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**Guest Editor: Marcella Palese**



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# Editorial

*Marcella Palese*

It is a pleasure to introduce this issue of *Communications in Mathematics* dedicated to research papers treating various geometric aspects and structures involved in the Calculus of Variations. Most of them were presented in a meeting which obtained the official status of a Satellite Thematic Session of the 6th European Congress of Mathematics held in Krakow, July 2012. The Satellite Thematic Session on *Geometric Methods in Calculus of Variations* was held on the 6th of July at the AGH University of Science and Technology in Krakow, in parallel with the scientific activities of the Congress. Topics covered global analysis, analysis on manifolds, differential geometry, mechanics of particles and systems and general relativity and gravitation.

In the contribution by Fatibene, Francaviglia and Garruto it is shown that, considering the range $3 \leq m \leq 20$, there exist reductive splittings of the spin group only in dimension $m = 4$ (and in this case in any signature), a result which is relevant for applications in loop quantum gravity. In fact, since reductive splittings allow to gobally define the standard Barbero-Immirzi connection, in dimension other than 4, for ensuring globality additional structures should be required.

The paper by Francaviglia, myself and Winterroth introduces the concept of conserved current variationally associated with locally variational invariant field equations. It is shown that the invariance of the variation of the corresponding local presentation is a sufficient condition for the current beeing variationally equivalent to a global one. The case of a Chern-Simons theory is worked out and a global current is variationally associated with a Chern-Simons local Lagrangian.

Havelková considers dynamical properties of singular Lagrangian systems by studing symmetries and conservation laws for a specific singular Lagrangian system of interest in physics. It is investigated whether to every point symmetry of a Euler-Lagrange form there exists a Lagrangian such that the symmetry is also a point symmetry of the Lagrangian itself. For the system under consideration the answer is affirmative and the corresponding Lagrangians are all of order one.

Muzsnay and Nagy characterized the 3-dimensional Heisenberg group with left invariant cubic metric as an example of Finsler manifold having infinite dimensional curvature algebra and holonomy group. The aim of their paper is to describe the algebraic structure of this curvature algebra; they prove that it is an infinite dimensional graded Lie subalgebra of the generalized Witt algebra of homogeneous vector fields generated by three elements.

The paper by Rossi and Musilová treats an important aspect of the inverse problem of the calculus of variations in a nonholonomic setting. The concept of constraint variationality is introduced in the context of first order mechanical systems with general nonholonomic constraints and it is shown that such a concept is equivalent with the existence of a closed representative in the class of 2-forms determining the nonholonomic system. Together with constraint Helmholtz conditions this result completes the basic geometric properties of constraint variational systems.

Saunders deals with the projective Finsler metrizability problem, precisely with the question whether a projective-equivalence class of sprays is the geodesic class of a (locally or globally defined) Finsler function; this paper reviews an interesting approach to the problem using an analogue of the multiplier approach to the inverse problem in Lagrangian mechanics. Conditions are determined for the existence of a global pseudo-Finsler function with Euler-Lagrange equations satisfied by the geodesics of the sprays.

Our meeting enjoyed a pleasant, friendly and stimulating atmosphere promoting interactions between various aspects and topics in the Calculus of Variations. My thanks to all who contributed to this intent and, particularly, to Prof. Olga Rossi for her fundamental help in the organization and the successful outcome of this event.

Torino, 18th November 2012

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# Do Barbero-Immirzi connections exist in different dimensions and signatures?

*L. Fatibene, M. Francaviglia, S. Garruto*

**Abstract.** We shall show that no reductive splitting of the spin group exists in dimension $3 \leq m \leq 20$ other than in dimension $m = 4$. In dimension 4 there are reductive splittings in any signature. Euclidean and Lorentzian signatures are reviewed in particular and signature $(2, 2)$ is investigated explicitly in detail.

Reductive splittings allow to define a global SU(2)-connection over spacetime which encodes in an weird way the holonomy of the standard spin connection. The standard Barbero-Immirzi (BI) connection used in LQG is then obtained by restriction to a spacelike slice. This mechanism provides a good control on globality and covariance of BI connection showing that in dimension other than 4 one needs to provide some other mechanism to define the analogous of BI connection and control its globality.

## 1 Introduction

Barbero-Immirzi (BI) connection is used in LQG to describe gravitational field on a spacelike slice of spacetime; see [1], [2]. In standard literature it is obtained by a canonical transformation on the phase space of the spatial Hamiltonian system describing classical GR; see [3].

The discussion about the possibility of defining a BI counterpart at the level of spacetime has been longly discussed in literature (see [4], [5]). The discussion mainly focused on the possibility of obtaining the BI space connection by *restricting* a suitable BI spin connection defined globally over spacetime as a spacetime object.

We recently showed that the standard spatial BI connection can be in fact obtained by restriction on space of a spacetime SU(2)-connection (see [6]) in spite of controversial opinions about such a possibility. Such a SU(2)-connection is not though simply related to the spacetime spin connection; it is obtained by a mechanism called *reduction* and its global properties can be controlled in view of an algebraic group-theoretical structure called a *reductive group splitting* (see [7]).

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*Key words:* Barbero-Immirzi connection, Global connections, Loop Quantum Gravity

When one defines connections by *restriction* then constraints on the holonomy group of the restricted connection apply (see [8], [9]) showing that standard spatial BI connection cannot be obtained directly by restriction from the spacetime spin connection. However, such holonomic constraints disappear when the connection is defined by *reduction*; as a matter of fact any $\mathrm{Spin}(\eta)$-connection can be reduced to a $\mathrm{SU}(2)$-connection. Unfortunately, reduction produces an encoding of the holonomy of the original spin connection into the holonomy of the reduced connection; such an encoding is far from being trivial and it needs to be further investigated.

The standard BI connection defined in LQG exists because of a number of coincidences; first of all there exist group embeddings $\iota\colon \mathrm{SU}(2) \to \mathrm{Spin}(4)$ and $\iota\colon \mathrm{SU}(2) \to \mathrm{Spin}(3,1)$ which are *reductive*. Second, in dimension 4 a number of topological coincidences guarantee that any spin bundle over spacetime can be reduced to a $\mathrm{SU}(2)$-bundle under the mild hypotheses which are equivalent to the existence of global Lorentzian metrics and global spin structures (see [10], [11], [7]). Finally, the dynamics can be written in terms of the BI connection by adding to the Hilbert action a term which is vanishing on-shell and not compromizing the classical sector; the modified action is called the *Holst* action ([12], [13], [14]) and it provides a dynamically equivalent formulation of standard GR.

Of course, standard BI approach is not the only way to work out LQG. Different frameworks have been proposed (see [15] and [16] just to mention some of them). Nor one can exclude other frameworks to control global properties of BI connection (see [17]). Still we have to stress that, to the best of our knowledge, the one based on *reductions* is the *only general framework known* (with the exception of some *ad hoc* method) to control global properties of standard BI connection at the full level of spacetime.

In this paper we shall consider possible extensions of BI construction by reduction to different signatures and dimensions. We shall show that the construction basically works only in dimension $m = 4$ in all signatures (at least for dimension $3 \leq m \leq 20$).

In Section 2 we shall briefly review the reduction framework. In Section 3 we shall briefly extend the framework to general dimensions. In Section 4 we shall report some result about non-existence of reductive splitting with groups relevant in dimension $m$ for $m \leq 20$. In Section 5 we shall check directly reductive splittings in all signatures in dimension 4. The Euclidean and Lorentzian signature are well known. Relatively new is the case of Kleinian signature $\eta = (2,2)$. BI connection has been proposed and used in signature $(2,2)$ (see [18]); however, to the best of our knowledge the global properties of BI connections for signature (2,2) and its relation to a reductive splitting is new.

## 2 Reductive splittings

In this Section we shall briefly consider the algebraic structure that enable us to reduce the connections. Let us consider a principal bundle $P$ with group $G$ and a subgroup $i\colon H \to G$. Let us then assume and fix any $H$-*reduction* $(Q, \iota)$ of $P$ given by

$$\begin{array}{ccc} Q & \xrightarrow{\iota} & P \\ \downarrow & & \downarrow \\ M & = & M \end{array} \tag{1}$$

The existence of such a reduction usually imposes topological conditions on spacetime. In the standard situation of $G = \mathrm{Spin}(3,1)$ and $H = \mathrm{SU}(2)$ the bundle reduction is automatically ensured by standard physical requirements (essentially by existence of global spinors).

The group embedding $i\colon H \to G$ induces an algebra embedding $T_e i\colon \mathfrak{h} \to \mathfrak{g}$. Let us define the vector space $V = \mathfrak{g}/\mathfrak{h}$ so to have the short sequence of vector spaces

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{h} \arrow[r, "T_e i"] & \mathfrak{g} \arrow[r, "p"] & V \arrow[r] \arrow[l, bend left, dashed, "\Phi"] & 0 \end{tikzcd} \tag{2}$$

where $\Phi\colon V \to \mathfrak{g}$ is a sequence splitting (i.e. $p \circ \Phi = \mathrm{id}_V$) which always exists for sequences of vector spaces. Accordingly, one has $\mathfrak{g} \simeq \mathfrak{h} \oplus \Phi(V)$.

We say that $H$ is *reductive* in $G$ if there is an action $\lambda\colon H \times V \to V$ such that $\mathrm{ad}(h)(\Phi(v)) \equiv \Phi \circ \lambda(h, v)$ where $\mathrm{ad}(h)\colon \mathfrak{g} \to \mathfrak{g}$ is the restriction to the subgroup $H$ of the *adjoint action* of $G$ onto its algebra $\mathfrak{g}$; see [10], [19], [20]. In other words, the subspace $\Phi(V) \subset \mathfrak{g}$ is invariant with respect to the adjoint action of $H \subset G$ on the algebra $\mathfrak{g}$.

Let us stress that the vector subspace $\Phi(V) \subset \mathfrak{g}$ is not required to be (and often it is not) a subalgebra; one just needs the group embedding $i\colon H \to G$. A bundle $H$-reduction $\iota\colon Q \to P$ with respect to a subgroup $H$ reductive in $G$ is enough to allow that *each* $G$-connection $\omega$ on $P$ induces an $H$-connection on $Q$, which will be called the *reduced connection* (see [6] and [7]).

## 3 Connections in Dimension $m > 2$

To fix notation let us consider here spacetimes with dimension $m \equiv n + 1 > 2$ and signature $\eta = (n, 1)$; the relevant spin groups are $\mathrm{Spin}(n)$ for space and $\mathrm{Spin}(n, 1)$ for spacetime. Accordingly, we are using signature $\mathrm{diag}(-1, 1, 1, \ldots, 1)$ on $M$ so that the first coordinate $x^0$ corresponds to time.

Here both the groups are thought as embedded within their relevant Clifford algebra; see [21]. The even Clifford algebras (where the groups' Lie algebras are embedded) are spanned by even products of Dirac matrices, here denoted by $\mathbb{I}, E_{\alpha\beta}, E_{\alpha\beta\gamma\delta}, \ldots$ with $\alpha, \beta, \cdots = 0..n$. The Clifford algebras are suitably embedded one into the other by

$$i_0\colon \mathrm{C}(n) \to \mathrm{C}(n, 1) : E_{i_1 \ldots i_{2l}} \mapsto E_{i_1 \ldots i_{2l}} \tag{3}$$

with $i_1, i_2 \cdots = 1..n$. In other words, the lower dimensional Clifford algebra $\mathrm{C}(n)$ is realized within the higher dimensional one $\mathrm{C}(n, 1)$ by means of even products of Dirac matrices, except $E_0$. Such an algebra embedding restricts to a group embedding

$$i\colon \mathrm{Spin}(n) \to \mathrm{Spin}(n,1) \tag{4}$$

The corresponding covering maps allow to define the embedding of $j\colon \mathrm{SO}(n) \to \mathrm{SO}(n,1)$ which corresponds to rotations that fix the first axis, i.e.

$$j\colon \mathrm{SO}(n) \to \mathrm{SO}(n,1) : \lambda \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \tag{5}$$

We have to show that the embedding (4) is reductive. Let us consider the sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{spin}(n) \arrow[r, "T_e i"] & \mathfrak{spin}(n,1) \arrow[r, "p"] & V \arrow[r] \arrow[l, dashed, bend left, "\Phi"] & 0 \end{tikzcd} \tag{6}$$

The complement vector space $V$ is spanned by $E_{0i}$ and we fix the splitting by setting

$$\Phi\colon V \to \mathfrak{spin}(n,1) : E_{0i} \mapsto E_{0i} + \frac{1}{2}\beta_i{}^{jk} E_{jk} \tag{7}$$

One can write down the condition for which such a splitting is reductive, i.e.

$$\lambda^l_i \beta_l{}^{jk} = \beta_i{}^{lm} \lambda^j_l \lambda^k_m \tag{8}$$

which must hold true for any $\lambda \in \mathrm{SO}(n)$. Then one can consider a 1-parameter subgroup $\lambda(t)$ based at the identity (i.e. $\lambda(0) = \mathbb{I}$) and the corresponding Lie algebra element $\dot\lambda = \dot\lambda(0)$; the infinitesimal form of (8) is then

$$\dot\lambda^l_i \beta_l{}^{jk} = \beta_i{}^{lk} \dot\lambda^j_l + \beta_i{}^{jm} \dot\lambda^k_m \tag{9}$$

which must hold for any $\dot\lambda \in \mathfrak{so}(n) \simeq \mathfrak{spin}(n)$, i.e. for any skew–symmetric matrix.

Then one should try to look for solutions of condition (9) that correspond to reductive splittings, besides the trivial case $\beta_i{}^{jk} = 0$ which corresponds to no Immirzi parameter. Before searching for explicit solutions for $2 \le n \le 19$ (i.e. spacetime dimension $3 \le m \le 20$) let us consider few simple examples.

For $n = 2$, Latin indices range in $i, j, \dots = 1, 2$. The condition (9) specifies to

$$\begin{cases} \beta_1{}^{12} = \beta_2{}^{12} \\ \beta_2{}^{12} = -\beta_1{}^{12} \end{cases} \tag{10}$$

Hence one has $\beta_1{}^{12} = \beta_2{}^{12} = 0$, so that there is no reductive splitting other then $\beta_i{}^{jk} = 0$.

For $n = 3$ (i.e. $m = 4$), Latin indices range in $i, j, \dots = 1, 2, 3$. The condition (9) has the only solution is $\beta_i{}^{jk} = \beta \epsilon_i{}^{jk}$ which spans reductive splittings (see [6] and [7]). The constant parameter $\beta$ is related to the standard Immirzi parameter.

One can immediately generalize that constructions in two classes of embeddings. In both cases let us fix on $M$ signature $\eta = (r, s)$ (with $r + s = m$). In the first case we take signature $\eta_{ab} = \mathrm{diag}(\underbrace{-1, \dots, -1}_{s\ \text{times}}, \underbrace{1, \dots, 1}_{r\ \text{times}})$ and consider the embedding

$$i\colon \mathrm{Spin}(r, s-1) \to \mathrm{Spin}(r, s) \tag{11}$$

Accordingly, one is left with a signature $\hat\eta = (r, s-1) = (k, l)$ on the *"spatial" leaf* of dimension $n = m - 1 = k + l$. The standard canonical form of signature $\hat\eta = (k, l)$ is fixed to be $\hat\eta_{ij} = \mathrm{diag}(\underbrace{-1, \dots, -1}_{l=s-1 \text{ times}}, \underbrace{1, \dots, 1}_{k=r \text{ times}})$. For notation convenience, in the second case we take signature $\eta_{ab} = \mathrm{diag}(\underbrace{1, \dots, 1}_{r \text{ times}}, \underbrace{-1, \dots, -1}_{s \text{ times}})$ and consider the embedding

$$i\colon \mathrm{Spin}(r-1, s) \to \mathrm{Spin}(r, s) \tag{12}$$

Accordingly, one is left with a signature $\hat\eta = (r-1, s) = (k, l)$ on the *"spatial" leaf* of dimension $n = m - 1 = k + l$. The standard canonical form of signature $\hat\eta = (k, l)$ is fixed to be $\hat\eta_{ij} = \mathrm{diag}(\underbrace{1, \dots, 1}_{k=r-1 \text{ times}}, \underbrace{-1, \dots, -1}_{l=s \text{ times}})$.

In both cases we select the first axis as a fixed rotational axis and denote by $\eta_{ij}$ the standard canonical form of signature $\hat\eta = (k, l)$.

## 4 Non-existence of reductive splittings in dimension different from $m = 4$

In order to verify whether a reductive splitting occurs in an arbitrary dimension we must solve equations (9), or better said the system obtained from (9) fixing and arbitrary $\dot\lambda \in \mathfrak{spin}(n)$. Since the number of equations increases with the dimension of the space, it is difficult to find solutions by direct calculations. However, one can use Maple tensor package (see [22]) to easily compute the solution of linear system (9) for any arbitrary (but fixed) dimension and signature.

First of all, one should look for the general expression of the generators $\dot\lambda^l_i$ of the Lie algebra $\mathfrak{spin}(k, l) \simeq \mathfrak{so}(k, l)$. Let us fix the standard bilinear form $\hat\eta_{ij} = \mathrm{diag}(\underbrace{1, \dots, 1}_{k \text{ times}}, \underbrace{-1, \dots, -1}_{l \text{ times}})$ of signature $\hat\eta = (k, l)$; then the corresponding orthogonal group $\mathrm{SO}(\hat\eta)$ is the set of matrices defined by the relation:

$$\lambda^i_k \hat\eta_{ij} \lambda^j_l = \hat\eta_{kl} \tag{13}$$

The relation above can be read in the algebra as:

$$\dot\lambda^i_k \hat\eta_{ij} + \hat\eta_{ki} \dot\lambda^i_j = 0 \tag{14}$$

It is easy to see that conditions (14) tell us that $\dot\lambda^l_i$ is a block matrix:

$$\dot\lambda = \begin{pmatrix} A_1 & B \\ {}^tB & A_2 \end{pmatrix} \tag{15}$$

where $A_1$ and $A_2$ are skew-symmetric matrices, of dimension $k \times k$ and $l \times l$ respectively, while $B$ is an arbitrary $k \times l$ matrix. One can set generators of $\mathfrak{so}(\hat\eta)$ to be matrices with all zero entries but two where $\pm 1$ is set according to (15).

Then equation (9) can be expanded along this basis of $\mathfrak{so}(\eta)$ obtaining a system of $\frac{n^3}{4}(n-1)^2$ equations. The unknowns $\beta_k{}^{ij}$ are $\frac{n^2}{2}(n-1)$. For any $n > 2$ one has more equations than unknowns and has to compute the rank of the system to discuss solutions. Of course computing the rank of the system obtained from (9) is rather difficult in general thus we shall analyze each case separately.

Of course, since the system is homogeneous, it cannot be inconsistent but it must have at least the trivial solution. We aim to discuss whether, in some dimension, there are solutions other than the trivial one.

As we have seen above, in a fixed dimension $m$ and signature $\eta = (r, s)$ there are two ways of defining group embeddings, one fixing a time axis and one fixing a space axis. So we have to check both of them.

We have obtained a computer-aided solution for the system in all spacetime dimensions from $m = 3$ up to $m = 20$; in each dimension we considered any signature of spacetime $\eta = (r, s)$ with $0 \leq r \leq m$ and $s = m - r$; in each such dimension and signature we consider both cases, i.e. fixing a time axis or a space axis.

[Of course, if $r = 0$ one can only fix a time axis. Analogously, if $r = m$ (and $s = 0$) one can only fix a space case.]

In all these cases (except for case $m = 4$ which will be analyzed in the next section) none of the group splitting considered is reductive, besides the trivial case $\beta_k{}^{ij} = 0$. Regardless the existence of bundle reductions, in these cases there is no canonical way of defining BI connections and one has to find out different mechanism (e.g. resorting to embeddings involving different groups) to control global properties and covariance of BI connections (possibly changing the groups involved) and to proceed to quantize *á la* loop.

## 5 Reductive splittings in dimension $m = 4$

Among the considered dimensions ($3 \leq m \leq 20$), we found that only in $m = 4$ there are non-trivial reductive splittings. In dimension $m = 4$ one has five signatures, three of them with 2 embeddings to be analyzed and two with one embedding only, for a total of 8 embeddings to be considered. In all these cases, it turns out to be that the splitting coefficients $\beta_l{}^{jk}$ are proportional to the Levi-Civita symbol:

$$\beta_l{}^{jk} = \beta\epsilon_l^{\cdot jk} := \beta\hat{\eta}_{lm}\epsilon^{mjk} \qquad (\gamma \in \mathbb{R}) \tag{16}$$

each using the relevant standard form $\hat{\eta}_{lm}$ according to the notation explained above.

Once $\beta_l{}^{jk}$ are calculated we can directly verify from the definition that splittings in dimension four are all reductive.

First of all we shall define some useful notation: let us set $\tau_i = \frac{1}{2}\epsilon_i{}^{jk}_{\cdot\cdot}E_{jk}$ and $\sigma_i = E_{0i}$. Since we shall have to compute products of $\tau_i$ it is convenient to write them in a closed form. One can verify that:

$$\tau_i\tau_j = -\eta_{00}\eta\eta_{ij}\mathbb{I} - \epsilon_{ij}{}^{k}_{\cdot}\tau_k \tag{17}$$

where, by an abuse of language, we denote by $\eta$ the determinant of $\eta_{ab}$.

Furthermore we can write the splitting $e_k = (-\alpha^3 E + \hat{\beta})\tau_k$, where $\hat{\beta} = \beta\hat{\eta}$ is a constant simply related to $\beta$ and we set $\alpha := \sqrt{\eta}$ (possibly imaginary) and $E := \alpha E_{0123}$. Let us remark also that if $S \in \mathrm{Spin}(k, l)$, than it can be written as a linear combination of $\mathrm{Spin}(k,l)$ generators, namely

$$S = a^0\mathbb{I} + a^i\tau_i \tag{18}$$

with inverse:

$$S^{-1} = a^0\mathbb{I} - a^i\tau_i \tag{19}$$

under the constraint:

$$(a_0)^2 + \eta_{00}\eta\,|\vec{a}|^2 = 1 \tag{20}$$

which is the condition that defines spin group in $\mathrm{C}^+(\eta)$.

