = \sum_{k=1}^{n} R_{ikjk}(t) = R_{ij} + r_{ij}t + o(t)$$

and the Scalar curvature

$$R(t) = \sum_{i=1}^{n} R_{ii}(t) = R + rt + o(t)$$

of $g(t)$ respectively the above two formulas, then by making contraction from (3) one obtain immediately

$$\frac{\partial R_{ij}(t)}{\partial t}\Big|_{t=0} = r_{ij} = -\Delta a_{ij} - a_{kk,ij} + a_{ik,kj} + a_{kj,ik} - R_{ik}a_{kj} - R_{ikjl}a_{kl} \tag{4}$$

$$\frac{\partial R(t)}{\partial t}\Big|_{t=0} = r = 2(a_{ij,ij} - \Delta a_{ii} - a_{ij}R_{ij}) \tag{5}$$

where Δa_{ij} denotes the Laplacian of a_{ij} with respect to the original metric g. For more details see [13].

3 Proof of the theorem 2

By (4) and (5) we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}F(t)\big|_{t=0} &= \int_{M^4} \Big\{2\Big(aR(t)\frac{dR(t)}{dt} + bR_{ij}\frac{dR_{ij}(t)}{dt}\Big) + (aR^2 + bR_{ij}^2)a_{mm}\Big\}\,\mathrm{d}v_g\big|_{t=0} \\
&= \int_{M^4} \big\{2aR\cdot 2(a_{ij,ij} - \Delta a_{ii} - a_{ij}R_{ij}) \\
&\quad + 2bR_{ij}(-\Delta a_{ij} - a_{kk,ij} + a_{ik,kj} + a_{kj,ik} - R_{im}a_{mj} - R_{ikjl}a_{kl}) \\
&\quad + (aR^2 + bR_{ij}^2)a_{mm}\big\}\,\mathrm{d}v_g \\
&= \int_{M^4} a_{ij}(\nabla F)_{ij}\,\mathrm{d}v_g
\end{aligned}$$

where

$$\begin{aligned}
(\nabla F)_{ij} = 4aR_{,ij} &- 4a\Delta R\delta_{ij} - 4aRR_{ij} - 2b\Delta R_{ij} \\
&- 2bR_{kl,kl}\delta_{ij} + 2bR_{ik,kj} + 2bR_{kj,ik} - 2bR_{im}R_{mj} \\
&- 2bR_{kl}R_{ik,jl} + (aR^2 + bR_{kl}^2)\delta_{ij}\,.
\end{aligned} \tag{6}$$

Since g is a critical point of the functional (2), we have

$$(\nabla F)_{ij} = 0\,. \tag{7}$$

Taking trace of (7) and making use of the following identities which are obtained from the second Bianchi identity and the Ricci identity respectively

$$2R_{ij,i} = R_{,j},$$

$$2R_{ij,ij} = \Delta R,$$
$$R_{ij,kl} - R_{ij,lk} = R_{mj}R_{mikl} + R_{im}R_{mjkl},$$
$$R_{kj,ik} = \frac{1}{2}R_{,ij} + R_{ik}R_{kj} + R_{kl}R_{iklj},$$

then we have

$$4a\Delta R - 4\cdot 4a\Delta R - 4aR^2 - 2b\Delta R - 4b\Delta R + b\Delta R + b\Delta R \\ - 2bR_{ij}^2 - 2bR_{ij}^2 + 4(aR^2 + bR_{ij}^2) = 0$$

or after simplifying we arrive at

$$(3a+b)\Delta R = 0\,.$$

By the assumptions of the theorem, $3a+b \neq 0$. This gives $\Delta R = 0$. Since M^4 is compact, R must be a constant. In this case, from (7) and (6) we have

$$-4aRR_{ij} - 2b\Delta R_{ij} + 2b(R_{in}R_{nj} + R_{kl}R_{iklj}) \\ - 2bR_{im}R_{mj} + 2bR_{kl}R_{iklj} + (aR^2 + bR_{kl}^2)\delta_{ij} = 0\,. \quad (8)$$

If (M^4, g) is locally conformally flat, then

$$R_{ijkl} = \frac{1}{2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + \delta_{ik}R_{jl} - \delta_{il}R_{jk}) - \frac{1}{6}R(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})\,.$$

Substituting this expression into (8) we have

$$\Big(4a + \frac{2}{3}b\Big)RE_{ij} + 2b\Delta E_{ij} - 4bE_{ik}E_{kj} + bE_{kl}^2\delta_{ij} = 0 \quad (9)$$

where

$$E_{ij} = R_{ij} - \frac{1}{4}R\delta_{ij},$$

is the traceless part of the Ricci tensor. If $b \neq 0$, then

$$\Delta E_{ij} = 2E_{ik}E_{kj} - \frac{1}{b}\Big(2a + \frac{b}{3}\Big)RE_{ij} - \frac{1}{2}E_{kl}^2\delta_{ij}\,. \quad (10)$$

Comparing the standard result in [7]

$$\Delta E_{ij} = 2E_{ik}E_{kj} - \frac{1}{3}RE_{ij} - \frac{1}{2}E_{kl}^2\delta_{ij}$$

on a locally conformally flat 4-manifold. We have

$$-\frac{1}{b}\Big(2a + \frac{1}{3}\Big)RE_{ij} = -\frac{1}{3}RE_{ij}$$

or equivalently

$$(6a - b + 1)RE_{ij} = 0\,.$$

Again by the assumption of the theorem, $6a - b + 1 \neq 0$, then

$$RE_{ij} = 0\,. \tag{11}$$

So $R = 0$ or $E_{ij} = 0$. In the first case, (M^4, g) is scalar flat and in the second case, considering g is also locally conformally flat we see that (M^4, g) has constant sectional curvature. If $b = 0$, then $a \neq 0$ by the assumption. From (9) we still have $RE_{ij} = 0$, and the same conclusion remains true. This completes the proof of theorem 2.

Remark 1.

1. If $3a+b = 0$ and $6a-b+1 = 0$, then $(a,b) = (-\frac{1}{9}, \frac{1}{3})$. It can be checked that

$$R_{ijkl}^2 - 4R_{ij}^2 + R^2 = -6\big(-\frac{1}{9}R^2 + \frac{1}{3}R_{ij}^2\big)$$

   that is, the integrand of our functional is a multiple of the integrand of the Gauss-Bonnet formula. In this case, the variation is identically zero.

2. All points (a, b) considered in our functional fall into four regions. It would be interesting to study further property of the functional.

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On a problem of Bednarek

Florian Luca

Abstract. We answer a question of Bednarek proposed at the 9th Polish, Slovak and Czech conference in Number Theory.

There are several problems in the literature concerning various arithmetic properties of the digit sum number function, see e.g. [1] and the references given there. In this paper, we deal with a particular problem. Namely, at the 9th Polish, Slovak and Czech Conference on Number Theory, June 11–14, 2012, W. Bednarek (via A. Schinzel) asked the following question.

Question *Is there a positive integer n divisible by $\underbrace{\overline{11\dots1}}_{k\ \text{times}}$ whose digit sum is less than k?*

Here, we prove that the answer is no in a slightly more general setting. For integers $N \geq 1$ and $b \geq 2$, let $N = \overline{d_m d_{m-1} \dots d_0}_{(b)}$ be the base b representation of N, where $d_0, \dots, d_m \in \{0, 1, \dots, b-1\}$ with $d_m \neq 0$. We have the following result.

Theorem *If $n \geq 2$ is a multiple of $\underbrace{\overline{11\dots1}}_{k\ \text{times}}{}_{(b)}$, then the sum of its base b digits is greater than or equal to k.*

Proof. We may assume that $k \geq 2$, otherwise there is nothing to prove. Write

$$n = \sum_{i=0}^{m} d_i b^i \,,$$

where $d_0, \dots, d_m$ are in $\{0, 1, \dots, b-1\}$ with $d_m \neq 0$. We may also assume that $d_0 \neq 0$. Put $N = (b^k - 1)/(b - 1)$. Then $b^k \equiv 1 \pmod{N}$. Thus,

$$n \equiv \sum_{j=0}^{k-1} c_j b^j \pmod{N}\,,$$

where

$$c_j = \sum_{\substack{0 \leq i \leq m \\ i \equiv j \pmod{k}}} d_i \,.$$

2010 MSC: 11A63
Key words: sum of digits

It is clear that $c_0 + \cdots + c_{k-1}$ is the sum of the digits of n.
For each $\ell \in \{0, 1, \ldots, k-1\}$, put

$$r_{j,\ell} = j + \ell - k \left\lfloor \frac{j+\ell}{k} \right\rfloor ,$$

and consider the integer

$$m_\ell = \sum_{j=0}^{k-1} c_j b^{r_{j,\ell}} .$$

Note that since $b^k \equiv 1 \pmod N$, it follows that

$$m_\ell \equiv \sum_{j=0}^{k-1} b^{j+\ell} c_j \pmod N \equiv b^\ell n \pmod N \equiv 0 \pmod N ,$$

and since $c_j > 0$ for some j, we get that $m_\ell \geq N$. Summing this up for all $\ell \in \{0, 1, \ldots, k-1\}$, we get

$$kN \leq \sum_{\ell=0}^{k-1} \sum_{j=0}^{k-1} b^{r_{j,\ell}} c_j = \sum_{j=0}^{k-1} c_j \sum_{\ell=0}^{k-1} b^{r_{j,\ell}} = N \sum_{j=0}^{k-1} c_j ,$$

so $\sum_{j=0}^{k-1} c_j \geq k$, which is what we wanted to prove. $\square$

Note After this paper was submitted, we learned that Bednarek's question was also asked by Zhi–Wei Sun in [3], who solved the particular case when the modulus b is a prime. We also learned that the main result of this paper was obtained independently by Pan in [2].

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On the Diophantine equation $x^2 + 2^\alpha 5^\beta 17^\gamma = y^n$

Hemar Godinho, Diego Marques, Alain Togbé

Abstract. In this paper, we find all solutions of the Diophantine equation $x^2 + 2^\alpha 5^\beta 17^\gamma = y^n$ in positive integers $x, y \geq 1$, $\alpha, \beta, \gamma, n \geq 3$ with $\gcd(x, y) = 1$.

1 Introduction

There are many results concerning the generalized Ramanujan-Nagell equation

$$x^2 + C = y^n, \tag{1}$$

where $C > 0$ is a given integer and x, y, n are positive integer unknowns with $n \geq 3$. Results obtained for general superelliptic equations clearly provide effective finiteness results for this equation, too (see for example [8], [31], [32] and the references given there). The first result concerning the above equation was due to V. A. Lebesgue [23] and it goes back to 1850, where he proved that the above equation has no solutions for $C = 1$. More recently, other values of C were considered. Tengely [33] solved the equation with $C = b^2$, b odd and $3 \leq b \leq 501$. The case where $C = p^k$, a power of a prime number, was studied in [5], [21], [20] for $p = 2$, in [6], [4], [24] for $p = 3$, in [1], [2] for $p = 5$, and in [27] for $p = 7$. The case $C = p^{2k}$ with $2 \leq p < 100$ prime and $\gcd(x, y) = 1$ was solved by Bérczes and Pink [9]. For arbitrary primes, some advances can be found in [7]. In [13], the cases with $1 \leq C \leq 100$ were completely solved. The solutions for the cases $C = 2^a \cdot 3^b$, $C = 2^a \cdot 5^b$ and $C = 5^a \cdot 13^b$, when x and y are coprime, can be found in [3], [25], [26], respectively. Recent progress on the subject were made in the cases $C = 5^a \cdot 11^b$, $C = 2^a \cdot 11^b$, $C = 2^a \cdot 3^b \cdot 11^c$, $C = 2^a \cdot 5^b \cdot 13^c$ and can be found in [16], [15], [14], [18]. For related results concerning equation (1) see [10], [22], [29], [30] and the references given there. For a survey concerning equation (1) see [12].

In this paper, we are interested in solving the Diophantine equation

$$x^2 + 2^\alpha 5^\beta 17^\gamma = y^n, \quad \gcd(x, y) = 1\,,\ x, y \geq 1\,,\ \alpha, \beta, \gamma \geq 0\,,\ n \geq 3\,. \tag{2}$$

2010 MSC: Primary 11D61, Secondary 11Y50
Key words: Diophantine equation, exponential equation, primitive divisor theorem

Our result is the following.

Theorem 1. *The equation* (2) *has no solution except for:*

$n=3$ *the solutions given in Table 1;*
$n=4$ *the solutions given in Table 2;*
$n=5$ $(x,y,\alpha,\beta,\gamma)=(401,11,1,3,0)$;
$n=6$ $(x,y,\alpha,\beta,\gamma)=(7,3,3,1,1),(23,3,3,2,0)$;
$n=8$ $(x,y,\alpha,\beta,\gamma)=(47,3,8,0,1),(79,3,6,1,0)$.

One can deduce from the above result the following corollary.

Corollary 1. *The equation*

$$x^2+5^k17^l=y^n,\quad x\geq 1\,,\ y\geq 1\,,\ \gcd(x,y)=1\,,\ n\geq 3\,,\ k\geq 0\,,\ l\geq 0 \qquad (3)$$

has only the solutions

$$(x,y,k,l,n)=(94,21,2,1,3)\,,\ (2034,161,3,2,3)\,,\ (8,3,0,1,4)\,.$$

Therefore, our work extends that of Pink and Rábai [28]. We will follow the standard approach to work on equation (2) but with another version of MAGMA (V2.18-6) that gives better results when we deal with the corresponding elliptic curves.

2 The case $n=3$

Lemma 1. *When* $n=3$*, all the solutions to equation* (2) *are given in Table 1.*
For $n=6$*, we have* $(x,y,\alpha,\beta,\gamma)=(7,3,3,1,1)\,,\ (23,3,3,2,0)$.

Proof. Equation (2) can be rewritten as

$$\left(\frac{x}{z^3}\right)^2+A=\left(\frac{y}{z^2}\right)^3, \qquad (4)$$

where A is sixth-power free and defined implicitly by $2^\alpha 5^\beta 17^\gamma=Az^6$. One can see that $A=2^{\alpha_1}5^{\beta_1}17^{\gamma_1}$ with $\alpha_1,\ \beta_1,\ \gamma_1,\ \in\{0,1,2,3,4,5\}$. We thus get

$$V^2=U^3-2^{\alpha_1}5^{\beta_1}17^{\beta_1}, \qquad (5)$$

with $U=y/z^2,\ V=x/z^3$ and $\alpha_1,\ \beta_1,\ \gamma_1\ \in\{0,1,2,3,4,5\}$. We need to determine all the $\{2,5,17\}$-integral points on the above 216 elliptic curves. Recall that if $\mathcal{S}$ is a finite set of prime numbers, then an $\mathcal{S}$-integer is rational number a/b with coprime integers a and b, where the prime factors of b are in $\mathcal{S}$. We use the command *SIntegralPoints* of `MAGMA` [17] to determine all the $\{2,5,17\}$-integer points on the above elliptic curves. Here are a few remarks about the computations:

1. We eliminate the solutions with $UV=0$ because they yield to $xy=0$.

Table 1: Solutions for $n = 3$.

α_1	β_1	γ_1	z	α	β	γ	x	y
1	0	0	1	1	0	0	5	3
1	0	0	$2 \cdot 5$	7	6	0	383	129
2	0	0	1	2	0	0	11	5
4	0	1	5	4	6	1	5369	321
3	0	2	5	3	6	2	167589	3041
1	1	1	2^2	13	1	1	93	89
1	1	1	5	1	7	1	1531	171
1	1	1	1	1	1	1	453	59
3	1	1	1	3	1	1	7	9
1	1	2	1	1	1	2	63	19
2	1	2	1	2	1	2	59	21
1	1	3	2	7	1	3	5471	321
1	1	3	5	1	7	3	17052501	66251
3	2	0	1	3	2	0	23	9
3	2	0	2	9	2	0	17771	681
5	2	0	1	5	2	0	261	41
0	2	1	1	0	2	1	94	21
0	2	1	2	6	2	1	55157	1449
3	3	1	2	9	3	1	10763	489
3	3	1	2^2	15	3	1	4617433	27729
0	3	2	1	0	3	2	2034	161
3	3	5	2^5	33	3	5	2037783243169	160733121
1	4	0	1	1	4	0	9	11
4	4	1	$2 \cdot 5$	10	10	1	3274947	22169
5	4	2	$2 \cdot 5$	11	10	2	699659581	788121
1	5	0	17	1	5	6	916769	9971
1	5	1	17	1	5	7	846227	14859
1	5	1	2	7	5	1	17579	681

2. We consider only solutions such that the numerators of U and V are coprime.

3. If U and V are integers then $z = 1$. So $\alpha_1 = \alpha$, $\beta_1 = \beta$, and $\gamma_1 = \gamma$.

4. If U and V are rational numbers which are not integers, then z is determined by the denominators of U and V. The numerators of these rational numbers give x and y. Then α, β, γ are computed knowing that $2^\alpha 5^\beta 17^\gamma = Az^6$.

Therefore, we first determine $(U, V, \alpha_1, \beta_1, \gamma_1)$ and then we use the relations

$$U = \frac{y}{z^2}, \quad V = \frac{x}{z^3}, \quad 2^\alpha 5^\beta 17^\gamma = Az^6,$$

to find the solutions $(x, y, \alpha, \beta, \gamma)$ listed in Table 1.

For $n = 6$, equation

$$x^2 + 2^\alpha 5^\beta 17^\gamma = y^6 \tag{6}$$

becomes equation

$$x^2 + 2^\alpha 5^\beta 17^\gamma = \left(y^2\right)^3 . \tag{7}$$

We look in the list of solutions of Table 1 and observe that y is a perfect square only when $y = 9$ corresponding to two solutions. Therefore, the only solutions to equation (2) for $n = 6$ are the two solutions listed in Theorem 1. This completes the proof of Lemma 1. □

Remark 1. Notice that with the old version of MAGMA, it was difficult to determine the rational points of certain elliptic curves when $2^\alpha 5^\beta 17^\gamma$ is very high. That is the case of the following elliptic curves:

$$V^2 = U^3 - 2^3 \cdot 5^5 \cdot 17^5, \quad V^2 = U^3 - 2^5 \cdot 5^1 \cdot 17^4.$$

We thank the team MAGMA, particularly Steve Donnelly for the new version (Magma V2.18-6) and their help.

3 The case $n = 4$

Here, we have the following result.

Lemma 2. *If $n = 4$, then the only solutions to equation (2) are given in Table 2.*

If $n = 8$, then the only solution to equation (2) is $(x, y, \alpha, \beta, \gamma) = (47, 3, 8, 0, 1)$, $(79, 3, 6, 1, 0)$.

Table 2: Solutions for $n = 4$.

α_1	β_1	γ_1	z	α	β	γ	x	y
1	0	0	2	5	0	0	7	3
0	1	0	2	4	1	0	1	3
0	0	1	2^2	8	0	1	1087	33
0	0	1	1	0	0	1	8	3
0	0	1	2^2	8	0	1	47	9
1	0	1	2	5	0	1	9	5
3	0	1	2	7	0	1	15	7
3	0	1	2^2	11	0	1	495	23
2	1	0	2	6	1	0	79	9
2	2	1	2	6	2	1	409	21
3	2	2	2	7	2	2	511	33
1	0	3	2^2	9	0	3	4785	71

Proof. Equation (2) can be written as

$$\left(\frac{x}{z^2}\right)^2 + A = \left(\frac{y}{z}\right)^4 , \tag{8}$$

where A is fourth-power free and defined implicitly by $2^\alpha\, 5^\beta\, 17^\gamma = Az^4$. One can see that $A = 2^{\alpha_1}\, 5^{\beta_1}\, 17^{\gamma_1}$ with $\alpha_1,\ \beta_1,\ \gamma_1 \in \{0,1,2,3\}$. Hence, the problem consists of determining the $\{2,5,17\}$-integer points on the totality of the 64 elliptic curves

$$V^2 = U^4 - 2^{\alpha_1}\, 5^{b_1}\, 17^{\gamma_1}, \tag{9}$$

with $U = y/z,\ V = x/z^2$ and $\alpha_1,\ \beta_1,\ \gamma_1 \in \{0,1,2,3\}$. Here, we use the command *SIntegralQuarticPoints* of `MAGMA` [17] to determine the $\{2,5,17\}$-integer points on the above elliptic curves. As in Section 2, we first find $(U, V, \alpha_1, \beta_1, \gamma_1)$, and then using the coprimality conditions on x and y and the definition of U and V, we determine all the corresponding solutions $(x, y, \alpha, \beta, \gamma)$ listed in Table 2.

Looking in the list of solutions of equation Table 2, we observe the 2 solutions in Table 2 whose values for y are perfect squares. Thus, the only solutions to equation (2) with $n = 8$ are those listed in Theorem 1. This concludes the proof of Lemma 2. □

4 The case $n \geq 5$

The aim of this section is to determine all solutions of equation (2), for $n \geq 5$ and to prove its unsolubility for $n = 7$ and $n \geq 9$. The cases when n is of the form $2^a 3^b$ were treated in previous sections. So, apart from these cases, in order to prove that (2) has no solution for $n \geq 7$, it suffices to consider n prime. In fact, if $(x, y, \alpha, \beta, \gamma, n)$ is a solution for (2) and $n = pk$, where $p \geq 7$ is prime and $k > 1$, then $(x, y^k, \alpha, \beta, \gamma, p)$ is also a solution. So, from now on, n will denote a prime number.