With this notation we are ready to verify the splitting by applying directly the definition. We have then to compute the adjoint action, restricted to $\mathrm{Spin}(k,l)$, on the bases $e_k$ of $\Phi(V) \subset \mathfrak{spin}(r,s)$. One has:

$$\begin{aligned}
Se_kS^{-1} &= (a^0\mathbb{I} + a^i\tau_i)(\alpha\eta E + \hat\beta)\tau_k(a^0\mathbb{I} - a^j\tau_j) = \\
&= (\alpha\eta E + \hat\beta)(a^0\mathbb{I} + a^i\tau_i)(a^0\tau_k - a^j\tau_k\tau_j) = \\
&= (\alpha\eta E + \hat\beta)(a^0\mathbb{I} + a^i\tau_i)\big(a^0\tau_k - a^j(-\eta_{00}\eta\hat\eta_{kj}\mathbb{I} - \epsilon_{kj\cdot}{}^{l}\tau_l)\big) = \\
&= (\alpha\eta E + \hat\beta)\big((a^0)^2\tau_k + \eta_{00}\eta a_{\cdot k}a^0\mathbb{I} + a^0a^j\epsilon_{kj\cdot}{}^{l}\tau_l + \\
&\quad + a^0a^i\tau_i\tau_k + a^ia_{\cdot k}\eta_{00}\eta\tau_i + a^ia^j\epsilon_{kj\cdot}{}^{l}\tau_i\tau_l\big) = \\
&= (\alpha\eta E + \hat\beta)((a^0)^2\tau_k + \underline{\eta_{00}\eta a_{\cdot k}a^0\mathbb{I}} + \underline{\underline{a^0a^j\epsilon_{kj\cdot}{}^{l}\tau_l}} + \\
&\quad + \underline{a^0a_{\cdot k}(-\eta_{00}\eta\mathbb{I})} - \underline{\underline{a^0a^i\epsilon_{ik\cdot}{}^{l}\tau_l}} + a^ia_{\cdot k}\eta_{00}\eta\tau_i + \\
&\quad + a^ia^j\epsilon_{kji}(-\eta_{00}\eta) - a^ia^j\epsilon_{kj\cdot}{}^{l}\epsilon_{il\cdot}{}^{m}\tau_m) = \\
&= (\alpha\eta E + \hat\beta)\big((a^0)^2\tau_k - 2a^0a^j\epsilon_{jk\cdot}{}^{l}\tau_l + \\
&\quad + a^ma_{\cdot k}\eta_{00}\eta\tau_m - a^ia^j\epsilon_{kj\cdot}{}^{l}\epsilon_{il\cdot}{}^{m}\tau_m\big)
\end{aligned} \tag{21}$$

By using the contraction formula $\epsilon_{kjl}\epsilon_{i}^{\cdot\,ml} = \hat\eta_{ki}\hat\eta^m_j - \hat\eta^m_k\hat\eta_{ji}$ we can re-write $Se_kS^{-1}$ as:

$$Se_kS^{-1} = l^m_k e_m \tag{22}$$

where

$$l^m_k = \big((a^0)^2 - \eta\eta_{00}|\vec{a}|^2\big)\,\delta^m_k + 2\eta\eta_{00}a^m a_{\cdot k} - 2a^0a^i\epsilon_{ik\cdot}{}^{m} \tag{23}$$

If one uses (20) it is easy to see that (23) is an orthogonal transformation for $\hat\eta_{ab}$, namely, $l^m_i\hat\eta_{mn}l^m_j = \hat\eta_{ij}$. In this way we have been able to show that in dimension $m = 4$ the splittings are reductive in all signatures.

## 6 Conclusions and Perspectives

We showed that for any dimension $3 \le m = r + s \le 20$ all the embeddings

$$\begin{aligned}
i\colon \mathrm{Spin}(r-1,s) &\to \mathrm{Spin}(r,s) \\
i\colon \mathrm{Spin}(r,s-1) &\to \mathrm{Spin}(r,s)
\end{aligned} \tag{24}$$

are *not* reductive except when $m = 4$.

In $m = 4$ they are all reductive for any choice of the signature, i.e. for $0 \leq r \leq m$. In Euclidean signature the reductive splitting $i\colon \mathrm{Spin}(3) \to \mathrm{Spin}(4)$ reproduces the standard BI connection used in the Euclidean sector. In Lorentzian signature the reductive splitting $i\colon \mathrm{Spin}(3) \to \mathrm{Spin}(3,1)$ reproduces the standard BI connection used in the Lorentzian sector.

The other signatures in dimension $m = 4$ allow us to define a BI SU(2)-connection on spacetime which produces the BI in Hamiltonian formalism by restriction. By this mechanism the global properties of the BI are under control and the holonomy encoding of the spin connection into the holonomy of the BI connection is manifest, though it surely deserves further investigations.

In dimension other than 4 this mechanism cannot be used in order to guarantee the existence of global BI connections (or fields which behaves as connections under gauge transformations enforcing covariance of holonomic variables) and one needs to rely on some other construction to quantize gravity as in LQG, possibly relying on some other group as suggested e.g. in [17].

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# Locally variational invariant field equations and global currents: Chern-Simons theories

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**Abstract.** We introduce the concept of conserved current variationally associated with locally variational invariant field equations. The invariance of the variation of the corresponding local presentation is a sufficient condition for the current beeing variationally equivalent to a global one. The case of a Chern-Simons theory is worked out and a global current is variationally associated with a Chern-Simons local Lagrangian.

## 1 Introduction

We are interested in the study of the relation between symmetries (i.e. invariance properties) of field equations and corresponding conservation laws. More precisely, the topic of this paper is the investigation of some aspects concerning the interplay between symmetries, conservation laws and *variational principles*. We shall consider Noether conservation laws associated with the invariance of global Euler-Lagrange morphisms generated by local variational problems of a given type.

We shall characterize symmetries of field equations having 'variational' meaning. In order to understand the *structure* of a phenomenon described by field equations, one should be interested in conservation laws more precisely characterized than those directly associated with invariance properties of field equations. Thus, we will look for conservation laws coming from invariance properties of a (*possibly local*) variational problem in its whole (rather than a field equation solely) to find a way of associating global conservation laws with a local Lagrangian *field* theory generating global Euler-Lagrange equations.

From a physical point of view, field equations appear to be a fundamental object, since they describe the changing of the field in base space. Somehow, we are well disposed to give importance to symmetries of equations, because they are transformations of the space leaving invariant the description of such a change

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provided by means of field equations. On the other hand the possibility of formulating a variational principle (i.e. a principle of *stationary* action) – from which both changing of fields and associated conservation laws (i.e. quantities not changing in the base space) could be obtained – has been one of the most important achievements in the history of mathematical and physical sciences in modern age. It allows, in fact, to keep account of both what (*and how*) changes and what (*and how*) is conserved. In the variational calculus perspective, we could say that *Euler--Lagrange field equations are 'adjoint' to stationary principles up to conservation laws*.

In line with Lepage's cornerstone papers [23], which pointed out the fact that the Euler-Lagrange operator is a quotient morphism of the exterior differential, we shall consider a geometric formulation of the calculus of variations on fibered manifolds for which the Euler-Lagrange operator is a morphism of a finite order exact sequence of sheaves according to [20]. The module in degree $(n+1)$, contains so-called (variational) dynamical forms; a given equation is globally an Euler--Lagrange equation if its dynamical form is closed in the complex of global sections (Helmholtz conditions) and its cohomolgy class is trivial. Dynamical forms which are only *locally variational*, i.e. closed in the complex and defining a non trivial cohomology class, admit a system of local Lagrangians, one for each open set in a suitable covering, which satisfy certain relations among them.

In her celebrated paper *Invariante Variationsprobleme* [25], Emmy Noether clearly pointed out how, considering invariance of *variational* problems, a major refinement in the description of associated conserved quantities is achieved. A formulation in modern language of Noether's results would say that symmetry properties of the Euler-Lagrange expressions introduce *a cohomology class which adds up to Noether currents*; it is important to stress that they are related with invariance properties of the first variation. Global projectable vector fields on prolongations of fibered manifold which are symmetries of dynamical forms, in particular of locally variational dynamical forms, and corresponding formulations of Noether theorem II can be considered in order to determine obstructions to the globality of associated conserved quantities [16]. The concept of global (and local) variationally trivial Lagrangians and in general of variationally trivial currents (i.e. $(n-1)$-forms) will be taken in consideration and for simplicity, in the sequel, a *locally variational form* will be any closed $p$-form in the variational sequence; inverse problems at any degree of variational forms will be considered.

In the present paper, we introduce the concept of conserved current variationally associated with locally variational invariant field equations. The invariance of the variation of the corresponding local presentation is a sufficient condition for the current beeing variationally equivalent to a global one. The case of a Chern-Simons gauge theory is worked out and a global current is variationally associated with a Chern-Simons local Lagrangian.

Chern-Simons theories exhibit in fact many interesting and important properties: they are based on secondary characteristic classes and can be associated with new topological invariants for knots and three-manifolds; they appeared in physics as natural mass terms for gauge theories and for gravity in dimension three, and after quantization they lead to a quantized coupling constant as well as a mass

[10]. In particular, Chern-Simons gauge theory is also an example of a topological field theory [32]. Furthermore, as it was remarked in [2], obstructions to the construction of natural Lagrangians are in a one-to-one correspondence with the conformally invariant characteristic forms discovered by Chern and Simons in [9]. Finally, the Chern-Simons term is related to the anomaly cancellation problem in 2-dimensional conformal field theories [7].

## 2 Locally variational invariant field equations and variationally equivalent problems

We shall consider the variational sequence [20] defined on a fibered manifold $\pi\colon \boldsymbol{Y} \to \boldsymbol{X}$, with $\dim \boldsymbol{X} = n$ and $\dim \boldsymbol{Y} = n+m$. For $r \geq 0$ we have the $r$-jet space $J_r\boldsymbol{Y}$ of jet prolongations of sections of the fibered manifold $\pi$. We have also the natural fiberings $\pi^r_s\colon J_r\boldsymbol{Y} \to J_s\boldsymbol{Y}$, $r \geq s$, and $\pi^r\colon J_r\boldsymbol{Y} \to \boldsymbol{X}$; among these the fiberings $\pi^r_{r-1}$ are *affine bundles* which induce the natural fibered splitting

$$J_r\boldsymbol{Y} \times_{J_{r-1}\boldsymbol{Y}} T^*J_{r-1}\boldsymbol{Y} \simeq J_r\boldsymbol{Y} \times_{J_{r-1}\boldsymbol{Y}} (T^*\boldsymbol{X} \oplus V^*J_{r-1}\boldsymbol{Y}) \,,$$

which, in turn, induces also a decomposition of the exterior differential on $\boldsymbol{Y}$ in the *horizontal* and *vertical differential*, $(\pi^{r+1}_r)^* \circ d = d_H + d_V$. By $(j_r\Xi, \xi)$ we denote the jet prolongation of a *projectable vector field* $(\Xi, \xi)$ on $\boldsymbol{Y}$, and by $j_r\Xi_H$ and $j_r\Xi_V$ the horizontal and the vertical part of $j_r\Xi$, respectively.

We have the *sheaf splitting* $\mathcal{H}^p_{(s+1,s)} = \bigoplus_{t=0}^p \mathcal{C}^{p-t}_{(s+1,s)} \wedge \mathcal{H}^t_{s+1}$ where $\mathcal{H}^p_{(s,q)}$ and $\mathcal{H}^p_s$ $(q \leq s)$ are sheaves of *horizontal forms*, and $\mathcal{C}^p_{(s,q)} \subset \mathcal{H}^p_{(s,q)}$ are subsheaves of *contact forms* [20]. Let us denote by $h$ the projection onto the nontrivial summand with the higest value of $t$ and by $d\ker h$ the sheaf generated by the corresponding presheaf and set then $\Theta^*_r \equiv \ker h + d\ker h$; the quotient sequence

$$0 \to I\!R_{\boldsymbol{Y}} \to \dots \xrightarrow{\mathcal{E}_{n-1}} \Lambda^n_r/\Theta^n_r \xrightarrow{\mathcal{E}_n} \Lambda^{n+1}_r/\Theta^{n+1}_r \xrightarrow{\mathcal{E}_{n+1}} \Lambda^{n+2}_r/\Theta^{n+2}_r \xrightarrow{\mathcal{E}_{n+2}} \dots \xrightarrow{d} 0$$

defines the *$r$–th order variational sequence* associated with the fibered manifold $\boldsymbol{Y} \to \boldsymbol{X}$; here $\Lambda^p_s$ is the standard sheaf of $p$–forms on $J_s\boldsymbol{Y}$. The quotient sheaves (the sections of which are classes of forms modulo contact forms) in the variational sequence can be represented as sheaves $\mathcal{V}^k_r$ of $k$-forms on jet spaces of higher order. In particular, currents are classes $\nu \in (\mathcal{V}^{n-1}_r)_{\boldsymbol{Y}}$; Lagrangians are classes $\lambda \in (\mathcal{V}^n_r)_{\boldsymbol{Y}}$, while $\mathcal{E}_n(\lambda)$ is called a Euler-Lagrange form (being $\mathcal{E}_n$ the Euler-Lagrange morphism); dynamical forms are classes $\eta \in (\mathcal{V}^{n+1}_r)_{\boldsymbol{Y}}$ and $\mathcal{E}_{n+1}(\eta)$ is a Helmohltz form (being $\mathcal{E}_{n+1}$ the corresponding Helmholtz morphism).

Since the variational sequence is a soft resolution of the constant sheaf $I\!R_{\boldsymbol{Y}}$ over $\boldsymbol{Y}$, the cohomology of the complex of global sections, denoted by $H^*_{VS}(\boldsymbol{Y})$, is naturally isomorphic to both the Čech cohomology of $\boldsymbol{Y}$ with coefficients in the constant sheaf $I\!R$ and the de Rham cohomology $H^k_{dR}\boldsymbol{Y}$ [20].

Let $\boldsymbol{K}^p_r \equiv Ker\ \mathcal{E}_p$. We have the short exact sequence of sheaves

$$0 \to \boldsymbol{K}^p_r \xrightarrow{i} \mathcal{V}^p_r \xrightarrow{\mathcal{E}_p} \mathcal{E}_p(\mathcal{V}^p_r) \to 0 \,.$$

For any global section $\beta \in (\mathcal{V}^{p+1}_r)_{\boldsymbol{Y}}$ we have $\beta \in (\mathcal{E}_p(\mathcal{V}^p_r))_{\boldsymbol{Y}}$ if and only if $\mathcal{E}_{p+1}(\beta) = 0$, which are conditions of local variationality. A global inverse problem is to find necessary and sufficient conditions for such a locally variational $\beta$

to be globally variational. In particular $\mathcal{E}_n(\mathcal{V}^n_r)$ is the sheave of Euler-Lagrange morphisms and $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}}$ if and only if $\mathcal{E}_{n+1}(\eta) = 0$, which are Helmholtz conditions.

The above exact sequence gives rise to the long exact sequence in Čech cohomology

$$0 \to (\boldsymbol{K}^p_r)_{\boldsymbol{Y}} \to (\mathcal{V}^p_r)_{\boldsymbol{Y}} \to (\mathcal{E}_p(\mathcal{V}^p_r))_{\boldsymbol{Y}} \xrightarrow{\delta_p} H^1(\boldsymbol{Y}, \boldsymbol{K}^p_r) \to 0\,,$$

where the *connecting homomorphism* $\delta_p = i^{-1} \circ \mathfrak{d} \circ \mathcal{E}_p^{-1}$ is the mapping of cohomologies in the corresponding diagram of cochain complexes. In particular, every $\eta \in (\mathcal{E}_n(\mathcal{V}^n_r))_{\boldsymbol{Y}}$ (i.e. locally variational) defines a cohomology class $\delta(\eta) \equiv \delta_n(\eta) \in H^1(\boldsymbol{Y}, \boldsymbol{K}^n_r)$. Furthermore, every $\mu \in (d_H(\mathcal{V}^{n-1}_r))_{\boldsymbol{Y}}$ (i.e. locally variationally trivial) defines a cohomology class $\delta'(\mu) \equiv \delta_{n-1}(\mu) \in H^1(\boldsymbol{Y}, \boldsymbol{K}^{n-1}_r)$.

Note that $\eta$ is globally variational if and only if $\delta(\eta) = 0$. In the following we will be interested in the non trivial case $\delta(\eta) \neq 0$ whereby $\eta = \mathcal{E}_n(\lambda)$ can be solved only locally, i.e. for any countable good covering of $\boldsymbol{Y}$ there exists a local Lagrangian $\lambda_i$ over each subset $U_i \subset \boldsymbol{Y}$ such that $\eta_i = \mathcal{E}_n(\lambda_i)$.

A *local variational problem* is a system of local sections $\lambda_i$ of $(\mathcal{V}^n_r)_{U_i}$ such that $\mathcal{E}_n((\lambda_i - \lambda_j)|_{U_i \cap U_j}) = 0$. Note that $\mathfrak{d}\lambda = 0$ implies $\mathfrak{d}\eta_\lambda = 0$, while $\mathfrak{d}\eta_\lambda = 0$ only implies $\eta_{\mathfrak{d}\lambda} = 0$ i.e. $\mathfrak{d}\lambda \in C^1(\mathfrak{U}, \boldsymbol{K}^n_r)$ in Čech cohomology [5]. We call $(\{\boldsymbol{U}_i\}_{i\in\mathbb{Z}}, \lambda_i)$ a *presentation* of the local variational problem. Two local variational problems of degree $p$ are *equivalent* if and only if they give rise to the same variational class of forms as the image of the corresponding morphism $\mathcal{E}_p$ in the variational sequence. This means that the coboundary is variationally trivial.

The concept of a *variational Lie derivative* operator $\mathcal{L}_{j_r\Xi}$ which is a local differential operator enables us to define symmetries of classes of forms of any degree in the variational sequence and the corresponding conservation theorems [19]. We notice that the variational Lie derivative acts on cohomology classes: closed variational forms defining nontrivial cohomology classes are trasformed in variational forms with trivial cohomology classes [29], [30]. Note, however, that an infinitesimal symmetry of a local presentation is not necessarily a symmetry of another local presentation [16].

In particular, if we have a 0-cocycle of currents $\nu_i$ ($\mathfrak{d}\nu_i \neq 0$) such that $\mu = d_H\nu_i$ and $\mathfrak{d}\mu_\nu = 0$, then by using the representation of the Lie derivative of classes of variational forms of degree $p \leq n-1$ given in [19], we have $\mu_{\mathcal{L}_\Xi \nu_i} = d_H(\Xi_H \rfloor d_H\nu_i + \Xi_V \rfloor d_V\nu_i)$; Since we also have

$$\mathcal{L}_\Xi \mu_\nu = \mu_{\mathcal{L}_\Xi \nu_i} = d_H(\Xi_H \rfloor \mu_\nu + \Xi_V \rfloor p_{d_V\mu_\nu})\,,$$

from the definition of an equivalent variational problem, we can state that the local problem defined by $\mathcal{L}_\Xi\nu_i$ is variationally equivalent to the global problem defined by $\Xi_H \rfloor \mu_\nu + \Xi_V \rfloor p_{d_V\mu_\nu}$.

Moreover, if we have a 0-cocycle of Lagrangians (case $p = n+1$) or of variational forms of higher degree (in case $p = n+2$ we have a 0-cocycle of dynamical forms) $\lambda_i$ ($\mathfrak{d}\lambda_i \neq 0$) such that $\eta = \mathcal{E}_p(\lambda_i)$, then by linearity $\eta_{\mathcal{L}_\Xi\lambda_i} = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda)$; again, as a consequence of the fact that $\eta_{\mathcal{L}_\Xi\lambda_i} = \mathcal{E}_n(\Xi_V \rfloor \eta_\lambda)$, we have that the local problem defined by the local presentation $\mathcal{L}_\Xi\lambda_i$ is variationally equivalent to the global problem defined by $\Xi_V \rfloor \eta_\lambda$.

Resorting to the naturality of the variational Lie derivative we stated the following important result for the calculus of variations [29], [30], [18].

**Lemma 1.** *Let $\mu \in \mathcal{V}^p_r$, with $p \leq n$, be a locally variationally trivial p-form, i.e. such that $\mathcal{E}_p(\mu) = 0$ and let $\delta_p(\mu_\nu) \neq 0$. We have $\delta_p(\mathcal{L}_\Xi \mu_\nu) = 0$. Analogously, let $\eta \in \mathcal{V}^p_r$, with $p \geq n+1$, be a locally variational p-form, i.e. such that $\mathcal{E}_p(\eta) = 0$ and let $\delta_p(\eta_\lambda) \neq 0$. We have $\delta_p(\mathcal{L}_\Xi \eta_\lambda) = 0$.*

Geometric definitions of conserved quantities in field theories have been proposed within formulations based on symmetries of Euler-Lagrange operator rather than of the Lagrangian, see e.g. [31], [15], and strictly related with such an approach are also papers proposing the concept of *relative conservation laws*; see e.g. [12]. Accordingly, let us now consider the case of invariance of field equations, i.e. the case in which we will assume $\Xi$ to be a generalized symmetry, i.e. a symmetry of a class of $(n+1)$-forms $\eta$ in the variational sequence.

Let then $\eta_\lambda$ be the global Euler-Lagrange morphism of a local variational problem. It is a well known fact that $\Xi$ being a generalized symmetry implies that $\mathcal{E}_n(\Xi_V \rfloor \eta) = 0$, thus locally $\Xi_V \rfloor \eta = d_H \nu_i$, then there exists a 0-cocycle $\nu_i$, defined by $\mu_\nu = \Xi_V \rfloor \eta_\lambda \equiv d_H \nu_i$. Notice that $\mathfrak{d} \Xi_V \rfloor \eta_\lambda = 0$, but in general $\delta_n(\Xi_V \rfloor \eta_\lambda) \neq 0$ [16]. Along critical sections this implies the conservation law $d_H \nu_i = 0$ [1].

Noether's Theorem II implies that locally $\mathcal{L}_\Xi \lambda_i = d_H \beta_i$, thus we can write $\Xi_V \rfloor \eta_\lambda + d_H(\epsilon_i - \beta_i) = 0$, where $\epsilon_i$ is the usual *canonical* Noether current; the current $\epsilon_i - \beta_i$ is a *local* object and it is conserved along the solutions of Euler--Lagrange equations (critical sections). We stress that when $\Xi$ is only a symmetry of a dynamical form and not a symmetry of the Lagrangian, the current $\nu_i + \epsilon_i$ *is not a conserved current* and it is such that $d_H(\nu_i + \epsilon_i)$ is locally equal to $d_H \beta_i$; see also [31]. We shall call $(\nu_i + \epsilon_i)$ a *strong Noether current*. Notice that if $\Xi$ would be also a symmetry of the cochain of Lagrangians a strong Noether current would turn out to be a *conserved current* along any sections, not only along critical sections. Thus in this specific case we get the following.

**Corollary 1.** *Divergence expressions of the local problem defined by $\mathcal{L}_\Xi \nu_i$ coincide with divergence expressions for the global current $\Xi_H \rfloor \Xi_V \rfloor \eta_\lambda + \Xi_V \rfloor p_{d_V(\Xi_V \rfloor \eta_\lambda)}$.*

## 3 Currents variationally associated with locally variational field equations

We shall study variations of conserved currents in a quite general setting by determining the condition for the variation of a system of local strong Noether current to be equivalent to a system of *global conserved currents*. We now introduce the concept of conserved current variationally associated with locally variational invariant field equations and show that the invariance of the variation of the corresponding local presentation is a sufficient condition for the current beeing variationally equivalent to a global one.

**Definition 1.** We say a conserved current for an invariant field equation to be *variationally associated* if the symmetry of the field equation is also a symmetry for the variation of the local problem generating such a field equation.

[1] In this particular case $\nu_i$ is more precisely fixed, since $d_H \nu_i = \Xi_V \rfloor \eta$.

In other words, if $\lambda_i$ is a local presentation we look for currents associated to a variation vector field $\Xi$ satisfying $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$.

Suppose that, on the intersection of any two open sets, $\mathfrak{d}\lambda_i = d_H \gamma_{ij}$. By linearity, we have $\mathcal{L}_\Xi \mathfrak{d} \mathcal{L}_\Xi \lambda_i = \mathfrak{d} \mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i$; thus the condition $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$ implies $\mathcal{L}_\Xi \mathcal{L}_\Xi \mathfrak{d}\lambda_i = 0$. By Noether's Theorem II we must have $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = d_H \zeta_{ij}$, where $\zeta_{ij}$ is the sum of the Noether current associated with $\mathfrak{d}\lambda_i$ and a form locally given as $\mathfrak{d}\nu_i + d_H \rho_{ij}$. On the other hand $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = \mathfrak{d} d_H \epsilon_i$, where $\epsilon_i$ is the Noether current associated with $\lambda_i$. Of course, we have $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = \mathcal{L}_\Xi d_H \gamma_{ij}$ and again by Noether's Theorem II $\mathcal{L}_\Xi \lambda_i = d_H \beta_i$, hence by linearity we get, locally, $\mathfrak{d} d_H \beta_i = \mathcal{L}_\Xi d_H \gamma_{ij}$, where $\beta_i = \nu_i + \epsilon_i + d_H \omega_i$.