Lemma 3. *The Diophantine equation* (2) *has no solution with* $n \geq 5$ *prime except for*

$$n = 5 \qquad (x, y, \alpha, \beta, \gamma) = (401, 11, 1, 3, 0)\,.$$

Proof. Let $(x, y, \alpha, \beta, \gamma, n)$ be a solution for (2). We claim that y is odd. In fact, if y is even and since $\gcd(x, y) = 1$, one has that x is odd, and then $-2^\alpha 5^\beta 17^\gamma \equiv x^2 - y^n \equiv 1 \pmod 4$, but this contradicts the fact that $-2^\alpha 5^\beta 17^\gamma \equiv 0, 2$ or $3 \pmod 4$ (according to $\alpha \geq 2, \alpha = 1$ or $\alpha = 0$, respectively). Now, write equation (2) as $x^2 + dz^2 = y^n$, where

$$d = 2^{\alpha - 2\lfloor \alpha/2 \rfloor} 5^{\beta - 2\lfloor \beta/2 \rfloor} 17^{\gamma - 2\lfloor \gamma/2 \rfloor}\,,$$

and $z = 2^{\lfloor \alpha/2 \rfloor} 5^{\lfloor \beta/2 \rfloor} 17^{\lfloor \gamma/2 \rfloor}$. Since $x - 2\lfloor x/2 \rfloor \in \{0, 1\}$, we have

$$d \in \{1, 2, 5, 10, 17, 34, 85, 170\}\,.$$

We then factor the previous equation over $\mathbb{K} = \mathbb{Q}[\mathrm{i}\sqrt{d}] = \mathbb{Q}[\sqrt{-d}]$ as

$$(x + \mathrm{i}\sqrt{d}z)(x - \mathrm{i}\sqrt{d}z) = y^n.$$

Now, we claim that the ideals $(x + \mathrm{i}\sqrt{d}z)\mathcal{O}_\mathbb{K}$ and $(x - \mathrm{i}\sqrt{d}z)\mathcal{O}_\mathbb{K}$ are coprime. If this is not the case, there must exist a prime ideal $\mathfrak{p}$ containing these ideals. Therefore, $x \pm \mathrm{i}\sqrt{d}z$ and y^n (and so y) belong to $\mathfrak{p}$. Thus $2x \in \mathfrak{p}$ and hence either 2 or x belongs to $\mathfrak{p}$. Since $\gcd(2,y)=\gcd(x,y)=1$, then 1 belongs to the ideals $\langle 2,y\rangle$ and $\langle x,y\rangle$, then $1\in\mathfrak{p}$ leading to an absurdity of $\mathfrak{p}=\mathcal{O}_{\mathbb{K}}$. By the unique factorization of ideals, it follows that $(x+\mathrm{i}\sqrt{d}z)\mathcal{O}_{\mathbb{K}}=\mathfrak{j}^n$, for some ideal $\mathfrak{j}$ of $\mathcal{O}_{\mathbb{K}}$. Using Mathematica's command *NumberFieldClassNumber[Sqrt[-d]]*, we obtain that the class number of $\mathbb{K}$ is either $1,2,4$ or 12 and so coprime to n, then $\mathfrak{j}$ is a principal ideal yielding

$$x+\mathrm{i}\sqrt{d}z=\varepsilon\eta^n, \tag{10}$$

for some $\eta\in\mathcal{O}_{\mathbb{K}}$ and ε a unit of $\mathbb{K}$. Since the group of units of $\mathbb{K}$ is a subset of $\{\pm1,\pm\mathrm{i}\}$ and n is odd, then ε is a n-th power. Thus, (10) can be reduced to $x+\mathrm{i}\sqrt{d}z=\eta^n$. Since $\mathbb{K}$ is an imaginary quadratic field and $-d\not\equiv 1 \pmod 4$, then $\{1,\mathrm{i}\sqrt{d}\}$ is an integral basis and we can write $\eta=u+\mathrm{i}\sqrt{d}v$, for some integers u and v. We then get

$$\frac{\eta^n-\overline{\eta}^n}{\eta-\overline{\eta}}=\frac{2^{\lfloor\alpha/2\rfloor}5^{\lfloor\beta/2\rfloor}17^{\lfloor\gamma/2\rfloor}}{v}, \tag{11}$$

where, as usual, $\overline{w}$ denotes the complex conjugate of w.

Let $(L_m)_{m\geq 0}$ be the Lucas sequence given by

$$L_m=\frac{\eta^m-\overline{\eta}^m}{\eta-\overline{\eta}}, \quad \text{for} \quad m\geq 0\,.$$

We recall that the Primitive Divisor Theorem for Lucas sequences ensures for primes $n\geq 5$, that there exists a *primitive divisor* for L_n, except for the finitely many (*defective*) pairs $(\eta,\overline{\eta})$ given in Table 1 of [11] (a primitive divisor of L_n is a prime that divides L_n but does not divide $(\eta-\overline{\eta})^2\prod_{j=1}^{n-1}L_j$). And a helpful property of a primitive divisor p is that $p\equiv\pm1 \pmod n$.

For $n=5$, we find in Table 1 in [11] that L_5 has a primitive divisor except for $(u,d,v)=(1,10,1)$ which leads to a number $\eta=1+\mathrm{i}\sqrt{10}\in\mathbb{Q}[\mathrm{i}\sqrt{10}]$ ($d=10$ is one of the possible values of d described in the beginning of this proof), which gives the solution with $n=5$.

Apart from this case, let p be a primitive divisor of L_n, $n\geq 7$. The identity (11) implies that $p\in\{2,5,17\}$ and so $p=17$, since $p\not\equiv\pm1 \pmod n$, for $p=2{,}5$. Hence, n is a prime dividing 17 ± 1 and so $n=2$ or 3 which contradicts the fact that $n\geq 7$. This completes the proof of Theorem 1. $\square$

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The principle of stationary action in the calculus of variations

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Abstract. We review some techniques from non-linear analysis in order to investigate critical paths for the action functional in the calculus of variations applied to physics. Our main intention in this regard is to expose precise mathematical conditions for critical paths to be minimum solutions in a variety of situations of interest in Physics. Our claim is that, with a few elementary techniques, a systematic analysis (including the domain for which critical points are genuine minima) of non-trivial models is possible. We present specific models arising in modern physical theories in order to make clear the ideas here exposed.

1 Introduction

The calculus of variations is one of the oldest techniques of differential calculus. Ever since its creation by Johann and Jakob Bernoulli in 1696–97, to solve the problem of the brachistochrone (others solved it, too: Newton, Leibniz, Tschirnhaus and L'Hôpital, but their methods were different), it has been applied to a variety of problems both in pure and applied mathematics. While occupying a central place in modern engineering techniques (mainly in control theory, see [9], [41], [50], [68], [78]), it is in physics where its use has been promoted to the highest level, that of the basic principle to obtain the equations of motion, both in the dynamics of particles and fields: the principle of stationary action (see [5], [10], [35], [59]). Accordingly to that point of view, almost every text on mechanics include a chapter on the calculus of variations although, surprisingly enough, the treatment in these texts is expeditious and superficial, directly oriented towards the obtention of Euler-Lagrange's equations, leaving aside the question of whether the solutions are true minima or maxima, despite the importance of this distinction (for instance, while

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in fields such as optics one is interested in the minimal optical length, in stochastic dynamics one seeks to maximize the path entropy [74]. On the other hand, while the principle of stationary action just selects critical paths, experimentally an actual minimum is detected in some systems, see [27]).

It is interesting to note that the principle was once called the principle of least action, although it was soon realized that many physical phenomena does not follow a trajectory that realizes a minimum of the action, but just a critical path (the main example is the harmonic oscillator, whose trajectory in phase space only minimizes the action for a time interval of length which depends on its frequency, see Sec. V of [37], which we recommend to get details about the physical meaning of the action integral and its extremals). In view of these phenomena, and because the emphasis was on the equations of motion, the elucidation of the true nature of the critical paths of the action functional was omitted, and the interest focused on the stationary property. However, some recent papers have made a “call to action” (the pun is not ours, see [36], [37], [54], [69]), renewing the interest in their extremal properties, not only their character of paths rendering the action stationary.

Our aim in this paper is twofold: on the one hand, to offer a concise, yet rigorous and self-contained, overview of some elementary techniques of non-linear analysis to investigate the extremals of an action functional. On the other hand, we intend to show several non trivial examples of physical interest illustrating the use of these techniques. We have avoided the well-known cases, so our examples go beyond oscillators and central potentials, and are taken from modern theories, ranging from astrophysics (Lane-Emden equations) to relativistic particles with energy dissipation. Each one of these examples has been chosen to illustrate some particular feature. Thus, example 6.1 shows a Lagrangian for a dissipative system; in example 6.2.3, as a bonus of the theory developed, we explicitly compute the solution (and its zeros) of an equation of the form $y'' + q(x)^2 y = 0$ with $q(x)$ rational; example 6.6 contains a justification of the use of Lagrange multipliers in the maximum entropy principle, etc.

We also differ from previous works, such as [36] or [37], in the flavour of the treatment: we feel that discussions trying to explain some plain analytic effects in physical terms are too lengthy, and sometimes add confusion instead of enlightenment when it comes to explicit computations. Thus, we center our exposition around the analytic definition and properties of the Gâteaux derivatives of functionals defined by integration (Lagrange functionals) and the techniques for the study of the behaviour of solutions of differential equations such as convexity or the comparison theorems of the Sturmian theory. This will be particularly patent in Section 5, where we show that the main result in [37] is a direct consequence of well-known properties of the zeros of the Jacobi equation (see Proposition 4 and comments).

We offer short proofs for those results that seem to be not common in the physics literature. The bibliography, although by no means complete, is somewhat lengthy as a result of our efforts to make it useful.

2 Calculus of variations

In this section we will briefly describe some basic concepts in the calculus of variations in order to set up our notation and conventions, and also in order to introduce the Jacobi equation and conjugate points as explicit criteria for a given extremal solution to be a minimum. We will start by discussing Gâteaux derivatives and extrema of functionals. For general references on the topics of functional analysis and calculus of variations, see [16], [19], [28], [30], [63], [65], [72], [73], [79]. Note that we deal with the local aspects of the theory, exclusively. There are several approaches to the global setting, some of these were developed in [26], [32], [62]; more modern versions are developed in [48]. For detailed accounts of the theory involved in the global analysis, see [29] and [47].

2.1 Gâteaux derivatives

Let $(E, \|\cdot\|)$ be a Banach space, $D \subseteq E$ an open subset of E and $y_0 \in D$. Given a functional $J\colon D \to \mathbb{R}$, if $v \in E$ is a non zero vector and $|t|$ small enough, $y_0 + tv$ will lie in D so the following definition makes sense.

Definition 1. Whenever it exists, the limit

$$\delta J(y_0, v) := \lim_{t\to 0} \frac{J(y_0 + tv) - J(y_0)}{t}$$

is called the Gâteaux derivative (or first variation) of J at y_0 in the direction $v \in E$. This defines a mapping $\delta J(y_0, \cdot)\colon E \to \mathbb{R}$. If this mapping is linear and continuous, we denote it by $J'(y_0)$ and say that J is Gâteaux differentiable at y_0. Thus, under these conditions, $\delta J(y_0, v) = J'(y_0)(v)$. Another common notation is $\delta J(y_0, v) = \delta_{y_0} J(v)$.
The $y_0 \in D$ such that $J'(y_0) = 0$ are called critical points of the functional J.

The extension to higher-order derivatives is immediate. If, for a fixed $v \in E$, $\delta J(z, v)$ exists for every $z \in D$, we have a mapping $D\colon \to \mathbb{R}$ and we can compute its Gâteaux derivative. Given an $y_0 \in D$ and $z, v \in E$, the second Gâteaux derivative (or second variation) of J at y_0 in the directions v and z (in that order) is

$$\delta^2 J(y_0, v, z) := \lim_{t\to 0} \frac{\delta J_{y_0+tz}(v) - \delta_{y_0} J(v)}{t}\,.$$

If $\delta^2 J(y_0, v, z)$ exists for any $z, v \in E$, and $(v, z) \mapsto \delta^2 J(y_0, v, z)$ is bilinear and continuous, we say that J is twice Gâteaux differentiable at $y_0 \in D$, and write $J''(y_0)$ for this mapping. With these notations, we will write

$$\delta^2_{y_0} J(v) = \delta^2 J(y_0, v) := \delta^2 J(y_0, v, v)\,.$$

Remark 1. For fixed $y_0 \in D$ and $v \in E$, if we consider the function $j_{(y_0,v)}\colon \mathbb{R} \to \mathbb{R}$ by $j_{(y_0,v)}(t) = J(y_0 + tv)$, it is obvious that it is defined in some open neighborhood of 0, $]-\varepsilon, \varepsilon[$, and the higher-order variations of J are given by

$$\delta^n J(y_0, v) = j^{(n)}_{(y_0,v)}(0)\,.$$

We will be interested in a particular class of functionals. To introduce it, we first need a technical observation: given an open subset $U \subset \mathbb{R}^3$, the set

$$D_U = \{y \in \mathcal{C}^1([a,b]) : \forall x \in [a,b]\,,\ (x, y(x), y'(x)) \in U\}$$

(the prime denotes derivation, although we will also make free use of the physicist's dot notation for derivatives) is evidently contained in the Banach space $\big(\mathcal{C}^1([a,b]), \|\cdot\|\big)$, endowed with the norm

$$\|y\| = \|y\|_0 + \|y'\|_0\,,$$

where $\|\cdot\|_0$ is the supremum norm. Moreover, $D_U \subset \mathcal{C}^1([a,b])$ is an open subset. This follows from the fact that for a given $y_0 \in D_U$ the set $\{x, y_0(x), y_0'(x) : x \in [a,b]\}$ is compact, so it has an open neighborhood contained in U.

Definition 2. A function $L \in \mathcal{C}^2(U)$ is called a Lagrangian. To every Lagrangian it corresponds a functional $J\colon D_U \to \mathbb{R}$, called its action, defined by

$$J(y) = \int_a^b L\big(x, y(x), y'(x)\big)\,\mathrm{d}x\,.$$

Proposition 1. *For any $U \subset \mathbb{R}^3$, the action J is Gâteaux differentiable on D_U.*

Proof. Let $y \in D_U$. Taking into account the remark 1, note that for any $t \in\,]-\varepsilon, \varepsilon[$ (applying Leibniz's theorem of derivation under the integral):

$$j'_{(y,v)}(t) = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\Big(L\big(s, y(s) + tv(s), y'(s) + tv'(s)\big)\Big)\,\mathrm{d}s\,.$$

Evaluating the derivative at $t = 0$, we get

$$\delta J(y,v) = \int_a^b \Big(v(s) D_2 L\big(s, y(s), y'(s)\big) + v'(s) D_3 L\big(s, y(s), y'(s)\big)\Big)\,\mathrm{d}s\,. \qquad (1)$$

Note that $\delta J(y,v)$ is linear in v. Moreover,

$$\big|\delta J(y,v)\big| \le \|v\| \int_a^b \Big(\big|D_2 L\big(s, y(s), y'(s)\big)\big| + \big|D_3 L\big(s, y(s), y'(s)\big)\big|\Big)\,\mathrm{d}s\,,$$

where the integral exists (because the $D_i L$, $i \in \{1,2\}$, are continuous on U and y, y' on $[a,b]$) and it is a number depending only on y, thus constant for fixed y. Then, $\delta J(y,v)$ is also continuous in v. □

2.2 Local extrema of functionals

Definition 3. Let $J\colon D \to \mathbb{R}$ be a functional and let $y_0 \in D$. We will say that J has a local maximum (local minimum, respectively) in y_0 if for all $y \in G$, where $G \subset D$ is a convex neighborhood of the point y_0, it follows that

$$J(y_0) \ge J(y)$$

($J(y_0) \le J(y)$, respectively).

Theorem 1. *Let $J\colon D \to \mathbb{R}$ be a functional and $y_0 \in D$. Then:*

1. *(Necessary condition) If J has a local extremal and the variation $\delta J(y_0, v)$ exists for some $v \in E$, then $\delta J(y_0, v) = 0$. Thus, if J is Gâteaux differentiable at y_0, $J'(y_0) = 0$.*
2. *(Sufficient condition for a minimum) The functional J has a local minimum at y_0 whenever the following hold:*
   (a) *For each $v \in E$, $\delta J(y_0, v) = 0$.*
   (b) *(Coercivity) For any y in a convex neighborhood of y_0, the second variation $\delta^2 J(y, v)$ exists for each $v \in E$. Moreover, there exists a $c > 0$ such that*
   $$\delta^2 J(y_0, v) \geq c\|v\|^2,$$
   for all $v \in E$.
   (c) *(Weak continuity) Given $\varepsilon > 0$, there exists an $\eta > 0$ such that*
   $$\left|\delta^2 J(y, v) - \delta^2 J(y_0, v)\right| \leq \varepsilon\|v\|^2,$$
   for any $v \in E$ and y satisfying $\|y - y_0\| < \eta$.

Proof.

1. Consider $j_{(y_0,v)}(t) = J(y_0+tv)$, so if J has a local extremal at y_0, $j_{(y_0,v)}$ has a local extremum at $t = 0$. Then, it must be (recall remark 1) $0 = j'_{(y_0,v)}(0) = \delta J(y_0, v)$.
2. Suppose each of (2a), (2b), (2c) holds. As before, we have $j'_{(y_0,v)}(t) = \delta J(y_0+tv, v)$ and $j''_{(y_0,v)}(t) = \delta^2 J(y_0+tv, v)$. The hypothesis on the second derivatives of J allows us to develop $j_{(y_0,v)}(t)$ by Taylor in the interval $[0, 1]$, and there exists a $\xi \in \,]0, 1[$ such that
   $$J(y_0 + v) - J(y_0) = j_{(y_0,v)}(1) - j_{(y_0,v)}(0) = \frac{1}{2} j''_{(y_0,v)}(\xi)\,.$$
   As $j''_{(y_0,v)}(\xi) = \delta^2 J(y_0 + \xi v, v)$, we have the following bound:
   $$\begin{aligned} J(y_0 + v) - J(y_0) &= \frac{1}{2}\delta^2 J(y_0, v) + \frac{1}{2}\Big(\delta^2 J(y_0 + \xi v, v) - \frac{1}{2}\delta^2 J(y_0, v)\Big) \\ &\geq \frac{1}{2}c\|v\|^2 + \frac{1}{2}\Big(\delta^2 J(y_0 + \xi v, v) - \frac{1}{2}\delta^2 J(y_0, v)\Big) \end{aligned}$$
   Taking $\varepsilon = c/4$ in (2c), there exists an $\eta > 0$ such that
   $$\left|\delta^2 J(y, v) - \delta^2 J(y_0, v)\right| \leq \frac{c}{4}\|v\|^2,$$
   for $\|y - y_0\| < \eta$. Choosing now $\|v\| < c\eta/2$, it is $\|y_0 + \xi v - y_0\| \leq \|v\| < c\eta/2$, so
   $$\left|\delta^2 J(y_0 + \xi v, v) - \delta^2 J(y_0, v)\right| \leq \frac{c}{4}\|v\|^2.$$

Substituting:

$$J(y_0+v)-J(y_0)\geq \frac{1}{2}c\|v\|^2-\frac{1}{4}c\|v\|^2=\frac{1}{4}c\|v\|^2>0\,,$$

so J has a local minimum.

$\square$

Remark 2. Writing $c<0$ and reversing the inequality for $\delta^2 J(y_0,v)$ in (2b), we get sufficient conditions for a local maximum.

Remark 3. Note that condition (2a) alone does not guarantee the existence of a local minimum, see counter-examples in [56], §2.10.

It is interesting to particularize the condition $\delta J(y_0,v)=0$ to the case of an action functional. For this, we need a couple of technical results whose proof is straightforward, but anyway can be found in any of the references cited at the beginning of this section. We will denote

$$\mathcal{C}_0^k([a,b])=\{f\in\mathcal{C}^k([a,b]):f(a)=0=f(b)\}\,.$$

Lemma 1 (Lagrange). *Let $f\in\mathcal{C}([a,b])$ be a continuous real-valued function over the interval $[a,b]$ such that*

$$\int_a^b f(x)\mu(x)\,\mathrm{d}x=0$$

for all $\mu\in\mathcal{C}_0([a,b])$. Then it follows $f\equiv 0$.

Lemma 2 (DuBois-Reymond). *Let $f\in\mathcal{C}([a,b])$ and $g\in\mathcal{C}^1([a,b])$ such that*

$$\int_a^b\big(f(x)\mu(x)+g(x)\mu'(x)\big)\,\mathrm{d}x=0$$

for all $\mu\in\mathcal{C}_0^1([a,b])$. Then it follows $g'=-f$.

Now, a simple integration by parts in (1), and the application of Lemmas 1, 2 and Theorem 1, leads directly to the following result.

Theorem 2 (Euler-Lagrange). *If $y\in D$ is an extremal (maximum or minimum) for the action functional $J\colon D\to\mathbb{R}$ given as in Definition 2, then y must satisfy the Euler-Lagrange equations*

$$D_2L\big(x,y(x),y'(x)\big)-\frac{\mathrm{d}}{\mathrm{d}x}D_3L\big(x,y(x),y'(x)\big)=0\,. \qquad (2)$$

Remark 4. In physics literature, it is common to commit a slight abuse of notation and to write the Euler-Lagrange equations in the form

$$\frac{\partial}{\partial y}L\big(x,y(x),y'(x)\big)-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial y'}L\big(x,y(x),y'(x)\big)=0\,.$$

Note that, for the case at hand, writing $L\big(s, y(s)+tv(s), y'(s)+tv'(s)\big) = L(s)$ for simplicity:

$$j''_{(y,v)}(t) = \int_a^b \big(v^2(s)D_{22}L(s) + 2v(s)v'(s)D_{23}L(s) + (v')^2(s)D_{33}L(s)\big)\,\mathrm{d}s\,,$$

so, evaluating at $t = 0$,

$$\begin{aligned}\delta^2 J(y,v) = \int_a^b \Big(&v^2(s)D_{22}L\big(s,y(s),y'(s)\big) + 2v(s)v'(s)D_{23}L\big(s,y(s),y'(s)\big)\\ &+ (v')^2(s)D_{33}L\big(s,y(s),y'(s)\big)\Big)\,\mathrm{d}s\,.\end{aligned} \tag{3}$$

It is now a routine computation (continuity arguments and Schwarz inequality) to prove that for an action functional $J(y) = \int_a^b L(x,y,y')\,\mathrm{d}x$ with $L \in \mathcal{C}^2(U)$ such that its second partial derivatives are bounded on U, under the hypothesis (2a) and (2b) of Theorem 1, the condition (2c) is satisfied ([30], pg. 224). Thus, a path $y_0 \in D$ is a minimum if it satisfies Euler-Lagrange's equations (2) *and* the second Gâteaux differential at y_0 is coercive, that is, there exists a $c > 0$ such that, for all $v \in E$:

$$\delta^2 J(y_0, v) \geq c\|v\|^2. \tag{4}$$

2.3 Problems with constraints

The calculus of variation is frequently applied when there are constraints. The problem can be reduced to that of extremizing a single functional constructed out from the original one and the constraints.

Definition 4. Let J be a functional defined on a Banach space $(X, \|\cdot\|)$. We say that δJ is weakly continuous at $y \in X$ if:

(a) The domain of J contains an open neighborhood $D \ni y$, and, for each $h \in X$, the variation $\delta J(y,h)$ is defined.

(b) $\lim\limits_{z \to y} \delta J(z,h) = \delta J(y,h)$.

If there exists an $r > 0$ such that δJ is weakly continuous for every $z \in B(y;r)$, we say that δJ is locally weakly continuous at y, or simply weakly continuous near y.

When we have an open subset $U = \,]a,b[\, \times \mathbb{R} \times \mathbb{R} \subset \mathbb{R}^3$ and an action functional $J\colon D_U \to \mathbb{R}$, $J(y) = \int_a^b L(x,y,y')\,\mathrm{d}x$, it is easy to see that imposing some mild conditions on the Lagrangian $L \in \mathcal{C}^2(U)$ we obtain a weakly continuous functional. For instance, it is enough to require that the second partial derivatives of L be bounded on U, or that its first partial derivatives be uniformly continuous. For most of the actions appearing in physics, however, it is usually easier to prove the weak continuity directly from the definition (cfr. Example 6.6).

Let now $J =: K_0, K_1, \ldots, K_r$ be functionals defined on D, all of them Gâteaux differentiables at each point $y \in D$. We will assume that the set

$$S = \{y \in D : K_i(y) = k_i,\ 1 \leq i \leq r\}$$

is not empty, and that y_0 is an interior point of S such that $J|_S$ has a local extremal at y_0.

Proposition 2. *Let δK_j be weakly continuous near y_0, $0 \leq j \leq r$. Then, for any $h =: h_0, h_1, \dots, h_r \in X$ we have:*

$$\det(\delta K_j(y_0, h_m)) = 0, \quad 0 \leq j, m \leq r\,.$$

Proof. By reduction to the absurd. Let us assume that there exist $h, h_1, \dots, h_r$ such that the determinant is non zero. As y_0 has an open neighborhood D, there exist a set of scalars $\alpha, \beta_1, \dots, \beta_r$ such that $y_0 + \alpha h + \beta_1 h_1 + \cdots + \beta_r h_r \in D$, and the variations $\delta J, \delta K_1, \dots, \delta K_r$ are continuous at $y_0 + \alpha h + \beta_1 h_1 + \cdots + \beta_r h_r$. Let us define the function $F\colon \mathbb{R}^{r+1} \to \mathbb{R}^{r+1}$, on a neighborhood G of the origin $(0, \dots, 0) \in \mathbb{R}^{r+1}$, by

$$F_{p+1}(\alpha, \beta_1, \dots, \beta_r) = K_p(y_0 + \alpha h + \beta_1 h_1 + \cdots + \beta_r h_r)\,,$$

for $0 \leq p \leq r$ (remember $K_0 = J$ and $h_0 = h$). It is immediate that

$$D_{q+1}F_{p+1}(\alpha, \beta_1, \dots, \beta_r) = \delta K_p(y_0 + \alpha h + \beta_1 h_1 + \cdots + \beta_r h_r, h_q)\,,$$

for $0 \leq q \leq r$. Now, as $\delta J, \delta K_1, \dots, \delta K_r$ are continuous near y_0, by shrinking G if necessary we can assume $F \in \mathcal{C}^1(G)$, with Jacobian at the origin:

$$\det(\delta K_j(y_0, h_m)) \neq 0, \quad 0 \leq j, m \leq r$$

by hypothesis. Applying to F the inverse function theorem in the neighborhood of the origin, we get that there exists an open subset $V \subset \mathbb{R}^{r+1}$, containing $F(0, \dots, 0) = \big(J(y_0), k_1, \dots, k_r\big)$, and a local diffeomorphism $\varphi\colon V \to G$ such that $\tilde{G} = \varphi(V) \subset G$ is an open neighborhood of the origin in $\mathbb{R}^{r+1}$, and for all $(x, y_1, \dots, y_r) \in V$:

$$(x, y_1, \dots, y_r) = F\big(\varphi(x, y_1, \dots, y_r)\big)\,.$$

In particular, $\varphi\big(J(y_0), k_1, \dots, k_r\big) = (0, 0, \dots, 0)$. As $\big(J(y_0), k_1, \dots, k_r\big)$ is a point of the open set $V \subset \mathbb{R}^{r+1}$, we can find in V two different points $(x_1, k_1, \dots, k_r)$ and $(x_2, k_1, \dots, k_r)$ such that $x_1 < J(y_0) < x_2$. Their corresponding images by φ are $(\alpha^1, \beta^1_1, \dots, \beta^1_r) = \varphi(x_1, k_1, \dots, k_r)$ and $(\alpha^2, \beta^2_1, \dots, \beta^2_r) = \varphi(x_2, k_1, \dots, k_r)$. Thus, the vectors $u = y_0 + \alpha^1 h + \beta^1_1 h_1 + \cdots + \beta^1_r h_r$ and $v = y_0 + \alpha^2 h + \beta^2_1 h_1 + \cdots + \beta^2_r h_r$ belong to D. Moreover,

$$F(\alpha^1, \beta^1_1, \dots, \beta^1_r) = (x_1, k_1, \dots, k_r)\,,$$
$$F(\alpha^2, \beta^2_1, \dots, \beta^2_r) = (x_2, k_1, \dots, k_r)\,,$$

so, equating components:

$$J(u) = K_0(u) = F_0(\alpha^1, \beta^1_1, \dots, \beta^1_r) = x_1\,,$$
$$K_p(u) = F_p(u) = F_p(\alpha^1, \beta^1_1, \dots, \beta^1_r) = k_p\,,$$

for $1 \leq p \leq r$, and

$$J(v) = K_0(v) = F_0(\alpha^2, \beta_1^2, \dots, \beta_r^2) = x_2\,,$$
$$K_p(v) = F_p(v) = F_p(\alpha^2, \beta_1^2, \dots, \beta_r^2) = k_p\,,$$

so $u, v \in S$ too. But, because of our choices, $J(u) = x_1 < J(y_0) < x_2 = J(v)$, and this construction can be repeated for x_1, x_2 arbitrarily close to $J(y_0)$, so (by the continuity of φ) the corresponding vectors u, v will be arbitrarily close to y_0, contradicting the assumption that y_0 is a local extremal of $J|_S$. □

Theorem 3. *Let δK_j be weakly continuous near y_0. Then, either*

$$\det(\delta K_i(y_0, h_l)) = 0, \quad 1 \leq i, l \leq r\,,$$

or there exist a set of real numbers (the Lagrange multipliers) $\lambda_1, \dots, \lambda_r$ such that

$$\delta J(y_0, h) = \sum_{i=1}^{m} \lambda_i \delta K_i(y_0, h)\,.$$

Proof. By Proposition 2, $\det(\delta K_j(y_0, h_j)) = 0, \quad 0 \leq j \leq r$; developing by the first column (where $K_0 = J$ and $h_0 = h$), we get

$$\delta J(y_0, h) \det(\delta K_i(y_0, h_l)) + \sum_{i=1}^{m} \mu_i \delta K_i(y_0, h) = 0\,, \tag{5}$$

for some set of scalars $\mu_1, \dots, \mu_r$ (which depend on the vectors $(h_1, \dots, h_r)$). Then, either

$$\det\big(\delta K_i(y_0, h_l)\big) = 0, \quad 1 \leq i, l \leq r\,,$$

or there exist some set of vectors $v_1, \dots, v_r \in X$ such that $\det(\delta K_i(y_0, v_l)) \neq 0$. In this case, substituting in (5), we obtain

$$\delta J(y_0, h) = \sum_{i=1}^{m} \lambda_i \delta K_i(y_0, h)\,,$$

where

$$\lambda_i = -\frac{\mu_i}{\det\big(\delta K_i(y_0, h_l)\big)}\,.$$

□

3 Conjugate points

Let us rewrite the conditions for a minimum of the action J, found in the previous section, in terms of a differential equation involving the derivatives of the Lagrangian L.

Proposition 3. *Let the action J be given as in Definition 2, and let $y_0 \in D$. Then, the second variation of J at y_0, in the direction of a $v \in \mathcal{C}_0^1([a,b])$, $\delta^2 J(y_0, v)$, reads*

$$\delta^2 J(y_0, v) = \frac{1}{2} \int_a^b \left(P v'^2 + Q v^2 \right) \mathrm{d}x \,, \tag{6}$$

where the functions $P(x)$ and $Q(x)$ are explicitly given by

$$P(x) = D_{33} L\big(x, y(x), y'(x)\big) \tag{7}$$

$$Q(x) = D_{22} L\big(x, y(x), y'(x)\big) - \frac{\mathrm{d}}{\mathrm{d}x} D_{23} L\big(x, y(x), y'(x)\big) \tag{8}$$

Proof. Just make an integration by parts in the middle term of the integrand in (3), taking into account the boundary conditions on $v \in \mathcal{C}_0^1([a,b])$. $\square$

Remark 5. The notation used in physics is:

$$P = \frac{\partial^2 L}{\partial (y')^2} \,,$$
$$Q = \frac{\partial^2 L}{\partial y^2} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial^2 L}{\partial y \partial y'} \right) .$$

Lagrange considered equation (6) already in 1786. He thought that a sufficient condition to have a minimum would be the positivity of the second variation $\delta^2 J(y_0, v) > 0$ (which is *not* true: coercivity is needed), so he tried to "complete the square" in (6) by introducing a boundary term of the form $gv^2/2$, where $g \in \mathcal{C}([a,b])$ is to be determined. In this way, we have

$$\begin{aligned} \delta^2 J(y_0, v) &= \int_a^b \left(P (v')^2 + Q v^2 \right) \mathrm{d}x + \int_a^b \frac{\mathrm{d}}{\mathrm{d}x} (g v^2) \, \mathrm{d}x \\ &= \int_a^b \left(P (v')^2 + 2 g v v' + (g' + Q) v^2 \right) \mathrm{d}x \\ &= \int_a^b P \left(v' + \frac{g}{P} v \right)^2 \mathrm{d}x + \int_a^b \left(g' + Q - \frac{g^2}{P} \right) v^2 \, \mathrm{d}x \,. \end{aligned} \tag{9}$$

Thus, it is straightforward that $\delta^2 J(y_0, v)$ will be positive definite if the following conditions are satisfied

$$P = D_{33} L(x, y(x), y'(x)) > 0 \,, \tag{10}$$

$$g' + Q - \frac{g^2}{P} = 0 \quad \text{has a solution } g \,. \tag{11}$$

Condition (10) is known as the Legendre condition. Also, note that equation (11) is of Riccati type. This equation is basic to determine the extremality properties of critical points of the action.

Definition 5. Let J be an action functional, and let $f \in \mathcal{C}([a,b])$. The differential equation for f

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(P\frac{\mathrm{d}f}{\mathrm{d}x}\right) + Qf = 0\,, \tag{12}$$

where P and Q are given in (7) and (8), respectively, is called the Jacobi equation.

Notice that Jacobi equation is simply obtained by introducing the change of variable $g = -P\mathrm{d}(\ln f)/\mathrm{d}x$ in equation (11), which renders it linear. Once we solve the equation for f, we get g and then we can assure that, if $P > 0$, then $\delta^2 J(y_0, v) > 0$. Although we know that this is not enough to guarantee a minimum (recall Remark 3), the properties of the solutions to the Jacobi equation (12) will lead us to an equivalent condition for a minimum of the action, given in terms of quantities determined by the Lagrangian.

Definition 6. Two points $p, q \in \mathbb{R}$ (with $p < q$) are called conjugate with respect to the Jacobi equation (12) if there is a solution $f \in \mathcal{C}^2([a,b])$ of (12) such that $f|_{]p,q[} \neq 0$ and $f(p) = 0 = f(q)$.

The following result is just a particular case of K. Friedrichs' inequalities for the one-dimensional case (see [1]). Its proof can also be done directly, as an application of the Schwarz inequality.

Lemma 3. *For any $v \in \mathcal{C}_0^1([a,b])$, we have*

$$\int_a^b v^2(x)\mathrm{d}x \leq \frac{(b-a)^2}{2}\int_a^b (v')^2(x)\,\mathrm{d}x$$

Theorem 4. *Let $J(y)$ be an action functional as in Definition 2. Sufficient conditions for a critical point y_0 of $J(y)$ to be a local minimum in the interval $[a,b]$ are given by*

(a) For all $x \in [a,b]$

$$P(x) = D_{33}L(x, y(x), y'(x)) > 0\,.$$

(b) The interval $[a,b]$ does not contain conjugate points at $x = a$ with respect to Jacobi equation (12).

Proof. Recall from (9) the expression

$$\delta^2 J(y_0, v) = \int_a^b \left(P(v')^2 + 2gvv' + (g' + Q)v^2\right)\mathrm{d}x\,.$$

Because of the assumptions made on the continuity of the derivatives of L and the compactness of $[a,b]$, we can choose a number σ such that $0 \leq \sigma < \min_{[a,b]}\{P(x)\}$. Inserting $\sigma P(v')^2 - \sigma P(v')^2$ in the equation above, and repeating the computation in (9) gives

$$\begin{aligned}\delta^2 J(y_0, v) = &\int_a^b (P-\sigma)\Big(v' + \frac{g}{P-\sigma}\Big)^2\mathrm{d}x \\ &+ \int_a^b \Big(g' + Q - \frac{g^2}{P-\sigma}\Big)v^2\,\mathrm{d}x + \sigma\int_a^b (v')^2\,\mathrm{d}x\,.\end{aligned} \tag{13}$$

As $P(x) > 0$ and we have chosen σ such that $P(x)-\sigma > 0$ on $[a,b]$, the first integral is positive, as it is the third one. In order to cancel out the second integral, we must take a $g \in \mathcal{C}^1([a,b])$ such that

$$g' + Q - \frac{g^2}{P-\sigma} = 0\,.$$

Introducing a function $f \in \mathcal{C}^2([a,b])$ through

$$g = -\frac{f'}{f}(P-\sigma)\,, \tag{14}$$

we arrive at the equation for f:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left((P-\sigma)\frac{\mathrm{d}f}{\mathrm{d}x}\right) + Qf = 0\,. \tag{15}$$

By the theorem on the dependence on parameters of the solutions to a second order differential equation, the general solution of (15) can be written as $f(x,\sigma)$, with $f(x,0) = f(x)$. Note that, by hypothesis, $f(x,0)$ does not admit points conjugate to a in $[a,b]$, so (by continuity), neither does $f(x,\sigma)$ for $\sigma > 0$ but close enough to 0. If $\tilde{f}(x) = f(x,\sigma)$ is such a solution, by substituting the corresponding $\tilde{g}$ of equation (14) into (13), we get:

$$\begin{aligned}\delta^2 J(y_0,v) &= \int_a^b (P-\sigma)\Big(v' + \frac{\tilde{g}}{P-\sigma}\Big)^2 \mathrm{d}x + \sigma\int_a^b (v')^2\,\mathrm{d}x \\ &\geq \sigma\int_a^b (v')^2\,\mathrm{d}x\,.\end{aligned}$$

Now, applying Lemma 3,

$$\delta^2 J(y_0,v) \geq \frac{\sigma}{1+\frac{(b-a)^2}{2}}\|v\|^2 := c\|v\|^2\,.$$

The statement follows then from Theorem 1 (see also the comments at the end of Subsection 2.2). □

Remark 6. We can obtain a criterion for a local maximum just by considering the condition $P(x) = D_{33}L\big(x,y(x),y'(x)\big) < 0$ and repeating the computations in the theorem with the inequalities reversed.

Remark 7. In order to apply the criterion of conjugate points in practice, it is desirable to have at our disposal some tools for explicitly computing solutions of the Jacobi equation (12). An old (but useful) method ([56] pp. 56–57, [24] pp. 42–43) is the following: the general solution of Euler-Lagrange's equations (which are second order) has the form $y = y(x;\alpha,\beta)$, where α,β are constants of integration (on which the y dependence is differentiable, under some mild conditions. In the examples this will be obvious). Then, $D_2y(x;\alpha,\beta) \equiv \frac{\partial y}{\partial\alpha}$ and $D_3y(x;\alpha,\beta) \equiv \frac{\partial y}{\partial\beta}$

are two independent solutions of the Jacobi equation[1]. We will use this method in Example 6.2.3.

4 Convex functionals

In this section we will discuss the particular case of convex Lagrangians. We will start by recalling that a subset $S \subset \mathbb{R}^n$ is said to be convex if for all $p, q \in S$ the interval $[p, q]$ lies entirely inside of S. This is equivalent to say that

$$[p, q] := \{p + t(q - p) = tq + (1 - t)p : 0 \leq t \leq 1\} \subset S\,. \tag{16}$$

Definition 7. A function $f : S \subset \mathbb{R}^2$ defined over a convex set, is said to be convex if

$$f([p, q]) \leq [f(p), f(q)]$$

or, equivalently, for all $t \in [0, 1]$:

$$f\big(p + t(q - p)\big) \leq f(p) + t\big(f(q) - f(p)\big)\,. \tag{17}$$

Let $f \colon S \to \mathbb{R}$ be a convex *differentiable* function. For any $t \in [0, 1]$, $p, q \in S$ we have (17), and, on the other hand, by the intermediate value theorem, there exists a $w_t \in [p, p + t(q - p)]$ such that

$$f\big(p + t(q - p)\big) = f(p) + t\mathrm{d}_{w_t} f(q - p)\,, \tag{18}$$

where $\mathrm{d}_{w_t} f$ denotes the differential of f at w_t. From (17) and (18) we get $f(p) + t\mathrm{d}_{w_t} f(q - p) \leq tf(q) + (1 - t)f(p)$, that is:

$$f(p) + \mathrm{d}_{w_t} f(q - p) \leq f(q)\,.$$

Taking the limit $t \to 0$ implies $w_t \to p$, so:

$$f(p) + \mathrm{d}_p f(q - p) \leq f(q)\,. \tag{19}$$

Remark 8. There exist well-known criteria (in terms of the Hessian matrix) to decide whether a function $f \colon S \to \mathbb{R}$ is convex or not, see for instance [73] Sec. 10.7. We will apply one such criterion in Example 6.2.1.

As a straightforward application of these results, we have the following theorem, stating that the critical points of an action with a convex Lagrangian are always minimals.

Theorem 5. *Consider a set $U = [a, b] \times S \subset \mathbb{R}^3$, such that for each fixed $x \in [a, b]$ the set $S_x = \{(x, u, v) \in U\} \subset \mathbb{R}^2$ is convex. Suppose that for any $x \in [a, b]$, the Lagrangian function $L(x, \cdot, \cdot) \colon S_x \to \mathbb{R}$ is convex. Then, any critical path $y_0 = y_0(x)$ is a minimal solution, among the paths with the same endpoints, for the corresponding action functional $J(y) = \int_a^b L\big(x, y(x), y'(x)\big)\,\mathrm{d}x$.*

[1] The proof is extremely simple: just take derivatives with respect to, say, α in Euler-Lagrange's equations (applying the chain rule) and collect terms, taking into account that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial^2 L}{\partial y \partial y'}\frac{\partial y}{\partial \alpha}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial^2 L}{\partial y \partial y'}\right)\frac{\partial y}{\partial \alpha} + \frac{\partial^2 L}{\partial y \partial y'}\frac{\partial y'}{\partial \alpha}.$$

Proof. The hypothesis of convexity implies, by (19),

$$\begin{aligned} L(x,u_2,v_2) &\geq L(x,u_1,v_1) + \mathrm{d}_{(x,u_1,v_1)} L((u_2,v_2)-(u_1,v_1)) \\ &= L(x,u_1,v_1) + D_1 L(x,u_1,v_1)(u_2-u_1) + D_2 L(x,u_1,v_1)(v_2-v_1)\,. \end{aligned} \tag{20}$$

Now we compute the action on a critical path, $y_0(x)$, and an arbitrary nearby one $y(x)$, with the same endpoints ($y_0(a)=y(a)$ and $y_0(b)=y(b)$), and compare them. From (20):

$$\begin{aligned} J(y)-J(y_0) &= \int_a^b \Big(L\big(x,y(x),y'(x)\big) - L\big(x,y_0(x),y_0'(x)\big)\Big)\,\mathrm{d}x \\ &\geq \int_a^b \big(D_1 L(x,y_0,y_0')(y-y_0) + D_2 L(x,y_0,y_0')(y'-y_0')\big)\,\mathrm{d}x\,. \end{aligned} \tag{21}$$

The second term in the last integrand can be –as usual– integrated by parts, we then get:

$$J(y)-J(y_0) \geq \int_a^b \Big(D_1 L(x,y_0,y_0') - \frac{\mathrm{d}}{\mathrm{d}x} D_2 L(x,y_0,y_0')\Big)(y-y_0)\,\mathrm{d}x\,.$$

But, by hypothesis, $y_0(x)$ is a critical path; equivalently, for each $x\in[a,b]$ it satisfies the Euler-Lagrange equations (2), and this implies $J(y)\geq J(y_0)$. □

Remark 9. Reversing the inequalities, we obtain the corresponding result for concave functionals.

5 Sturmian Theory

Sturmian theory is concerned with the analysis of the zeros that a solution of a linear second order differential equation, of the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0, \tag{22}$$

(with $p(x),q(x)$ piecewise continuous) has in a given interval of the independent variable. This theory is an invaluable tool to check the properties of critical points, as we will see in the examples of the next section.
The results stated here without proof are well-known (see [2], [64]). We write them just for easy reference.
As it is well-known, the differential equation (22) may be written, through the change of variable $v = y\exp(\frac{1}{2}\int p\,\mathrm{d}x)$, in its normal form

$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + r(x)v = 0\,, \tag{23}$$

where $r(x) = q(x) - \frac{1}{4}p^2(x) - \frac{1}{2}p'(x)$, which clearly preserves the zeros of the solutions to (22) if $\int^x p(s)\,\mathrm{d}s$ is finite for finite x. The first observation is that the zeros of such an equation cannot accumulate.

Proposition 4. *Let $y(x)$ be a non trivial solution of* (22) *or* (23)*. Then, its zeros are simple and the set they form does not have accumulation points. Thus, on each closed interval $[a,b]$, $y(x)$ only possess a finite number of zeros.*

Proof. If x_0 were a double zero, $y(x_0) = 0 = y'(x_0)$, so $y(x)$ would be the trivial solution by uniqueness.
If x_0 were an accumulation point for the zeros of $y(x)$, there would be a sequence of zeros (x_n) such that $x_n \to x_0$. By Rolle's theorem, there is a zero of the derivative between any two consecutive zeros of the function, so there would be a sequence (u_m) of zeros of $y'(x)$ such that $u_m \to x_0$. Then, by continuity of both y and y', we would have $y(x_0) = 0 = y'(x_0)$, which, as we have just seen, is impossible. □

As a corollary, the zeros of a non trivial solution of (22) or (23) are: either a finite set, or a sequence diverging to $+\infty$, or a sequence diverging to $-\infty$, or a sequence diverging to $\pm\infty$.
This applies in particular to the Jacobi equation (12). Thus, if we have a solution $f(x)$ on the interval $[a,b]$ such that $f(a) = 0$, its first zero after $x = a$ must be located at a point $c > a$. In other words: there exists a $c > a$ such that there are no conjugate points in the interval $[a,c]$. A direct consequence of this fact is that for a short enough interval, critical points of the action $J(y) = \int_a^b L(x,y,y')\,\mathrm{d}x$ *such that* $P(x) = D_{33}L(x,y,y') > 0$, are local minimizers.
Some authors state, erroneously, that *for any* Lagrangian the critical points are local minimizers. The origin of the confusion can be traced back to the fact that this is true for *natural* Lagrangians[2], in particular it is true for free Lagrangians ($V = 0$), for which the trajectories are geodesics. However, not every system of interest in Physics is natural. Of all the examples presented in this note, only Example 6.5 is natural; and in Example 6.6 we present a case for which the critical points are maximizers.

Definition 8. If every solution y of (22) or (23) has arbitrarily large (in absolute value) zeros, then the equation (and all its solutions) are called oscillatory. Otherwise, the equation and all of its solutions are called non-oscillatory.

Theorem 6. *Let be $y_1(x)$ and $y_2(x)$ denote two linear independent solutions of equation* (23)*. Then, the zeros of both functions are distinct and alternating in the sense that $y_1(x)$ has a zero between any two consecutive zeros of $y_2(x)$, and vice-versa.*

Theorem 7. *If $r(x) \leq 0$ on $[a,b]$, then no non-trivial solution of* (23) *can have two zeros on $[a,b]$.*

Theorem 8 (Sturm's Comparison Theorem). *Let $y_1(x)$ and $y_2(x)$ be non-trivial solutions to the differential equation* (23) *with $r_1(x)$ and $r_2(x)$, respectively. If $r_1 > r_2$ in a certain interval $[a,b]$, then $y_1(x)$ has at least a zero between two consecutive zeros of $y_2(x)$, unless $y_1 = y_2$ on $[a,b]$.*

[2] That is, those of the form $L(y,y') = K(y') - V(y)$ where K is a positive-definite quadratic form associated to some metric (usually $K(y') = (y')^2/2$, that is, the metric is the euclidean one) and V is some $\mathcal{C}^1$ function. Note that in this case $P \geq 0$.