More precisely, we can immediately see that the condition $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = 0$ implies only $\mathfrak{d} d_H \nu_i = 0$, i.e. $d_H \nu_i$ is global [2]. In order to get $\mathfrak{d} d_H \beta_i = 0$, i.e. the divergence of the strong Noether current, $d_H \beta_i$, to be *global* we must require a stronger condition, which is $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = 0$. This condition, by linearity, means that the Lie derivative must drag the local problems in such a way that they coincide on the intersections of two open sets [3]. Under this condition we have the conservation law $d_H \mathcal{L}_\Xi (\nu_i + \epsilon_i) = 0$, where $\mathcal{L}_\Xi (\nu_i + \epsilon_i)$, the *variation of the strong Noether currents*, is a *local representative* of a global conserved current.

In fact, $\Xi$ being a *generalized symmetry*, we have $\mathcal{L}_\Xi \mathcal{L}_\Xi \lambda_i = d_H \mathcal{L}_\Xi (\nu_i + \epsilon_i)$. If the second variational derivative is vanishing, then we have the conservation law $d_H \mathcal{L}_\Xi (\nu_i + \epsilon_i) = 0$, where $\mathcal{L}_\Xi (\nu_i + \epsilon_i)$ is a local representative of the current given by

$$\Xi_H \rfloor \mu_{\nu+\epsilon} + \Xi_V \rfloor p_{d_V \mu_{\nu+\epsilon}} \equiv \Xi_H \rfloor d_H(\nu_i + \epsilon_i) + \Xi_V \rfloor p_{d_V (d_H(\nu_i+\epsilon_i))} \, .$$

This current is global if $d_H \mathfrak{d}(\nu_i + \epsilon_i) = 0$; a sufficient condition for this to hold true is $\mathcal{L}_\Xi \mathfrak{d}\lambda_i = 0$.

The conserved current associated with a generalized symmetry, assumed to be also a symmetry of the variational derivative of the corresponding local inverse problem, is variationally equivalent to the variation of the strong Noether currents for the corresponding local system of Lagrangians. Moreover, if the variational Lie derivative of the local system of Lagrangians is a global object, such a variation is variationally equivalent to a global conserved current [18]. In this paper, we make explicit the latter result in the case of Chern-Simons equations.

### 3.1 Chern-Simons gauge theory

It is well known that Chern-Simons field theories [8], [9] constitute a model for classical and quantum gravitational fields and that gravity can be considered as a gauge theory: in all odd dimensions and particularly in dimension three, where the field equations reproduce exactly the Einstein field equations, a Chern-Simons Lagrangian can be considered (instead of the Hilbert–Einstein Lagrangian) in which the gauge potential is a linear combination of a frame and a spin connection; in particular, $2+1$ gravity with a negative cosmological constant can be formulated as

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[2] Notice that the symmetry $\Xi$, besides beeing a generalized symmetry, is *also a symmetry of the variational problem* $\mathcal{L}_\Xi \lambda_i$.

[3] This is also equivalent to $\mathfrak{d}(\nu_i + \epsilon_i) = d_H(\psi_{ij} - \rho_{ij})$, i.e. the coboundary of the strong Noether currents is locally exact; for details see [18].

a Chern-Simons theory (see, e.g. [32], as well as [7] for higher dimensional Chern--Simons gravity). Developing a 3-dimensional Chern-Simons theory as a possible and simpler model to analyse $(2+1)$-dimensional gravity brought in particular results concerned with thermodynamics of higher dimensional black holes [3], which in turn produced a renewed interest in Chern-Simons theories and, consequently, in the problem of gauge symmetries and gauge charges for Chern-Simons theories.

Let us then take in consideration the 3-dimensional Chern-Simons Lagrangian

$$\lambda_{CS}(A) = \frac{\kappa}{4\pi}\epsilon^{\mu\nu\rho}\mathrm{Tr}(A_\mu d_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho)ds\,,$$

where $ds$ is a 3-dimensional volume density, $\kappa = \frac{l}{4G}$ (being $G$ the Newton's constant and setting $c=1$), while $A_\mu = A^i_\mu\, J_i$ are the coefficients of the connection 1-form $A = A_\mu\, dx^\mu$ taking their values in any Lie algebra $\mathfrak{g}$ with generators $J_i$. By fixing $\mathfrak{g} = sl(2,\mathbb{R})$ and choosing the generators $J_k = \frac{1}{2}\sigma_k$, whith $\sigma_k$ Pauli matrices, we have $[J_i, J_j] = \eta^{lk}\epsilon_{kij}\, J_l$ and $\mathrm{Tr}(J_i J_j) = \frac{1}{2}\eta_{ij}$, with $\eta = \mathrm{diag}(-1,1,1)$ and $\epsilon_{012} = 1$. Hence, we can explicitly write $\lambda_{CS}(A) = \frac{\kappa}{16\pi}\epsilon^{\mu\nu\rho}(\eta_{ij}F^i_{\mu\nu}A^j_\rho - \frac{1}{3}\epsilon_{ijk}A^i_\mu A^j_\nu A^k_\rho)ds$, where $F^i_{\mu\nu} = d_\mu A^i_\nu - d_\nu A^i_\mu + \epsilon^i_{jk}A^j_\mu A^k_\nu$ is the so-called field strength.

Note that, if we consider two independent $sl(2,I\!R)$ connections $A$ and $\bar{A}$, on the intersection of two open sets, the Lagrangian $\lambda_{CS}(A,\bar{A}) := \mathfrak{d}\lambda_{CS}(A) = \lambda_{CS}(A) - \lambda_{CS}(\bar{A})$ is a divergence. We recall for the sake of completeness that it is possible to perform a change of fiber coordinates, i.e. to define two new dynamical fields, $e^i$ and $\omega^i$, setting $A^i = \omega^i + \frac{e^i}{l}$ and $\bar{A}^i = \omega^i - \frac{e^i}{l}$, with $l$ a constant, and in terms of these new variables we can write $\lambda_{CS}(A(\omega,e),\bar{A}(\omega,e)) = \frac{\kappa}{4\pi l}\sqrt{g}(g^{\mu\nu}R_{\mu\nu} + \frac{2}{l^2}) + d_\mu\{\frac{k}{4\pi l}\eta_{ij}\epsilon^{\mu\nu\rho}e^i_\nu\omega^j_\rho\}$, with $g_{\mu\nu} = \eta_{ij}\, e^i_\mu\, e^j_\nu$ and $R_{\mu\nu} = R^i_{j\rho\nu}\, e^j_\mu\, e^\rho_i$ the Ricci tensor of the metric $g$. In this expression, the non invariant term is under the total derivative and subtracting such a term from $\lambda_{CS}(A,\bar{A})$ one can get a global covariant Chern--Simons Lagrangian $\lambda_{CScov}(A(\omega,e),\bar{A}(\omega,e)) = \frac{\kappa}{4\pi\, l}\sqrt{g}(g^{\mu\nu}R_{\mu\nu} + \frac{2}{l^2})ds$, which can be recasted as $\lambda_{CScov}(A,\bar{A}) = \frac{k}{8\pi}\epsilon^{\mu\nu\rho}(\eta_{ij}\bar{F}^i_{\mu\nu}\, B^j_\rho + \eta_{ij}\bar{\nabla}_\mu B^i_\nu B^j_\rho + \frac{1}{3}\epsilon_{ijk}B^i_\mu B^j_\nu\, B^k_\rho)ds$, where $\bar{\nabla}_\mu$ is the covariant derivative with respect to the connection $\bar{A}$ and we set $B^i_\mu = A^i_\mu - \bar{A}^i_\mu$. As just explained, the procedure of writing the gauge potential $A$ as a linear combination of the frame $e$ and the spin connection $\omega$ enables one to split the non invariant divergence $\mathfrak{d}\lambda_{CS}(A)$ into a global piece plus a non covariant divergence and to generate a new Lagrangian. The latter is the difference of two local Lagrangians and it is covariant up to a divergence. It is well known that such a procedure can be applied to each Chern-Simons Lagrangian in dimension three, independently on the relevant gauge group of the theory, and it has been exploited in order to find Noether covariant charges (see e.g. [1], [4] and references therein). It should be noticed that such charges are associated with invariance properties of the thus obtained and above mentioned new Lagrangian; the interpretation of the relation with the conserved quantities associated with the original Euler-Lagrange equations for the Chern-Simons Lagrangian must be deeper investigated, see the discussion in [1].

The concept of a conserved current variationally associated with locally variational invariant Chern-Simons field equations provides a global conserved current directly related with the Euler-Lagrange equations. Chern-Simons equations of motion are manifestly covariant with respect to spacetime diffeomorphism as well as with respect to gauge transformations, the Chern-Simons Lagrangian instead is not gauge invariant. Let $\nu_i + \epsilon_i$ be a 0-cocycle of strong Noether currents for the Chern-Simons Lagrangian and let $\beta_i = \nu_i + \epsilon_i + d_H\omega_i$ as above. We have the following important result.

**Proposition 1.** *Let $\Xi$ be a symmetry of the Chern-Simons dynamical form, the global conserved current*

$$\Xi_H \rfloor \mathcal{L}_\Xi \lambda_{CS_i} + \Xi_V \rfloor p_{d_V \mathcal{L}_\Xi \lambda_{CS_i}} \,,$$

*is associated with the invariance of the Chern-Simons equations and it is* variationally equivalent *to the variation of the strong Noether currents $\nu_i + \epsilon_i$.*

*Proof.* Let $\mathcal{L}_\Xi \lambda_{CS_i}$ be a local Lagrangian presentation of the inverse problem associated with the Chern-Simons dynamical form, we have $\mathcal{L}_\Xi \lambda_{CS_i} = d_H \beta_i$ and it is easy to verify that for a Chern-Simons Lagrangian the relation $\mathfrak{d}\mathcal{L}_\Xi \lambda_{CS_i} = \mathcal{L}_\Xi d_H \gamma_{ij}$ holds. Comparing these equations, we have that $\mathfrak{d} d_H \beta_i = \mathcal{L}_\Xi d_H \gamma_{ij}$, thus, in particular, $\mathfrak{d} d_H(\nu_i + \epsilon_i) = \mathcal{L}_\Xi d_H \gamma_{ij}$. As stated in the section above it is clear that, if $\mathcal{L}_\Xi d_H \gamma_{ij} = 0$, then $d_H(\nu_i + \epsilon_i)$ is global. Generators of such a global current lie in the kernel of the second variational derivative and are symmetries of the variationally trivial Lagrangian $d_H \gamma_{ij}$. □

**Example 1.** From to the relation $\mathcal{L}_\Xi A^i_\mu = \nabla_\mu \Xi^i_v$, where $\Xi^i_v$ is the component of the vertical part of $\Xi$ with respect to a principal connection $\omega$ on the bundle of frames, we have $\mathcal{L}_\Xi \lambda_{CS}(A) = d_\mu(\frac{\kappa}{8\pi}\epsilon^{\mu\nu\rho}\eta_{ij}A^i_\nu d_\rho \Xi^j_v)$, therefore, writing $\Xi^i_v = \Xi^i_V + (A^i_\mu - \omega^i_\mu)\Xi^\mu$, for each $\Xi_V \in \mathfrak{k}$ the local expression of a global current associated with the gauge invariance of the Chern-Simons dynamical form is given by

$$\big[\Xi^\gamma d_\mu(\frac{\kappa}{8\pi}\epsilon^{\mu\nu\rho}\eta_{ij}A^i_\nu d_\rho(\Xi^j_V + (A^j_\tau - \omega^j_\tau)\Xi^\tau)) + \\ (\Xi^k - A^k_\lambda \Xi^\lambda) d_\mu\big(\frac{\kappa}{8\pi}\epsilon^{\mu\gamma\rho}\eta_{kj} d_\rho(\Xi^j_V + (A^j_\zeta - \omega^j_\zeta)\Xi^\zeta)\big)\big] ds_\gamma \,.$$

As a final remark, we mention that local conserved currents can be derived by using Lepagian equivalent of local systems of Lagrangians [6]. Therefore, a study of inverse problems within a sequence of Lepage equivalent forms following [21], [22], [24] is of great interest and will be the object of future investigations.

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# Symmetries of a dynamical system represented by singular Lagrangians

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**Abstract.** Dynamical properties of singular Lagrangian systems differ from those of classical Lagrangians of the form $L = T - V$. Even less is known about symmetries and conservation laws of such Lagrangians and of their corresponding actions. In this article we study symmetries and conservation laws of a concrete singular Lagrangian system interesting in physics. We solve the problem of determining all point symmetries of the Lagrangian and of its Euler-Lagrange form, i.e. of the action.

It is known that every point symmetry of a Lagrangian is a point symmetry of its Euler-Lagrange form, and this of course happens also in our case. We are also interested in the converse statement, namely if to every point symmetry $\xi$ of the Euler-Lagrange form $E$ there exists a Lagrangian $\lambda$ for $E$ such that $\xi$ is a point symmetry of $\lambda$. In the case studied the answer is affirmative, moreover we have found that the corresponding Lagrangians are all of order one.

## 1 Introduction

The aim of this paper is to investigate symmetry properties of a singular (degenerate) Lagrangian system, when the Euler-Lagrange equations cannot be put a normal form and rather form a system of implicit second order ordinary differential equations. While dynamical and symmetry properties of classical Lagrangians of the form $L = T - V$ (kinetic minus potential energy) have been in the focus of analytical mechanics and the calculus of variations from the very beginning of the art, there is still not much known about singular Lagrangian systems.

The problem of investigating singular Lagrangian systems goes back to the pioneer work of Dirac [4]. Nowadays there are two approaches to this topic, the first one coming from a generalization of the symplectic geometry [1], [2], [3], [6],

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[7], [8], [9], [10] to mention only a few, and the second (more recent) one based on the Lepagian theory of Lagrangian systems in jet bundles [12], [13], [14].

The Dirac algorithm is rather heuristic and it is known that applied to a concrete Lagrangian system different authors sometimes obtain confusing or even contradictory results. Also the application of the symplectic constraint algorithm in principle cannot provide complete information on the dynamics of the system and its symmetries – due to the fact that the image of the constrained dynamics in the symplectic manifold is not in one-to-one correspondence with the Lagrangian dynamics.

In this paper we apply the second way to singular systems, based on a model of a Lagrangian dynamics and the corresponding Hamiltonian dynamics defined in the same jet bundle [13]. We study a concrete singular Lagrangian interesting in physics, namely

$$L = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3. \tag{1}$$

Hamilton equations and symmetries of this Lagrangian were studied e.g. in [15], [16], [17], with incomplete results. With help of a "direct method" by Krupková we described the dynamics of this Lagrangian system completely in [3]. Here we continue in investigating the symmetry properties. Contrary to the preceding authors looking for symmetries of this Lagrangian [5] our approach to the problem is to find a complete solution of the symmetry conditions which take the form of a system of partial differential equations for the components of the invariance vector field. In this way we get all point symmetries of the Lagrangian (1) by solving the Noether equation, and all its generalized symmetries (that is point symmetries of the corresponding action) by solving the Noether–Bessel-Hagen equation.

It is known that every point symmetry of a Lagrangian is a point symmetry of its Euler-Lagrange form, and this of course happens also in our case. We are also interested in the converse statement, namely if to every point symmetry $\xi$ of the Euler-Lagrange form $E$ there exists a Lagrangian $\lambda$ for $E$ such that $\xi$ is a point symmetry of $\lambda$. In the case studied the answer is affirmative, moreover we have found that the corresponding Lagrangians are *all of order one*.

In this context it is worth mention that in the well-known case of the kinetic energy Lagrangian, surprisingly, the situation is not so simple. The invariance group of the action – the Galilei group – also has the property that to every symmetry there is an invariant Lagrangian, however, not in all the cases the Lagrangian is of the first order. Namely, for Galilei transformations, one has only a *second order Lagrangian providing the same equations of motion as the kinetic energy* [14].

## 2 Symmetries and conservation laws

We consider a fibred manifold $\pi\colon Y \to X$, $\dim X = 1$, $\dim Y = m + 1$ and its jet prolongations $\pi_1\colon J^1Y \to X$, $\pi_2\colon J^2Y \to X$. In what follows, we denote by $\partial_\xi$ the Lie derivative with respect to a vector field $\xi$.

**Definition 1.** Let $\xi$ be a $\pi$-projectable vector field on $Y$. Let $\lambda$ be Lagrangian on $J^1Y$. A vector field $\xi$ on $Y$ is called point symmetry of $\lambda$, if

$$\partial_{J^1\xi}\lambda = 0. \tag{2}$$

This equation is called Noether equation.

In fibred coordinates, the Noether equation reads:

$$L\frac{\mathrm{d}\xi^0}{\mathrm{d}t} + \frac{\partial L}{\partial t}\xi^0 + \frac{\partial L}{\partial q^\sigma}\xi^\sigma + \frac{\partial L}{\partial \dot{q}^\sigma}\widetilde{\xi}^\sigma = 0, \tag{3}$$

where

$$\widetilde{\xi^\sigma} = \frac{\mathrm{d}\xi^\sigma}{\mathrm{d}t} - \dot{q}^\sigma\frac{\mathrm{d}\xi^0}{\mathrm{d}t}.$$

In this paper we shall use the Noether equation to solve the following problem:

*Given a Lagrangian, find all its point symmetries and the corresponding first integrals.* In this case one has to solve the Noether equation with respect to the vector field. First integrals are then found on the basis of the Noether Theorem:

**Theorem 1.** *Let $\lambda$ be a Lagrangian defined on an open subset $W \subset J^1Y$, let $\theta_\lambda$ be its Cartan form. Let a $\pi$-projectable vector field $\xi$ on $Y$ be a point symmetry of the Lagrangian $\lambda$. Let $\gamma$ be an extremal of $\lambda$ defined on $\pi_1(W) \subset X$. Then*

$$i_{J^1\xi}\theta_\lambda \circ J^1\gamma = const.$$

**Definition 2.** A vector field $\xi$ is called point symmetry of the Euler-Lagrange form $E_\lambda$, if

$$\partial_{J^2\xi}E_\lambda = 0. \tag{4}$$

This equation is called the Noether–Bessel-Hagen equation.

We shall use the Noether–Bessel-Hagen equation to solve the following problem:

*Find (all) infinitesimal transformations of $Y$ which leave given Euler-Lagrange expressions (a given Euler-Lagrange form) invariant.* In this case (4) is considered as a system of PDE's for symmetries $\xi$ of the given Euler-Lagrange form. (Solving this problem one gets all point symmetries of the corresponding action).

## 2.1 Symmetries of a singular Lagrangian

In what follows we shall be interested in symmetry properties of the following Lagrangian

$$L = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3.$$

With help of the Noether equation we shall find point symmetries of the Lagrangian (1).

We get:

$$0 = (\dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3)\dot{\xi_0} + q^3\xi^1 - \dot{q}^3\xi^2 + q^1\xi^3 + \dot{q}^3\widetilde{\xi^1} + (\dot{q}^1 - q^2)\widetilde{\xi^3}$$

and then:

$$0 = (\dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3)\frac{\mathrm{d}\xi^0}{\mathrm{d}t} + q^3\xi^1 - \dot{q}^3\xi^2 + q^1\xi^3 + \dot{q}^3\cdot\left(\frac{\mathrm{d}\xi^1}{\mathrm{d}t} - \dot{q}^1\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right)$$
$$+ (\dot{q}^1 - q^2)\cdot\left(\frac{\mathrm{d}\xi^3}{\mathrm{d}t} - \dot{q}^3\frac{\mathrm{d}\xi^0}{\mathrm{d}t}\right)$$

where $\xi^0 = \xi^0(t)$ a $\xi^\sigma = \xi^\sigma(t, \xi^1, \xi^2, \xi^3)$.
More explicitly:

$$\begin{aligned}
0 = {} & \dot{q}^1 \dot{q}^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t} - q^2 \dot{q}^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t} + q^1 q^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t} + q^3 \xi^1 - \dot{q}^3 \xi^2 + q^1 \xi^3 + \dot{q}^3 \frac{\partial \xi^1}{\partial t} + \dot{q}^1 \dot{q}^3 \frac{\partial \xi^1}{\partial q^1} + \\
& + \dot{q}^2 \dot{q}^3 \frac{\partial \xi^1}{\partial q^2} + (\dot{q}^3)^2 \frac{\partial \xi^1}{\partial q^3} - \dot{q}^1 \dot{q}^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t} + \dot{q}^1 \frac{\partial \xi^3}{\partial t} + (\dot{q}^1)^2 \frac{\partial \xi^3}{\partial q^1} + \dot{q}^1 \dot{q}^2 \frac{\partial \xi^3}{\partial q^2} + \dot{q}^1 \dot{q}^3 \frac{\partial \xi^3}{\partial q^3} \\
& - \dot{q}^1 \dot{q}^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t} - q^2 \frac{\partial \xi^3}{\partial t} - q^2 \dot{q}^1 \frac{\partial \xi^3}{\partial q^1} - q^2 \dot{q}^2 \frac{\partial \xi^3}{\partial q^2} - q^2 \dot{q}^3 \frac{\partial \xi^3}{\partial q^3} + q^2 \dot{q}^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t}
\end{aligned}$$

From this equation we obtain a system of equations for components of $\xi$ as follows:

$$\begin{aligned}
\frac{\partial \xi^1}{\partial q^1} + \frac{\partial \xi^3}{\partial q^3} - \frac{\mathrm{d}\xi^0}{\mathrm{d}t} &= 0 \\
\frac{\partial \xi^1}{\partial t} - q^2 \frac{\partial \xi^3}{\partial q^3} - \xi^2 &= 0 \\
q^1 q^3 \frac{\mathrm{d}\xi^0}{\mathrm{d}t} + q^3 \xi^1 + q^1 \xi^3 &= 0
\end{aligned}$$

where $\xi^0 = \xi^0(t),\ \xi^1 = \xi^1(t, q^1),\ \xi^2 = \xi^2(t, q^1, q^2, q^3),\ \xi^3 = \xi^3(q^3)$.

Solving these equations we get:

**Theorem 2.** *Point symmetries of Lagrangian* (1) *are generated by two vector fields:*

$$\frac{\partial}{\partial t}; \qquad q^1 \frac{\partial}{\partial q^1} + q^2 \frac{\partial}{\partial q^2} - q^3 \frac{\partial}{\partial q^3}.$$

*To the time translation generated by $\frac{\partial}{\partial t}$ there corresponds the first integral*

$$F_1 = (q^1 q^3 - \dot{q}^1 \dot{q}^3) = -H,$$

*where $H$ is the Hamiltonian. To $q^1 \frac{\partial}{\partial q^1} + q^2 \frac{\partial}{\partial q^2} - q^3 \frac{\partial}{\partial q^3}$ there corresponds the conserved function*

$$F_2 = \dot{q}^3 q^1 - (\dot{q}^1 + q^2) \cdot q^3 = p_1 \cdot q^1 - p_3 \cdot q^3,$$

*where*

$$p_1 = \dot{q}^3, \quad p_2 = 0, \quad p_3 = \dot{q}^1 - q^2$$

*are momenta.*

### 2.2 Symmetries of the Euler-Lagrange form of $L$

Now let us determine point symmetries of the action of $L$, that is solutions of the Noether–Bessel-Hagen equation. Since

$$E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^\sigma},$$

where $\sigma = 1, 2, 3$ we get from $L$ (1) the following Euler-Lagrange expressions:

$$E_1 = q^3 - \ddot{q}^3 \quad E_2 = -\dot{q}^3 \quad E_3 = q^1 + \dot{q}^2 - \ddot{q}^1. \tag{5}$$

Substituting the second jet prolongation of $\xi$:

$$J^2\xi = \xi^0 \frac{\partial}{\partial t} + \xi^\sigma \frac{\partial}{\partial q^\sigma} + \widetilde{\xi^i} \frac{\partial}{\partial \dot{q}^\sigma} + \widetilde{\widetilde{\xi^\sigma}} \frac{\partial}{\partial \ddot{q}^\sigma},$$

where

$$\widetilde{\xi^\sigma} = \dot{\xi^\sigma} - \dot{q}^\sigma \dot{\xi}^0, \quad \widetilde{\widetilde{\xi^\sigma}} = \ddot{\xi^\sigma} - 2\ddot{q}^\sigma \dot{\xi}^0 - \dot{q}^\sigma \ddot{\xi}^0,$$

we obtain the Noether–Bessel-Hagen equation in the following form:

$$E_\nu \frac{\partial \xi^\nu}{\partial q^\sigma} + E_\sigma \dot{\xi}^0 + \frac{\partial E_\sigma}{\partial t}\xi^0 + \frac{\partial E_\sigma}{\partial q^\nu}\xi^\nu + \frac{\partial E_\sigma}{\partial \dot{q}^\nu}(\dot{\xi}^\nu - \dot{q}^\nu \dot{\xi}^0) + \frac{\partial E_\sigma}{\partial \ddot{q}^\nu}(\ddot{\xi}^\nu - 2\ddot{q}^\nu \dot{\xi}^0 - \dot{q}^\nu \ddot{\xi}^0) = 0.$$

Substituting the Euler-Lagrange expressions (5), we get:

$$\begin{aligned}
&(q^3 - \ddot{q}^3)\frac{\partial \xi^1}{\partial q^1} - \dot{q}^3 \frac{\partial \xi^2}{\partial q^1} + (q^1 - \ddot{q}^1 + \dot{q}^2)\frac{\partial \xi^3}{\partial q^1} + (q^3 - \ddot{q}^3)\frac{\partial \xi^0}{\partial t} + \\
&\qquad + \xi^3 - \left( \frac{\mathrm{d}^2 \xi^3}{\mathrm{d}t^2} - 2\ddot{q}^3 \frac{\partial \xi^0}{\partial t} - \dot{q}^3 \frac{\partial^2 \xi^0}{\partial t^2} \right) = 0 \\
&(q^3 - \ddot{q}^3)\frac{\partial \xi^1}{\partial q^2} - \dot{q}^3 \frac{\partial \xi^2}{\partial q^2} + (q^1 - \ddot{q}^1 + \dot{q}^2)\frac{\partial \xi^3}{\partial q^2} - \dot{q}^3 \frac{\partial \xi^0}{\partial t} - \left( \frac{\mathrm{d}\xi^3}{\mathrm{d}t} - \dot{q}^3 \frac{\partial \xi^0}{\partial t} \right) = 0 \\
&(q^3 - \ddot{q}^3)\frac{\partial \xi^3}{\partial q^1} - \dot{q}^3 \frac{\partial \xi^2}{\partial q^3} + (q^1 - \ddot{q}^1 + \dot{q}^2)\frac{\partial \xi^3}{\partial q^3} + (q^1 - \ddot{q}^1 + \dot{q}^2)\frac{\partial \xi^0}{\partial t} + \xi^1 + \\
&\qquad + \left( \frac{\mathrm{d}\xi^2}{\mathrm{d}t} - \dot{q}^2 \frac{\partial \xi^0}{\partial t} \right) - \left( \frac{d^2 \xi^1}{\mathrm{d}t^2} - 2\ddot{q}^1 \frac{\partial \xi^0}{\partial t} - \dot{q}^1 \frac{\partial^2 \xi^0}{\partial t^2} \right) = 0
\end{aligned}$$

where $\xi^1 = \xi^1(t, q^1, q^2, q^3)$, $\xi^2 = \xi^2(t, q^1, q^2, q^3)$, $\xi^3 = \xi^3(t, q^1, q^2, q^3)$. More explicitly,

$$\begin{aligned}
&q^3 \frac{\partial \xi^1}{\partial q^1} - \ddot{q}^3 \frac{\partial \xi^1}{\partial q^1} - \dot{q}^3 \frac{\partial \xi^2}{\partial q^1} + q^1 \frac{\partial \xi^3}{\partial q^1} - \ddot{q}^1 \frac{\partial \xi^3}{\partial q^1} + \dot{q}^2 \frac{\partial \xi^3}{\partial q^1} + q^3 \frac{\partial \xi^0}{\partial t} - \ddot{q}^3 \frac{\partial \xi^0}{\partial t} - \\
&- \frac{\partial^2 \xi^3}{\partial t^2} - 2\dot{q}^1 \frac{\partial^2 \xi^3}{\partial t \partial q^1} - 2\dot{q}^2 \frac{\partial^2 \xi^3}{\partial t \partial q^2} - 2\dot{q}^3 \frac{\partial^2 \xi^3}{\partial t \partial q^3} - \frac{\partial^2 \xi^3}{\partial (q^1)^2}(\dot{q}^1)^2 - \frac{\partial^2 \xi^3}{\partial (q^2)^2}(\dot{q}^2)^2 - \\
&- \frac{\partial^2 \xi^3}{\partial (q^3)^2}(\dot{q}^3)^2 - 2\frac{\partial^2 \xi^3}{\partial q^1 \partial q^2}\dot{q}^1 \dot{q}^2 - 2\frac{\partial^2 \xi^3}{\partial q^1 \partial q^3}\dot{q}^1 \dot{q}^3 - 2\frac{\partial^2 \xi^3}{\partial q^2 \partial q^3}\dot{q}^2 \dot{q}^3 + 2\ddot{q}^3 \frac{\partial \xi^0}{\partial t} + \\
&\qquad + \xi^3 + \dot{q}^3 \frac{\partial^2 \xi^0}{\partial t^2} - \frac{\partial \xi^3}{\partial q^1}\ddot{q}^1 - \frac{\partial \xi^3}{\partial q^2}\ddot{q}^2 - \frac{\partial \xi^3}{\partial q^3}\ddot{q}^3 = 0 \\
&q^3 \frac{\partial \xi^1}{\partial q^2} - \ddot{q}^3 \frac{\partial \xi^1}{\partial q^2} - \dot{q}^3 \frac{\partial \xi^2}{\partial q^2} + q^1 \frac{\partial \xi^3}{\partial q^2} - \ddot{q}^1 \frac{\partial \xi^3}{\partial q^2} + \dot{q}^2 \frac{\partial \xi^3}{\partial q^2} - \dot{q}^3 \frac{\partial \xi^0}{\partial t} - \frac{\partial \xi^3}{\partial t} - \\
&\qquad - \frac{\partial \xi^3}{\partial q^1}\dot{q}^1 - \frac{\partial \xi^3}{\partial q^2}\dot{q}^2 - \frac{\partial \xi^3}{\partial q^3}\dot{q}^3 + \dot{q}^3 \frac{\partial \xi^0}{\partial t} = 0
\end{aligned}$$

$$q^3\frac{\partial\xi^1}{\partial q^3}-\ddot q^3\frac{\partial\xi^1}{\partial q^3}-\dot q^3\frac{\partial\xi^2}{\partial q^3}+q^1\frac{\partial\xi^3}{\partial q^3}-\ddot q^1\frac{\partial\xi^3}{\partial q^3}+\dot q^2\frac{\partial\xi^3}{\partial q^3}+q^1\frac{\partial\xi^0}{\partial t}-\ddot q^1\frac{\partial\xi^0}{\partial t}+$$
$$+\dot q^2\frac{\partial\xi^0}{\partial t}+\frac{\partial\xi^2}{\partial t}+\frac{\partial\xi^2}{\partial q^1}\dot q^1+\frac{\partial\xi^2}{\partial q^2}\dot q^2+\frac{\partial\xi^2}{\partial q^3}\dot q^3-\dot q^2\frac{\partial\xi^0}{\partial t}-\frac{\partial^2\xi^1}{\partial t^2}-2\dot q^1\frac{\partial^2\xi^1}{\partial t\partial q^1}-$$
$$-2\dot q^2\frac{\partial^2\xi^1}{\partial t\partial q^2}+\xi^1-2\dot q^3\frac{\partial^2\xi^1}{\partial t\partial q^3}-\frac{\partial^2\xi^1}{\partial(q^1)^2}(\dot q^1)^2-\frac{\partial^2\xi^1}{\partial(q^2)^2}(\dot q^2)^2-$$
$$-\frac{\partial^2\xi^1}{\partial(q^3)^2}(\dot q^3)^2-2\frac{\partial^2\xi^1}{\partial q^1\partial q^2}\dot q^1\dot q^2-2\frac{\partial^2\xi^1}{\partial q^1\partial q^3}\dot q^1\dot q^3-2\frac{\partial^2\xi^1}{\partial q^2\partial q^3}\dot q^2\dot q^3+2\ddot q^1\frac{\partial\xi^0}{\partial t}+$$
$$+\dot q^1\frac{\partial^2\xi^0}{\partial t^2}-\frac{\partial\xi^1}{\partial q^1}\ddot q^1-\frac{\partial\xi^1}{\partial q^2}\ddot q^2-\frac{\partial\xi^1}{\partial q^3}\ddot q^3=0$$

From these equations we get conditions for components of $\xi$ as follows:

$$\begin{aligned}\xi^1&=k_1\cdot\mathrm{e}^t+k_2\cdot\mathrm{e}^{-t}-C_2q^1\\ \xi^2&=C_3-C_2\cdot q^2+b(q^3)\\ \xi^3&=C_2\cdot q^3\\ \xi^0&=C_1\end{aligned}$$

Hence, we obtained:

$$\xi=C_1\frac{\partial}{\partial t}+(k_1\cdot\mathrm{e}^t+k_2\cdot\mathrm{e}^{-t}-C_2q^1)\frac{\partial}{\partial q^1}+\big(C_3-C_2\cdot q^2+b(q^3)\big)\frac{\partial}{\partial q^2}+C_2\cdot q^3\frac{\partial}{\partial q^3}$$

where $C_1,C_2,C_3,k_1,k_2$ are constants and $b(q^3)$ is a function depending only on $q^3$. This result can be formulated as follows:

**Theorem 3.** *Point symmetries of the Euler-Lagrange form of Lagrangian L (4) are generated by the following vector fields:*

$$\frac{\partial}{\partial t};\quad q^1\frac{\partial}{\partial q^1}+q^2\frac{\partial}{\partial q^2}-q^3\frac{\partial}{\partial q^3};\quad \mathrm{e}^t\frac{\partial}{\partial q^1};\quad \mathrm{e}^{-t}\frac{\partial}{\partial q^1};\quad b(q^3)\frac{\partial}{\partial q^2},\tag{6}$$

*where $b(q^3)$ is an arbitrary function of the variable $q^3$.*
*The corresponding first integrals are*

$$F_1=-H$$

*for the transformation $\frac{\partial}{\partial t}$,*

$$F_2=p_1q^1-p_3q^3$$

*for the transformation $q^1\frac{\partial}{\partial q^1}+q^2\frac{\partial}{\partial q^2}-q^3\frac{\partial}{\partial q^3}$,*

$$F_3=p_1\cdot\mathrm{e}^t$$

*for the transformation $\mathrm{e}^t\frac{\partial}{\partial q^1}$,*

$$F_4=p_1\cdot\mathrm{e}^{-t}$$

*for the transformation $\mathrm{e}^{-t}\frac{\partial}{\partial q^1}$ and*

$$F_5=0$$

*for the transformation $b(q^3)\frac{\partial}{\partial q^2}$.*

Theorem 2 and 3 demonstrate the known fact that every symmetry of Lagrangian $\lambda$ is a symmetry of its Euler-Lagrange form (or the corresponding action). In the sequel we shall be interested in the converse problem, namely, if to every point symmetry of the action (that is of $E$) there exists a Lagrangian $\lambda$ for $E$ such that $\xi$ is a point symmetry of $\lambda$.

To this end we shall represent our Lagrangian system in the form of the equivalence class of Lagrangians for $E$; the equivalence relation is given by

$$L \sim L' \qquad \text{iff} \qquad L' = L + \frac{\mathrm{d}f}{\mathrm{d}t},$$

where $f$ is a function.

This means that

$$L' = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3 + \frac{\mathrm{d}f}{\mathrm{d}t},$$

where $f = f(t, q^1, q^2, q^3)$.

Substituting $L'$ to the Noether equation we shall try to determine $f$ for every symmetry (6)

(a) If $\xi = \mathrm{e}^t \frac{\partial}{\partial q^1} \Rightarrow \xi^1 = \mathrm{e}^t$
we get

$$q^3\xi^1 + \frac{\partial}{\partial q^1}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\xi^1 + \dot{q}^3\left(\dot{\xi}^1 - \dot{q}^1\dot{\xi}^0\right) + \frac{\partial}{\partial \dot{q}^1}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\left(\dot{\xi}^1 - \dot{q}^1\dot{\xi}^0\right) = 0.$$

From this equation we get equations:

$$\begin{aligned}
\frac{\partial^2 f}{(\partial q^1)^2} &= 0 \\
\frac{\partial^2 f}{\partial q^1 \partial q^2} &= 0 \\
\frac{\partial^2 f}{\partial q^1 \partial q^3} &= -1 \\
\frac{\partial^2 f}{\partial q^1 \partial t} &= -q^3 - \frac{\partial f}{\partial q^1}
\end{aligned}$$

having the solution

$$f = a(t, q^2, q^3) - q^1q^3 + C_1 e^{-t} q^1,$$

where $a(t, q^2, q^3)$ is arbitrary function of variables $t$, $q^2$ and $q^3$. $C_1$ is a constant.

(b) If $\xi = \mathrm{e}^{-t} \frac{\partial}{\partial q^1} \Rightarrow \xi^1 = \mathrm{e}^{-t}$
we get

$$q^3\xi^1 + \frac{\partial}{\partial q^1}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\xi^1 + \dot{q}^3\left(\dot{\xi}^1 - \dot{q}^1\dot{\xi}^0\right) + \frac{\partial}{\partial \dot{q}^1}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)\left(\dot{\xi}^1 - \dot{q}^1\dot{\xi}^0\right) = 0.$$

From this equation we get equations:

$$\begin{aligned}\frac{\partial^2 f}{(\partial q^1)^2} &= 0\\ \frac{\partial^2 f}{\partial q^1 \partial q^2} &= 0\\ \frac{\partial^2 f}{\partial q^1 \partial q^3} &= 1\\ \frac{\partial^2 f}{\partial q^1 \partial t} &= -q^3 + \frac{\partial f}{\partial q^1}\end{aligned}$$

having the solution

$$f = a(t, q^2, q^3) + q^1 q^3 + C_2 \mathrm{e}^t q^1,$$

where $a(t, q^2, q^3)$ is arbitrary function of variables $t$, $q^2$ and $q^3$. $C_2$ is a constant.

(c) If $\xi = b(q^3)\frac{\partial}{\partial q^2} \Rightarrow \xi^2 = b(q^3)$
we get

$$-\dot{q}^3 \cdot b(q^3) + \frac{\partial}{\partial q^2}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) b(q^3) = 0$$

and from this equation we get a equations:

$$\begin{aligned}\frac{\partial^2 f}{\partial q^2 \partial q^1} &= 0\\ \frac{\partial^2 f}{(\partial q^2)^2} &= 0\\ \frac{\partial^2 f}{\partial q^2 \partial q^3} &= 1\\ \frac{\partial^2 f}{\partial q^2 \partial t} &= 0.\end{aligned}$$

having the solution

$$f = a(t, q^1, q^3) + q^2 q^3,$$

where $a(t, q^1, q^3)$ is arbitrary function of variables $t$, $q^1$ and $q^3$.

Summarizing, we obtained the following result:

**Theorem 4.** *For every point symmetry $\xi$ of the Euler-Lagrange form*

$$E = (q^3 - \ddot{q}^3)\mathrm{d}q^1 \wedge \mathrm{d}t - \dot{q}^3 \mathrm{d}q^2 \wedge \mathrm{d}t + (q^1 + \dot{q}^2 - \ddot{q}^1)\mathrm{d}q^3 \wedge \mathrm{d}t \tag{7}$$

*there exists a first order Lagrangian $\lambda'$ such that $\xi$ is a point symmetry of $\lambda'$.*

*Explicitly:*

- *For $\xi = \mathrm{e}^t \frac{\partial}{\partial q^1}$ it holds*

$$f = a(t, q^2, q^3) - q^1 q^3 + C_1 \mathrm{e}^{-t} q^1,$$

*i.e.* $L' = \dot{q}^1 \dot{q}^3 - q^2 \dot{q}^3 + q^1 q^3 + \frac{\mathrm{d}}{\mathrm{d}t}\left(a(t, q^2, q^3) - q^1 q^3 + C_1 \mathrm{e}^{-t} q^1\right).$

- *For* $\xi = \mathrm{e}^{-t}\frac{\partial}{\partial q^1}$ *it holds*

$$f = a(t, q^2, q^3) + q^1 q^3 + C_2 \mathrm{e}^t q^1,$$

*i.e.* $L' = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1 q^3 + \frac{\mathrm{d}}{\mathrm{d}t}\left(a(t, q^2, q^3) + q^1 q^3 + C_2 \mathrm{e}^t q^1\right).$

- *For* $\xi = b(q^3)\frac{\partial}{\partial q^2}$ *it holds*

$$f = a(t, q^1, q^3) + q^2 q^3,$$

*i.e.* $L' = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1 q^3 + \frac{\mathrm{d}}{\mathrm{d}t}\left(a(t, q^1, q^3) + q^2 q^3\right).$

By the above theorem, *the set of point symmetries of the class of equivalent first order Lagrangians for E locally coincides with the set of point symmetries of the Euler-Lagrange form* (7).

Remarkably, in this case, all the invariant Lagrangians are of *order one*. It is worth note that in the most often considered case of the Lagrangian $L = \frac{1}{2}mv^2$ (free particle of classical mechanics) this is *not* the case. Namely, one can prove that [14]:

– to every point symmetry $\xi$ of the Euler-Lagrange form $E$ ($E_i = m\ddot{x}^i$) there exists a Lagrangian $L$ for $E$ such that $\xi$ is a point symmetry of $L$

– *such a Lagrangian need not be of order one:* in case of Galilei transformations $L$ is a second order Lagrangian, equivalent with the kinetic energy.

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# Witt algebra and the curvature of the Heisenberg group

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**Abstract.** The aim of this paper is to determine explicitly the algebraic structure of the curvature algebra of the 3-dimensional Heisenberg group with left invariant cubic metric. We show, that this curvature algebra is an infinite dimensional graded Lie subalgebra of the generalized Witt algebra of homogeneous vector fields generated by three elements.

## 1 Introduction

The notion of curvature algebra of a Finsler manifold is introduced in a previous paper [4] of the authors and it is proved that this algebra is tangent to the holonomy group. This property used for the proof that the holonomy group of Finsler manifolds of constant non-zero curvature cannot be a compact Lie group, if the dimension of the manifold is greater than 2. The 3-dimensional Heisenberg group with left invariant cubic metric was given as an example of Finsler manifolds having infinite dimensional curvature algebra and holonomy group. The aim of this paper is to describe explicitly the algebraic structure of this curvature algebra. We show, that it is a filtered subalgebra of the generalized Witt algebra of Laurent polynomial vector fields defined on a 3-dimensional vector space, which is generated by three elements. We determine the generators of the curvature algebra in this Witt algebra.

## 2 Preliminaries

### Generalized Witt algebras

Let $A$ be an abelian group, $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) = 0$ and $T$ a vector space over $\mathbb{F}$. The group algebra $\mathbb{F}A$ of $A$ over $\mathbb{F}$ generated by the basis elements $t^J$, $J \in A$, and the multiplication of $\mathbb{F}A$ is defined by $t^J t^K = t^{J+K}$. The unit of $\mathbb{F}A$ is the element $1 = t^0$.

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Let us consider the tensor product

$$W = \mathbb{F}A \otimes_{\mathbb{F}} T = \mathrm{Span}_{\mathbb{F}} \left\{ t^J \otimes \partial \,\middle|\, J \in A, \partial \in T \right\}.$$

The element of $W$ is also denoted as $t^J \partial := t^J \otimes \partial$. Now, if a given map $(\partial, J) \mapsto \partial(J) : T \otimes A \to \mathbb{F}$ is $\mathbb{F}$-linear in the first variable and additive in the second variable, then the bracket

$$[t^J \partial_1, t^K \partial_2] := t^{J+K}(\partial_1(K)\partial_2 - \partial_2(J)\partial_1), \qquad J, K \in A, \quad \partial_1, \partial_2 \in T, \tag{1}$$

defines an infinite dimensional Lie algebra on the tensor product $W$. The Lie algebra $W$ with the Lie multiplication (1) is called a *generalized Witt algebra over the vector space $T$ graded by the abelian group $A$.*

#### Witt algebra $W_n(\mathbb{F})$ over the vector space $\mathbb{F}^n$

If $A$ is the additive group of $\mathbb{Z}^n$ with $n > 0$, then the group algebra $\mathbb{F}A$ is isomorphic to the Laurent polynomial algebra $\mathbb{F}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ over $\mathbb{F}$. For an $n$-tuple $J = (j_1, \dots, j_n) \in \mathbb{Z}^n$ we write $t^J = t_1^{j_1} \cdots t_n^{j_n}$. Let $T$ be the linear span $T = \oplus_{i=1}^n \mathbb{F}\partial_i$ of the operators $\partial_i = t_i \frac{\partial}{\partial t_i}$. If the map $(\partial, J) \mapsto \partial(J) : T \otimes A \to \mathbb{F}$ satisfies $\partial_i(J) = j_i$ then the corresponding generalized Witt algebra $W =: W_n(\mathbb{F})$ can be identified with the Lie algebra $\mathrm{Der}_{\mathbb{F}}(\mathbb{F}[t_1^{\pm 1}, \dots, t_n^{\pm 1}])$ of derivations of the Laurent polynomial algebra $\mathbb{F}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ over $\mathbb{F}$, consisting of the Laurent polynomial vector fields

$$w(J; i) = w(j_1, \dots, j_n; i) = t_1^{j_1} \cdots t_n^{j_n} t_i \frac{\partial}{\partial t_i},$$

where $(t_1, \dots, t_n) \in \mathbb{F}^n$ are the canonical coordinates in $\mathbb{F}^n$ (c. f. [2], [1]). A Lie algebra isomorphic to the Lie algebra $W_n(\mathbb{F})$ of Laurent polynomial vector fields is called *Witt algebra over the vector space $\mathbb{F}^n$.*

#### Lie subalgebras of $W_n(\mathbb{F})$

Let $\omega\colon \mathbb{Z}^{n-1} \to \mathbb{Z}$ be an additive map. We consider the linear subspace $\overline{W}_\omega$ of the Witt algebra $W_n(\mathbb{F})$ generated by the basis consisting of the elements

$$\bar{w}(\kappa; i) := w\big(\kappa, \omega(\kappa); i\big)$$

with $\kappa = (k_1, \dots, k_{n-1}) \in \mathbb{Z}^n$ and $i \in \{1, \dots, n\}$. The Lie multiplication of $W_n(\mathbb{F})$ induces a Lie multiplication

$$\big[\bar{w}(\kappa; i), \bar{w}(\lambda; j)\big] = \big[w\big(\kappa, \omega(\kappa); i\big), w\big(\lambda, \omega(\lambda); j\big)\big]$$

on $\overline{W}_\omega$ which makes it a Lie subalgebra of $W_n(\mathbb{F})$.

**Definition 1.** *If $\mathbb{F} = \mathbb{R}$ and $\omega(\kappa) = -(k_1 + \cdots + k_{n-1})$, then we denote the corresponding Lie algebra by $W_n^0(\mathbb{R})$, and $W_n^0(\mathbb{R}) \subseteq W_n(\mathbb{R})$ will be called the* Lie algebra of homogeneous vector fields on the vector space $\mathbb{R}^n$.