6 Examples

In this section we will develop some physically motivated examples in order to elucidate the ideas analysed so far. In each case, the regularity conditions on the Lagrangian are trivially satisfied. Unless otherwise explicitly stated, we will work on the space $X = \mathcal{C}^1([a,b])$. Also, in some examples we will follow the notation common in physics, taking t as the independent variable and x, $\dot{x} = \mathrm{d}x/\mathrm{d}t$ as the dependent ones.

6.1 Driven harmonic oscillator

As it is well known, the differential equation for a driven damped harmonic oscillator under a sinusoidal external force is given by [33], [39], [44], [70], [71]:

$$\ddot{x} + \beta\dot{x} + \omega_0^2 x = \sin(\omega t)\,. \tag{24}$$

Its solutions are well-known. They have the form

$$x(t) = \frac{1}{\omega Z}\sin(\omega t + \varphi)\,, \tag{25}$$

where $Z = \sqrt{\beta^2 + \frac{1}{\omega^2}(\omega_0^2 - \omega^2)}$ is the impedance and $\varphi = \arctan\left(\frac{\beta\omega}{\omega_0^2 - \omega^2}\right)$ is the phase.

The corresponding Lagrangian function is:

$$L(t, x, \dot{x}) = \frac{1}{2}\mathrm{e}^{\beta t}\Big(\dot{x}^2 - \frac{\beta\sin(\omega t) - \omega\cos(\omega t)}{\omega^2 + \beta^2}\dot{x} - \omega_0^2 x^2\Big)\,.$$

Here we follow the standard conventions and denote $\omega_0 \in \mathbb{R}$ as the natural oscillation frequency, $\beta \in \mathbb{R}^+$ is the damping parameter, and $\omega \in \mathbb{R}$ stands for the frequency of the driving force. For this Lagrangian, we straightforwardly note that the functions $P = \mathrm{e}^{\beta t} > 0$ and $Q = -\omega_0^2\mathrm{e}^{\beta t}$, given by (7) and (8), respectively, lead to the Jacobi equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}f}{\mathrm{d}t} + \omega_0^2 f = 0 \tag{26}$$

which is the damped harmonic oscillation equation for the function f. Note that this equation is damped by the same amount as the driven equation (24).

This is a second-order linear equation of constant coefficients, so it can be analytically solved and we get the general solution (for the underdamped[3] case $\beta < \omega_0$) in the form

$$f(t) = \mathrm{e}^{-\frac{\beta\, t}{2}}\left(k_1\sin\left(\frac{\sqrt{4\,\omega_0^2 - \beta^2}\,t}{2}\right) + k_2\cos\left(\frac{\sqrt{4\,\omega_0^2 - \beta^2}\,t}{2}\right)\right),$$

where k_1, k_2 are constants of integration. The solution verifying $f(0) = 0$ can be written as

$$f(t) = C\frac{2}{\sqrt{4\,\omega_0^2 - \beta^2}}\,\mathrm{e}^{-\frac{\beta\, t}{2}}\sin\left(\frac{\sqrt{4\,\omega_0^2 - \beta^2}\,t}{2}\right),$$

[3]The other cases (critical damping and overdamping) are treated similarly.

from which we clearly see that its zeros are located at the values $t = \frac{2k\pi}{\sqrt{4\,\omega_0^2 - \beta^2}}$, for $k \in \mathbb{Z}$. The first zero after $t = 0$ is located at $t = \frac{2\pi}{\sqrt{4\,\omega_0^2 - \beta^2}}$, so the solution (25) is a minimum on the interval $[0, \frac{2\pi}{\sqrt{4\,\omega_0^2 - \beta^2}}[$. When $\beta = 0$ this interval becomes $[0, \frac{\pi}{\omega_0}[$, which is the particular case of the harmonic oscillator (see [37], pg. 446).

6.2 Lane-Emden Equations

The Lane-Emden second-order differential equation was originally proposed by Lane [49], and studied in detail by Emden [17] and Fowler [21], [20], [22], [23], in order to understand equilibrium configurations of spherical clouds of gas (self-gravitating polytropic gas spheres) [7], [13], [15]. Lane-Emden equations also appears in several other contexts such as viscous fluid dynamics, radiation, condensed matter, relativistic mechanics, and even for systems under chemical reactions (see [31], [77], and references therein for an account of its applications; for a mathematical treatment of their zeros, see [14]). The Lane-Emden equation is characterized by a non-linear term $y^n(x)$, where the non-negative parameter $n \in \mathbb{Z}$ (the polytropic index, in its original context) defines the nature of the second-order differential equation

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}y}{\mathrm{d}x}\right) + y^n = 0\,, \tag{27}$$

which may be obtained from the associated Lagrangian

$$L(x, y, y') = x^2\left(\frac{y'^{\,2}}{2} - \frac{y^{n+1}}{n+1}\right)\,, \tag{28}$$

in the sense that its Euler-Lagrange equations reduce to (27). For this Lagrangian, the functions (7) and (8) are given by $P = x^2$ and $Q = -nx^2y^{n-1}$, respectively. As in the preceding example, the function P is always positive definite on any interval of the form $]0, b]$, $b \in \mathbb{R}$. We also note that the function Q depends on the parameter n. In this way, we obtain the Jacobi equation (12)

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}f}{\mathrm{d}x}\right) + ny^{n-1}f = 0\,. \tag{29}$$

As the most frequent analytical solutions to the Lane-Emden equation are those corresponding to the values $n = 0, 1, 5$, [77] (but see [31] for other cases), we will focus next on solution of both, Euler-Lagrange and Jacobi equations, for these cases.

6.2.1 $n = 0$

Consider any interval $[0, b]$. For this case, the general solution $y(x)$ of Euler--Lagrange's equation (27) reads

$$y(x) = -\frac{x^2}{6} + \frac{k_2}{x} + k_1\,,$$

where $k_1,\ k_2$ are arbitrary integration constants. Note that this function is singular at the origin. The physical origin of the problem demands that the solution verify $y(0) = 1$, $y'(0) = 0$, so we must take $k_2 = 0$, $k_1 = 1$. The solution is then

$$y(x) = -\frac{x^2}{6} + 1\,.$$

In this very simple case, it is not necessary to deal with the Jacobi equation, as the corresponding Lagrangian

$$L(x, y, y') = x^2 \left(\frac{y'^2}{2} - y \right)\,,$$

determines, for each $x \in [a, b]$ fixed, a function $L(x, \cdot, \cdot)$ which is convex on the convex set $\mathbb{R} \times \mathbb{R}$, as it has a semi-definite Hessian: for any $u, v \in \mathbb{R}$,

$$\begin{aligned} \det \operatorname{Hess} L(x, u, v) &= 0 \\ D_{22} L(x, u, v) &= 0 \\ D_{33} L(x, u, v) &= x^2 \geq 0\,. \end{aligned}$$

Thus, the solution is minimal on $]0, b]$, for any $b > 0$.

6.2.2 $n = 1$

The general solution the Lane-Emden equation (27) is given by

$$y(x) = k_1 \frac{\sin(x)}{x} + k_2 \frac{\cos(x)}{x} \tag{30}$$

where $k_1,\ k_2$ are constants. Again, the physical meaning of the problem imposes the condition that the solutions be defined at $x = 0$ (where it must be $y(0) = 1$), so we must take $k_1 = 1, k_2 = 0$, getting the sinc function

$$y(x) = \frac{\sin(x)}{x}\,. \tag{31}$$

As the functions $P = x^2$, $Q = -x^2$ in this case, Jacobi equation (29) results again a Lane-Emden equation with $n = 1$ for the function $f(x)$. Hence, for $n = 1$ solutions of the Jacobi equation and the Lane-Emden equation are of an identical nature. Thus, in view of (30), the solutions of the Jacobi equation defined for $x = 0$ are those of the form $f(x) = C \operatorname{sinc}(x)$. They have zeros located at the points $x = k\pi$, $k \in \mathbb{Z} - \{1\}$. So, on any interval of the form $](k-1)\pi, k\pi[$, $k \in \mathbb{Z} - \{0, 1\}$, the solutions (31) are minimal. A minimum is also obtained on $]-\pi, 0[$ and $]0, \pi[$.

6.2.3 $n = 5$

Analogously, in this case, the general solution to (27) reads (see [13], [42][4]):

$$y(x; \alpha, \beta) = \sqrt{\frac{\alpha}{\frac{(\alpha\, x)^2}{\beta} + \frac{\beta}{3}}}\,. \tag{32}$$

[4]Although in these references only a 1-parameter family of solutions is given, it is easy to trace back the missing parameter α from the calculations presented there (it is fixed at certain point to make the output of an integral more manageable).

We further note that if we impose the boundary conditions $y(0) = 1$ and $y'(0) = 0$ (which set $\alpha = 1$, $\beta = 3$) our solution becomes the common one

$$y(x; 1, 3) = \frac{\sqrt{3}}{\sqrt{3 + x^2}} \,. \tag{33}$$

We will work on this simplified solution. For this case, the function $Q = -5x^2y^4$, thus yielding the Jacobi equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{45}{(3 + x^2)^2} f = 0 \,, \tag{34}$$

which may be simplified to its normal form (23) by the change $f(x) = u(x)/x$:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{45}{(3 + x^2)^2} u = 0 \,. \tag{35}$$

Let us apply the method outlined in Remark 7. The derivative of the general solution with respect to the parameter α, evaluated at the values that give the solution we are considering, is

$$D_2 L(x; 1, 3) = -\frac{\sqrt{3}\,\left(x^2 - 3\right)}{2\left(x^2 + 3\right)^{\frac{3}{2}}} \,.$$

It is immediate to check that this is indeed a solution of Jacobi's equation (34). The derivative with respect to β gives nothing new (a multiple of $D_2L(x; 1, 3)$). By making the change of variables stated above, we get:

$$u(x) = -\frac{\sqrt{3}\,x\left(x^2 - 3\right)}{2\left(x^2 + 3\right)^{\frac{3}{2}}} .$$

Note that $u(0) = 0$. The first (and only) zero of $u(x)$ after $x = 0$ is given by $x = \sqrt{3}$. Thus, the solution to the Lane-Emden equation for $n = 5$ (33), is a minimum for $x \in [0, \sqrt{3}[$.

6.3 Quantum gravity in one dimension

In modern physics, spin foams models have been introduced in order to analyse certain generalizations of path integrals appearing in gauge theories. In particular, in quantum gravity the spin foam approach has been developed as a tool to understand dynamical issues of the theory by the introduction of discretizations describing the metric properties of spacetime [3]. To some extent, spin foams for quantum gravity were motivated by a particular discretization of general relativity known as Regge calculus [76]. In this context, a discrete model for a scalar field representing gravity in one temporal dimension has been studied in detail in [38]. Here we present the continuum analogue of this model. The general action functional is:

$$J(y) = \frac{1}{2}\int \sqrt{g(x)}\,\left(g^{-1}(x)y'(x)^2 + \omega y^2(x)\right)\mathrm{d}x \,,$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is a positive function which acts as the metric on $\mathbb{R}$, and will be taken in what follows as $g(x) = \exp(x)$, for simplicity. Thus our model Lagrangian will be

$$L(x, y, y') = \frac{1}{2}\exp(x/2)\left(\exp(-x)(y')^2 + \omega y^2\right) .$$

The Euler-Lagrange equations readily follow:

$$\omega \exp(x/2) y + \frac{1}{2}\exp(-x/2) y' - \exp(-x/2) y'' = 0 ,$$

or, as $\exp(-x/2) > 0$,

$$y'' - \frac{1}{2} y' - \omega e^x y = 0 . \tag{36}$$

By making the change of variable $u = \exp(x/2)$, we can put (36) in the form

$$u^2 \left(\frac{1}{4}\frac{\mathrm{d}y}{\mathrm{d}u} - \omega y(u) \right) = 0 ,$$

which, as $u = \exp(x/2) > 0$, reduces to

$$\frac{\mathrm{d}y}{\mathrm{d}u} - 4\omega y(u) = 0 .$$

This equation is integrated by elementary techniques; its solutions are:

$$y(u) = k_1 \exp(2\sqrt{\omega} u) + k_2 \exp(-2\sqrt{\omega} u) ,$$

(with k_1, k_2 constants of integration) or, in the original variable x:

$$y(x) = k_1 \exp(2\sqrt{\omega}\exp(x/2)) + k_2 \exp\left(-2\sqrt{\omega}\exp(x/2)\right) . \tag{37}$$

The coefficients of the Jacobi equation are $P = \exp(-x/2) > 0$ and $Q = \omega \exp(x/2)$, so the Jacobi equation is (after simplifying an $\exp(x/2) > 0$ factor):

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - \frac{1}{2}\frac{\mathrm{d}f}{\mathrm{d}x} - \omega e^x f = 0 ,$$

which has the same form as the Euler-Lagrange equation (36) (a phenomenon already encountered in the case of the Lane-Emden equation for $n = 1$, recall Example 6.2.2). Thus, the general solution to the Jacobi equation is

$$f(x) = c_1 \exp\left(2\sqrt{\omega}\exp(x/2)\right) + c_2 \exp\left(-2\sqrt{\omega}\exp(x/2)\right),$$

with c_1, c_2 constants of integration which can be fixed by the initial conditions $f(0) = 0$, $f'(0) = 1$, giving

$$c_1 = \frac{e^{-2\sqrt{\omega}}}{2\sqrt{\omega}}, \quad c_2 = -\frac{e^{2\sqrt{\omega}}}{2\sqrt{\omega}} .$$

Substituting in the expression for $f(x)$ above, the solution to the Jacobi equation can be written as

$$\begin{aligned} f(x) &= \frac{1}{2\sqrt{\omega}}\Big(\exp\big((\mathrm{e}^{x/2}-1)2\sqrt{\omega}\big)-\exp\big(-(\mathrm{e}^{x/2}-1)2\sqrt{\omega}\big)\Big) \\ &= \frac{1}{\sqrt{\omega}}\sinh(2\sqrt{\omega}(\mathrm{e}^{x/2}-1))\,, \end{aligned}$$

which clearly shows that there are no conjugate points to $x=0$. The solutions (37) are thus minimals on their interval of definition.

6.4 Square root Hamiltonian with a dissipation term

Square-root Hamiltonians (or Lagrangians) are a standard feature of many reparametrization invariant field theories [11], [57]. The action of a relativistic particle [18], [25], [51], and the action of the Nambu string are familiar examples [4], [81]. Further examples of physical theories where this sort of Hamiltonians appear include general relativity [25], [53], as well as certain approaches to quantum gravity [12], [53], and also, they appear in brane motivated models [4], [61]. Aspects on the quantization of these kind of Hamiltonian theories may be found in [6] and [57], to mention some. In this section, we will study the Hamiltonian for a free particle under relativistic motion with a linear dissipation term, as proposed in [34]. This Hamiltonian reads

$$H(p,x,t) := \mathrm{e}^{\gamma t}\sqrt{1+p^2\mathrm{e}^{-2\gamma t}}\,, \tag{38}$$

where p stands for the canonical momentum associated to the dependent variable $x=x(t)$, and γ is the dissipation term. As discussed in [34], in the low velocity regime, this Hamiltonian reduces to the Caldirola-Kanai Hamiltonian which describes the motion of a non-relativistic particle with a linear dissipation term [60]. The Lagrangian associated to (38) is given by

$$L(t,x,\dot{x}) = -\mathrm{e}^{\gamma t}\sqrt{1-(\dot{x})^2}\,, \tag{39}$$

and thus the solution to Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{e}^{\gamma t}\dot{x}}{\sqrt{1-(\dot{x})^2}}\right)=0 \tag{40}$$

is given by

$$x(t) = -\frac{A_0\,\mathrm{arcsinh}\,(\mathrm{e}^{-\gamma\, t}\ |A_0|)}{\gamma\ |A_0|}\,, \tag{41}$$

where A_0 is an integration constant. The functions (7) and (8) are $P=\mathrm{e}^{\gamma t}/(1-(\dot{x})^2)^{3/2}$ and $Q=0$, respectively, and thus the Jacobi equation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{e}^{\gamma t}}{(1-(\dot{x})^2)^{3/2}}\frac{\mathrm{d}f}{\mathrm{d}t}\right)=0\,. \tag{42}$$

In this last equation, the term $\dot{x}$ must be understood as the time-depending function $\dot{x}(t) = A_0 \mathrm{e}^{-\gamma t}/\sqrt{1 + A_0^2 \mathrm{e}^{-2\gamma t}}$. Therefore, for this model we are able to explicitly find the solution to Jacobi equation (42)

$$f(t) = c_0 - \frac{c_1}{\gamma} \frac{\mathrm{e}^{-\gamma t}}{\sqrt{1 + A_0^2 \mathrm{e}^{-2\gamma t}}}\,, \tag{43}$$

being c_0 and c_1 integration constants. We then note that this solution has a unique zero at the value $t = (1/\gamma)\log\left(\sqrt{c_1^2 - A_0^2\gamma^2 c_0^2}/\gamma c_0\right)$, so there are no conjugate points for the function $f(t)$, and the solution (41) is a minimum for the action on any interval $[0, b]$, $b \in \mathbb{R}$.

6.5 Quartic potential model

In this section we will develop a model inspired by the static kink of the well-known ϕ^4 model in quantum field theory [45], [58]. The model can be resolved both on classical and quantum grounds, and contains soliton solutions (see below). In the context of brane theories, the so-called kink model also appears by the inclusion of rigidity terms associated to the intrinsic curvature in their effective actions [55], [80].

The Lagrangian for our model is

$$L(t, x, \dot{x}) = \frac{1}{2}(\dot{x})^2 + \frac{\lambda}{4}\left(x^2 - \frac{m^2}{\lambda}\right)^2, \tag{44}$$

where m and $\lambda > 0$ are arbitrary real constants. The Euler-Lagrange equation associated to this Lagrangian reads

$$\ddot{x} - \lambda x\left(x^2 - \frac{m^2}{\lambda}\right) = 0\,. \tag{45}$$

This equation is of Lénard type: $\ddot{x} + f(x)\dot{x} + g(x) = 0$, where $f \equiv 0$ and $g(x) = -\lambda x\left(x^2 - \frac{m^2}{\lambda}\right)$. The change of variables $u(x) = \dot{x}$ converts it in the first-order equation $uu' = \lambda x\left(x^2 - \frac{m^2}{\lambda}\right)$, which is immediately integrated to give

$$\int \frac{\mathrm{d}x}{\sqrt{\lambda x^4 - 2m^2x^2 - 2b}} = \frac{1}{\sqrt{2}}\int \mathrm{d}t = \frac{t-a}{\sqrt{2}}\,, \tag{46}$$

where $a, b \in \mathbb{R}$ are integration constants. The solutions commonly found in the literature (cited above) are obtained by taking $b = -\frac{m^4}{2\lambda}$, so to get a perfect square in the radicand of (46). In this way, the resulting solutions are:

$$y(t) = \pm\frac{m}{\sqrt{\lambda}}\tanh\left(\frac{m(t-a)}{\sqrt{2}}\right).$$

The solution with the plus sign is commonly termed the kink solution, while the one with the minus sign is called the anti-kink solution. Both solutions are bounded by the values $\pm m/\sqrt{\lambda}$. In particular, the energy density of the kink solution goes as the fourth power of the hyperbolic secant, and is localised within a width characterised by the quantity l/m [58]. However, other solutions exist. For instance, we could as well take $b = 0$ in (46) to get (through an obvious change of variable):

$$\frac{t-a}{\sqrt{2}} = \int \frac{\mathrm{d}x}{\sqrt{\lambda x^4 - 2m^2x^2}} = \frac{1}{m\sqrt{2}} \int \frac{\mathrm{d}\eta}{\eta\sqrt{\eta^2-1}} = \frac{1}{m\sqrt{2}} \operatorname{arcsin}\eta\,,$$

and hence the solution to the Euler-Lagrange equation (45):

$$x(t) = m\sqrt{\frac{2}{\lambda}}\sec\big(m(t-a)\big)\,. \tag{47}$$

In order to get the Jacobi equation (12), we consider the functions $P = 1 > 0$ and $Q = 3\lambda x^2 - m^2$, which set the equation for the function $f(t)$:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + m^2\Big(1 - 6\sec^2\big(m(t-a)\big)\Big)f = 0\,. \tag{48}$$

We can take $a = 0$ and $m = 1$ without loss of generality (these are just re-scalings). Then, the Jacobi equation has the form $\ddot{f} + \phi(t)f = 0$, where $\phi(t) = 1 - \sec^2 t \leq 0$ in the interval $]-\pi/2, \pi/2[$. At the points $t = \pm\pi/2$, the solutions have a blow-up and are not defined (so they can not be extended beyond these points). Thus, the solutions to the Jacobi equation are defined on $]-\pi/2, \pi/2[$, do not possess conjugate points in this interval (see Theorem 7) and the solution (48) is a true minimum on $]-\pi/2, \pi/2[$.

6.6 Probability density and maximal entropy

In this section we implement a constrained Lagrangian system related to a probability density function [67]. In Bayesian probability theory and in statistical mechanics, this system is related to the principle of maximum entropy [43], which also appears in other branches of physics, and in chemistry and biology [8], [46], [52], [74]. The model is defined as follows. Let Z be a random variable, and let $\rho(x)$ its associated density function, so $\rho\colon \mathbb{R} \to\,]0, +\infty[$. Suppose that we know the second order momentum

$$\sigma^2 = \int_{\mathbb{R}} x^2 \rho(x)\,\mathrm{d}x\,, \tag{49}$$

and that we want to obtain the least biased density function $\rho(x)$. This may be written as the problem of finding the maximals for the entropy functional (defined in terms of the information theory):

$$S(\rho) = -\int_{\mathbb{R}} \rho(x)\log\rho(x)\,\mathrm{d}x\,, \tag{50}$$

and subject to the constraints

$$\int_{\mathbb{R}} \rho(x)\,\mathrm{d}x = 1\,, \tag{51}$$

$$\int_{\mathbb{R}} x^2\rho(x)\,\mathrm{d}x = \sigma^2. \tag{52}$$

Notice that the Lagrangian here, is defined on $U = \mathbb{R} \times \,]0, +\infty[\, \times \mathbb{R}$, although its dependence on the first and third variables is trivial.
Thus, we have the three functionals (in the notation of subsection 2.3)

$$\begin{aligned} S(\rho) &= -\int_{\mathbb{R}} \rho(x) \log \rho(x)\, \mathrm{d}x\,, \\ K_1(\rho) &= \int_{\mathbb{R}} \rho(x)\, \mathrm{d}x\,, \\ K_2(\rho) &= \int_{\mathbb{R}} x^2 \rho(x)\, \mathrm{d}x\,. \end{aligned}$$

It is immediate to compute the variations:

$$\begin{aligned} \delta S(\rho, h) &= -\int_{\mathbb{R}} h(x)(1 + \log \rho(x))\, \mathrm{d}x\,, \\ \delta K_1(\rho, h) &= \int_{\mathbb{R}} h(x)\, \mathrm{d}x\,, \\ \delta K_2(\rho, h) &= \int_{\mathbb{R}} x^2 h(x)\, \mathrm{d}x\,, \end{aligned}$$

so it is obvious that they are weakly continuous. Let us apply the theorem 3 on Lagrange multipliers. The case $\det(\delta K_i(y, h_l)) = 0$ $(1 \leq i, l \leq 2)$, would lead to

$$\det \begin{pmatrix} \int_{\mathbb{R}} h_1(x)\, \mathrm{d}x & \int_{\mathbb{R}} h_2(x)\, \mathrm{d}x \\ \int_{\mathbb{R}} x^2 h_1(x)\, \mathrm{d}x & \int_{\mathbb{R}} x^2 h_2(x)\, \mathrm{d}x \end{pmatrix} = 0$$

for arbitrary $h_1, h_2 \in X$, or:

$$\frac{\int_{\mathbb{R}} x^2 h_1(x)\, \mathrm{d}x}{\int_{\mathbb{R}} h_1(x)} = \frac{\int_{\mathbb{R}} x^2 h_2(x)\, \mathrm{d}x}{\int_{\mathbb{R}} h_2(x)}\,,$$

which is absurd. Thus, we can introduce two Lagrange multipliers λ_1, λ_2 and consider the Lagrangian

$$L(x, \rho, \rho') = -\rho(x) \log \rho(x) + \lambda_1 \rho(x) + \lambda_2 x^2 \rho(x)\,.$$

The Euler-Lagrange equation yields

$$-\log \rho(x) - 1 + \lambda_1 + \lambda_2 x^2 = 0$$

with solution $\rho(x) = \mathrm{e}^{-1+\lambda_1+\lambda_2 x^2}$. Substitution of this solution into the constraints (51) and (52) implies that the Lagrange multipliers are equal to $\lambda_1 = 1 + \log \frac{1}{\sqrt{2\pi}\sigma}$ and $\lambda_2 = -1/(2\sigma^2)$, respectively. Thus, the solution of the Euler--Lagrange equation reads

$$\rho(x) = (\sqrt{2\pi}\,\sigma)^{-1} \exp\left(-x^2/(2\sigma^2)\right). \tag{53}$$

Finally, we see that the original Lagrangian $L_0(x, \rho, \rho') = -\rho(x) \log \rho(x)$, can actually be seen as a real function of a single variable on $]0, +\infty[$, for which the second derivative $L_0''(\rho) = -1/\rho$ is always negative. Therefore, L_0 is concave and, due to Theorem 5 and Remark 9, the solution obtained is a (global) maximal in $]0, +\infty[$.

Remark 10. *A posteriori*, we see that the solution we have found, (53), belongs to $\mathcal{C}^1(\mathbb{R})$. However, this is not obvious *a priori*. Indeed, the method of Lagrange multipliers is *not* the best one to deal with the problem involving higher order moments, precisely because the eventual solution may lie outside the space from which we start, see [40].

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Associative and Lie deformations of Poisson algebras

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Abstract. Considering a Poisson algebra as a nonassociative algebra satisfying the Markl-Remm identity, we study deformations of Poisson algebras as deformations of this nonassociative algebra. We give a natural interpretation of deformations which preserve the underlying associative structure and of deformations which preserve the underlying Lie algebra and we compare the associated cohomologies with the Poisson cohomology parametrizing the general deformations of Poisson algebras.

1 Introduction

The Poisson bracket is a multiplication which naturally appears when studying deformations of associative commutative algebras. For instance the algebra $\mathcal{C}^\infty(\mathbb{R}^2)$ with its ordinary multiplication $\mu = \mu_0$ admits a formal deformation $\sum_0^\infty t^n \mu_n$ such that the skew-symmetric bracket $\{a, b\} = \mu_1(a, b) - \mu_1(b, a)$ is the classical Poisson bracket (recalled in Section 2). This deformation is connected to the star product and then to the theory of deformation quantization (see Section 1 of [10]). This naturally leads to study deformations of Poisson algebras. But a Poisson algebra is usually defined by two multiplications, an associative commutative one $a * b$ and a Lie bracket $\{a, b\}$ (also called Poisson bracket) which are linked by the Leibniz rule $\{a * b, c\} = a * \{b, c\} + \{a, c\} * b$. The deformations of Poisson algebras which are classically considered consist of those deforming the Lie bracket while the associative product remains unchanged. The first studied Poisson algebras were defined on associative algebras of functions whose product is undeformable. This explains why this type of deformations, that we call Lie deformations of Poisson algebras, were first studied. They are parametrized by the Poisson-Lichnerowicz cohomology. Here we want to give a general approach of deformations of Poisson algebras, that is, we make deformations where both products are deformed. We then use the presentation of Poisson algebras in [13] with a single nonassociative multiplication which capture all informations. Then we find the Lie deformations

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as a particular case of deformations of this single multiplication but also the associative deformations obtained by deforming the associative product and letting the Lie bracket unchanged. We call Poisson-Hochschild the cohomology parametrizing the associative deformations (see Section 4.2). We then describe the Poisson cohomology parametrizing the general deformations of Poisson algebras and study the interactions between Poisson, Poisson-Lichnerovicz and Poisson-Hochschild cohomologies.

2 Generalities on Poisson algebras

2.1 Definition

Let $\mathbb{K}$ be a field of characteristic 0. A $\mathbb{K}$-Poisson algebra is a $\mathbb{K}$-vector space $\mathcal{P}$ equipped with two bilinear products denoted by $x * y$ and $\{x, y\}$, with the following properties:

1. The couple $(\mathcal{P}, *)$ is an associative commutative $\mathbb{K}$-algebra.
2. The couple $(\mathcal{P}, \{\cdot, \cdot\})$ is a $\mathbb{K}$-Lie algebra.
3. The products $*$ and $\{\cdot, \cdot\}$ satisfy the Leibniz rule:
$$\{x * y, z\} = x * \{y, z\} + \{x, z\} * y$$
for any $x, y, z \in \mathcal{P}$.

The product $\{\cdot, \cdot\}$ is usually called Poisson bracket and the Leibniz identity means that the Poisson bracket acts as a derivation of the associative product.

Classical examples: Poisson structures on the polynomial algebra. The polynomial algebra $A_n = \mathbb{C}[x_1, \dots, x_n]$ is provided with several Poisson algebra structures. These examples are well studied, see, for example, [2], [8], [20] for results on classifications, or [16] for the study of the Poisson-Lichnerowicz cohomology.

2.2 Non standard example: Poisson algebras defined by a contact structure

The first Poisson structures appeared in classical mechanics. In 1809 Siméon Denis Poisson introduced a bracket in the algebra of smooth functions on $\mathbb{R}^{2r}$:
$$\{f, g\} = \sum_{i=1}^{r} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right).$$

This classical example has a natural generalization in symplectic geometry ([3]): Let (M^{2p}, θ) be a symplectic manifold. For any Pfaffian form α on M^{2p}, we will denote by X_α the vector field defined by $\alpha = i(X_\alpha)\theta$, where $i(X)$ is the interior product by X: $(i(X)\theta)(Y) = \theta(X, Y)$. The Poisson bracket of two Pfaffian forms α, β on M^{2p} is the Pfaffian form $\{\alpha, \beta\} = i([X_\alpha, X_\beta])\theta$. If $\mathcal{D}(M^{2p})$ denotes the associative commutative algebra of smooth functions on M^{2p}, we provide it with a Poisson algebra structure letting $\{f, g\} = -\theta(X_{df}, X_{dg})$. This Poisson bracket satisfies $d(\{f, g\}) = \{df, dg\}$.

We can also define a Poisson bracket in contact geometry ([5]). Let (M^{2p+1}, α) be a contact manifold, that is, α is a Pfaffian form on the $(2p + 1)$-dimensional differential manifold M^{2p+1} satisfying $(\alpha \wedge (d\alpha)^p)(x) \neq 0$ for any $x \in M^{2p+1}$. There exists one and only one vector field Z_α on M^{2p+1}, called the Reeb vector field of α, such that $\alpha(Z_\alpha) = 1$ and $i(Z_\alpha)d\alpha = 0$ at any point of M^{2p+1}. Let $\mathcal{D}_\alpha(M^{2p+1})$ be the set of first integrals of Z_α, that is,

$$\mathcal{D}_\alpha(M^{2p+1}) = \{f \in \mathcal{D}(M^{2p+1}),\ Z_\alpha(f) = 0\}\,.$$

Since we have $Z_\alpha(f) = i(Z_\alpha)df = 0$, then df is invariant by Z_α.

Lemma 1. *$\mathcal{D}_\alpha(M^{2p+1})$ is a commutative associative subalgebra of $\mathcal{D}(M^{2p+1})$.*

Proof. This is a consequence of the classical formulae

$$Z_\alpha(f+g) = Z_\alpha(f) + Z_\alpha(g) \text{ and } Z_\alpha(fg) = \big(Z_\alpha(f)\big)g + f\big(Z_\alpha(g)\big)\,.$$

□

Lemma 2. *For any non zero Pfaffian form β on M^{2p+1} satisfying $\beta(Z_\alpha) = 0$, there exists a vector field X_β with $\beta(Y) = d\alpha(X_\beta, Y)$ for any vector field Y. Two vector fields X_β and X'_β with this property satisfy $i(X_\beta - X'_\beta)d\alpha = 0$.*

This means that X_β is uniquely defined up to a vector field belonging to the distribution given by the characteristic space of $d\alpha$,

$$A(d\alpha)_x = \{X_x \in T_xM^{2p+1}, i(X_x)d\alpha(x) = 0\}\,.$$

In any Darboux open set, the contact form writes as $\alpha = x_1dx_2 + \cdots + x_{2p-1}dx_{2p} + dx_{2p+1}$. The Reeb vector field is $Z_\alpha = \partial/\partial x_{2p+1}$ and the form β satisfying $\beta(Z_\alpha) = 0$ writes as $\beta = \sum_{i=1}^{2p} \beta_i dx_i$. Then we have

$$X_\beta = \sum_{i=1}^{p} (\beta_{2i}\partial/\partial x_{2i-1} - \beta_{2i-1}\partial/\partial x_{2i})\,.$$

For any $f \in \mathcal{D}_\alpha(M^{2p+1})$, we writes X_f for X_{df}.

Theorem 1. *The algebra $\mathcal{D}_\alpha(M^{2p+1})$ is a Poisson algebra.*

Proof. (see [5]). Let f_1, f_2 be in $\mathcal{D}_\alpha(M^{2p+1})$. Since we have

$$d\alpha(X_{f_1}, X_{f_2}) = d\alpha(X_{f_1} + U_1, X_{f_2} + U_2)$$

for any $U_1, U_2 \in A(d\alpha)$, the bracket

$$\{f_1, f_2\} = d\alpha(X_{f_1}, X_{f_2})$$

is well defined. It is a Poisson bracket. □

2.3 Poisson algebra viewed as nonassociative algebra

In [13], we prove that any Poisson structure on a $\mathbb{K}$-vector space is also given by a nonassociative product denoted by xy and satisfying the nonassociative identity

$$3A(x,y,z) = (xz)y + (yz)x - (yx)z - (zx)y\,, \tag{1}$$

where $A(x,y,z)$ is the associator $A(x,y,z) = (xy)z - x(yz)$. In fact, if $\mathcal{P}$ is a Poisson algebra with associative product $x * y$ and Poisson bracket $\{x,y\}$, then xy is given by $xy = \{x,y\} + x * y$. Conversely, the Poisson bracket and the associative product of $\mathcal{P}$ are the skew-symmetric part and the symmetric part of the product xy. Thus it is equivalent to present a Poisson algebra classically or by this nonassociative product.

If $\mathcal{P}$ is a Poisson algebra given by the nonassociative product (1), we denote by $\mathfrak{g}_{\mathcal{P}}$ the Lie algebra on the same vector space $\mathcal{P}$ whose Lie bracket is $\{x,y\} = \frac{xy-yx}{2}$ and by $\mathcal{A}_{\mathcal{P}}$ the commutative associative algebra, on the same vector space, whose product is $x * y = \frac{xy+yx}{2}$.

In [7], we have studied algebraic properties of the nonassociative algebra $\mathcal{P}$. In particular we have proved that this algebra is flexible, power-associative, and admits a Pierce decomposition.

Remark 1. A class of Poisson algebras is already defined with a single noncommutative multiplication but starting with a Jordan algebra. In [19], a noncommutative Jordan algebra is viewed as a Jordan commutative algebra J with an additional skew-symmetric operator $[\cdot,\cdot] : J \times J \to J$ such that

$$[x^2, y] = 2[x,y] \cdot x\,.$$

This definition is equivalent to consider only one multiplication satisfying

$$(xy)x - x(yx) = (x^2y)x - x^2(yx).$$

A particular class of such algebras for which $A^{(+)}$ is associative corresponds to Poisson algebras.

2.4 Classification of complex Poisson algebras of dimension 2 and 3

If e is an idempotent of the associative algebra, then the Leibniz rule implies that it is in the center of the Lie algebra corresponding to the Poisson bracket. In fact if e satisfies $e * e = e$, thus $\{e * e, x\} = 2e * \{e,x\} = \{e,x\}$. But if y is a non zero vector with $e * y = \lambda y$, then

$$(e * e) * y = e * y = \lambda y = e * (e * y) = \lambda^2 y.$$

This gives $\lambda^2 = \lambda$, that is, $\lambda = 0$ or 1. Since we have $e * \{e,x\} = 2^{-1}\{e,x\}$, the vector $\{e,x\}$ is zero for any x and e is in the center of the Lie algebra corresponding to the Poisson bracket. This remark simplifies the determination of all possible Poisson brackets when the associative product is fixed. In the following, we give the associative and Lie products in a fixed basis $\{e_i\}$ and the null products or the products which are deduced by commutativity or skew-symmetry are often not written.

Dimension 2

algebra	associative product	Lie product
$\mathcal{P}_1^2$	$e_1 * e_i = e_i\,,\ i = 1,2$ $e_2 * e_2 = e_2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_2^2$	$e_1 * e_i = e_i\,,\ i = 1,2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_3^2$	$e_1 * e_1 = e_2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_4^2$	$e_1 * e_1 = e_1$	$\{e_i, e_j\} = 0$
$\mathcal{P}_5^2$	$e_i * e_j = 0$	$\{e_1, e_2\} = e_2$
$\mathcal{P}_6^2$	$e_i * e_j = 0$	$\{e_i, e_j\} = 0$

Dimension 3

$\mathcal{P}_1^3$	$e_1 * e_i = e_i\,,\ i = 1,2,3$ $e_2 * e_2 = e_2$ $e_3 * e_3 = e_3$	$\{e_i, e_j\} = 0$
$\mathcal{P}_2^3$	$e_1 * e_i = e_i\,,\ i = 1,2,3$ $e_2 * e_2 = e_2$ $e_3 * e_3 = e_2 - e_1$	$\{e_i, e_j\} = 0$
$\mathcal{P}_3^3$	$e_1 * e_i = e_i\,,\ i = 1,2,3$ $e_2 * e_2 = e_2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_4^3$	$e_1 * e_i = e_i\,,\ i = 1,2,3$ $e_3 * e_3 = e_2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_5^3$	$e_1 * e_i = e_i\,,\ i = 1,2,3$	$\{e_2, e_3\} = e_3$
$\mathcal{P}_6^3$	$e_1 * e_i = e_i\,,\ i = 1,2,3$	$\{e_i, e_j\} = 0$
$\mathcal{P}_7^3$	$e_1 * e_i = e_i\,,\ i = 1,2$ $e_2 * e_2 = e_2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_8^3$	$e_1 * e_1 = e_1$	$\{e_2, e_3\} = e_3$
$\mathcal{P}_9^3$	$e_1 * e_1 = e_1$	$\{e_i, e_i\} = 0$
$\mathcal{P}_{10}^3$	$e_1 * e_i = e_i\,,\ i = 1,2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_{11}^3$	$e_1 * e_1 = e_1$ $e_2 * e_2 = e_3$	$\{e_i, e_j\} = 0$
$\mathcal{P}_{12}^3(b)$	$e_1 * e_1 = e_2$	$\{e_1, e_3\} = e_2 + be_3$
$\mathcal{P}_{13}^3$	$e_1 * e_1 = e_2$	$\{e_1, e_3\} = e_3$
$\mathcal{P}_{14}^3$	$e_1 * e_1 = e_2$	$\{e_i, e_j\} = 0$
$\mathcal{P}_{15}^3$	$e_1 * e_1 = e_2$ $e_1 * e_2 = e_3$	$\{e_i, e_j\} = 0$
$\mathcal{P}_{16}^3(a)$	$e_i * e_j = 0$	any Lie algebra

It is also possible to establish this classification in small dimension starting from the nonassociative product. We can use, for example, technics used in [4] where we classify all the complex 2-dimensional algebras (and in particular the Poisson algebras).

3 Deformations of Poisson algebras

In this section we recall briefly the classical notion of formal deformations of a $\mathbb{K}$-algebra. These deformations are parametrized by a cohomology, called deformation cohomology, which is often difficult to define globally and to compute explicitly. But using the operadic approach, we can sometimes obtain this cohomology using the associated operad: when the operad is Koszul, which is the case for the operad associated to Poisson algebras. When the operad is non Koszul the operadic and deformation cohomologies differ and the last one is even more complicated to describe see [14]. Using the Markl-Remm definition of a Poisson algebra, we describe the formal deformations. So in this section, we mean by Poisson algebra a $\mathbb{K}$-algebra defined by a nonassociative product satisfying Identity (1).

3.1 Formal deformations of a Poisson algebra

Let R be a complete local augmented ring such that the augmentation ε takes values in $\mathbb{K}$. If B is an R-Poisson algebra, we consider the $\mathbb{K}$-Poisson algebra $\overline{B} = \mathbb{K} \otimes_R B$ given by $\alpha(\beta \otimes b) = \alpha\beta \otimes b$, with $\alpha, \beta \in \mathbb{K}$ and $b \in B$. It is clear that $\overline{B}$ satisfies (1). An R-deformation of a $\mathbb{K}$-Poisson algebra A is an R-Poisson algebra B with a $\mathbb{K}$-algebra homomorphism

$$\varrho \colon \overline{B} \to A\,.$$

A formal deformation of A is an R-deformation with $R = \mathbb{K}[[t]]$, the local ring of formal series on $\mathbb{K}$. We assume also that B is an R-free module isomorphic to $R \otimes A$.

Let $\mathbb{K}[\Sigma_3]$ be the $\mathbb{K}$-group algebra of the symmetric group Σ_3. We denote by τ_{ij} the transposition exchanging i and j and by c the cycle $(1,2,3)$. Every $\sigma \in \Sigma_3$ defines a natural action on any $\mathbb{K}$-vector space W by:

$$\begin{array}{cccc} \Phi_\sigma \colon & W^{\otimes^3} & \longrightarrow & W^{\otimes^3} \\ & x_1 \otimes x_2 \otimes x_3 & \longrightarrow & x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)}. \end{array}$$

We extend this action of Σ_3 to an action of the algebra $\mathbb{K}[\Sigma_3]$. If $v = \Sigma_i a_i \sigma_i \in \mathbb{K}[\Sigma_3]$, then

$$\Phi_v = \Sigma_i a_i \Phi_{\sigma_i}.$$

Consider v_P the vector of $\mathbb{K}[\Sigma_3]$

$$v_P = 3Id - \tau_{23} + \tau_{12} - c + c^2.$$

Let $\mathcal{P}$ be a Poisson algebra and μ_0 its (nonassociative) multiplication. Identity (1) writes as

$$(\mu_0 \circ_1 \mu_0) \circ \Phi_{v_P} - 3(\mu_0 \circ_2 \mu_0) = 0$$

where $\circ_1$ and $\circ_2$ are the *comp*$_i$ operations given by

$$(\mu \circ_1 \mu')(x,y,z) = \mu(\mu'(x,y),z)\,,$$
$$(\mu \circ_2 \mu')(x,y,z) = \mu(x,\mu'(y,z))$$

for any bilinear maps μ and μ'.

Theorem 2. *A formal deformation B of the $\mathbb{K}$-Poisson algebra A is given by a family of linear maps*

$$\{\mu_i : A \otimes A \to A,\ i \in \mathbb{N}\}$$

satisfying

(i) μ_0 *is the multiplication of A,*

(ii) (D_k): $\displaystyle\sum_{\substack{i+j=k,\\ i,j\geq 0}} (\mu_i \circ_1 \mu_j) \circ \Phi_{v_P} = 3 \sum_{\substack{i+j=k,\\ i,j\geq 0}} \mu_i \circ_2 \mu_j$ *for each $k \geq 1$.*

Proof. The multiplication in B is determined by its restriction to $A \otimes A$ ([1]). We expand $\mu(x,y)$ for x, y in A into the power series

$$\mu(x,y) = \mu_0(x,y) + t\mu_1(x,y) + t^2\mu_2(x,y) + \cdots + t^n\mu_n(x,y) + \cdots$$

then μ is a Poisson product if and only if the family $\{\mu_i\}$ satisfies condition (D_k) for each k. $\square$

Remark 2. As R is a complete ring, this formal expansion is convergent. It is also the case if R is a valued local ring (see [6]).

Let $\mathbb{K} = \mathbb{C}$ or an algebraically closed field. If $\{e_1, \ldots, e_n\}$ is a fixed basis of $\mathbb{K}^n$, we denote by $\mathcal{P}_n$ the set of all Poisson algebra structures on $\mathbb{K}^n$, that is, the set of structure constants $\{\Gamma_{ij}^k\}$ given by $\mu(e_i, e_j) = \sum_{k=1}^n \Gamma_{ij}^k e_k$. Relation (1) is equivalent to

$$\sum_{l=1}^{n} \left(3\Gamma_{ij}^l \Gamma_{lk}^s - 3\Gamma_{il}^s \Gamma_{jk}^l - \Gamma_{ik}^l \Gamma_{lj}^s - \Gamma_{jk}^l \Gamma_{li}^s + \Gamma_{ji}^l \Gamma_{lk}^s + \Gamma_{ki}^l \Gamma_{lj}^s\right) = 0.$$

Thus $\mathcal{P}_n$ is an affine algebraic variety. If we replace $\mathcal{P}_n$ by a differential graded scheme, we call *Deformation Cohomology,* the cohomology of the tangent space of this scheme.

Remark 3. This cohomology of deformation is defined in same manner for any $\mathbb{K}$-algebra and more generally for any n-ary algebra. If we denote by $H_{\text{def}}(A) = \bigoplus_{n\geq 0} H_{\text{def}}^n(A)$ the deformation cohomology of the algebra A, then $H_{\text{def}}^0(A) = \mathbb{K}$, $H_{\text{def}}^1(A)$ is the space of outer derivations of A and the coboundary operator δ_{def}^1 corresponds to the operator of derivation, and the space of 2-cocycles is determined by the linearization of the identities defining A. Thus, in any case, the three first spaces of cohomology are easy to compute. But the determination of the spaces $H_{\text{def}}^n(A)$ for $n \geq 3$ is usually not easy; we cannot deduce for example $H_{\text{def}}^3(A)$ directly from the knowledge of $H_{\text{def}}^2(A)$. However we have the following result:

Proposition 1. *Let $\mathcal{P}_A$ be the quadratic operad related to A. If $\mathcal{P}_A$ is a Koszul operad, then $H_{\mathrm{def}}(A)$ coincides with the natural operadic cohomology.*

For example, if A is a Lie algebra or an associative algebra, the corresponding operads $\mathcal{L}ie$ and $\mathcal{A}ss$ are Koszul and $H_{\mathrm{def}}(A)$ coincides with the operadic cohomology, that is, respectively, the Chevalley-Eilenberg cohomology and the Hochschild cohomology. Examples of determination of $H_{\mathrm{def}}(A)$ in the non-Koszul cases can be found in [9], [17]. A theory of deformations on non-Koszul operads in presented in [14].