## 3 Curvature algebra of Finsler manifolds

A *Finsler manifold* $(M,\mathcal{F})$ is a pair of an $n$-dimensional manifold $M$ and of a continuous function $\mathcal{F}\colon TM \to \mathbb{R}$ is (called *Finsler functional*) defined on the tangent bundle of $M$, smooth on $\hat{T}M := TM\backslash\{0\}$ and for any $x\in M$ the restriction $\mathcal{F}_x = \mathcal{F}|_{T_xM}$ of $\mathcal{F}$ to the tangent space $T_xM$ is a 1-homogeneous continuous function such that for all $y\in \hat{T}_xM = TM_x\setminus\{0\}$ the symmetric bilinear form $g_y\colon T_xM\times T_xM\to\mathbb{R}$ defined by

$$g_y\colon (u,v) \ \mapsto\ g_{ij}(y)u^iv^j = \frac{1}{2}\frac{\partial^2\mathcal{F}^2(y+su+tv)}{\partial s\,\partial t}\Big|_{t=s=0} \tag{2}$$

is non-degenerate. $(M,\mathcal{F})$ is called a *singular Finsler manifold* if the condition (2) is assumed to be satisfied on an open dense cone in $T_xM$. In the following we will use the name *Finsler manifold* also for *singular Finsler manifolds*.

*Geodesics* of Finsler manifolds are determined by a system of 2nd order ordinary differential equation $\ddot{x}^i + 2G^i(x,\dot{x}) = 0$, $i=1,\dots,n$ in a local coordinate system. The functions $G^i(x,y)$ are called the *spray coefficients* belonging to the coordinate system, which are given by

$$G^i(x,y) := \frac{1}{4}g^{ij}(x,y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big)y^jy^k.$$

A vector field $X(t)$ is *parallel along a curve* $c(t)$ if and only if it is a solution of the differential equation

$$\nabla_{\dot{c}}X(t) := \Big(\frac{dX^i(t)}{dt} + \Gamma^i_j(c(t),X(t))\dot{c}^j(t)\Big)\frac{\partial}{\partial x^i} = 0, \tag{3}$$

where $\Gamma^i_j = \frac{\partial G^i}{\partial y^j}$ are the parameters of the *associated non-linear connection*. The *curvature tensor* field

$$R_{(x,y)} = \left(\frac{\partial\Gamma^k_i}{\partial x^j} - \frac{\partial\Gamma^k_j}{\partial x^i} + \Gamma^m_i\frac{\partial\Gamma^k_j}{\partial y^m} - \Gamma^m_j\frac{\partial\Gamma^k_i}{\partial y^m}\right)dx^i\otimes dx^j\otimes\frac{\partial}{\partial y^k}.$$

characterizes the integrability of the horizontal distribution $\mathcal{H}TM\subset TTM$, which is locally generated by the vector fields $\frac{\partial}{\partial x^i}+\Gamma^k_i(x,y)\frac{\partial}{\partial y^k}$, $i=1,\dots,n$. The *indicatrix* $\mathfrak{I}_xM$ $(M,\mathcal{F})$ at $x\in M$ is defined by the hypersurface of $T_xM$:

$$\mathfrak{I}_xM := \{y\in T_xM;\ \mathcal{F}(y)=\pm1\}.$$

Since the *parallel translation* $\tau_c\colon T_{c(0)}M\to T_{c(1)}M$ is a differentiable map between $\hat{T}_{c(0)}M$ and $\hat{T}_{c(1)}M$ preserving the value of the Finsler functional, it induces a map

$$\tau^{\mathfrak{I}}_c\colon \mathfrak{I}_{c(0)}M \longrightarrow \mathfrak{I}_{c(1)}M \tag{4}$$

between the indicatrices. The *holonomy group* $\mathsf{Hol}_x(M)$ of $(M,\mathcal{F})$ at $x\in M$ is the subgroup of the group of diffeomorphisms $\mathsf{Diff}(\mathfrak{I}_xM)$ of the indicatrix $\mathfrak{I}_xM$ determined by parallel translation of $\mathfrak{I}_xM$ along piece-wise differentiable closed curves initiated at the point $x\in M$.

**Definition 2.** *A vector field $\xi \in \mathfrak{X}(\mathfrak{I}_x M)$ on the indicatrix $\mathfrak{I}_x M$ is called a* curvature vector field *of the Finsler manifold $(M, \mathcal{F})$ at $x \in M$, if there exists $X, Y \in T_x M$ such that $\xi = r_x(X, Y)$, where*

$$r_x(X,Y)(y) := R_{(x,y)}(X,Y). \tag{5}$$

*The Lie subalgebra $\mathfrak{R}_x := \langle r_x(X,Y);\ X, Y \in T_x M\rangle$ of $\mathfrak{X}(\mathfrak{I}_x M)$ generated by the curvature vector fields is called the* curvature algebra *of the Finsler manifold $(M, \mathcal{F})$ at the point $x \in M$.*

The following assertion is proved in [4]:
**Theorem** *The curvature algebra $\mathfrak{R}_x$ of a Finsler manifold $(M, \mathcal{F})$ is tangent to the holonomy group $\mathsf{Hol}_x(M)$ for any $x \in M$.*

## 4 Heisenberg group with left invariant cubic metric

The Finsler functional $\mathcal{F}$ of a Finsler manifold $(M, \mathcal{F})$ is called *cubic metric* if it has the form $\mathcal{F}(x,y)^3 = a_{pqr}(x) y^p y^q y^r$, where $a_{pqr}(x)$ are components of a symmetric covariant tensor field. The tensor field $a_{ij}(x,y)$ defined by $\mathcal{F}(x,y) a_{ij}(x,y) = a_{ijr}(x) y^r$ is called the *basic tensor* of $(M, \mathcal{F})$. Let us denote

$$\{ijk, r\} = \frac{1}{4}\left(\frac{\partial a_{ijr}}{\partial x^k} + \frac{\partial a_{jkr}}{\partial x^i} + \frac{\partial a_{kir}}{\partial x^j} - \frac{\partial a_{ijk}}{\partial x^r}\right).$$

According to equation (1.6.2.6) in [3], p. 595, the spray coefficients $G^i(x,y)$ satisfy the system of linear equations

$$3\mathcal{F}(x,y) a_{ir}(x,y) G^r(x,y) = \{jkl, i\} y^j y^k y^l. \tag{6}$$

Let us consider the Heisenberg group $H_3$ consisting of $3 \times 3$-matrices

$$x = \begin{pmatrix} 1 & x^1 & x^3 \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix}, \quad (x^1, x^2, x^3) \in \mathbb{R}^3.$$

The vector $(x^1, x^2, x^3) \in \mathbb{R}^3$ is the coordinate representation of the element $x \in H_3$. The unit element of $H_3$ in this coordinate representation is $0 = (0,0,0) \in \mathbb{R}^3$ and the group multiplication has the form

$$(x^1, x^2, x^3) \cdot (x'^1, x'^2, x'^3) = (x^1 + x'^1,\ x^2 + x'^2,\ x^3 + x'^3 + x^1 x'^2, ).$$

The Lie algebra $\mathfrak{h}_3 = T_0 H_3$ of $H_3$ has the matrix representation

$$y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} \quad \longrightarrow \quad \begin{pmatrix} 0 & y^1 & y^3 \\ 0 & 0 & y^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The left-invariant *Berwald-Moór* cubic Finsler functional $\mathcal{F}$ (c. f. [5], Example 1.1.5, p. 8) on the Heisenberg group $H_3$ is determined by the function $\mathcal{F}_0 : \mathfrak{h}_3 \to \mathbb{R}$ satisfying $\mathcal{F}_0(y)^3 := y^1 y^2 y^3$. If $y = (y^1, y^2, y^3)$ is a tangent vector at $x \in H_3$, then $\mathcal{F}(x,y) := \mathcal{F}_0(x^{-1} y)$, and hence its coordinate expression is of the form

$$\mathcal{F}(x,y)^3 = y^1 y^2 \left(y^3 - x^1 y^2\right).$$

Since $\mathcal{F}$ is left-invariant, the associated geometric structures (connection, geodesics, curvature) are also left-invariant and the curvature algebras at different points are isomorphic. The coefficients $a_{pqr}(x)$ are the following:

$$a_{122} = a_{212} = a_{221} = -\frac{x_1}{3}, \quad a_{123} = a_{231} = a_{312} = a_{321} = a_{213} = a_{132} = \frac{1}{6},$$

$$a_{ijj} = a_{jij} = a_{jji} = 0 \quad \text{with} \quad i,j \in \{1,2,3\} \quad \text{and} \quad (i,j) \neq (1,2).$$

Hence the right hand side of (6) gives $\{jkl,1\}y^j y^k y^l = \{jkl,3\}y^j y^k y^l = 0$ and

$$\{jkl,2\}y^j y^k y^l = \frac{3}{4}\frac{\partial a_{jk2}(x)}{\partial x^1} y^j y^k y^1 = \frac{3}{2}\frac{\partial a_{122}(x)}{\partial x^1} {y^1}^2 y^2 = -\frac{1}{2}{y^1}^2 y^2.$$

The matrix of the basic tensor field $a_{ij}(x,y)$ is

$$(a_{ij}(x,y)) = \frac{1}{\mathcal{F}(x,y)} \begin{pmatrix} 0 & -\frac{x^1}{3}y^2 + \frac{1}{6}y^3 & \frac{1}{6}y^2 \\ -\frac{x^1}{3}y^2 + \frac{1}{6}y^3 & -\frac{x^1}{3}y^1 & \frac{1}{6}y^1 \\ \frac{1}{6}y^2 & \frac{1}{6}y^1 & 0 \end{pmatrix}.$$

This matrix is non-singular on the open dense cone determined by $y^1 y^2 \left(y^3 - x^1 y^2\right) \neq 0$ in $T_x M$. In the following we will investigate on this domain. We obtain from equations (6)

$$\frac{1}{6}y^2 G^1(x,y) + \frac{1}{6}y^1 G^2(x,y) = 0.$$

$$\left(-\frac{x^1}{3}y^2 + \frac{1}{6}y^3\right) G^2(x,y) + \frac{1}{6}y^2 G^3(x,y) = 0,$$

$$\left(-\frac{x^1}{3}y^2 + \frac{1}{6}y^3\right) G^1(x,y) - \frac{x^1}{3}y^1 G^2(x,y) + \frac{1}{6}y^1 G^3(x,y) = -\frac{1}{6}{y^1}^2 y^2,$$

The solution yields

$$G^1(x,y) = -\frac{{y^1}^2 y^2}{2(y^3 - x^1 y^2)}, \; G^2(x,y) = \frac{y^1 {y^2}^2}{2(y^3 - x^1 y^2)}, \; G^3(x,y) = \frac{y^1 y^2 y^3}{2(y^3 - x^1 y^2)} - y^1 y^2.$$

The matrix of the parameters $\Gamma^i_j$ of the associated non-linear connection is

$$(\Gamma^i_j(x,y)) = \begin{pmatrix} -\frac{y^1 y^2}{y^3 - x^1 y^2} & -\frac{{y^1}^2 y^3}{2(y^3 - x^1 y^2)^2} & \frac{{y^1}^2 y^2}{2(y^3 - x^1 y^2)^2} \\ \frac{{y^2}^2}{2(y^3 - x^1 y^2)} & \frac{2y^1 y^2 y^3 - y^1 {y^2}^2 x^1}{2(y^3 - x^1 y^2)^2} & -\frac{y^1 {y^2}^2}{2(y^3 - x^1 y^2)^2} \\ \frac{y^2 y^3}{2(y^3 - x^1 y^2)} - y^2 & \frac{y^1 {y^3}^2}{2(y^3 - x^1 y^2)^2} - y^1 & -\frac{x^1 y^1 {y^2}^2}{2(y^3 - x^1 y^2)^2} \end{pmatrix}.$$

The derivatives of $\Gamma^i_j$ by $x^2$ and $x^3$ vanish, we compute their derivatives by $x^1$:

$$\left(\frac{\partial \Gamma^i_j}{\partial x^1}(x,y)\right) = \begin{pmatrix} -\frac{y^1 {y^2}^2}{(y^3 - x^1 y^2)^2} & -\frac{{y^1}^2 y^2 y^3}{(y^3 - x^1 y^2)^3} & \frac{{y^1}^2 {y^2}^2}{(y^3 - x^1 y^2)^3} \\ \frac{{y^2}^3}{2(y^3 - x^1 y^2)^2} & \frac{y^1 {y^2}^2 (3y^3 - y^2 x^1)}{2(y^3 - x^1 y^2)^3} & -\frac{y^1 {y^2}^3}{(y^3 - x^1 y^2)^3} \\ \frac{{y^2}^2 y^3}{2(y^3 - x^1 y^2)^2} & \frac{y^1 y^2 {y^3}^2}{(y^3 - x^1 y^2)^3} & -\frac{y^1 {y^2}^2 (y^3 + x^1 y^2)}{2(y^3 - x^1 y^2)^3} \end{pmatrix}.$$

In the following we put $x=0$ and we get

$$
(\Gamma^i_j)=\begin{pmatrix} -\frac{y^1y^2}{y^3} & -\frac{{y^1}^2}{2y^3} & \frac{{y^1}^2y^2}{2{y^3}^2} \\ \frac{{y^2}^2}{2y^3} & \frac{y^1y^2}{y^3} & -\frac{y^1{y^2}^2}{2{y^3}^2} \\ -\frac{y^2}{2} & -\frac{y^1}{2} & 0 \end{pmatrix}, \quad \left(\frac{\partial\Gamma^i_j}{\partial x^1}\right)=\begin{pmatrix} -\frac{y^1{y^2}^2}{{y^3}^2} & -\frac{{y^1}^2y^2}{{y^3}^2} & \frac{{y^1}^2{y^2}^2}{{y^3}^3} \\ \frac{{y^2}^3}{2{y^3}^2} & \frac{3}{2}\frac{y^1{y^2}^2}{{y^3}^2} & -\frac{y^1{y^2}^3}{{y^3}^3} \\ \frac{{y^2}^2}{2y^3} & \frac{y^1y^2}{y^3} & -\frac{y^1{y^2}^2y^3}{2{y^3}^3} \end{pmatrix}.
$$

Moreover

$$
\left(\frac{\partial\Gamma^i_1}{\partial y^j}\right)=\begin{pmatrix} -\frac{y^2}{y^3} & -\frac{y^1}{y^3} & \frac{y^1y^2}{{y^3}^2} \\ 0 & \frac{y^2}{y^3} & -\frac{{y^2}^2}{2{y^3}^2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \quad \left(\frac{\partial\Gamma^i_2}{\partial y^j}\right)=\begin{pmatrix} -\frac{y^1}{y^3} & 0 & \frac{{y^1}^2}{2{y^3}^2} \\ \frac{y^2}{y^3} & \frac{y^1}{y^3} & -\frac{y^1y^2}{{y^3}^2} \\ -\frac{1}{2} & 0 & 0 \end{pmatrix}
$$

and

$$
\left(\frac{\partial\Gamma^i_3}{\partial y^j}\right)=\begin{pmatrix} \frac{y^1y^2}{{y^3}^2} & \frac{{y^1}^2}{2{y^3}^2} & -\frac{{y^1}^2y^2}{{y^3}^3} \\ -\frac{{y^2}^2}{2{y^3}^2} & -\frac{y^1y^2}{{y^3}^2} & \frac{y^1{y^2}^2}{{y^3}^3} \\ 0 & 0 & 0 \end{pmatrix}.
$$

We obtain for $\frac{\partial\Gamma^i_a}{\partial x^b}-\frac{\partial\Gamma^i_a}{\partial y^m}\Gamma^m_b$, $(a,b)=(1,2),(2,1),(1,3),(3,1),(2,3),(3,2)$ the following expressions:

$$
\left(\frac{\partial\Gamma^i_1}{\partial x^2}-\frac{\partial\Gamma^i_1}{\partial y^m}\Gamma^m_2\right)=\begin{pmatrix} \frac{{y^1}^2y^2}{{y^3}^2} \\ -\frac{5}{4}\frac{y^1{y^2}^2}{{y^3}^2} \\ \frac{1}{2}\frac{y^1y^2}{y^3} \end{pmatrix}, \quad \left(\frac{\partial\Gamma^i_2}{\partial x^1}-\frac{\partial\Gamma^i_2}{\partial y^m}\Gamma^m_1\right)=\begin{pmatrix} -\frac{7}{4}\frac{{y^1}^2y^2}{{y^3}^2} \\ \frac{3}{2}\frac{y^1{y^2}^2}{{y^3}^2} \\ \frac{1}{2}\frac{y^1y^2}{y^3} \end{pmatrix},
$$

$$
\left(\frac{\partial\Gamma^i_1}{\partial x^3}-\frac{\partial\Gamma^i_1}{\partial y^m}\Gamma^m_3\right)=\begin{pmatrix} 0 \\ \frac{1}{2}\frac{y^1{y^2}^3}{{y^3}^3} \\ -\frac{1}{4}\frac{y^1{y^2}^2}{{y^3}^2} \end{pmatrix}, \quad \left(\frac{\partial\Gamma^i_3}{\partial x^1}-\frac{\partial\Gamma^i_3}{\partial y^m}\Gamma^m_1\right)=\begin{pmatrix} \frac{5}{4}\frac{{y^1}^2{y^2}^2}{{y^3}^3} \\ -\frac{1}{2}\frac{y^1{y^2}^3}{{y^3}^3} \\ -\frac{1}{2}\frac{y^1{y^2}^2}{{y^3}^2} \end{pmatrix},
$$

$$
\left(\frac{\partial\Gamma^i_2}{\partial x^3}-\frac{\partial\Gamma^i_2}{\partial y^m}\Gamma^m_3\right)=\begin{pmatrix} \frac{1}{2}\frac{{y^1}^3y^2}{{y^3}^3} \\ 0 \\ \frac{1}{4}\frac{{y^1}^2y^2}{{y^3}^2} \end{pmatrix}, \quad \left(\frac{\partial\Gamma^i_3}{\partial x^2}-\frac{\partial\Gamma^i_3}{\partial y^m}\Gamma^m_2\right)=\begin{pmatrix} -\frac{1}{2}\frac{{y^1}^3y^2}{{y^3}^3} \\ \frac{5}{4}\frac{{y^1}^2{y^2}^2}{{y^3}^3} \\ 0 \end{pmatrix}.
$$

Hence we can obtain the following curvature vector fields on the indicatrix $\mathfrak{I}_0(M)$ at $x=0$:

$$
r_0(1,2)=\frac{11}{4}\begin{pmatrix} \frac{{y^1}^2y^2}{{y^3}^2} \\ -\frac{y^1{y^2}^2}{{y^3}^2} \\ 0 \end{pmatrix}, \quad r_0(1,3)=\begin{pmatrix} -\frac{5}{4}\frac{{y^1}^2{y^2}^2}{{y^3}^3} \\ \frac{y^1{y^2}^3}{{y^3}^3} \\ \frac{1}{4}\frac{y^1{y^2}^2}{{y^3}^2} \end{pmatrix}, \quad r_0(2,3)=\begin{pmatrix} \frac{{y^1}^3y^2}{{y^3}^3} \\ -\frac{5}{4}\frac{{y^1}^2{y^2}^2}{{y^3}^3} \\ \frac{1}{4}\frac{{y^1}^2y^2}{{y^3}^2} \end{pmatrix}.
$$

where we use the notation $r_0(i,j)=r_0\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)$. These vector fields $r_0(i,j)$, $i<j$, $i,j=1,2,3$ generate the curvature algebra $\mathfrak{r}_0$ at $x=0$.

Let us consider the vector fields

$$A^{k,m}(a_1, a_2, a_3) := a_1 Y^{k+1,m} E_1 + a_2 Y^{k,m+1} E_2 + a_3 Y^{k,m} E_3,$$

defined on $\mathfrak{h}_3 = T_0 H_3$, where

$$(a_1, a_2, a_3) \in \mathbb{R}^3, \qquad E_i = \frac{\partial}{\partial y^i}\Big|_0, \; i = 1, 2, 3,$$

and

$$Y^{k,m} := \frac{{y^1}^k {y^2}^m}{{y^3}^{k+m-1}}, \quad k, m \in \mathbb{N}.$$

Then the curvature vector fields $r_0(i,j), \; i = 1, 2, 3$ can be written in the form

$$r_0(1,2) = \frac{11}{4} A^{1,1}(1, -1, 0), \quad r_0(1,3) = \frac{1}{4} A^{1,2}(-5, 4, 1), \quad r_0(2,3) = \frac{1}{4} A^{2,1}(4, -5, 1).$$

We have that

$$\left[A^{k,l}(a_1, a_2, a_3), A^{p,q}(b_1, b_2, b_3)\right] = A^{k+p,l+q}(c_1, c_2, c_3),$$

where

$$\begin{aligned} c_1 &= b_1\big((p+1)a_1 + q a_2 - (p+q)a_3\big) - a_1\big((k+1)b_1 + l b_2 - (k+l)b_3\big), \\ c_2 &= b_2\big(p a_1 + (q+1)a_2 - (p+q)a_3\big) - a_2\big(k b_1 + (l+1)b_2 - (k+l)b_3\big), \\ c_3 &= b_3\big(p a_1 + q a_2 - (p+q-1)a_3\big) - a_3\big(k b_1 + l b_2 - (k+l-1)b_3\big). \end{aligned}$$

With these preparations we are able to completely describe the structure of the curvature algebra of the Heisenberg group as a Lie subalgebra of the Witt algebra $W_3(\mathbb{R})$. We have the following

**Theorem 1.** *The curvature algebra $\mathfrak{r}_0$ of the Berwald-Moór left-invariant cubic metric $\mathcal{F}$ on the Heisenberg group $H_3$ is isomorphic to the Lie subalgebra $W_{\langle h_1,h_2,h_3\rangle}$ of $W_3^0(\mathbb{R})$ generated by the elements*

$$\begin{aligned} h_1 &= \quad \bar{w}(1,1;1) - \bar{w}(1,1;2), \\ h_2 &= -5\bar{w}(1,2;1) + 4\bar{w}(1,2;2) + \bar{w}(1,2;3), \\ h_3 &= \quad 4\bar{w}(2,1;1) - 5\bar{w}(2,1;2) + \bar{w}(2,1;3). \end{aligned}$$

*In particular, $\mathfrak{r}_0$ is infinite dimensional, and we have the following sequence of Lie algebras:*

$$\mathfrak{r}_0 \;\cong\; W_{\langle h_1,h_2,h_3\rangle} \;\subset\; W_3^0(\mathbb{R}) \;\subset\; W_3(\mathbb{R}).$$

*where $W_3^0(\mathbb{R})$ is the Lie algebra of homogeneous vector fields and $W_3(\mathbb{R})$ is the Witt algebra of Laurent polynomial vector fields on the vector space $\mathfrak{h}_3 \cong \mathbb{R}^3$.*

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# On the inverse variational problem in nonholonomic mechanics

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**Abstract.** The inverse problem of the calculus of variations in a nonholonomic setting is studied. The concept of constraint variationality is introduced on the basis of a recently discovered nonholonomic variational principle. Variational properties of first order mechanical systems with general nonholonomic constraints are studied. It is shown that constraint variationality is equivalent with the existence of a closed representative in the class of 2-forms determining the nonholonomic system. Together with the recently found constraint Helmholtz conditions this result completes basic geometric properties of constraint variational systems. A few examples of constraint variational systems are discussed.