3.2 The operadic cohomology of a Poisson algebra

Let $\mathcal{P}oiss$ be the quadratic binary operad associated with Poisson algebras. Recall briefly its definition. Let $E = \mathbb{K}[\Sigma_2]$ be the $\mathbb{K}$-group algebra of the symmetric group on two elements. The basis of the free $\mathbb{K}$-module $\mathcal{F}(E)(n)$ consists of the "n-parenthesized products" of n variables $\{x_1, \dots, x_n\}$. Let R be the $\mathbb{K}[\Sigma_3]$-submodule of $\mathcal{F}(E)(3)$ generated by the vector

$$u = 3x_1(x_2x_3) - 3(x_1x_2)x_3 + (x_1x_3)x_2 + (x_2x_3)x_1 - (x_2x_1)x_3 - (x_3x_1)x_2 \,.$$

Then $\mathcal{P}oiss$ is the binary quadratic operad with generators E and relations R. It is given by

$$\mathcal{P}oiss(n) = (\mathcal{F}(E)/\mathcal{R})(n) = \frac{\mathcal{F}(E)(n)}{\mathcal{R}(n)}$$

where $\mathcal{R}$ is the operadic ideal of $\mathcal{F}(E)$ generated by R satisfying $\mathcal{R}(1) = \mathcal{R}(2) = 0$, $\mathcal{R}(3) = R$. The dual operad $\mathcal{P}oiss^!$ is equal to $\mathcal{P}oiss$, that is, $\mathcal{P}oiss$ is self-dual. In [18] we defined, for a binary quadratic operad $\mathcal{E}$, an associated quadratic operad $\widetilde{\mathcal{E}}$ which gives a functor

$$\mathcal{E} \otimes \widetilde{\mathcal{E}} \to \mathcal{E} \,.$$

In the case $\mathcal{E} = \mathcal{P}oiss$, we have $\widetilde{\mathcal{E}} = \mathcal{P}oiss^! = \mathcal{P}oiss$. All these properties show that the operad $\mathcal{P}oiss$ is a Koszul operad (see also [12]). In this case the cohomology of deformation of $\mathcal{P}oiss$-algebras coincides with the natural operadic cohomology. An explicit presentation of the space of k-cochains is given in [15]:

$$\mathcal{C}^k(\mathcal{P}, \mathcal{P}) = \mathcal{L}in(\mathcal{P}oiss(n)^! \otimes_{\Sigma_n} V^{\otimes n}, V) = End(\mathcal{P}^{\otimes k}, \mathcal{P})$$

where V is the underlying vector space (here $\mathbb{C}^n$). The cohomology associated with the complex $(\mathcal{C}^k(\mathcal{P}, \mathcal{P}), \delta^k_P)_k$ where δ^k_P denotes the coboundary operator

$$\delta^k_P : \mathcal{C}^k(\mathcal{P}, \mathcal{P}) \to \mathcal{C}^{k+1}(\mathcal{P}, \mathcal{P}) \,,$$

is denoted by $H^*_P(\mathcal{P}, \mathcal{P})$. We will describe the coboundary operators δ^2_P in Subsection 3.3 and δ^k_P in Section 5.

Consequence: The deformation cohomology of a Poisson algebra. If $\mathcal{P}$ is a Poisson algebra, then $H_{\mathrm{def}}(\mathcal{P})$ is the operadic cohomology $H^*_P(\mathcal{P}, \mathcal{P})$ or briefly $H^*_P(\mathcal{P})$.

3.3 Some relations on the coboundary operator δ^2_P

Let $\mathcal{P}$ be a Poisson algebra whose nonassociative product $\mu_0(X,Y)$ is denoted by $X\cdot Y$. Let $\mathfrak{g}_{\mathcal{P}}$ and $\mathcal{A}_{\mathcal{P}}$ be its corresponding Lie and associative algebras. We denote by $H^*_C(\mathfrak{g}_{\mathcal{P}},\mathfrak{g}_{\mathcal{P}})$ the Chevalley-Eilenberg cohomology of $\mathfrak{g}_{\mathcal{P}}$ and by $H^*_H(\mathcal{A}_{\mathcal{P}},\mathcal{A}_{\mathcal{P}})$ the Hochschild cohomology of $\mathcal{A}_{\mathcal{P}}$. A important part of this work devoted to describe the coboundary operator and its links with the Chevalley-Eilenberg and Hochschild coboundary operators. We focus in this section on the degree 2 because it is related to the parametrization of deformations. The condition (D_1) writes as

$$(\mu_0 \circ_1 \mu_1 + \mu_1 \circ_1 \mu_0)\circ \Phi_{v_P} = 3(\mu_0 \circ_2 \mu_1 + \mu_1 \circ_2 \mu_0)\,,$$

that is,

$$\begin{aligned}&3\mu_1(\mu_0(x,y),z) - 3\mu_1(x,\mu_0(y,z)) - \mu_1(\mu_0(x,z),y) - \mu_1(\mu_0(y,z),x)\\ &\quad + \mu_1(\mu_0(y,x),z) + \mu_1(\mu_0(z,x),y) + 3\mu_0(\mu_1(x,y),z) - 3\mu_0(x,\mu_1(y,z))\\ &\qquad - \mu_0(\mu_1(x,z),y) - \mu_0(\mu_1(y,z),x) + \mu_0(\mu_1(y,x),z) + \mu_0(\mu_1(z,x),y) = 0\end{aligned}$$

for any $x,y,z\in\mathcal{P}$. If φ is a 2-cocycle of $H^2_{\mathrm{def}}(\mathcal{P})$, this implies

$$\delta^2_P\varphi = (\varphi \circ_1 \mu + \mu \circ_1 \varphi)\circ\Phi_{v_P} - 3(\varphi\circ_2\mu + \mu\circ_2\varphi)\circ\Phi_{Id}.$$

Recall that $v_P = 3Id - \tau_{23} + \tau_{12} - c + c^2$.

Let $\varphi\colon \mathcal{P}^{\otimes^2} \to \mathcal{P}$ be a bilinear map and μ be the nonassociative multiplication of the Poisson algebra $\mathcal{P}$. We denote by $\varphi_a = \frac{\varphi-\widetilde{\varphi}}{2}$ and $\varphi_s = \frac{\varphi+\widetilde{\varphi}}{2}$ the skew-symmetric and symmetric parts of φ with $\widetilde{\varphi}(X,Y) = \varphi(Y,X)$. We consider the following trilinear maps:

$$\mathcal{L}_C(\varphi) = \frac{1}{2}[\varphi\circ_1\mu\circ\Phi_{Id+c+c^2-\tau_{12}-\tau_{13}-\tau_{23}} + (\mu\circ_1\varphi - \mu\circ_2\varphi)\circ\Phi_{Id+c+c^2}]\,,$$

$$\mathcal{L}_H(\varphi) = \frac{1}{2}[\mu\circ_1\varphi\circ\Phi_{Id-c} - \mu\circ_2\varphi\circ\Phi_{Id-c^2} + \varphi\circ_1\mu\circ\Phi_{Id+\tau_{12}} - \varphi\circ_2\mu\circ\Phi_{Id+\tau_{13}}].$$

If $\varphi=\varphi_a$, that is, if φ is skew-symmetric, then $\mathcal{L}_C(\varphi_a) = \delta^2_{C,\{\,,\}}\varphi_a$ where $\delta^2_{C,\{\cdot,\cdot\}}$ is the Chevalley-Eilenberg coboundary operator of the cohomology of the Lie algebra $\mathfrak{g}_{\mathcal{P}}$ associated with $\mathcal{P}$. Similarly if $\varphi=\varphi_s$, that is, if φ is symmetric, then $\mathcal{L}_H(\varphi_s) = \delta^2_{H,*}\varphi_s$ where $\delta^2_{H,*}$ is the Hochschild coboundary operator of the cohomology of the associative algebra $\mathcal{A}_{\mathcal{P}}$ associated with $\mathcal{P}$. Since no confusions are possible we will write δ^*_C and δ^*_H in place of $\delta^*_{C,\{\cdot,\cdot\}}$ $\delta^*_{H,*}$. Then for any bilinear map φ on $\mathcal{P}^{\otimes^2}$ with skew-symmetric part φ_a and symmetric part φ_s, we obtain

$$\begin{aligned}4\delta^2_C\varphi_a &= (\mu\circ_1\varphi + \varphi\circ_1\mu - \mu\circ_2\varphi + \varphi\circ_2\mu)\circ\Phi_V\\ &\qquad \text{with}\quad V = Id - \tau_{12} - \tau_{13} - \tau_{23} + c + c^2\,,\\ 4\mathcal{L}_C(\varphi_s) &= (\mu\circ_1\varphi - \mu\circ_2\varphi)\circ\Phi_W + (\varphi\circ_1\mu + \varphi\circ_2\mu)\circ\Phi_V\\ &\qquad \text{with}\quad W = Id + \tau_{12} + \tau_{13} + \tau_{23} + c + c^2\,,\\ 4\mathcal{L}_H(\varphi_a) &= \mu\circ_1\varphi\circ\Phi_{Id-\tau_{12}+\tau_{13}-c} + \mu\circ_2\varphi\circ\Phi_{-Id-\tau_{13}+\tau_{23}+c^2}\\ &\quad + \varphi\circ_1\mu\circ\Phi_{Id+\tau_{12}+\tau_{13}+c} + \varphi\circ_2\mu\circ\Phi_{-Id-\tau_{13}-\tau_{23}-c^2}\,,\\ 4\delta^2_H\varphi_s &= \mu\circ_1\varphi\circ\Phi_{Id+\tau_{12}-\tau_{13}-c} + \mu\circ_2\varphi\circ\Phi_{-Id+\tau_{13}-\tau_{23}+c^2}\\ &\quad + \varphi\circ_1\mu\circ\Phi_{Id+\tau_{12}-\tau_{13}-c} + \varphi\circ_2\mu\circ\Phi_{-Id+\tau_{13}-\tau_{23}+c^2}\,.\end{aligned}$$

At least we introduce the following operators, $\mathcal{L}_1$ which acts on the space of skew-symmetric bilinear maps and $\mathcal{L}_2$ which acts on the space of symmetric bilinear maps on $\mathcal{P}$:

$$\begin{aligned} 4\mathcal{L}_1(\varphi_a) &= \mu \circ_1 \varphi \circ \Phi_{\tau_{13}-\tau_{23}-c+c^2} + \mu \circ_2 \varphi \circ \Phi_{-Id-\tau_{12}+\tau_{23}+c} \\ &\quad + \varphi \circ_1 \mu \circ \Phi_{Id+\tau_{12}} + \varphi \circ_2 \mu \circ \Phi_{-\tau_{13}-c^2}, \\ 4\mathcal{L}_2(\varphi_s) &= \mu \circ_1 \varphi \circ \Phi_{2Id+2\tau_{12}-\tau_{13}-\tau_{23}-c-c^2} + \mu \circ_2 \varphi \circ \Phi_{-Id+\tau_{12}-\tau_{13}+c} \\ &\quad + \varphi \circ_1 \mu \circ \Phi_{Id+\tau_{12}+2\tau_{13}-4c} + \varphi \circ_2 \mu \circ \Phi_{-4Id+\tau_{13}+2\tau_{23}+c^2}. \end{aligned}$$

Lemma 3. *We have $\mathcal{L}_1(\varphi_a) = 0$ if and only if φ_a is a skew derivation of the associative product associated with μ, that is:*

$$\varphi_a(x * y, z) = x * \varphi_a(y, z) + \varphi_a(x, z) * y\,.$$

Proof.

$$\begin{aligned} &\varphi_a(x * y, z) - x * \varphi_a(y, z) - \varphi_a(x, z) * y \\ &\quad = \tfrac{1}{2}\big(\varphi_a(xy + yx, z) - x\varphi_a(y, z) - \varphi_a(y, z)x - \varphi_a(x, z)y - y\varphi_a(x, z)\big) \\ &\quad = \tfrac{1}{2}\big(\varphi_a(xy, z) + \varphi_a(yx, z) + x\varphi_a(z, y) + \varphi_a(z, y)x + \varphi_a(z, x)y + y\varphi_a(z, x)\big) \\ &\quad = \tfrac{1}{4}\mathcal{L}_1(\varphi_a)(x, y, z). \end{aligned}$$

□

Proposition 2. *For every bilinear map φ on $\mathcal{P}$, we have*

$$\delta_P^2\varphi = 2\big(\delta_C^2\varphi_a + \mathcal{L}_C(\varphi_s) + \delta_H^2\varphi_s + \mathcal{L}_H(\varphi_a) + \mathcal{L}_1(\varphi_a) + \mathcal{L}_2(\varphi_s)\big)\,. \tag{2}$$

Corollary 1. *Let φ be a bilinear map and φ_a and φ_s the skew-symmetric and the symmetric parts of φ. We have:*

$$12\delta_C^2\varphi_a = \delta_P^2\varphi \circ \Phi_{Id-\tau_{12}-\tau_{13}-\tau_{23}+c+c^2} \tag{3}$$

and

$$12\delta_H^2\varphi_s = \delta_P^2\varphi \circ \Phi_{Id-\tau_{13}+\tau_{23}-c^2}. \tag{4}$$

4 Particular deformations: Lie and associative deformations of a Poisson algebra

In this section we study two particular types of deformations. Usually, only Lie deformations of Poisson algebras are considered. This is a consequence of the classical problem of considering Poisson algebras on the associative commutative algebra of differential functions on a manifold. In this context, the associative algebra is preserved when we consider deformations of Poisson structures on this algebra, for example in problems of deformation quantization. Moreover, such an associative structure is rigid, so it is not appropriate to consider deformations of this multiplication. As consequence, the corresponding deformation cohomology is the Poisson-Lichnerowicz cohomology [11]. So the first particular type we consider, the Lie deformations, is when we deform the Poisson bracket and let the associative product unchanged. We study a second special case which is non classical, the associative deformations. It consists in deformations of the associative product with a preserved Poisson bracket. Such deformations appear naturally when the Poisson bracket is a rigid Lie bracket. These deformations are parametrized by a cohomology defined by a subcomplex of the Poisson complex. We called it Poisson--Hochschild cohomology and describe it explicitely.

4.1 Lie deformations

Definition 1. We say that the formal deformation μ of the Poisson multiplication μ_0 is a Lie formal deformation if the corresponding commutative associative multiplication is conserved, that is, if

$$\mu_0(x,y)+\mu_0(y,x)=\mu(x,y)+\mu(y,x)$$

for any x,y.

As $\mu(x,y)=\mu_0(x,y)+\sum_{n\geq 1}t^n\mu_n(x,y)$, if μ is a Lie deformation of μ_0, then

$$\mu(x,y)+\mu(y,x)=\mu_0(x,y)+\mu_0(y,x)+\sum_{n\geq 1}t^n\left(\mu_n(x,y)+\mu_n(y,x)\right).$$

So

$$\sum_{n\geq 1}t^n\left(\mu_n(x,y)+\mu_n(y,x)\right)=0$$

and

$$\mu_n(x,y)+\mu_n(y,x)=0$$

for any $n\geq 1$. Each bilinear maps μ_n is skew-symmetric. In particular μ_1 is skew-symmetric and $(\mu_1)_s=0$. As $\delta^2_P\mu_1=0$, Relation (2) writes as

$$\delta^2_C\mu_1+\mathcal{L}_H(\mu_1)+\mathcal{L}_1(\mu_1)=0\,.$$

But, from (3), $\delta^2_P\mu_1=0$ implies $\delta^2_C\mu_1=0$. Thus we have $\mathcal{L}_H(\mu_1)+\mathcal{L}_1(\mu_1)=0$. Since μ_1 is skew-symmetric:

$$\begin{aligned}\mathcal{L}_H(\mu_1)(x,y,z)&=\mu_1(x,y)*z-x*\mu_1(y,z)+\mu_1(x*y,z)-\mu_1(x,y*z)\\&=-\mu_1(x,y*z)+\mu_1(x,y)*z+y*\mu_1(x,z)+\mu_1(x*y,z)\\&\quad -x*\mu_1(y,z)-\mu_1(x,z)*y\\&=\mathcal{L}_1(\mu_1)(x,y,z)+\mathcal{L}_1(\mu_1)(y,z,x)\,.\end{aligned}$$

So

$$\mathcal{L}_H(\mu_1)=\mathcal{L}_1(\mu_1)\circ\Phi_{Id+c}\,.$$

We deduce that

$$\mathcal{L}_H(\mu_1)+\mathcal{L}_1(\mu_1)=\mathcal{L}_1(\mu_1)\circ\Phi_{2Id+c}$$

and $\mathcal{L}_H(\mu_1)+\mathcal{L}_1(\mu_1)=0$ implies $\mathcal{L}_1(\mu_1)=0$.

Theorem 3. *If $\mu(x,y) = \mu_0(x,y) + \sum_{n\geq 1} t^n \mu_n(x,y)$ is a Lie deformation of the Poisson product μ_0, then μ_1 is a skew-symmetric map satisfying*

$$\begin{cases} \delta_C^2 \mu_1 = 0, \\ \mathcal{L}_1(\mu_1) = 0. \end{cases}$$

Recall that Poisson-Lichnerowicz cohomology [11] is associated with the complex

$$(\mathcal{C}^*_{PL}(\mathcal{P},\mathcal{P}), \delta_C^*)$$

where the cochains are the skew-symmetric multilinear maps $\mathcal{P} \times \cdots \times \mathcal{P} \to \mathcal{P}$ satisfying the Leibniz rule in each of their arguments (such maps are called skew-symmetric multiderivations of the algebra $\mathcal{P}$). The coboundary operators coincide with the Chevalley-Eilenberg coboundary operator denoted by δ_C^*. Of course $\mathcal{C}^n_{PL}(\mathcal{P},\mathcal{P})$ is a vector subspace of $\mathcal{C}^n_P(\mathcal{P},\mathcal{P})$. The previous theorem shows that if φ is a 2-cochain of $\mathcal{C}^2_{PL}(\mathcal{P},\mathcal{P})$, thus its classes of cohomology in $H^2_{PL}(\mathcal{P},\mathcal{P})$ and $H^2_P(\mathcal{P},\mathcal{P})$ are equal.

4.2 Associative deformations of Poisson algebras

Definition 2. We say that the formal deformation μ of the Poisson multiplication μ_0 is an associative formal deformation if the corresponding Lie multiplication is conserved, that is, if

$$\mu_0(x,y) - \mu_0(y,x) = \mu(x,y) - \mu(y,x)$$

for any x, y.

As $\mu(x,y) = \mu_0(x,y) + \sum_{n\geq 1} t^n \mu_n(x,y)$, if μ is an associative deformation of μ_0, then

$$\mu(x,y) - \mu(y,x) = \mu_0(x,y) - \mu_0(y,x) + \sum_{n\geq 1} t^n \big(\mu_n(x,y) - \mu_n(y,x)\big)\,.$$

Thus

$$\sum_{n\geq 1} t^n \big(\mu_n(x,y) - \mu_n(y,x)\big) = 0$$

and

$$\mu_n(x,y) - \mu_n(y,x) = 0$$

for any $n \geq 1$. Each bilinear maps μ_n is symmetric. In particular μ_1 is symmetric and $(\mu_1)_a = 0$. Since $\delta_P^2 \mu_1 = 0$, Relation (2) writes as

$$\mathcal{L}_C(\mu_1) + \delta_H^2 \mu_1 + \mathcal{L}_2(\mu_1) = 0\,.$$

But, from (4), $\delta_P^2 \mu_1 = 0$ implies $\delta_H^2 \mu_1 = 0$. Thus

$$\mathcal{L}_C(\mu_1) + \mathcal{L}_2(\mu_1) = 0\,.$$

Lemma 4. *When φ is a symmetric map with $\delta_H^2\varphi = 0$,*

$$\begin{aligned}\mathcal{L}_C(\varphi)(x,y,z) &= \{\varphi(x,y),z\} + \{\varphi(y,z),x\} + \{\varphi(z,x),y\} \\ &\quad + \varphi(\{x,y\},z) + \varphi(\{y,z\},x) + \varphi(\{z,x\},y)\,, \\ \mathcal{L}_2(\varphi)(x,y,z) &= \{y,\varphi(x,z)\} - \{z,\varphi(x,y)\} + 3\varphi(x,\{z,y\})\,.\end{aligned}$$

This is a direct consequence of the definition of $\mathcal{L}_C(\varphi_s)$ and $\mathcal{L}_2(\varphi_s)$ when φ is a symmetric bilinear map, replacing $\mu_0(x,y) - \mu_0(y,x)$ by $2\{x,y\}$.

We deduce

$$\begin{aligned}\big(\mathcal{L}_C(\mu_1) + \mathcal{L}_2(\mu_1)\big)(x,y,z) &= 2\{\mu_1(x,y),z\} + \{\mu_1(y,z),x\} + \mu_1(\{x,y\},z) \\ &\quad + \mu_1(\{z,x\},y) + 2\mu_1(\{z,y\},x) \\ &= 2\{\mu_1(x,y),z\} - 2\mu_1(\{y,z\},x) - 2\mu_1(\{x,z\},y) \\ &\quad + \{\mu_1(y,z),x\} - \mu_1(\{y,x\},z) - \mu_1(\{z,x\},y) \\ &= 2\Delta\mu_1(x,y,z) + \Delta\mu_1(y,z,x)\end{aligned}$$

with

$$\Delta\mu_1(x,y,z) = \{\mu_1(x,y),z\} - \mu_1(\{y,z\},x) - \mu_1(\{x,z\},y)\,.$$

We deduce that

$$\big(\mathcal{L}_C(\mu_1) + \mathcal{L}_2(\mu_1)\big) = \Delta\mu_1 \circ \Phi_{2Id+c}\,.$$

But Φ_{2Id+c} is an invertible map on $\mathcal{P}^{\otimes^3}$. Then $\big(\mathcal{L}_C(\mu_1) + \mathcal{L}_2(\mu_1)\big) = 0$ if and only if

$$\Delta\mu_1(x,y,z) = \{\mu_1(x,y),z\} - \mu_1\big(\{y,z\},x\big) - \mu_1\big(\{x,z\},y\big) = 0\,.$$

Definition 3. Let $\mathcal{P}$ be a Poisson algebra and let $\{x,y\}$ be its Poisson bracket. A bilinear map φ on $\mathcal{P}$ is called a Lie biderivation if

$$\{\varphi(x_1,x_2),x_3\} - \varphi\big(x_1,\{x_2,x_3\}\big) - \varphi\big(\{x_1,x_3\},x_2\big) = 0$$

for any $x_1,x_2,x_3 \in \mathcal{P}$.

We deduce that μ_1, which is a symmetric map, is a Lie biderivation.

Theorem 4. *If $\mu(x,y) = \mu_0(x,y) + \sum_{n\geq 1} t^n\mu_n(x,y)$ is an associative deformation of the Poisson product μ_0, then μ_1 is a symmetric map such that*

1. $\delta_H^2\mu_1 = 0$.
2. *μ_1 is a Lie biderivation.*

In case of Lie deformation of the Poisson product μ_0, we have seen that the relations concerning μ_1 can be interpreted in terms of Poisson-Lichnerowicz cohomology. We propose a similar approach for the Lie deformations of μ_0.

Recall that $x * y$ the associative commutative product associated with the Poisson product μ_0, that is $x * y = \dfrac{\mu_0(x,y) + \mu_0(y,x)}{2}$.

Lemma 5. *Let φ be a symmetric bilinear map on $\mathcal{P}$ which is a Lie biderivation. If $\delta^2_H\varphi$ is the Hochschild coboundary operator, we have*

$$\begin{aligned}\delta^2_H\varphi(x_1,x_2,x_3) = x_1 * \varphi(x_2,x_3) - \varphi(x_1 * x_2, x_3)\\ + \varphi(x_1, x_2 * x_3) - \varphi(x_1,x_2) * x_3\end{aligned}$$

and

$$\begin{aligned}\{\delta^2_H\varphi(x_1,x_2,x_3),x_4\} = \delta^2_H\varphi(\{x_1,x_4\},x_2,x_3) + \delta^2_H\varphi(x_1,\{x_2,x_4\},x_3)\\ + \delta^2_H\varphi(x_1,x_2,\{x_3,x_4\})\end{aligned}$$

for any $x_1,x_2,x_3,x_4 \in \mathcal{P}$.

Proof. As φ is a Lie biderivation, we have

$$\{\varphi(x_1,x_2),x_3\} - \varphi(x_1,\{x_2,x_3\}) - \varphi(\{x_1,x_3\},x_2) = 0\,.$$

Thus, using the definition of $\delta^2_H\varphi$, we obtain

$$\begin{aligned}\{\delta^2_H\varphi(x_1,x_2,x_3),x_4\} &= \{x_1 * \varphi(x_2,x_3),x_4\} - \{\varphi(x_1 * x_2,x_3),x_4\}\\ &\quad + \{\varphi(x_1,x_2 * x_3),x_4\} - \{\varphi(x_1,x_2) * x_3,x_4\}\\ &= x_1 * \{\varphi(x_2,x_3),x_4\} - x_3 * \{\varphi(x_1,x_2),x_4\}\\ &\quad + \varphi(x_2,x_3) * \{x_1,x_4\} - \varphi(x_1,x_2) * \{x_3,x_4\}\\ &\quad - \{\varphi(x_1 * x_2,x_3),x_4\} + \{\varphi(x_1,x_2 * x_3),x_4\}\end{aligned}$$

As φ is a Lie biderivation,

$$\begin{aligned}\{\delta^2_H\varphi(x_1,x_2,x_3),x_4\} &= x_1 * \varphi(\{x_2,x_4\},x_3) + x_1 * \varphi(x_2,\{x_3,x_4\})\\ &\quad - x_3 * \varphi(\{x_1,x_4\},x_2) - x_3 * \varphi(x_1,\{x_2,x_4\})\\ &\quad + \varphi(x_2,x_3) * \{x_1,x_4\} - \varphi(x_1,x_2) * \{x_3,x_4\}\\ &\quad - \varphi(x_1 * \{x_2,x_4\},x_3) - \varphi(x_2 * \{x_1,x_4\},x_3)\\ &\quad - \varphi(x_1 * x_2,\{x_3,x_4\}) + \varphi(\{x_1,x_4\},x_2 * x_3)\\ &\quad + \varphi(x_1,x_2 * \{x_3,x_4\}) + \varphi(x_1,x_3 * \{x_2,x_4\})\,.\end{aligned}$$

But

$$\begin{aligned}\delta^2_H\varphi(\{x_1,x_4\},x_2,x_3) &= \{x_1,x_4\} * \varphi(x_2,x_3) - \varphi(\{x_1,x_4\} * x_2,x_3)\\ &\quad + \varphi(\{x_1,x_4\},x_2 * x_3) - \varphi(\{x_1,x_4\},x_2) * x_3\\ \delta^2_H\varphi(x_1,\{x_2,x_4\},x_3) &= x_1 * \varphi(\{x_2,x_4\},x_3) - \varphi(x_1 * \{x_2,x_4\},x_3)\\ &\quad + \varphi(x_1,\{x_2,x_4\} * x_3) - \varphi(x_1,\{x_2,x_4\}) * x_3\\ \delta^2_H\varphi(x_1,x_2,\{x_3,x_4\}) &= x_1 * \varphi(x_2,\{x_3,x_4\}) - \varphi(x_1 * x_2,\{x_3,x_4\})\\ &\quad + \varphi(x_1,x_2 * \{x_3,x_4\}) - \varphi(x_1,x_2) * \{x_3,x_4\}\,.\end{aligned}$$

As the product $*$ is commutative, we deduce

$$\begin{aligned}\{\delta^2_H\varphi(x_1,x_2,x_3),x_4\} &= \delta^2_H\varphi(\{x_1,x_4\},x_2,x_3) + \delta^2_H\varphi(x_1,\{x_2,x_4\},x_3)\\ &\quad + \delta^2_H\varphi(x_1,x_2,\{x_3,x_4\})\,.\end{aligned}$$

□

Observe that the last identity is not a consequence of the symmetry of φ. It is satified for any bilinear Lie biderivation. Now, we can generalize these identities.

Definition 4. Let ϕ be a k-linear map on $\mathcal{P}$. We say that ϕ is a Lie k-derivation if

$$\{\phi(x_1,\ldots,x_k),x_{k+1}\} = \sum_{i=1}^{k} \phi(x_1,\ldots,\{x_i,x_{k+1}\},\ldots,x_k)$$

for any $x_1,\ldots,x_{k+1} \in \mathcal{P}$, where $\{x,y\}$ denotes the Lie bracket associated with the Poisson product.

For example, from the previous lemma, if φ is a Lie 2-derivation (or biderivation), then $\delta^2_H\varphi$ is a Lie 3-derivation.

For any $(k-1)$-linear map on $\mathcal{P}$, let $\delta^{k-1}_H\varphi$ the k-linear map given by

$$\begin{aligned}\delta^{k-1}_H\varphi(x_1,\ldots,x_k) = x_1 * \varphi(x_2,\cdots,x_k) - \varphi(x_1 * x_2,\cdots,x_k) + \varphi(x_1, x_2 * x_3,\cdots,x_k)\\ + \cdots + (-1)^{k-1}\varphi(x_1,x_2,\cdots,x_{k-1} * x_k)\\ + (-1)^k\varphi(x_1,x_2,\cdots,x_{k-1}) * x_k.\end{aligned}$$

This operator is the coboundary operator of the Hochschild complex related to the associative operad $\mathcal{A}ss$.

Theorem 5. *If φ is a Lie k-derivation of $\mathcal{P}$, then $\delta^k_H\varphi$ is a Lie $(k+1)$-derivation of $\mathcal{P}$.*

Proof. It is analogous to the proof detailed for $k=3$. It depends only of the symmetry of the associative product $x * y$. □

Recall that a k-linear map φ on a vector space is called commutative if it satisfies $\varphi \circ \phi_{V_k} = 0$ where $V_k = \sum_{\sigma\in\Sigma_k} \varepsilon(\sigma)\sigma = 0$.

Lemma 6. *For any k-linear commutative map φ on $\mathcal{P}$, the $(k+1)$-linear map $\delta^k_H\varphi$ is commutative.*

Proof. In fact, consider the first term of $\delta^k_H\varphi(x_1,\ldots,x_{k+1})$, that is,

$$x_1 * \varphi(x_2,\ldots,x_{k+1})\,.$$

We have

$$\sum_{\sigma\in\Sigma^i_{k+1}} \varepsilon(\sigma)x_i * \varphi(x_{\sigma(1)},\ldots,x_{\sigma(i-1)},x_{\sigma(i+1)},\ldots,x_{\sigma(k+1)}) = 0$$

because φ is commutative, where $\Sigma^i_{k+1} = \{\sigma \in \Sigma_{k+1}, \sigma(i) = i\}$. The same trick vanishes the last terms, that is,

$$\sum_{\sigma\in\Sigma k+1} \varphi\big(x_\sigma(1),x_\sigma(2),\cdots,x_{\sigma(k)}\big) * x_{\sigma(k+1)}\,.$$

The terms in between vanishes two by two when we compose with Φ_{V_k}. □

Let $C^k_{PH}(\mathcal{P},\mathcal{P})$ be the vector space constituted by k-linear maps on $\mathcal{P}$ which are commutative and which are Lie k-derivations. From the previous result, the image of the $C^k_{PH}(\mathcal{P},\mathcal{P})$ by the map δ^k_H is contained in $C^{k+1}_{PH}(\mathcal{P},\mathcal{P})$. As these maps coincide with the coboundary operators of the complex, we obtain a complex $(C^k_{PH}(\mathcal{P},\mathcal{P}),\delta^k_H)$ whose associated cohomology is called the Poisson-Hochschild cohomology.

Theorem 6. *Let $\mathcal{P}$ be a Poisson algebra whose (nonassociative) product is denoted μ_0. For any associative deformation $\mu = \sum_{n\geq 0} t^i\mu_i$ of μ_0, the linear term μ_1 is a 2-cocycle for the Poisson-Hochschild cohomology.*

4.3 Example: Poisson structures on rigid Lie algebras

Such Poisson structures have been studied in [8], [7]. We will study these structures in terms of Poisson-Hochschild cohomology. Consider, for example, the 3-dimensional complex Poisson algebra given, in a basis $\{e_1, e_2, e_3\}$, by

$$e_1e_2 = 2e_2\,, \quad e_1e_3 = -2e_3\,, \quad e_2e_3 = e_1\,.$$

If $\{\cdot,\cdot\}$ and $*$ denote respectively the Lie bracket and the commutative associative product attached with the Poisson product, we have

$$\{e_1, e_2\} = 2e_2\,, \quad \{e_1, e_3\} = -2e_3\,, \quad \{e_2, e_3\} = e_1$$

and

$$e_i * e_j = 0\,,$$

for any i, j. If φ is a Lie biderivation, it satisfies

$$\{\varphi(e_i, e_j), e_k\} = \varphi(\{e_i, e_k\}, e_j) + \varphi(\{e_j, e_k\}, e_i)\,.$$

This implies $\varphi = 0$ and the Poisson algebra is rigid.

5 Coboundary operators of the general Poisson cohomology

In this section, we describe relations between the coboundary operators δ^k_P of the Poisson cohomology (the operadic cohomology or the deformation cohomology) of a Poisson algebra $\mathcal{P}$ and the corresponding operators of the Poisson-Lichnerowicz and Poisson-Hochschild cohomology of $\mathcal{P}$.

5.1 The cases $k = 0$ and $k = 1$

- $k = 0$. We put

$$H^0_P(\mathcal{P},\mathcal{P}) = \{X \in \mathcal{P} \;\text{ such that }\; \forall Y \in \mathcal{P},\; X \cdot Y = 0\}\,.$$

- $k = 1$. For $f \in End(\mathcal{P},\mathcal{P})$, we put

$$\delta^1_P f(X,Y) = f(X)\cdot Y + X \cdot f(Y) - f(X \cdot Y)$$

for any $X, Y \in \mathcal{P}$. Then we have

$$H^1_P(\mathcal{P},\mathcal{P}) = H^1_C(\mathfrak{g}_{\mathcal{P}},\mathfrak{g}_{\mathcal{P}}) \cap H^1_H(\mathcal{A}_{\mathcal{P}},\mathcal{A}_{\mathcal{P}})\,.$$

5.2 Description of δ^2_P

In Section 4, we have seen that

$$\begin{aligned}\delta^2_P\varphi(x,y,z) &= 3\varphi(x\cdot y,z) - 3\varphi(x,y\cdot z) - \varphi(x\cdot z,y) - \varphi(y\cdot z,x)\\ &\quad + \varphi(y\cdot x,z) + \varphi(z\cdot x,y) + 3\varphi(x,y)\cdot z - 3x\cdot\varphi(y,z)\\ &\quad - \varphi(x,z)\cdot y - \varphi(y,z)\cdot x + \varphi(y,x)\cdot z + \varphi(z,x)\cdot y\end{aligned}$$

and

$$\delta^2_P\varphi = 2\left(\delta^2_C\varphi_a + \mathcal{L}_C(\varphi_s) + \delta^2_H\varphi_s + \mathcal{L}_H(\varphi_a) + \mathcal{L}_1(\varphi_a) + \mathcal{L}_2(\varphi_s)\right).$$

Let us compare this operator with the corresponding Poisson-Lichnerowicz and Poisson-Hochschild ones.

Example 1. Assume that the Poisson product is skew-symmetric. Then $\{x,y\} = x\cdot y$ and $x * y = 0$. If $\varphi \in \mathcal{C}^2_P(\mathcal{P},\mathcal{P})$ is also skew-symmetric, then

$$\begin{aligned}\delta^2_P\varphi(x,y,z) &= 2\varphi(x\cdot y,z) + 2\varphi(y\cdot z,x) - 2\varphi(x\cdot z,y)\\ &\quad + 2\varphi(x,y)\cdot z + 2\varphi(y,z)\cdot x - 2\varphi(x,z)\cdot y\\ &= \delta^2_{PL}\varphi(x,y,z),\end{aligned}$$

that is, the coboundary operator of the Poisson-Lichnerowicz cohomology.

The results of the previous sections imply:

Theorem 7. *Let φ be in $C^2_P(\mathcal{P},\mathcal{P})$, φ_s and φ_a be its symmetric and skew-symmetric parts. Then the following propositions are equivalent:*

1. $\delta^2_P\varphi = 0$.
2. $\begin{cases} i)\ \delta^2_C\varphi_a = 0,\ \ \delta^2_H\varphi_s = 0,\\ ii)\ \mathcal{L}_C(\varphi_s) + \mathcal{L}_H(\varphi_a) + \mathcal{L}_1(\varphi_a) + \mathcal{L}_2(\varphi_s) = 0.\end{cases}$

Applications. Suppose that φ is skew-symmetric. Then $\varphi = \varphi_a$ and $\varphi_s = 0$. Then $\delta^2_P\varphi = 0$ if and only if $\delta^2_C\varphi = 0$ and $\mathcal{L}_H(\varphi) + \mathcal{L}_1(\varphi) = 0$. Morever if we suppose than φ is a biderivation on each argument, that is, $\mathcal{L}_1(\varphi) = 0$, then $\delta^2_P\varphi = 0$ if and only if $\mathcal{L}_H(\varphi) = 0$. But we have seen in Section 3 that

$$\mathcal{L}_H(\varphi) = \mathcal{L}_1(\varphi)\circ\Phi_{Id+c}.$$

Thus $\mathcal{L}_H(\varphi) = 0$ as soon as $\mathcal{L}_1(\varphi) = 0$.

Proposition 3. *Let φ be a skew-symmetric map which is a biderivation, that is, φ is a Poisson-Lichnerowicz 2-cochain. Then $\varphi \in Z^2_{PL}(\mathcal{P},\mathcal{P})$ if and only if $\varphi \in Z^2_P(\mathcal{P},\mathcal{P})$.*

Similarly, if φ is symmetric, then $\delta^2_P\varphi = 0$ if and only if $\delta^2_H\varphi = 0$ and $\mathcal{L}_C(\varphi) + \mathcal{L}_2(\varphi) = 0$. If φ be a skew-symmetric map which is a Lie biderivation, that is, if φ is a Poisson-Hochschild 2-cochain, then $\varphi \in Z^2_{PH}(\mathcal{P},\mathcal{P})$ if and only if $\varphi \in Z^2_P(\mathcal{P},\mathcal{P})$.

5.3 The case $k \geq 3$

Let $\mathcal{P}$ be a Poisson algebra and $H^*_{def}(\mathcal{P})$ or $H^*_P(\mathcal{P},\mathcal{P})$ its operadic cohomology. We propose here to describe $H^n_P(\mathcal{P},\mathcal{P})$ for $n \geq 3$. Let φ be a n-cochain of $C^n_P(\mathcal{P},\mathcal{P})$, that is, a n-linear map on $\mathcal{P}$. Its skew-symmetric part is the skew-symmetric n-linear map

$$\varphi_a = \frac{1}{n!}\varphi \circ \Phi_{V_n}$$

with $V_n = \sum_{\sigma\in\Sigma_n} \varepsilon(\sigma)\sigma$; its symmetric part is the symmetric n-linear map

$$\varphi_s = \frac{1}{n!}\varphi \circ \Phi_{W_n}$$

with $W_n = \sum_{\sigma\in\Sigma_n} \sigma$. We denote by δ^n_P, δ^n_C and δ^n_H respectively the coboundary operators associated with the Poisson cohomology of $\mathcal{P}$, the Chevalley-Eilenberg cohomology of $\mathfrak{g}_{\mathcal{P}}$ and the Hochschild cohomology of $\mathcal{A}_{\mathcal{P}}$.

The formulae (3) and (4) can be generalized as follows

$$2(n+1)!\delta^n_C\varphi_a = \delta^n_p\varphi \circ \Phi_{V_n}\,, \tag{5}$$

$$2(n+1)!\delta^n_H\varphi_s = \delta^n_p\varphi \circ \Phi_{U_{H,n}}\,, \tag{6}$$

where $U_{H,n} = \sum_{\sigma\in\Sigma_{1,n}} \sigma + (-1)^n \sum_{\sigma\in\Sigma_{n,n}} \sigma$ with $\Sigma_{i,n} = \{\sigma \in \Sigma_n, \sigma(1) = i\}$.

Proposition 4. *Let φ be a n-cochain of the Poisson complex of the Poisson algebra $\mathcal{P}$. Then*

$$\delta^n_P\varphi = 0 \Rightarrow \begin{cases} \delta^n_C\varphi_a = 0\,, \\ \delta^n_H\varphi_s = 0\,. \end{cases}$$

Let us consider $\mathcal{L}_{1,n}$ acting on the skew-symmetric n-linear map by

$$\begin{aligned} 2(n-1)!\mathcal{L}_{1,n}\varphi_a = & \sum_{\sigma^{-1}\in\Sigma_{i,i+1,n}} \varepsilon(\sigma)\varphi \circ_{\sigma^{-1}(1)} \mu \circ \Phi_{(Id+\tau_{12})\circ\sigma} \\ & + (-1)^{n-1} \sum_{\sigma^{-1}\in\Sigma_{n,n}} \varepsilon(\sigma)\mu \circ_1 \varphi \circ \Phi_{(Id+\tau_{12})\circ\sigma} \\ & - \sum_{\sigma^{-1}\in\Sigma_{1,n}} \varepsilon(\sigma)\mu \circ_2 \varphi \circ \Phi_{(Id+\tau_{12})\circ\sigma} \end{aligned}$$

where $\Sigma_{i,i+1,n} = \{\sigma \in \Sigma_n, \sigma(1) = i, \sigma(2) = i+1\}$.

Lemma 7. *φ_a is a skew-symmetric n-derivation, that is, a skew-symmetric n-linear map which is a derivation for the associative product $x * y$ on each argument, if and only if $\mathcal{L}_{1,n}\varphi_a = 0$.*

Now we define the operator $\mathcal{L}_{H,n}$which acts on the the skew-symmetric n-linear map by

$$\mathcal{L}_{H,n}\varphi_a = \mathcal{L}_{1,n}\varphi_a \circ \Phi_{Id+c_n+c_n^2+\cdots+c_n^{n-2}}$$

where $c_n \in \Sigma_n$ is the cycle $(1, 2, \ldots, n)$.

Proposition 5. *Let φ be a skew-symmetric linear map on $\mathcal{P}^{\otimes^n}$. Then $\delta^n_P\varphi = 0$ if and only if $\delta^n_C\varphi = 0$ and $\mathcal{L}_{1,n}\varphi = 0$.*

We find again the classical result: the associative deformations of a Poisson algebra are parametrized by the Poisson-Lichnerowciz cohomology.

Assume now that φ is a symmetric n-linear map. We have seen that:

$$\delta^n_P\varphi = 0 \Rightarrow \delta^n_H\varphi_s = \delta^n_H\varphi = 0\,.$$

Consider the operator ∇^n acting on the symmetric n-linear maps by:

$$\begin{aligned}\nabla^n\varphi_s(x_1,\dots,x_{n+1}) = \{\varphi(x_1,\dots,x_n),x_{n+1}\} &- \varphi(\{x_1,x_{n+1}\},x_2,\dots,x_n)\\ &- \varphi(x_1,\{x_2,x_{n+1}\},x_3,\dots,x_n)\\ &- \cdots - \varphi(x_1,x_2,\dots,x_{n-1},\{x_n,x_{n+1}\})\,.\end{aligned}$$

Then $\varphi = \varphi_s$ is a Lie n-derivation if and only if $\nabla^n\varphi_s = 0$.

Now we consider the following operator acting also on the symmetric n-linear maps by:

$$\begin{aligned}\mathcal{L}^n_C\varphi_s = \mu\circ_1\varphi\circ\Phi_{-c+c^2+\cdots+(-1)^{n+1}c^{n+1}} &+ \mu\circ_2\varphi\circ\Phi_{Id-c+c^2+\cdots+(-1)^nc^n}\\ + \varphi\circ_1\mu\circ\Phi_{\sum_{1\le i,j\le n+1}(-1)^{i+j+1}c_{ij}}\end{aligned}$$

where c_{ij} is the permutation $\begin{pmatrix} 1 & 2 & 3 & \cdots & \cdots & \cdots & \cdots & \cdots & n+1\\ i & j & 1 & \cdots & \check{i} & \cdots & \check{j} & \cdots & n+1\end{pmatrix}$ and $\mathcal{L}^n_2\varphi_s$ defined by:

$$\mathcal{L}^n_C\varphi_s + \mathcal{L}^n_2\varphi_s = \nabla^n\varphi_s\circ\Phi_u$$

with $u\in\mathbb{K}[\Sigma_n]$ equal to $\tau_{12}+\tau_{13}+\cdots+\tau_{1n}$. Since Φ_u is invertible, the equation $\mathcal{L}^n_C\varphi_s + \mathcal{L}^n_2\varphi_s = 0$ implies $\nabla^n\varphi_s = 0$ and we find that the Poisson-Hochschild cohomology coincides with the Poisson cohomology when $\varphi = \varphi_s$.

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Distinguished Riemann-Hamilton geometry in the polymomentum electrodynamics

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Abstract. In this paper we develop the distinguished (d-) Riemannian differential geometry (in the sense of d-connections, d-torsions, d-curvatures and some geometrical Maxwell-like and Einstein-like equations) for the polymomentum Hamiltonian which governs the multi-time electrodynamics.