## 1 Introduction

The covariant local inverse problem of the calculus of variations for second order ordinary differential equations means to find necessary and sufficient conditions under which a system of equations

$$A_\sigma(t, q^\nu, \dot{q}^\nu) + B_{\sigma\rho}(t, q^\nu, \dot{q}^\nu)\ddot{q}^\rho = 0\,, \quad 1 \le \sigma \le m \tag{1}$$

for curves $\mathbb{R} \ni t \to (q^\nu(t)) \in \mathbb{R}^m$, is variational "as it stands", i.e. to determine if there exists a Lagrangian $L(t, q^\nu, \dot{q}^\nu)$ such that the functions on the left-hand-sides are Euler-Lagrange expressions of $L$:

$$A_\sigma + B_{\sigma\rho}\ddot{q}^\rho = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma}\,, \tag{2}$$

and, moreover, in the affirmative case to find a formula for computing a Lagrangian.

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*Key words:* The inverse problem of the calculus of variations, Helmholtz conditions, nonholonomic constraints, the nonholonomic variational principle, constraint Euler-Lagrange equations, constraint Helmholtz conditions, constraint Lagrangian, constraint ballistic motion, relativistic particle.

Solution to this problem is very well known: conditions for variationality are the celebrated *Helmholtz conditions* [3], and a corresponding Lagrangian is then given by the famous *Vainberg-Tonti integral formula* [15], [16].

In this paper we are interested in a generalization of the inverse problem to nonholonomic mechanics. Some aspects based on an analogy with certain properties of unconstrained variational equations have already been studied (see e.g. [1], [9], [12]). However, only recently a variational principle for nonholonomic systems has been found [6], which opened a new way to formulate the problem and search for a solution in a parallel to the unconstrained case. Here we follow this way and introduce the concept of *constraint variationality* on the basis of the constraint variational principle, in a spirit as it is understood for unconstrained equations.

Namely, given a constraint $Q$ by $k$ first order ordinary differential equations

$$\dot{q}^{m-k+a} = g^a(t, q^\sigma, \dot{q}^l)\,, \quad 1 \le a \le k, \tag{3}$$

where $1 \le \sigma \le m$ and $1 \le l \le m-k$, the generalized ("nonholonomic") Euler--Lagrange equations represent a system of $m-k$ second order ordinary differential equations on the constraint submanifold $Q \subset J^1(\mathbb{R} \times \mathbb{R}^m)$. It is interesting that in this case one has $k+1$ "Lagrange functions" where $k$ is the number of constraint equations. This rather mysterious property of nonholonomic systems is related to the fact that the corresponding Lagrangian 1-form has $k+1$ generic components, and is not reduced to a horizontal form (which is determined by a single function) as happens by circumstance in the unconstrained case.

The nonholonomic inverse problem concerns a system of *mixed first order and second order* ordinary differential equations

$$\dot{q}^{m-k+a} - g^a(t, q^\sigma, \dot{q}^l) = 0\,, \quad 1 \le a \le k\,, \tag{4}$$

$$\bar{A}_s(t, q^\sigma, \dot{q}^l) + \bar{B}_{sr}(t, q^\sigma, \dot{q}^l)\ddot{q}^r = 0\,, \quad 1 \le s \le m-k\,. \tag{5}$$

The first order equations give rise to a nonholonomic constraint submanifold $Q \subset J^1(\mathbb{R} \times \mathbb{R}^m)$ of corank $k$, while the second order equations then represent the dynamics on the constraint submanifold $Q$. The problem now is to find necessary and sufficient conditions under which equations (5) "as they stand" become the constrained Euler-Lagrange equations, and in the affirmative case, to find a corresponding constraint Lagrange 1-form.

It is known that in the unconstrained case variationality is equivalent with the possibility to extend the Euler-Lagrange form to a closed 2-form. Helmholtz conditions then become nothing but the closedness conditions, and the Vainberg--Tonti formula appears by application of the Poincaré Lemma. The main result we achieve in this paper means that the solution of the inverse problem in the nonholonomic setting has the same geometric properties: namely, that the constraint variationality is equivalent with the property that the corresponding equations can be represented by a closed form defined on the constraint $Q$. The closedness conditions are the constraint Helmholtz conditions obtained in our older paper [9].

It is worth mention that given an unconstrained Lagrangian system, the corresponding constrained system is constraint variational for *any* nonholonomic constraint. On the other hand, however, a nonholonomic system which is constraint variational may arise from a *non-variational unconstrained system*. Moreover, such an unconstrained system *need not be unique* in the sense that the corresponding unconstrained systems are *generically different*.

Due to the above properties, the range of applications of constraint variationality conditions is broader than that of Helmholtz conditions for unconstrained systems. At the end of the paper we illustrate on a few examples some possible applications of constraint Helmholtz conditions with the stress on rather unexpected properties of constraint variationality.

## 2 Unconstrained mechanical systems and variationality

Throughout the paper we consider a fibred manifold $\pi\colon Y \to \mathbb{R}$, $\dim Y = m+1$, and the corresponding jet bundles $\pi_r\colon J^rY \to \mathbb{R}$ where $r = 1, 2$. We denote by $\pi_{1,0}\colon J^1Y \to Y$, $\pi_{2,0}\colon J^2Y \to Y$ and $\pi_{2,1}\colon J^2Y \to J^1Y$ the canonical projections. Recall that a section $\delta$ of $\pi_r$ is called *holonomic*, if it is of the form $\delta = J^r\gamma$ for a section $\gamma$ of $\pi$.

A form $\eta$ on $J^rY$ is called *horizontal*, if $i_\xi\eta = 0$ for every $\pi_r$-vertical vector field $\xi$, and is called *contact*, if $J^r\gamma^*\eta = 0$ for every section $\gamma$ of $\pi$. We shall use the following basis of 1-forms on $J^1Y$ and $J^2Y$ respectively, adapted to the contact structure:

$$(dt, \omega^\sigma, d\dot{q}^\sigma), \quad (dt, \omega^\sigma, \dot{\omega}^\sigma, d\ddot{q}^\sigma)\,,$$

where

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma dt, \quad \dot{\omega}^\sigma = d\dot{q}^\sigma - \ddot{q}^\sigma dt.$$

For every $k$-form $\eta$ on $J^1Y$ there exists a unique decomposition

$$\pi_{2,1}^*\eta = p_{k-1}\eta + p_k\eta,$$

where $p_{k-1}\eta$ and $p_k\eta$ is the $(k-1)$-contact component and the $k$-contact component of $\eta$, respectively, containing in every its term exactly $(k-1)$, respectively $k$, factors $\omega^\sigma$ and $\dot{\omega}^\sigma$. If $p_k\eta = 0$ we say that $\eta$ is $(k-1)$contact. Similarly, if $p_{k-1}\eta = 0$ we speak about a $k$-contact form.

A first order mechanical system is described by a dynamical form $E$ on $J^2Y$ with components affine in the second derivatives; in fibered coordinates,

$$E = E_\sigma(t, q^\nu, \dot{q}^\nu, \ddot{q}^\nu) dq^\sigma \wedge dt\,, \tag{6}$$

where

$$E_\sigma = A_\sigma(t, q^\lambda, \dot{q}^\lambda) + B_{\sigma\nu}(t, q^\lambda, \dot{q}^\lambda)\ddot{q}^\nu. \tag{7}$$

A section $\gamma$ of $\pi$ is called a *path of $E$* if $E_\sigma \circ J^2\gamma = 0$. This condition gives a system of $m$ second order ordinary differential equations

$$A_\sigma\Big(t, q^\lambda, \frac{dq^\lambda}{dt}\Big) + B_{\sigma\nu}\Big(t, q^\lambda, \frac{dq^\lambda}{dt}\Big)\frac{d^2q^\nu}{dt^2} = 0\,, \tag{8}$$

which have the meaning of the equations of motion.

If $E$ is a dynamical form with components affine in the second derivatives then in a neighborhood of every point in $J^1Y$ there exists a 2-form $\alpha$ such that

$$\pi_{2,1}^*\alpha = E + F, \tag{9}$$

where $F$ is a 2-contact 2-form. The $\alpha$ is not unique. In fibered coordinates

$$\alpha = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot{q}^\nu + F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu, \tag{10}$$

where $F_{\sigma\nu}(t, q^\lambda, \dot{q}^\lambda)$ are arbitrary functions, skew-symmetric in the indices.

With help of $\alpha$ equations for paths of $E$ (8) take the form

$$J^1\gamma^* i_\xi \alpha = 0 \quad \text{for every vertical vector field } \xi \text{ on } J^1Y \tag{11}$$

of equations for holonomic integral sections of a local Pfaffian system on $J^1Y$. It is to be stressed that the set of solutions of equations (11) does not depend upon a choice of the 2-form $F$, and that (for any $F$) equations (11) are *locally equivalent* with equations of paths of $E$ (8).

We denote the family of all the local 2-forms on $J^1Y$ associated with $E$ as above by $[\alpha]$ and call it the *Lepage class of $E$*. Note that forms belonging to the Lepage class of $E$ satisfy

$$\alpha_1 - \alpha_2 \text{ is a 2-contact 2-form}$$

(on the intersection of their domains) and

$$p_1\alpha = E.$$

A dynamical form $E$ is called *locally variational* if in a neighborhood of every point in $J^2Y$ there exists a Lagrangian such that $E$ coincides with its Euler--Lagrange form. It is known that if such a Lagrangian exists, there exists also an equivalent local first-order Lagrangian $\lambda = Ldt$ such that (7) coincide with the Euler-Lagrange expressions of $\lambda$

$$E_\sigma \equiv A_\sigma + B_{\sigma\nu}\ddot{q}^\nu = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma}. \tag{12}$$

Equations for paths of a locally variational form are Euler-Lagrange equations. For the Lepage class of a locally variational form we have $[\alpha] = [d\theta_\lambda]$ where $\theta_\lambda$ is the Cartan form of $\lambda$.

The following theorem shows the importance of the properties of the Lepage class for variationality of dynamical forms (see [4]).

**Theorem 1.** *A dynamical form $E$ is locally variational if and only if the corresponding Lepage class $[\alpha]$ contains a closed representative. In this case, moreover, the closed 2-form $\alpha_E \in [\alpha]$ is unique and global (defined on $J^1Y$).*

The form $\alpha_E$ is called *Lepage equivalent of $E$* and the corresponding mechanical system is called *Lagrangian system*.

A direct calculation of $d\alpha$ for a representative of the class $[\alpha]$ leads to the famous *Helmholtz conditions* (necessary and sufficient conditions of variationality).

**Theorem 2.** *A dynamical form $E$ is locally variational if and only if in fibered coordinates the following conditions hold:*

$$
\begin{aligned}
&(B_{\sigma\nu})_{\mathrm{alt}(\sigma\nu)} = 0\,, && \Big(\frac{\partial B_{\sigma\nu}}{\partial \dot q^{\lambda}}\Big)_{\mathrm{alt}(\nu\lambda)} = 0\,,\\
&\Big(-\frac{\partial A_\sigma}{\partial \dot q^{\nu}} + \frac{d' B_{\sigma\nu}}{dt}\Big)_{\mathrm{sym}(\sigma\nu)} = 0\,, && \Big(\frac{\partial A_\sigma}{\partial q^{\nu}} - \frac{1}{2}\frac{d'}{dt}\Big(\frac{\partial A_\sigma}{\partial \dot q^{\nu}}\Big)\Big)_{\mathrm{alt}(\sigma\nu)} = 0\,,
\end{aligned}
\tag{13}
$$

*where* sym *and* alt *means symmetrization and skew-symmetrization respectively, and*

$$\frac{d'}{dt} = \frac{\partial}{\partial t} + \dot q^{\sigma}\frac{\partial}{\partial q^{\sigma}}\,.$$

Note that for a locally variational form $E$ be globally variational (i.e. to arise as an Euler-Lagrange form from a global first-order Lagrangian) it is necessary and sufficient that the Lepage equivalent $\alpha_E$ of $E$ is exact.

## 3 Constrained mechanical systems

Our approach to the inverse variational problem for nonholonomically constrained systems is based on the model representing nonholonomic constraints as a submanifold $Q$ in $J^1Y$, naturally endowed with a nonintegrable distribution, and a constrained system as a dynamical form (an exterior differential system) defined on the constraint submanifold [4], [5]; here we follow the exposition of the survey article [7].

In what follows, greek indices $\sigma, \nu$ etc. run over $1, 2, \dots, m$ as above, and the latin indices $a, b, i, j$ (respectively $l, s$) run over $1, 2, \dots, k = \operatorname{codim} Q$ (respectively $1, 2, \dots, m-k$). Summation over repeated indices is understood.

Let us consider a submanifold $Q \subset J^1Y$ of codimension $k$, $1 \le k \le m-1$, fibred over $Y$, called a *constraint submanifold*. We denote by $\iota\colon Q \to J^1Y$ the canonical embedding. Locally, $Q$ is given by $k$ independent equations

$$f^a(t, q^\sigma, \dot q^\sigma) = 0\,, \quad 1 \le a \le k\,, \tag{14}$$

or, in normal form,

$$\dot q^{m-k+a} = g^a(t, q^\sigma, \dot q^l)\,, \quad 1 \le a \le k\,, \tag{15}$$

where $l = 1, 2, \dots, m-k$.

We shall consider also the first prolongation $\hat Q$ of the constraint $Q$, that is a submanifold in $J^2Y$, consisting of all points $J^2_x\gamma$ such that $J^1_x\gamma \in Q$, $x \in \mathbb{R}$. Locally $\hat Q$ is defined by the equations of the constraint and their derivatives:

$$f^a = 0\,, \quad \frac{df^a}{dt} = 0\,, \tag{16}$$

respectively, in normal form,

$$\dot q^{m-k+a} = g^a, \quad \ddot q^{m-k+a} = \frac{dg^a}{dt}\,. \tag{17}$$

We denote by $\hat\iota\colon \hat Q \to J^2Y$ the corresponding canonical embedding. The manifold $\hat Q$ is fibred over $Q$, $Y$ and $\mathbb{R}$, the fibred projections are simply restrictions of the corresponding canonical projections of the underlying fibred manifolds. We write $\bar\pi_2\colon \hat Q \to \mathbb{R}$, $\bar\pi_{2,1}\colon \hat Q \to Q$, $\bar\pi_{2,0}\colon \hat Q \to Y$, and $\bar\pi_1\colon Q \to \mathbb{R}$, $\bar\pi_{1,0}\colon Q \to Y$. Usually we shall use on $Q$ adapted coordinates $(t, q^\sigma, \dot q^s)$, and on $\hat Q$ associated coordinates $(t, q^\sigma, \dot q^s, \ddot q^s)$, where $1 \le \sigma \le m$, $1 \le s \le m-k$.

Similarly as in the unconstrained case, for every $q$-form $\eta$ on $Q$ one has a *unique decomposition* into a sum of a $\bar\pi_2$-horizontal form and $i$-contact forms, $i = 1, 2, \dots q$, on $\hat Q$ [6]; we write

$$\bar\pi^*_{2,1}\eta = \bar h\eta + \bar p_1\eta + \dots + \bar p_q\eta\,. \tag{18}$$

In particular, we get an invariant *splitting of the exterior derivative* $d$ to the horizontal and contact part, $\bar\pi^*_{2,1}d = \bar hd + \bar p_1 d$. The operator $\bar hd$ (the constraint total derivative) has the component

$$\frac{d_{\mathrm{c}}}{dt} = \frac{\partial}{\partial t} + \dot q^s \frac{\partial}{\partial q^s} + g^a \frac{\partial}{\partial q^{m-k+a}} + \ddot q^s \frac{\partial}{\partial \dot q^s}\,. \tag{19}$$

For convenience of notations we also put

$$\frac{d'_{\mathrm{c}}}{dt} = \frac{\partial}{\partial t} + \dot q^s \frac{\partial}{\partial q^s} + g^a \frac{\partial}{\partial q^{m-k+a}}\,. \tag{20}$$

Over every nonholonomic constraint there naturally arises a bundle, called the *canonical distribution* [4] or *Chetaev bundle* [11], giving a geometric meaning to *virtual displacements* in the space of positions and velocities, and to the concept of *reactive (Chetaev) forces*. It is a corank $k$ distribution $\mathcal{C}$ *on the manifold* $Q$, locally annihilated by the system of $k$ linearly independent 1-forms

$$\varphi^a = \left(\frac{\partial f^a}{\partial \dot q^\sigma} \circ \iota\right) \bar\omega^\sigma = \bar\omega^{m-k+a} - \frac{\partial g^a}{\partial \dot q^s}\bar\omega^s, \tag{21}$$

where

$$\bar\omega^\sigma = \iota^*\omega^\sigma, \tag{22}$$

or, equivalently, locally spanned by the following system of $2(m-k)+1$ independent vector fields

$$\begin{aligned} \frac{\partial_{\mathrm{c}}}{\partial t} &\equiv \frac{\partial}{\partial t} + \left(g^a - \frac{\partial g^a}{\partial \dot q^l}\dot q^l\right)\frac{\partial}{\partial q^{m-k+a}}\,, \\ \frac{\partial_{\mathrm{c}}}{\partial q^s} &\equiv \frac{\partial}{\partial q^s} + \frac{\partial g^a}{\partial \dot q^s}\frac{\partial}{\partial q^{m-k+a}}\,, \\ \frac{\partial}{\partial \dot q^s}&\,. \end{aligned} \tag{23}$$

Vector fields belonging to the canonical distribution are called *Chetaev vector fields*.

The annihilator of $\mathcal{C}$ is denoted by $\mathcal{C}^0$.

The ideal in the exterior algebra on $Q$ locally generated by the 1-forms $\varphi^a$, $1 \le a \le k$, is called the *constraint ideal*, and denoted by $\mathcal{I}(\mathcal{C}^0)$. Differential forms belonging to the constraint ideal are called *constraint forms*.

Let us recall the following theorem [4]:

**Theorem 3.** *The constraint $Q$ is given by equations affine in the first derivatives if and only if the canonical distribution $\mathcal{C}$ on $Q$ is $\bar\pi_{1,0}$-projectable (i.e. the projection of $\mathcal{C}$ is* a distribution on *$Y$).*

A nonholonomic constraint $Q$ is called *semiholonomic* if its canonical distribution $\mathcal{C}$ is completely integrable.

The canonical distribution is naturally lifted to the distribution $\hat{\mathcal{C}}$ on $\hat{Q}$, defined with help of its annihilator by $\hat{\mathcal{C}}^0 = \bar{\pi}^*_{2,1}\mathcal{C}^0$.

Now, let $E$ be a dynamical form on $J^2Y$ and $[\alpha]$ its Lepage class as above. According to [4], a *constrained mechanical system* associated with $[\alpha]$ is the class

$$[\bar{\alpha}] = \iota^*[\alpha] \mod \mathcal{I}(\mathcal{C}^0)\,. \tag{24}$$

This means that $[\bar{\alpha}]$ is defined on the constraint $Q$ and consists of all (possibly local) 2-forms on $Q$ such that

$$\bar{\alpha} = \bar{A}_l\omega^l \wedge dt + \bar{B}_{ls}\omega^l \wedge d\dot{q}^s + F + \varphi, \tag{25}$$

where $F$ is a 2-contact and $\varphi$ is a constraint 2-form on $Q$, and

$$\begin{aligned}
\bar{A}_l &= \Big(A_l + A_{m-k+b}\frac{\partial g^b}{\partial \dot{q}^l} + \Big(B_{l,m-k+a} + B_{m-k+b,m-k+a}\frac{\partial g^b}{\partial \dot{q}^l}\Big)\frac{d'g^a}{dt}\Big) \circ \iota,\\
\bar{B}_{ls} &= \Big(B_{ls} + B_{l,m-k+a}\frac{\partial g^a}{\partial \dot{q}^s} + B_{m-k+a,s}\frac{\partial g^a}{\partial \dot{q}^l} + B_{m-k+b,m-k+a}\frac{\partial g^b}{\partial \dot{q}^l}\frac{\partial g^a}{\partial \dot{q}^s}\Big) \circ \iota.
\end{aligned} \tag{26}$$

In place of a single dynamical form $E = p_1\alpha$, for the constrained system we get the class $[\bar{E}]$ on $\hat{Q}$,

$$\bar{E} = \bar{p}_1\bar{\alpha} = \hat{\iota}^*E + \varphi^a \wedge \nu_a \tag{27}$$

where $\varphi^a$ are the canonical constraint 1-forms defined above and $\nu_a$ are horizontal forms. Putting $\bar{E}^{\mathrm{c}} = (\hat{\iota}^*E)|_{\hat{\mathcal{C}}}$ we get an element of $\Lambda^2(\hat{\mathcal{C}})$, a 2-form along the canonical distribution, called *constrained dynamical form*; $\bar{E}^{\mathrm{c}}$ is the same for all $\bar{E} \in [\bar{E}]$. In coordinates

$$\bar{E}^{\mathrm{c}} = (\bar{A}_s + \bar{B}_{sr}\ddot{q}^r)\bar{\omega}^s \wedge dt\,. \tag{28}$$

By a *constrained section* of $\pi$ we shall mean a section $\gamma : I \to Y$, $I \subset \mathbb{R}$, such that $J^1\gamma(I) \subset Q$. Hence, constrained sections satisfy the first order ODE's of the constraint (14) resp. (15). In particular, constrained sections are integral sections of the canonical distribution $\mathcal{C}$.

We have the following theorem [4] providing equations of motion of nonholonomically constrained systems in both intrinsic and coordinate form:

**Theorem 4.** *Let $\gamma\colon I \to Y$ be a constrained section. The following conditions are equivalent:*

*(1) $\gamma$ is a path of $\bar{E}^{\mathrm{c}}$, i.e. it satisfies*

$$\bar{E}^{\mathrm{c}} \circ J^2\gamma = 0\,. \tag{29}$$

*(2) For every $\bar{\pi}_1$-vertical Chetaev vector field $Z$ on $Q$*

$$J^1\gamma^* i_Z\bar{\alpha} = 0 \tag{30}$$

*where $\bar{\alpha}$ is any representative of the class $[\bar{\alpha}]$.*

*(3) Along* $J^2\gamma$,

$$\bar{A}_s + \bar{B}_{sr}\ddot{q}^r = 0\,, \quad 1 \le s \le m-k\,. \tag{31}$$

The above equations are called *reduced nonholonomic equations* [4]. Remarkably, reduced equations do not contain Lagrange multipliers.

## 4 The nonholonomic variational principle

We shall briefly recall a variational principle proposed in [6], providing reduced nonholonomic equations as equations for extremals.