1 Introduction

Let M^n be a smooth real manifold of dimension n, whose local coordinates are $x = (x^i)_{i=\overline{1,n}}$, having the physical meaning of *"space of events"*. In order to justify the *"electrodynamics"* terminology used in this paper, we recall that, in the study of classical electrodynamics, the Lagrangian function $L\colon TM \to \mathbb{R}$ that governs the movement law of a particule of mass $m \neq 0$ and electric charge e, placed concomitantly into a gravitational field and an electromagnetic one, is expressed by

$$L(x, y) = mc\varphi_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i + \mathcal{P}(x)\,, \tag{1}$$

where the semi-Riemannian metric $\varphi_{ij}(x)$ represents the *gravitational potentials* of the space M, $A_i(x)$ are the components of an 1-form on M representing the *electromagnetic potential*, $\mathcal{P}(x)$ is a smooth *potential function* on M and c is the velocity of light in vacuum. The Lagrange space $L^n = (M, L(x, y))$, where L is given by (1), is known in the literature of specialty as the *autonomous Lagrange space of electrodynamics*. A deep geometrical study of the Lagrange space L^n is now completely done in Miron-Anastasiei's book [15]. More general, in the study of classical *time-dependent* electrodynamics, a central role is played by the

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autonomous time-dependent Lagrangian function of electrodynamics:

$$L(t,x,y) = mc\varphi_{ij}(x)y^iy^j + \frac{2e}{m}A_i(t,x)y^i + \mathcal{P}(t,x)\,, \tag{2}$$

where $L\colon \mathbb{R}\times TM \to \mathbb{R}$. Note that the non-dynamical character (i.e., the independence on the temporal coordinate t) of the spatial semi-Riemannian metric $\varphi_{ij}(x)$ determines the usage of the term *"autonomous"* in the preceding definition.

Let $(\mathcal{T}^m, h_{ab}(t))$ be a *"multi-time"* smooth Riemannian manifold of dimension m (please do not confuse with the mass $m \neq 0$), having the local coordinates $t = (t^c)_{c=\overline{1,m}}$, and let $J^1(\mathcal{T}, M)$ be the 1-jet space produced by the manifolds $\mathcal{T}$ and M.

Remark 1. The use in our work of the *"multi-time"* terminology was lent by us from Dickey's monograph [6]. However, it is important to note that "multi-time" does not mean a "multidimensional time", but has the sense of a *"multi-parameter"* or *"many parameters"*.

By a natural extension of the preceding examples of electrodynamics Lagrangian functions, we can consider the jet multi-time Lagrangian function

$$L(t^c, x^k, x^k_c) = mch^{ab}(t)\varphi_{ij}(x)x^i_ax^j_b + \frac{2e}{m}A^{(a)}_{(i)}(t,x)x^i_a + \mathcal{P}(t,x)\,, \tag{3}$$

where $A^{(a)}_{(i)}(t,x)$ is a d-tensor on $J^1(\mathcal{T}, M)$ and $\mathcal{P}(t,x)$ is a smooth function on the product manifold $\mathcal{T}\times M$.

Remark 2. Throughout this paper, the indices $a, b, c, \dots$ run from 1 to m, while the indices $i, j, k, \dots$ run from 1 to n. The Einstein convention of summation is also adopted all over this work.

The pair $\mathcal{EDML}^n_m = (J^1(\mathcal{T}, M), L)$, where L is given by (3), is called the *autonomous multi-time Lagrange space of electrodynamics*. The distinguished Riemannian geometrization of the multi-time Lagrange space $\mathcal{EDML}^n_m$ is now completely developed in the Neagu's works [17] and [18].

Via the classical Legendre transformation, the jet multi-time Lagrangian function of electrodynamics (3) leads us to the Hamiltonian function of polymomenta

$$H = \frac{1}{4mc}h_{ab}\varphi^{ij}p^a_ip^b_j - \frac{e}{m^2c}h_{ab}\varphi^{ij}A^{(b)}_{(j)}p^a_i + \frac{e^2}{m^3c}\left\|A\right\|^2 - \mathcal{P}\,, \tag{4}$$

where $H\colon J^{1*}(\mathcal{T}, M) \to \mathbb{R}$, and

$$\left\|A\right\|^2(t,x) = h_{ab}\varphi^{ij}A^{(a)}_{(i)}A^{(b)}_{(j)}\,.$$

Definition 1. The pair $\mathcal{EDMH}^n_m = (J^{1*}(\mathcal{T}, M), H)$, where H is given by (4), is called the *autonomous multi-time Hamilton space of electrodynamics*.

But, using as a pattern the Miron's geometrical ideas from [16], the distinguished Riemannian geometry for quadratic Hamiltonians of polymomenta (geometry in the sense of d-connections, d-torsions, d-curvatures and geometrical Maxwell-like and Einstein-like equations) is constructed on dual 1-jet spaces in the Oană-Neagu's paper [21]. Consequently, in what follows, we apply the general geometrical result from [21] for the particular Hamiltonian function of polymomenta (4), which governs the multi-time electrodynamics.

2 The geometry of the autonomous multi-time Hamilton space of electrodynamics $\mathcal{EDMH}^n_m$

To initiate our Hamiltonian geometrical development for multi-time electrodynamics, let us consider on the dual 1-jet space $E^* = J^{1*}(\mathcal{T}, M)$ the *fundamental vertical metrical d-tensor*

$$\Phi^{(i)(j)}_{(a)(b)} = \frac{1}{2}\frac{\partial^2 H}{\partial p^a_i \partial p^b_j} = h^*_{ab}(t^c)\varphi^{ij}(x^k),$$

where $h^*_{ab}(t) := (4mc)^{-1} \cdot h_{ab}(t)$. Let $\chi^a_{bc}(t)$ (respectively $\gamma^k_{ij}(x)$) be the Christoffel symbols of the metric $h_{ab}(t)$ (respectively $\varphi_{ij}(x)$). Obviously, if $\overset{*}{\chi}{}^a_{bc}$ are the Christoffel symbols of the Riemannian metric $h^*_{ab}(t)$, then we have $\overset{*}{\chi}{}^a_{bc} = \chi^a_{bc}$.

Using a general result from the geometrical theory of multi-time Hamilton spaces (see [2] and [21]), by direct computations, we find

Theorem 1. *The pair of local functions* $N_{\mathcal{ED}} = \left(\underset{1}{N}{}^{(a)}_{(i)b},\ \underset{2}{N}{}^{(a)}_{(i)j}\right)$ *on the dual* 1*-jet space* E^**, which are given by*

$$\underset{1}{N}{}^{(a)}_{(i)b} = \chi^a_{bf}p^f_i, \qquad \underset{2}{N}{}^{(a)}_{(i)j} = \gamma^r_{ij}\left[\frac{2e}{m}A^{(a)}_{(r)} - p^a_r\right] - \frac{e}{m}\left[\frac{\partial A^{(a)}_{(i)}}{\partial x^j} + \frac{\partial A^{(a)}_{(j)}}{\partial x^i}\right],$$

represents a nonlinear connection on E^**. This nonlinear connection is called the* canonical nonlinear connection of the multi-time Hamilton space of electrodynamics $\mathcal{EDMH}^n_m$.

Now, let

$$\left\{\frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p^a_i}\right\} \subset \chi(E^*), \qquad \left\{dt^a, dx^i, \delta p^a_i\right\} \subset \chi^*(E^*)$$

be the adapted bases produced by the nonlinear connection $N_{\mathcal{ED}}$, where

$$\frac{\delta}{\delta t^a} = \frac{\partial}{\partial t^a} - \underset{1}{N}{}^{(f)}_{(r)a}\frac{\partial}{\partial p^f_r}, \qquad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{2}{N}{}^{(f)}_{(r)i}\frac{\partial}{\partial p^f_r},$$
$$\delta p^a_i = dp^a_i + \underset{1}{N}{}^{(a)}_{(i)f}dt^f + \underset{2}{N}{}^{(a)}_{(i)r}dx^r. \tag{5}$$

Working with these adapted bases, by direct computations, we can determine the adapted components of the *generalized Cartan canonical connection* of the space $\mathcal{EDMH}^n_m$, together with its local d-torsions and d-curvatures (for details, see the general formulas from [21]).

Theorem 2. (1) *The generalized Cartan canonical linear connection of the autonomous multi-time Hamilton space of electrodynamics $\mathcal{EDMH}^n_m$ is given by*

$$C\Gamma(N) = \left(\chi^a_{bc},\ A^i_{jc},\ H^i_{jk},\ C^{i(k)}_{j(c)}\right),$$

where its adapted components are

$$H^c_{ab} = \chi^c_{ab}\,, \quad A^i_{jc} = 0\,, \quad H^i_{jk} = \gamma^i_{jk}\,, \quad C^{i(k)}_{j(c)} = 0\,. \tag{6}$$

(2) *The torsion $\mathbb{T}$ of the generalized Cartan canonical linear connection of the space $\mathcal{EDMH}^n_m$ is determined by* three *effective adapted components:*

$$\begin{aligned}
R^{(f)}_{(r)ab} &= \chi^f_{gab} p^g_r\,,\\
R^{(f)}_{(r)aj} &= -\frac{2e}{m}\gamma^s_{rj} A^{(f)}_{(s);a} + \frac{e}{m}\left[\frac{\partial A^{(f)}_{(r)}}{\partial x^j} + \frac{\partial A^{(f)}_{(j)}}{\partial x^r}\right]_{;a}\,,\\
R^{(f)}_{(r)ij} &= \mathfrak{R}^s_{rij}\left[\frac{2e}{m} A^{(f)}_{(s)} - p^f_s\right] - \frac{e}{m}\left[\frac{\partial A^{(f)}_{(i)}}{\partial x^j} - \frac{\partial A^{(f)}_{(j)}}{\partial x^i}\right]_{:r}\,,
\end{aligned} \tag{7}$$

where $\chi^c_{dab}(t)$ (respectively $\mathfrak{R}^k_{rij}(x)$) are the classical local curvature tensors of the Riemannian metric $h_{ab}(t)$ (respectively semi-Riemannian metric $\varphi_{ij}(x)$), and "$_{;a}$" and "$_{:k}$" represent the following generalized Levi-Civita covariant derivatives*:*

- *the $\mathcal{T}$-generalized Levi-Civita covariant derivative:*

$$\begin{aligned}
T^{bi(d)(r)\ldots}_{cj(l)(f)\ldots;a} \stackrel{\text{def}}{=} \frac{\partial T^{bi(d)(r)\ldots}_{cj(l)(f)\ldots}}{\partial t^a} + T^{gi(d)(r)\ldots}_{cj(l)(f)\ldots}\chi^b_{ga} + T^{bi(g)(r)\ldots}_{cj(l)(f)\ldots}\chi^d_{ga}\\
+\cdots - T^{bi(d)(r)\ldots}_{gj(l)(f)\ldots}\chi^g_{ca} - T^{bi(d)(r)\ldots}_{cj(l)(g)\ldots}\chi^g_{fa} - \cdots\,,
\end{aligned}$$

- *the M-generalized Levi-Civita covariant derivative:*

$$\begin{aligned}
T^{bi(d)(r)\ldots}_{cj(l)(f)\cdots:k} \stackrel{\text{def}}{=} \frac{\partial T^{bi(d)(r)\ldots}_{cj(l)(f)\ldots}}{\partial x^k} + T^{bs(d)(r)\ldots}_{cj(l)(f)\ldots}\gamma^i_{sk} + T^{bi(d)(s)\ldots}_{cj(l)(f)\ldots}\gamma^r_{sk}\\
+\cdots - T^{bi(d)(r)\ldots}_{cs(l)(f)\ldots}\gamma^s_{jk} - T^{bi(d)(r)\ldots}_{cj(s)(f)\ldots}\gamma^s_{lk} - \cdots\,.
\end{aligned}$$

(3) *The curvature $\mathbb{R}$ of the Cartan canonical connection of the space $\mathcal{EDMH}^n_m$ is determined by the following* four *effective adapted components:*

$$H^d_{abc} = \chi^d_{abc}\,, \quad R^l_{ijk} = \mathfrak{R}^l_{ijk}$$

and

$$-R^{(d)(i)}_{(l)(a)bc} = \delta^i_l \chi^d_{abc}\,, \quad -R^{(d)(l)}_{(i)(a)jk} = -\delta^d_a \mathfrak{R}^l_{ijk}\,.$$

3 Electromagnetic-like model on the multi-time Hamilton space of electrodynamics $\mathcal{EDM}H^n_m$

In order to describe our geometrical electromagnetic-like theory (depending on polymomenta) on the multi-time Hamilton space of electrodynamics $\mathcal{EDM}H^n_m$, we underline that, by a simple direct calculation, we obtain (see [21]).

Proposition 1. *The* metrical deflection d-tensors *of the space $\mathcal{EDM}H^n_m$ are expressed by the formulas:*

$$
\begin{gathered}
\Delta^{(i)}_{(a)b} = \left[h^*_{af}\varphi^{ir}p^f_r\right]_{/b} = 0\,, \qquad \vartheta^{(i)(j)}_{(a)(b)} = [h^*_{af}\varphi^{ir}p^f_r]\Big|^{(j)}_{(b)} = \frac{1}{4mc}h_{ab}\varphi^{ij}\,, \\
\Delta^{(i)}_{(a)j} = [h^*_{af}\varphi^{ir}p^f_r]_{|j} = \frac{e}{4m^2c}h_{af}\varphi^{ir}\left[A^{(f)}_{(r):j} + A^{(f)}_{(j):r}\right],
\end{gathered} \tag{8}
$$

where "$_{/b}$", "$_{|j}$" and "$\Big|^{(b)}_{(j)}$" are the local covariant derivatives induced by the generalized Cartan canonical connection $C\Gamma(N)$ (see [20] and [21]).

Moreover, taking into account some general formulas from [21], we introduce

Definition 2. The distinguished 2-form on $J^{1*}(\mathcal{T}, M)$, locally defined by

$$
\mathbb{F} = F^{(i)}_{(a)j}\delta p^a_i \wedge dx^j + f^{(i)(j)}_{(a)(b)}\delta p^a_i \wedge \delta p^b_j\,, \tag{9}
$$

where

$$
\begin{gathered}
F^{(i)}_{(a)j} = \frac{1}{2}\left[\Delta^{(i)}_{(a)j} - \Delta^{(j)}_{(a)i}\right] = \frac{e}{8m^2c}\cdot \underset{\{i,j\}}{\mathcal{A}}\left\{h_{af}\varphi^{ir}\left[A^{(f)}_{(r):j} + A^{(f)}_{(j):r}\right]\right\}, \\
f^{(i)(j)}_{(a)(b)} = \frac{1}{2}\left[\vartheta^{(i)(j)}_{(a)(b)} - \vartheta^{(j)(i)}_{(a)(b)}\right] = 0\,,
\end{gathered} \tag{10}
$$

is called the *polymomentum electromagnetic field attached to the multi-time Hamilton space of electrodynamics $\mathcal{EDM}H^n_m$.*

Now, particularizing the generalized Maxwell-like equations of the polymomentum electromagnetic field that govern a general multi-time Hamilton space MH^n_m, we obtain the main result of the polymomentum electromagnetism on the space $\mathcal{EDM}H^n_m$ (for more details, see [21]):

Theorem 3. *The polymomentum electromagnetic components* (10) *of the autonomous multi-time Hamilton space of electrodynamics $\mathcal{EDM}H^n_m$ are governed by* *the following* geometrical Maxwell-like equations*:*

$$\begin{cases} F^{(i)}_{(a)j/b} = F^{(i)}_{(a)j;b} = \dfrac{e \cdot h_{af}}{8m^2c} \cdot \underset{\{i,j\}}{\mathcal{A}} \left\{ \varphi^{ir} \left[\dfrac{\partial A^{(f)}_{(r)}}{\partial x^j} + \dfrac{\partial A^{(f)}_{(j)}}{\partial x^r} \right]_{;b} -2\varphi^{ir}\gamma^s_{rj} A^{(f)}_{(s);b} \right\} \\ \sum\limits_{\{i,j,k\}} F^{(i)}_{(a)j|k} = \sum\limits_{\{i,j,k\}} F^{(i)}_{(a)j:k} = -\dfrac{h_{af}}{8mc} \cdot \sum\limits_{\{i,j,k\}} \left\{ \left[\varphi^{sr} \mathfrak{R}^i_{rjk} - \varphi^{ir} \mathfrak{R}^s_{rjk} \right] p^f_s + \right. \\ \qquad \left. + \dfrac{e}{m} \varphi^{ir} \left[2\mathfrak{R}^s_{rjk} A^{(f)}_{(s)} - \left(\dfrac{\partial A^{(f)}_{(j)}}{\partial x^k} - \dfrac{\partial A^{(f)}_{(k)}}{\partial x^j} \right)_{:r} \right] \right\} \\ \sum\limits_{\{i,j,k\}} F^{(i)}_{(a)j} \Big|^{(k)}_{(c)} = 0\,, \end{cases} \tag{11}$$

where $\mathcal{A}_{\{i,j\}}$ *represents an alternate sum,* $\sum_{\{i,j,k\}}$ *represents a cyclic sum, and we have*

$$F^{(i)}_{(a)j} \Big|^{(k)}_{(c)} = \frac{\partial F^{(i)}_{(a)j}}{\partial p^c_k} = 0\,.$$

4 Gravitational-like geometrical model on the multi-time Hamilton space of electrodynamics

To expose our geometrical Hamiltonian polymomentum gravitational theory on the autonomous multi-time Hamilton space of electrodynamics $\mathcal{EDMH}^n_m$, we recall that the fundamental vertical metrical d-tensor

$$\Phi^{(i)(j)}_{(a)(b)} = h^*_{ab}(t)\varphi^{ij}(x)$$

and the canonical nonlinear connection

$$N_{\mathcal{ED}} = \left(\underset{1}{N}{}^{(a)}_{(i)b},\ \underset{2}{N}{}^{(a)}_{(i)j} \right)$$

of the multi-time Hamilton space $\mathcal{EDMH}^n_m$ produce a polymomentum gravitational h^*-potential $\mathbb{G}$ on $E^* = J^{1*}(\mathcal{T}, M)$, locally expressed by

$$\mathbb{G} = h^*_{ab} dt^a \otimes dt^b + \varphi_{ij} dx^i \otimes dx^j + h^*_{ab}\varphi^{ij} \delta p^a_i \otimes \delta p^b_j\,. \tag{12}$$

We postulate that the *geometrical Einstein-like equations*, which govern the multi-time gravitational h^*-potential $\mathbb{G}$ of the multi-time Hamilton space of electrodynamics $\mathcal{EDMH}^n_m$, are the abstract geometrical Einstein equations attached to the Cartan canonical connection $C\Gamma(N)$ and to the adapted metric $\mathbb{G}$ on E^*, namely

$$\mathrm{Ric}(C\Gamma) - \frac{\mathrm{Sc}(C\Gamma)}{2}\mathbb{G} = \mathcal{K}\mathbb{T}\,, \tag{13}$$

where $\mathrm{Ric}(C\Gamma)$ represents the *Ricci tensor* of the Cartan connection, $\mathrm{Sc}(C\Gamma)$ is the *scalar curvature*, $\mathcal{K}$ is the *Einstein constant* and $\mathbb{T}$ is an intrinsic d-tensor of matter, which is called the *stress-energy d-tensor of polymomenta*.

In order to describe the local geometrical Einstein-like equations (together with their generalized conservation laws) in the adapted basis

$$\{X_A\} = \left\{ \frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial p_i^a} \right\},$$

let $C\Gamma(N) = (\chi_{ab}^c, 0, \gamma_{jk}^i, 0)$ be the generalized Cartan canonical connection of the space $\mathcal{EDMH}_m^n$. Taking into account the expressions of its adapted curvature d-tensors on the space $\mathcal{EDMH}_m^n$, we immediately find (see [21]):

Theorem 4. *The Ricci tensor* $\mathrm{Ric}(C\Gamma)$ *of the autonomous multi-time Hamilton space of electrodynamics* $\mathcal{EDMH}_m^n$ *is characterized by* two *effective local Ricci d-tensors:*

$$\chi_{ab} = \chi_{abf}^f, \qquad \mathfrak{R}_{ij} = \mathfrak{R}_{ijr}^r.$$

These are exactly the classical Ricci tensors of the Riemannian temporal metric $h_{ab}(t)$ *and the semi-Riemannian spatial metric* $\varphi_{ij}(x)$*.*

Consequently, using the notations $\chi = h^{ab}\chi_{ab}$ and $\mathfrak{R} = \varphi^{ij}\mathfrak{R}_{ij}$, we get

Theorem 5. *The scalar curvature* $\mathrm{Sc}(C\Gamma)$ *of the generalized Cartan connection* $C\Gamma$ *of the space* $\mathcal{EDMH}_m^n$ *has the expression (for details, see [21])*

$$Sc(C\Gamma) = (4mc) \cdot \chi + \mathfrak{R},$$

where χ *and* $\mathfrak{R}$ *are the classical scalar curvatures of the semi-Riemannian metrics* $h_{ab}(t)$ *and* $\varphi_{ij}(x)$*.*

Particularizing the generalized Einstein-like equations and the generalized conservation laws of an arbitrary multi-time Hamilton space MH_m^n, we can establish the main result of the geometrical polymomentum gravitational theory on the autonomous multi-time Hamilton space of electrodynamics $\mathcal{EDMH}_m^n$ (for more details, see [21]):

Theorem 6. (1) *The local* geometrical Einstein-like equations, *that govern the polymomentum gravitational potential of the space* $\mathcal{EDMH}_m^n$*, have the form*

$$\begin{cases} \chi_{ab} - \dfrac{(4mc)\cdot\chi + \mathfrak{R}}{8mc} h_{ab} = \mathcal{K}\mathbb{T}_{ab} \\ \mathfrak{R}_{ij} - \dfrac{(4mc)\cdot\chi + \mathfrak{R}}{2} \varphi_{ij} = \mathcal{K}\mathbb{T}_{ij} \\ -\dfrac{(4mc)\cdot\chi + \mathfrak{R}}{8mc} h_{ab}\varphi^{ij} = \mathcal{K}\mathbb{T}_{(a)(b)}^{(i)(j)}, \end{cases} \tag{14}$$

$$\begin{cases} 0 = \mathbb{T}_{ai}, \quad 0 = \mathbb{T}_{ia}, \quad 0 = \mathbb{T}_{(a)b}^{(i)} \\ 0 = \mathbb{T}_{a(b)}^{(j)}, \quad 0 = \mathbb{T}_{i(b)}^{(j)}, \quad 0 = \mathbb{T}_{(a)j}^{(i)}, \end{cases} \tag{15}$$

where $\mathbb{T}_{AB}$, $A, B \in \left\{ a, i, {}^{(i)}_{(a)} \right\}$, *are the adapted components of the polymomentum stress-energy d-tensor of matter* $\mathbb{T}$*.*

(2) *The* polymomentum conservation laws *of the geometrical Einstein-like equations of the space $\mathcal{EDMH}^n_m$ are expressed by the formulas*

$$\begin{cases} \left[(4mc)\cdot\chi^f_b - \dfrac{(4mc)\cdot\chi+\mathfrak{R}}{2}\delta^f_b\right]_{/f} = 0 \\ \left[\mathfrak{R}^r_j - \dfrac{(4mc)\cdot\chi+\mathfrak{R}}{2}\delta^r_j\right]_{|r} = 0, \end{cases} \tag{16}$$

where $\chi^f_b = h^{fd}\chi_{db}$ *and* $\mathfrak{R}^r_j = \varphi^{rs}\mathfrak{R}_{sj}$.

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