Consider a Lagrangian $\lambda$ on $J^1Y$, let $\theta_\lambda$ be its Cartan form. Let $\iota\colon Q \to J^1Y$ be a nonholonomic constraint, $\mathcal{C}$ the canonical distribution. Denote by $\mathcal{S}_{[a,b]}(\bar{\pi}_1)$ the set of sections of $\bar{\pi}_1$, defined around an interval $[a,b] \subset \mathbb{R}$, $a<b$. By *constrained action* we mean the function

$$\mathcal{S}_{[a,b]}(\bar{\pi}_1) \ni \delta \to \int_a^b \delta^*\iota^*\theta_\lambda \in \mathbb{R}\,. \tag{32}$$

Given a $\bar{\pi}_1$-projectable vector field $Z \in \mathcal{C}$, denote by $\phi$ and $\phi_0$ the flows of $Z$ and its projection $Z_0$, respectively. The one-parameter family $\{\delta_u\}$ of sections of $\bar{\pi}_1$, where $\delta_u = \phi_u\,\delta\,\phi_{0u}^{-1}$, is called *constrained variation of* $\delta$ induced by $Z$. The function

$$\mathcal{S}_{[a,b]}(\bar{\pi}_1) \ni \delta \to \left(\frac{d}{du}\int_{\phi_{0u}([a,b])} \delta_u^*\,\iota^*\theta_\lambda\right)_{u=0} = \int_a^b \delta^*\,\mathcal{L}_Z\,\iota^*\theta_\lambda \in \mathbb{R} \tag{33}$$

is then the *first constrained variation* of the action function of $\lambda$ over $[a,b]$, *induced by* $Z$. Restricting the domain of definition $\mathcal{S}_{[a,b]}(\bar{\pi}_1)$ of the function (33) to the subset $\mathcal{S}^h_{[a,b]}(\bar{\pi}_1)$ of *holonomic* sections of the projection $\bar{\pi}_1$, i.e. $\delta = J^1\gamma$ where $\gamma \in \mathcal{S}_{[a,b]}(\pi)$, one can regard the first constrained variation (33) as a function

$$\mathcal{S}_{[a,b],Q}(\pi) \ni \gamma \to \int_a^b J^1\gamma^*\,\mathcal{L}_Z\,\iota^*\theta_\lambda \in \mathbb{R} \tag{34}$$

defined on a *subset of sections of the projection* $\pi\colon Y \to \mathbb{R}$. Applying to (34) Cartan's formula for the decomposition of Lie derivative we obtain the *nonholonomic first variation formula*

$$\int_a^b J^1\gamma^*\,\mathcal{L}_Z\iota^*\theta_\lambda = \int_a^b J^1\gamma^*\,i_Z\iota^*d\theta_\lambda + \int_a^b J^1\gamma^*\,di_Z\iota^*\theta_\lambda\,, \tag{35}$$

giving us the splitting of the first constrained variation to a "constrained Euler--Lagrange term" and a boundary term.

A section $\gamma$ of $\pi$ is called a *constrained extremal* of $\lambda$ on $[a,b]$ if $\mathrm{Im}\,J^1\gamma \subset Q$, and if the first constraint variation of the action on the interval $[a,b]$ vanishes for every "fixed endpoints" variation $Z$ over $[a,b]$. $\gamma$ is called a *constrained extremal of* $\lambda$ if it is its constrained extremal on every interval $[a,b] \subset \mathrm{Dom}\,\gamma$.

**Theorem 5.** *Consider a Lagrangian* $\lambda$ *on* $J^1Y$ *and a nonholonomic constraint. Let* $\gamma\colon I \to Y$ *be a constrained section. The following conditions are equivalent:*

*(1)* $\gamma$ *is a constrained extremal of* $\lambda$.

*(2) For every* $\bar{\pi}_1$*-vertical Chetaev vector field* $Z$ *on* $Q$

$$J^1\gamma^* i_Z \iota^* d\theta_\lambda = 0. \tag{36}$$

*(3) Along* $J^2\gamma$,

$$\frac{\partial_c \bar{L}}{\partial q^s} - \frac{d_c}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^s} - \bar{L}_a\Big(\frac{\partial_c g^a}{\partial q^s} - \frac{d_c}{dt}\frac{\partial g^a}{\partial \dot{q}^s}\Big) = 0\,, \quad 1 \le s \le m-k\,, \tag{37}$$

*where* $\bar{L} = L \circ \iota$ *and*

$$\bar{L}_a = \frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota\,. \tag{38}$$

The proof uses the same techniques as the proof of the similar assertion in the unconstrained case. Keeping the above notations, we can see that for a Lagrangian system the corresponding constrained system is

$$[\bar{\alpha}] = [\iota^* d\theta_\lambda], \tag{39}$$

the constrained dynamical form is

$$\bar{E}^c_\lambda = \Big(\frac{\partial_c \bar{L}}{\partial q^s} - \frac{d_c}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^s} - \bar{L}_a\Big(\frac{\partial_c g^a}{\partial q^s} - \frac{d_c}{dt}\frac{\partial g^a}{\partial \dot{q}^s}\Big)\Big)\bar{\omega}^s \wedge dt, \tag{40}$$

and

$$\bar{A}_s = \frac{\partial_c \bar{L}}{\partial q^s} - \frac{d'_c}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^s} - \bar{L}_a\Big(\frac{\partial_c g^a}{\partial q^s} - \frac{d'_c}{dt}\frac{\partial g^a}{\partial \dot{q}^s}\Big) \tag{41}$$

$$\bar{B}_{sr} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^r \partial \dot{q}^s} + \bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s}. \tag{42}$$

We call equations (36) or (37) *constrained Euler-Lagrange equations*, $\bar{E}^c_\lambda$ the *constrained Euler-Lagrange form*, and its components *constrained Euler-Lagrange expressions*.

In what follows, we use the following notations:

$$\varepsilon_s = \frac{\partial_c}{\partial q^s} - \frac{d_c}{dt}\frac{\partial}{\partial \dot{q}^s}\,, \qquad \varepsilon'_s = \frac{\partial_c}{\partial q^s} - \frac{d'_c}{dt}\frac{\partial}{\partial \dot{q}^s}\,. \tag{43}$$

Finally, let us recall the fundamental relation between well-known Chetaev equations (with Lagrange multipliers) [2] and reduced equations (without Lagrange multipliers) [4], [13]:

**Theorem 6.** *A constrained section* $\gamma$ *of* $\pi$ *is a solution of constrained Euler-Lagrange equations (36) or (37) if and only if it is a solution of Chetaev equations*

$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma} = \lambda_a \frac{\partial f^a}{\partial \dot{q}^\sigma}\,. \tag{44}$$

It is worth note that for *semiholonomic constraints* one has $\varepsilon_s(g^a) = 0$ [5], so that the constrained Euler-Lagrange equations simplify to

$$\frac{\partial_c \bar{L}}{\partial q^s} - \frac{d_c}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^s} = 0\,. \tag{45}$$

## 5 The local inverse variational problem for nonholonomic systems

Now we are prepared to generalize the inverse variational problem to nonholonomic mechanics. In what follows we consider a constraint $\iota : Q \to J^1Y$. Given a system of second order differential equations on $Q$, (31), the question is if the equations are *constraint variational*, i.e. if they come from a constrained variational functional as equations for constrained extremals. Similarly as in the unconstrained case, the problem has several different formulations: local and global, direct (covariant) and contravariant (variational multipliers). We shall deal with the *local inverse problem in covariant form* (for equations "as they stand"), so that in what follows, $Y = \mathbb{R} \times \mathbb{R}^m$ and $J^1Y = \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$.

More precisely, consider a system of mixed first order and second order ODE's

$$\begin{aligned} \dot q^{m-k+a} - g^a(t, q^\sigma, \dot q^l) &= 0\,, \quad 1 \le a \le k\,, \\ \bar A_s(t, q^\sigma, \dot q^l) + \bar B_{sr}(t, q^\sigma, \dot q^l)\ddot q^r &= 0\,, \quad 1 \le s \le m-k \end{aligned} \tag{46}$$

for sections $\gamma\colon I \to Y$. The equations give rise to a nonholonomic constraint $Q \subset J^1Y$ of corank $k$, with the canonical distribution $\mathcal{C}$, and a constrained system, represented either by a class of first order 2-forms

$$\bar\alpha = \bar A_s \bar\omega^s \wedge dt + \bar B_{sr}\bar\omega^s \wedge d\dot q^r + F + \nu \tag{47}$$

where $F$ is a 2-contact and $\nu$ is a constraint form on $Q$, or, by a constrained dynamical form

$$\bar E^{\mathrm{c}} = (\bar A_s + \bar B_{sr}\ddot q^r)\bar\omega^s \wedge dt \tag{48}$$

on $\hat Q$.

**Definition 1.** A constrained dynamical form $\bar E^c$ on $Q$ will be called *constraint variational* if there exist $m+1$ functions $\bar L$, $\bar L_a$ such that

$$\bar A_s + \bar B_{sr}\ddot q^r = \frac{\partial_{\mathrm{c}}\bar L}{\partial q^s} - \frac{d_{\mathrm{c}}}{dt}\frac{\partial \bar L}{\partial \dot q^s} - \bar L_a\Big(\frac{\partial_{\mathrm{c}} g^a}{\partial q^s} - \frac{d_{\mathrm{c}}}{dt}\frac{\partial g^a}{\partial \dot q^s}\Big). \tag{49}$$

A system of equations (46) is called *constraint variational* (as it stands) if the corresponding constrained dynamical form $\bar E^c = (\bar A_s + \bar B_{sr}\ddot q^r)\bar\omega^s \wedge dt$ is constraint variational.

Note that if a system of equations (a constrained dynamical form) is constraint variational, and $\bar L$, $\bar L_a$ are the corresponding "constraint Lagrange functions" then the *constraint Lagrangian* takes the form

$$\lambda_{\mathrm{c}} = \bar L\, dt + \bar L_a \varphi^a, \tag{50}$$

and the action is

$$\mathcal{S}_{[a,b]}(\bar\pi_1) \ni \delta \to \int_a^b \delta^*\theta_{\lambda_{\mathrm{c}}} \in \mathbb{R}\,, \tag{51}$$

where $\theta_{\lambda_{\mathrm{c}}}$ is the constraint Lepage equivalent of $\lambda_{\mathrm{c}}$ (constraint Cartan form) as introduced in [10]; in coordinates,

$$\theta_{\lambda_{\mathrm{c}}} = \bar L\, dt + \frac{\partial \bar L}{\partial \dot q^s}\bar\omega^s + \bar L_a \varphi^a. \tag{52}$$

Immediately from the definition we can see that *given an unconstrained Lagrangian system, the corresponding constrained system is constraint variational for any nonholonomic constraint.* Indeed, in this case,

$$\bar{L} = L \circ \iota, \quad \bar{L}_a = \frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota, \tag{53}$$

and

$$\theta_{\lambda_c} = \iota^* \theta_\lambda, \tag{54}$$

where $\lambda = L\, dt$ is a first order Lagrangian for the given variational dynamical form.

On the other hand, as we shall see below, a nonholonomic system which is constraint variational may arise from a *non-variational* unconstrained system on $J^1Y$. Moreover, such an unconstrained system need not be unique.

We have the following main theorem on constraint variationality of reduced equations on nonholonomic manifolds:

**Theorem 7.** *Let $\bar{E}^c$ be a constrained dynamical form, $[\bar{\alpha}]$ the corresponding class of 2-forms. $\bar{E}^c$ is constraint variational if and only if in a neighborhood of every point in $Q$ the class $[\bar{\alpha}]$ has a closed representative.*

*Proof.* If $\bar{E}^c$ is constraint variational, we have Lagrange functions $\bar{L}$, $\bar{L}_a$ such that

$$\bar{E}^c = \Big(\bar{A}_s + \bar{B}_{sr}\ddot{q}^r\Big)\bar{\omega}^s \wedge dt, \tag{55}$$

with

$$\bar{A}_s = \varepsilon'_s(\bar{L}) - \bar{L}_a \varepsilon'_s(g^a), \quad \bar{B}_{sr} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^s \partial \dot{q}^r} + \bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^s \partial \dot{q}^r}\,. \tag{56}$$

Putting

$$\rho = \bar{L}\, dt + \frac{\partial \bar{L}}{\partial \dot{q}^s}\bar{\omega}^s + \bar{L}_a \varphi^a \tag{57}$$

we obtain

$$d\rho \sim \Big(\varepsilon'_s(\bar{L}) - \bar{L}_a\, \varepsilon'_s(g^a)\Big)\bar{\omega}^s \wedge dt + \Big(\bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s} - \frac{\partial^2 \bar{L}}{\partial \dot{q}^r \partial \dot{q}^s}\Big)\bar{\omega}^s \wedge d\dot{q}^r \in [\bar{\alpha}] \tag{58}$$

as desired.

Let us show the converse. Given $\bar{E}^c = (\bar{A}_s + \bar{B}_{sr}\ddot{q}^r)\bar{\omega}^s \wedge dt$, let

$$\begin{aligned} \bar{\alpha} = \bar{A}_s \bar{\omega}^s \wedge dt + \bar{B}_{rs}\bar{\omega}^r \wedge d\dot{q}^s + \bar{F}_{rs}\bar{\omega}^r \wedge \bar{\omega}^s \\ + \varphi^a \wedge (b_a dt + b_{as}\bar{\omega}^s + c_{as} d\dot{q}^s) + \gamma_{ab}\varphi^a \wedge \varphi^b \end{aligned} \tag{59}$$

where $\bar{F}_{rs} = -\bar{F}_{sr}$ and $\gamma_{ab} = -\gamma_{ba}$, be a 2-form belonging to the class $[\bar{\alpha}]$ of $\bar{E}^c$, and assume that it is closed. Then $\bar{\alpha} = d\rho$ where $\rho$ is a local 1-form on $Q$, i.e. in coordinates it reads as follows:

$$\rho = \rho^0 dt + \rho^1_s \bar{\omega}^s + \rho^2_a \varphi^a + \rho^3_s d\dot{q}^s. \tag{60}$$

Computing $d\rho$ and equating its components with those of (59) we can immediately see that the term $d\dot{q}^r \wedge d\dot{q}^s$ is missing in $\bar{\alpha}$. Hence

$$\frac{\partial \rho^3_s}{\partial \dot{q}^r} = \frac{\partial \rho^3_r}{\partial \dot{q}^s}, \tag{61}$$

i.e.

$$\rho^3_s = \frac{\partial h}{\partial \dot{q}^s} + h_s(t, q^\sigma), \tag{62}$$

meaning that $\rho$ is of the form

$$\begin{aligned}\rho = \Big(\rho^0 - \frac{d'_{\mathrm{c}} h}{dt} - \dot{q}^s \frac{d'_{\mathrm{c}} h_s}{dt}\Big) dt + \Big(\rho^1_s - \frac{\partial_{\mathrm{c}} h}{\partial q^s} - \dot{q}^r \frac{\partial_{\mathrm{c}} h_r}{\partial q^s}\Big)\bar{\omega}^s \\ + \Big(\rho^2_a - \frac{\partial h}{\partial q^{m-k+a}} - \dot{q}^s \frac{\partial h_s}{\partial q^{m-k+a}}\Big)\varphi^a + d(h + h_s \dot{q}^s)\,.\end{aligned} \tag{63}$$

We conclude that without loss of generality we may assume $\bar{\alpha} = d\bar{\rho}$ where

$$\bar{\rho} = \bar{L}\, dt + f_s \bar{\omega}^s + \bar{L}_a \varphi^a. \tag{64}$$

Comparing now $d\bar{\rho}$ with $\bar{\alpha}$ and accounting that

$$\begin{aligned}d\varphi^a = -\varepsilon'_s(g^a)\bar{\omega}^s \wedge dt + \Big(\frac{\partial_{\mathrm{c}}}{\partial q^r}\frac{\partial g^a}{\partial \dot{q}^s}\Big)\bar{\omega}^s \wedge \bar{\omega}^r + \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s}\bar{\omega}^s \wedge d\dot{q}^r \\ - \frac{\partial g^a}{\partial q^{m-k+b}}\varphi^b \wedge dt - \Big(\frac{\partial}{\partial q^{m-k+b}}\frac{\partial g^a}{\partial \dot{q}^s}\Big)\varphi^b \wedge \bar{\omega}^s\end{aligned} \tag{65}$$

we obtain:

$$f_s = \frac{\partial \bar{L}}{\partial \dot{q}^s}\,, \tag{66}$$

and

$$\bar{A}_s = \frac{\partial_{\mathrm{c}} \bar{L}}{\partial q^s} - \frac{d'_{\mathrm{c}} f_s}{dt} - \bar{L}_a \varepsilon'_s(g^a), \quad \bar{B}_{rs} = -\frac{\partial f_r}{\partial \dot{q}^s} + \bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s} \tag{67}$$

proving that $\bar{E}^{\mathrm{c}}$ is constraint variational. Moreover, we find expressions for the other components of $\bar{\alpha}$ by means of $\bar{L}$ and $\bar{L}_a$ as follows:

$$\bar{F}_{rs} = \frac{1}{2}\bigg(\Big(\frac{\partial_{\mathrm{c}} f_s}{\partial q^r} - \frac{\partial_{\mathrm{c}} f_r}{\partial q^s}\Big) - \bar{L}_a\Big(\frac{\partial_{\mathrm{c}}}{\partial q^r}\frac{\partial g^a}{\partial \dot{q}^s} - \frac{\partial_{\mathrm{c}}}{\partial q^s}\Big(\frac{\partial g^a}{\partial \dot{q}^r}\Big)\Big)\bigg) \tag{68}$$

and

$$\begin{aligned}b_a &= \frac{\partial \bar{L}}{\partial q^{m-k+a}} - \frac{d'_{\mathrm{c}} \bar{L}_a}{dt} - \bar{L}_b \frac{\partial g^b}{\partial q^{m-k+a}} \\ b_{as} &= \frac{\partial f_s}{\partial q^{m-k+a}} - \frac{\partial_{\mathrm{c}} \bar{L}_a}{\partial q^s} - \bar{L}_b \frac{\partial}{\partial q^{m-k+a}}\Big(\frac{\partial g^b}{\partial \dot{q}^s}\Big) \\ c_{as} &= -\frac{\partial \bar{L}_a}{\partial \dot{q}^s} \\ \gamma_{ab} &= \frac{1}{2}\Big(\frac{\partial \bar{L}_b}{\partial q^{m-k+a}} - \frac{\partial \bar{L}_a}{\partial q^{m-k+b}}\Big)\end{aligned} \tag{69}$$

□

Notice that in the class $[\bar{\alpha}]$ we have three distinguished representatives: $\bar{\alpha}_1 = d\bar{\rho}$ with components as above,

$$\bar{\alpha}_2 = \Big(\varepsilon'_s(\bar{L}) - \bar{L}_a \varepsilon'_s(g^a)\Big)\bar{\omega}^s \wedge dt - \Big(\frac{\partial^2 \bar{L}}{\partial \dot{q}^r \partial \dot{q}^s} - \bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s}\Big)\bar{\omega}^r \wedge d\dot{q}^s \tag{70}$$

and

$$\begin{aligned}\bar{\alpha}_3 = \Big(\varepsilon'_s(\bar{L}) - \bar{L}_a \varepsilon'_s(g^a)\Big)\bar{\omega}^s \wedge dt + \Big(\frac{\partial_{\mathrm{c}}}{\partial q^r}\frac{\partial \bar{L}}{\partial \dot{q}^s} - \bar{L}_a \Big(\frac{\partial_{\mathrm{c}}}{\partial q^r}\frac{\partial g^a}{\partial \dot{q}^s}\Big)\Big)\bar{\omega}^r \wedge \bar{\omega}^s \\ - \Big(\frac{\partial^2 \bar{L}}{\partial \dot{q}^r \partial \dot{q}^s} - \bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s}\Big)\bar{\omega}^r \wedge d\dot{q}^s \,.\end{aligned} \tag{71}$$

The following theorem provides variationality conditions of reduced equations, called *constraint Helmholtz conditions*, first obtained in [9].

**Theorem 8.** *Let $\bar{E}^{\mathrm{c}}$ be a constrained dynamical form, $[\bar{\alpha}]$ the corresponding class of 2-forms. $\bar{E}^{\mathrm{c}}$ is constraint variational if and only if (locally) there exist functions $b_a$, $c_{as}$ and $\gamma_{ab}$ on $Q$ (i.e. functions of variables $(t, q^\sigma, \dot{q}^l)$) such that $\gamma_{ab} = -\gamma_{ba}$, the $\gamma$'s are solutions of the equations*

$$\Big(\frac{d'_{\mathrm{c}}\gamma_{ab}}{dt} - 2\gamma_{bc}\frac{\partial g^c}{\partial q^{m-k+a}} - \frac{\partial b_a}{\partial q^{m-k+b}}\Big)_{\mathrm{alt}(ab)} = 0\,, \tag{72}$$

*and the following conditions hold*

$$\begin{aligned}(\bar{B}_{ls})_{\mathrm{alt}(ls)} &= 0 \\ \Big(\frac{\partial \bar{B}_{ls}}{\partial \dot{q}^r} - \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^r} c_{as}\Big)_{\mathrm{alt}(sr)} &= 0 \\ \Big(\frac{\partial \bar{A}_l}{\partial \dot{q}^s} - \varepsilon'_l(g^a)c_{as} - \frac{d'_{\mathrm{c}}\bar{B}_{ls}}{dt} - \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s} b_a\Big)_{\mathrm{sym}(ls)} &= 0 \\ \Big(-\frac{\partial_{\mathrm{c}}\bar{A}_l}{\partial q^s} + \varepsilon'_l(g^a)b_{as} + \frac{1}{2}\frac{d'_{\mathrm{c}}}{dt}\Big(\frac{\partial \bar{A}_l}{\partial \dot{q}^s} - \varepsilon'_l(g^a)c_{as}\Big) + b_a \frac{\partial_{\mathrm{c}}}{\partial q^s}\Big(\frac{\partial g^a}{\partial \dot{q}^l}\Big)\Big)_{\mathrm{alt}(ls)} &= 0 \\ \frac{\partial \bar{A}_l}{\partial q^{m-k-a}} + 2\gamma_{ac}\varepsilon'_l(g^c) - \frac{\partial_{\mathrm{c}} b_a}{\partial q^l} - b_c \frac{\partial^2 g^c}{\partial \dot{q}^l \partial q^{m-k+a}} + \frac{d'_{\mathrm{c}} b_{al}}{dt} + \frac{\partial g^c}{\partial q^{m-k+a}} b_{cl} &= 0 \\ \frac{\partial \bar{B}_{ls}}{\partial q^{m-k+a}} - 2\gamma_{ab}\frac{\partial^2 g^b}{\partial \dot{q}^l \partial \dot{q}^s} + \frac{\partial b_{al}}{\partial \dot{q}^s} - \frac{\partial_{\mathrm{c}} c_{as}}{\partial q^l} - \frac{\partial^2 g^b}{\partial \dot{q}^l \partial q^{m-k+a}} c_{bs} &= 0\end{aligned} \tag{73}$$

*where*

$$b_{as} = \frac{\partial b_a}{\partial \dot{q}^s} - \frac{d'_{\mathrm{c}} c_{as}}{dt} - \frac{\partial g^b}{\partial q^{m-k+a}} c_{bs}\,. \tag{74}$$

*Proof.* By the preceding theorem the result comes from the condition $d\bar{\alpha} = 0$ where $\bar{\alpha}$ is given by (59). □

Notice that by the above computation we obtain for components of the 2-form $F$ the following formula

$$\bar{F}_{rs} = \frac{1}{4}\left(\Big(\frac{\partial \bar{A}_r}{\partial \dot{q}^s} - \frac{\partial \bar{A}_s}{\partial \dot{q}^r}\Big) - \big(\varepsilon'_r(g^a)c_{as} - \varepsilon'_s(g^a)c_{ar}\big)\right)\,, \tag{75}$$

which is just another expression of (68).

Compared with Helmholtz conditions, the constraint Helmholtz conditions have a rather surprising form. While the former are *identities* to be fulfilled by the components of a dynamical form (i.e. by the functions on the left-hand sides of the corresponding equations), the latter are rather *equations* for unknown functions $b_a$, $c_{as}$ and $\gamma_{ab}$. This means that for a system of equations (46), if the answer to the question on constraint variationality is affirmative, the corresponding constraint Lagrangian form need not be unique. This is closely related with the yet unsolved problem on the structure of constraint null Lagrangians.

## 6 Examples: Planar motions

In this section we shall study examples of various simple mechanical systems, namely planar systems subject to one nonholonomic constraint. This means that we have one reduced equation of motion in this case. In the notation used so far, $m = 2$, $k = 1$, $Y = \mathbb{R} \times \mathbb{R}^2$; coordinates in the plane will be denoted by $(x, y)$.

The unconstrained equations of motion are of the form

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -F_1\,, \quad \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = -F_2\,, \tag{76}$$

where the force on the right-hand side generally is not assumed variational. The functions $B_{\sigma\nu}$ are the same for any (variational and non-variational) force, obstructions to variationality may enter only through additional terms to $A_\sigma$, i.e. $A_\sigma \to \mathcal{A}_\sigma = A_\sigma + F_\sigma$.

A nonholonomic constraint in $J^1(\mathbb{R} \times \mathbb{R}^2)$ is given by equation

$$\dot{y} = g(t,\, x,\, y,\, \dot{x})\,, \tag{77}$$

so that

$$\varphi^1 = dy - \frac{\partial g}{\partial \dot{x}}\,dx - \Big(g - \dot{x}\frac{\partial g}{\partial \dot{x}}\Big)dt\,, \tag{78}$$

and the reduced equation of motion takes the form (37) modified by $\Phi$, i.e.

$$\frac{\partial_{\mathrm{c}}\bar{L}}{\partial x} - \frac{d_{\mathrm{c}}}{dt}\frac{\partial \bar{L}}{\partial \dot{x}} - \bar{L}_1\Big(\frac{\partial_{\mathrm{c}} g}{\partial x} - \frac{d_{\mathrm{c}}}{dt}\frac{\partial g}{\partial \dot{x}}\Big) = -\bar{\Phi}, \tag{79}$$

where

$$\bar{\Phi} = \bar{F}_1 + \bar{F}_2\frac{\partial g}{\partial \dot{x}}, \quad \bar{F}_\sigma = F_\sigma \circ \iota. \tag{80}$$

The constraint Helmholtz conditions (73) reduce to the following equations for functions $b_1$ and $c_{11}$ (due to skew symmetry, $\gamma_{11} = 0$):

$$\begin{gathered}
\frac{\partial \bar{A}_1}{\partial \dot{x}} - \varepsilon'_1(g)c_{11} - \frac{d'_{\mathrm{c}}\bar{B}_{11}}{dt} - \frac{\partial^2 g}{\partial \dot{x}^2}b_1 = 0 \\
\frac{\partial \bar{A}_1}{\partial y} - \frac{\partial_{\mathrm{c}} b_1}{\partial x} - \frac{\partial^2 g}{\partial \dot{x}\partial y}b_1 + \frac{d'_{\mathrm{c}} b_{11}}{dt} + \frac{\partial g}{\partial y}b_{11} = 0 \\
\frac{\partial \bar{B}_{11}}{\partial y} + \frac{\partial b_1}{\partial \dot{x}} - \frac{\partial_{\mathrm{c}} c_{11}}{\partial x} - \frac{\partial^2 g}{\partial \dot{x}\partial y}c_{11} = 0
\end{gathered} \tag{81}$$

where

$$b_{11} = \frac{\partial b_1}{\partial \dot{x}} - \frac{d'_{\mathrm{c}} c_{11}}{dt} - \frac{\partial g}{\partial y} c_{11}\,. \tag{82}$$

Recall that conditions (81) are fulfilled for every constrained system arising from an unconstrained Lagrangian one. Adding the force $(F_1, F_2)$ to equations of motion, the reduced equation changes by $\bar{\Phi}$, and the first two conditions (81) by $\frac{\partial \Phi}{\partial \dot{x}}$ and $\frac{\partial \Phi}{\partial y}$, respectively. Hence for a Lagrangian system in a force field $(F_1, F_2)$ the constraint Helmholtz conditions are fulfilled trivially for every constraint satisfying the following compatibility condition:

$$\bar{\Phi} = \quad \bar{F}_1 + \bar{F}_2 \frac{\partial g}{\partial \dot{x}} \quad = \quad \chi(t, x), \tag{83}$$

where $\chi(t, x)$ is an arbitrary function. For such a case equations (81) retain the same solution $(b_1, b_{11}, c_{11})$ as in the case without additional forces. Moreover, if $\chi(t, x) = 0$, the "free" Lagrangian system (i.e. with $F_1 = F_2 = 0$) and that (essentially different!) moving in a constraint-compatible force field $(F_1, F_2) \neq 0$ have *the same* reduced motion equation.

### 6.1 Motion in a homogeneous field

Let us consider the motion of a mass particle $m$ in a homogeneous field, for concreteness e.g. in the gravitational field $\vec{G}$. Such a particle moves in a plane $xOy$ along a parabolic trajectory (so called *parabolic* or *projectile motion*),

$$x(t) = vt\cos\alpha, \quad y(t) = vt\sin\alpha - \frac{1}{2}Gt^2,$$

where $\vec{v} = (v\cos\alpha, v\sin\alpha)$ is the initial velocity. The unconstrained system is variational, with the Lagrangian

$$\lambda = L\,dt, \quad L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mGy, \tag{84}$$

the corresponding dynamical form is

$$E_\lambda = -m\ddot{x}\,dx \wedge dt - m(\ddot{y} + G)dy \wedge dt. \tag{85}$$

Consider a constraint (77). Then

$$\bar{B}_{11} = -m\left(1 + \left(\frac{\partial g}{\partial \dot{x}}\right)^2\right), \quad \bar{A}_1 = -m\frac{\partial g}{\partial \dot{x}}\left(G + \frac{d'_c g}{dt}\right), \tag{86}$$

so that the reduced equation is of the form

$$-m\ddot{x}\left(1 + \left(\frac{\partial g}{\partial \dot{x}}\right)^2\right) - m\frac{\partial g}{\partial \dot{x}}\left(G + \frac{d'_c g}{dt}\right) \quad = \quad 0. \tag{87}$$

Since the unconstrained system is Lagrangian, the arising constrained system is constraint variational for any nonholonomic constraint. This means, of course, that the constraint Helmholtz conditions have a solution (certain functions $b_1$, $b_{11}$, $c_{11}$) for every fixed constraint (77).

Now, let us consider the question on constraint variationality of equation (87) from the other side: Let us try to find a solution of the inverse problem directly, by solving the constraint Helmholtz conditions as equations for $b_1$ and $c_{11}$.

Accounting commutation relations for constraint derivative operators, the constraint Helmholtz conditions take the form

$$-\frac{\partial^2 g}{\partial \dot{x}^2}\left(m\left(G+\frac{d'_{\mathrm{c}}g}{dt}\right)+b_1\right)-\varepsilon'_1(g)\left(c_{11}+m\frac{\partial g}{\partial \dot{x}}\right)=0\,, \tag{88}$$

$$-\frac{\partial^2 g}{\partial \dot{x}\partial y}\left(m\left(G+\frac{d'_{\mathrm{c}}g}{dt}\right)+b_1\right)-m\frac{\partial g}{\partial \dot{x}}\frac{\partial}{\partial y}\left(\frac{d'_{\mathrm{c}}g}{dt}\right)-\frac{\partial_{\mathrm{c}}b_1}{\partial x}+\frac{d'_{\mathrm{c}}b_{11}}{dt}+b_{11}\frac{\partial g}{\partial y}=0\,, \tag{89}$$

$$-\frac{\partial^2 g}{\partial \dot{x}\partial y}\left(c_{11}+2m\frac{\partial g}{\partial \dot{x}}\right)+\frac{\partial b_{11}}{\partial \dot{x}}-\frac{\partial_{\mathrm{c}}c_{11}}{\partial x}=0\,, \tag{90}$$

with

$$b_{11}=\frac{\partial b_1}{\partial \dot{x}}-\frac{d'_{\mathrm{c}}c_{11}}{dt}-\frac{\partial g}{\partial y}c_{11}\,. \tag{91}$$

Condition (88) can be fulfilled e.g. for functions $b_1$ and $c_{11}$ of the form

$$b_1=-m\left(G+\frac{d'_{\mathrm{c}}g}{dt}\right), \tag{92}$$

$$c_{11}=-m\frac{\partial g}{\partial \dot{x}}. \tag{93}$$

Then

$$b_{11}=-m\frac{\partial g}{\partial x}\,. \tag{94}$$

It can be verified by a direct calculation that with the above choice of functions $b_1$, $c_{11}$ and $b_{11}$ the remaining two constraint Helmholtz conditions (89) and (90) are satisfied. In this way we have obtained that the reduced equation (87) is indeed constraint variational.

We can ask the question if the above solution to the constraint Helmholtz conditions is in correspondence with the original (unconstrained) system, since, in principle, the obtained $b_1$, $b_{11}$, and $c_{11}$ could correspond to a different unconstrained Lagrangian system having the same reduced equation of motion. To this end let us compute the corresponding functions related with the Lagrangian (84); let us use notations $b_1(L)$, $c_{11}(L)$, and $b_{11}(L)$ to distinguish them from the $b_1$, $c_{11}$, and $b_{11}$ above.

We have

$$\bar{L}=\frac{1}{2}m\dot{x}^2+\frac{1}{2}mg^2-mGy\,,\quad \bar{L}_1=mg\,, \tag{95}$$

hence, by (69)

$$
\begin{aligned}
b_1(L) &= \frac{\partial \bar{L}}{\partial y} - \frac{d'_c \bar{L}_1}{dt} - \bar{L}_1 \frac{\partial g}{\partial y} = -m\left(G + \frac{d'_c g}{dt}\right) = b_1, \\
b_{11}(L) &= \frac{\partial^2 \bar{L}}{\partial y \partial \dot{x}} - \frac{\partial_c \bar{L}_1}{\partial x} - \bar{L}_1 \frac{\partial^2 g}{\partial y \partial \dot{x}} = -m\frac{\partial g}{\partial x} = b_{11}, \\
c_{11}(L) &= -\frac{\partial \bar{L}_1}{\partial \dot{x}} = -m\frac{\partial g}{\partial \dot{x}} = c_{11}.
\end{aligned}
$$

Finally, the constraint Lagrangian and the constraint Cartan form read

$$
\begin{aligned}
\lambda_{\mathrm{c}} &= \Big(\frac{1}{2}m(\dot{x}^2 + g^2) - mGy\Big)\, dt + mg\varphi^1, \\
\bar{\rho} &= \lambda_{\mathrm{c}} + m\Big(\dot{x} + g\frac{\partial g}{\partial \dot{x}}\Big)(dx - \dot{x}\, dt)\,.
\end{aligned}
\tag{96}
$$

**Remark 1.** An interesting constraint for the Lagrangian system (84) was considered in [14], namely

$$
\dot{y} = \sqrt{v^2 - \dot{x}^2}. \tag{97}
$$

In this case the reduced equation has the form

$$
\begin{aligned}
-\frac{mv^2}{v^2 - \dot{x}^2}\ddot{x} + \frac{mG\dot{x}}{\sqrt{v^2 - \dot{x}^2}} = 0 \quad &\Longrightarrow \\
\ddot{x} - \frac{G}{v^2}\dot{x}\sqrt{v^2 - \dot{x}^2} = 0 &
\end{aligned}
\tag{98}
$$

and it can be solved analytically (see [14] for the solution and conservation laws).

The functions $b_1$, $c_{11}$ and $b_{11}$ given by (92), (93) and (94).take the form

$$
b_1 = -mG, \quad c_{11} = \frac{m\dot{x}}{\sqrt{v^2 - \dot{x}^2}}, \quad b_{11} = 0,
$$

and a constraint Lagrangian is

$$
\lambda_{\mathrm{c}} = -mGy\, dt + m\sqrt{v^2 - \dot{x}^2}\varphi^1.
$$

One can easily verify that, indeed, $\varepsilon_1(\bar{L}) - \bar{L}_1\varepsilon_1(g)$ is the left-hand-side of the reduced equation (98).

There are, however, also other solutions $b_1$, $b_{11}$ and $c_{11}$ of the constraint Helmholtz conditions. One of them is $b_1 = -mG$, $b_{11} = 0$, $c_{11} = 0$ as can be easily verified substituting into (81). A corresponding constraint Lagrangian, leading to the same reduced equation (98), is then

$$
\lambda'_{\mathrm{c}} = \bar{L}'\, dt\,,
$$

where

$$
\bar{L}' = -mGy + L_0, \quad L_0 = \frac{1}{2}mv\left((v + \dot{x})\ln(v + \dot{x}) + (v - \dot{x})\ln(v - \dot{x})\right)\,.
$$

The 1-form

$$\tau = {\lambda_c}' - \lambda_c = L_0 dt - m\sqrt{v^2 - \dot{x}^2}\,\varphi^1$$

leads to identically zero left-hand side of the reduced equation and thus it is a null constraint Lagrangian.

We can see that the constraint Lagrangian $\bar{L}'\,dt$ can be extended e.g. to the Lagrangian $\left(\bar{L}' + L_0'\right)dt$, defined on $J^1(\mathbb{R} \times \mathbb{R}^2)$, where $L_0'$ is a polynomial of at least second degree in the variable $\dot{y} - \sqrt{v^2 - \dot{x}^2}$, for example, one can take simply $L_0' = \frac{1}{2}m\left(\dot{y} - \sqrt{v^2 - \dot{x}^2}\right)^2$. For such an additional Lagrangian it holds $L_0' \circ \iota = 0$ and $\frac{\partial L_0'}{\partial \dot{y}} \circ \iota = 0$. (In general, for a constraint $\dot{y} = g(t, x, y, \dot{x})$ the same is fulfilled for a polynomial of at least second degree in the variable $\dot{y} - g$.) If $L_0' = \frac{1}{2}m\left(\dot{y} - \sqrt{v^2 - \dot{x}^2}\right)^2$ then the corresponding unconstrained equations of motion of $\tilde{\lambda} = (\bar{L}' + L_0')dt$ take the form

$$\begin{aligned} -\frac{mv^2}{v^2 - \dot{x}^2}\left(\frac{\dot{x}^2}{v^2} + \frac{\dot{y}}{\sqrt{v^2 - \dot{x}^2}}\right)\ddot{x} - \frac{m\dot{x}}{\sqrt{v^2 - \dot{x}^2}}\ddot{y} = 0, \\ -mG - \frac{m\dot{x}}{\sqrt{v^2 - \dot{x}^2}}\ddot{x} - m\ddot{y} = 0 \end{aligned} \tag{99}$$

and apparently they are not equivalent with the motion equations of the Lagrangian (84).

### 6.2 Damped motion in a homogeneous field

Let us turn to the case when the unconstrained system is not variational.

Consider the same Lagrangian (84) as above, but now suppose that additionally the motion is damped by Stokes force $\vec{F} = -\beta\vec{v}$, i.e. $(F_\sigma) = (-\beta\dot{x}, -\beta\dot{y})$, where $\beta$ is a positive constant. (The trajectory of the particle is the well-known ballistic curve.)

The dynamical form

$$E = -(m\ddot{x} + \beta\dot{x})dx \wedge dt - (m\ddot{y} + mG + \beta\dot{y})dy \wedge dt \tag{100}$$

is not variational. Denote

$$\mathcal{A}_\sigma = A_\sigma + F_\sigma \tag{101}$$

where $A_\sigma$ corresponds to the undamped (variational) system above.

Given a nonholonomic constraint (77) we obtain

$$\begin{aligned} \bar{B}_{11} &= -m\left(1 + \left(\frac{\partial g}{\partial \dot{x}}\right)^2\right) \\ \bar{\mathcal{A}}_1 &= -m\frac{\partial g}{\partial \dot{x}}\left(G + \frac{d_c' g}{dt}\right) - \beta\left(\dot{x} + g\frac{\partial g}{\partial \dot{x}}\right) \end{aligned} \tag{102}$$

yielding the reduced equation

$$-m\ddot{x}\left(1 + \left(\frac{\partial g}{\partial \dot{x}}\right)^2\right) - m\frac{\partial g}{\partial \dot{x}}\left(G + \frac{d_c' g}{dt}\right) - \beta\left(\dot{x} + g\frac{\partial g}{\partial \dot{x}}\right) = 0 \tag{103}$$

which differs from the preceding (constraint variational) motion equation by an additional force term

$$\bar{\Phi} = -\beta \left( \dot{x} + g \frac{\partial g}{\partial \dot{x}} \right). \tag{104}$$

We shall be interested under what conditions equation (103) is *constraint variational.*

For additional (non-variational) forces $F_1$ and $F_2$ it is necessary to add to constraint Helmholtz conditions (89) and (90) additional terms $\frac{\partial \bar{F}_1}{\partial \dot{x}}$ and $\frac{\partial \bar{F}_1}{\partial y}$, respectively. Condition (88) remains unchanged. Then there is a possibility to fulfill the constraint Helmholtz conditions by a simple way, namely to find such a constraint $g$ for which equation (83) is satisfied. Integrating this equation we obtain

$$g = \sqrt{\phi(t,\, x,\, y) + \dot{x}\chi(t,\, x) - \dot{x}^2}\,, \tag{105}$$

where $\phi(t,\, x,\, y)$ and $\chi(t,\, x)$ are arbitrary functions of indicated variables. For every constraint of this type the constraint Helmholtz conditions are the same as for the undamped case. Let us emphasize that the family of solutions $b_1$, $b_{11}$ and $c_{11}$ of constraint Helmholtz conditions remains unchanged as well. One of these solutions is thus again given by (92), (93) and (94). A corresponding constraint Lagrangian is then, accordingly

$$\lambda_{\mathrm{c}} = \Big(\frac{1}{2} m(\phi + \dot{x}\chi) - mGy)\Big) dt + m\sqrt{\phi + \dot{x}\chi - \dot{x}^2}\,\varphi^1\,.$$

An interesting case occurs for $\phi = 2Gy$, $\chi = 0$. Then $\bar{L} = 0$, hence

$$\lambda_c = m\sqrt{2Gy - \dot{x}^2}\,\varphi^1.$$

So, we can see that there is a possibility to choose a constraint Lagrangian for which $\bar{L} = 0$: this Lagrangian belongs to the constraint ideal. Note that on the other hand, there is no possibility to get $\bar{L}_1 = 0$, i.e. $\lambda_{\mathrm{c}}$ of a form $\bar{L}dt$. The corresponding reduced equation reads

$$\frac{2mG\dot{x}}{\sqrt{2Gy - \dot{x}^2}} - \frac{2mGy}{2Gy - \dot{x}^2}\ddot{x} = 0.$$

**Remark 2.** It is worth note that condition $\bar{\Phi} = 0$ yields the same reduced equation, hence the same constraint dynamics (which, moreover is constraint variational) for essentially different unconstrained systems. In our example this concerns a variational system given by Lagrangian (84) and a non variational one, given by the same Lagrangian and a non-potential Stokes force. Recall that this happens subject a constraint

$$g = \sqrt{\phi - \dot{x}^2}\,. \tag{106}$$

## 7 Example: Relativistic particle

A physically highly interesting example of a constrained system subject to a nonlinear nonholomic costraint is a massive particle in the special relativity theory. It was studied in detail in [8]. It can be modeled with help of an initially variational unconstrained system on $Y = \mathbb{R} \times \mathbb{R}^4$ ($m = 4$, coordinates $(s,\, q^\sigma,\, \dot{q}^\sigma), 1 \leq \sigma \leq 4$) defined by the following Lagrangian

$$L = -\frac{1}{2} m_0 \sqrt{(\dot{q}^4)^2 - \sum_{l=1}^{3} (\dot{q}^l)^2} + \dot{q}^\sigma \phi_\sigma - \psi, \tag{107}$$

where $\phi(q^\sigma)$ and $\psi(q^\sigma)$ are functions on $Y$. The corresponding Euler-Lagrange form reads

$$\begin{aligned} E_\lambda &= \varepsilon_l(L)\, dq^l \wedge dt + \varepsilon_4(L) dq^4 \wedge dt, \quad 1 \leq l \leq 3\,, \\ \varepsilon_l(L) &= B_{ls} \ddot{q}^s + A_l = -m_0 \ddot{q}^l + \dot{q}^\sigma \left( \frac{\partial \phi_\sigma}{\partial q^l} - \frac{\partial \phi_l}{\partial q^\sigma} \right) - \frac{\partial \psi}{\partial q^l}\,, \\ \varepsilon_4(L) &= B_{4s} \ddot{q}^s + A_4 = m_0 \ddot{q}^4 + \dot{q}^\sigma \left( \frac{\partial \phi_\sigma}{\partial q^4} - \frac{\partial \phi_4}{\partial q^\sigma} \right) - \frac{\partial \psi}{\partial q^4}. \end{aligned}$$

The constraint is given by the standard condition for 4-velocity,

$$(\dot{q}^4)^2 - \sum_{p=1}^{3} (\dot{q}^p)^2 = 1 \implies \dot{q}^4 = \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}. \tag{108}$$

For coefficients of reduced equations we obtain (see (26))

$$\begin{aligned} \bar{A}_l = \dot{q}^a \left( \frac{\partial \phi_a}{\partial q^l} - \frac{\partial \phi_l}{\partial q^a} \right) - \frac{\partial \psi}{\partial q^l} + \left( \dot{q}^a \left( \frac{\partial \phi_a}{\partial q^4} - \frac{\partial \phi_4}{\partial q^a} \right) - \frac{\partial \psi}{\partial q^4} \right) \frac{\dot{q}^l}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}} \\ + \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2} \left( \frac{\partial \phi_4}{\partial q^l} - \frac{\partial \phi_l}{\partial q^4} \right), \end{aligned} \tag{109}$$

$$\bar{B}_{ls} = -m_0 \left( \delta_{ls} - \frac{\dot{q}^l \dot{q}^s}{1 + \sum_{p=1}^{3} (\dot{q}^p)^2} \right). \tag{110}$$

Our aim is to find a solution of constraint Helmholtz conditions for the corresponding reduced equations of motion

$$\bar{A}_l + \bar{B}_{ls} \ddot{q}^s = 0.$$

The first of conditions (73) is fulfilled because $\bar{B}_{ls}$ are symmetric. As for the second of conditions (73), it holds

$$\frac{\partial \bar{B}_{ls}}{\partial \dot{q}^r} - \frac{\partial \bar{B}_{lr}}{\partial \dot{q}^s} = m_0 \frac{\delta^l_r \dot{q}^s - \delta^l_s \dot{q}^r}{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}. \tag{111}$$

On the other hand, we have

$$c_{1s} \frac{\partial^2 g}{\partial \dot{q}^l \partial \dot{q}^r} - c_{1r} \frac{\partial^2 g}{\partial \dot{q}^l \partial \dot{s}^s} = \frac{c_{1s} \delta^l_r - c_{1r} \delta^l_s}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}} - \frac{c_{1s} \dot{q}^l \dot{q}^r - c_{1r} \dot{q}^l \dot{q}^s}{(1 + \sum_{p=1}^{3} (\dot{q}^p)^2)^{3/2}}. \tag{112}$$

Comparing (111) and (112) we find a solution

$$c_{1l} = \frac{m_0 \dot{q}^l}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}}, \quad 1 \le l \le 3. \tag{113}$$

Substituting $c_{1l}$ into the third condition in (73) and taking into account that $\varepsilon'_l(g) = 0$ ($g$ depends on $\dot{q}^l$, $1 \le l \le 3$, only), we obtain, after some calculations,

$$b_1 = \dot{q}^l \left( \frac{\partial \phi_l}{\partial q^4} - \frac{\partial \phi_4}{\partial q^l} \right) - \frac{\partial \psi}{\partial q^4}. \tag{114}$$

Finally, using (74), we get

$$b_{1l} = \frac{\partial \phi_l}{\partial q^4} - \frac{\partial \phi_4}{\partial q^l}. \tag{115}$$

The remaining constraint Helmholtz conditions of (73) are then fulfilled.

It can be easily verified that functions (113), (114) and (115) are the same as those calculated from Lagrangian (107) using (69).

The constraint Lagrangian is

$$\lambda_{\mathrm{c}} = \bar{L}\, ds + \bar{L}_1 \varphi^1, \quad \varphi^1 = - \frac{\dot{q}^l}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}}\, \omega^l + \iota^* \omega^4,$$

where

$$\bar{L} = L \circ \iota = -\frac{1}{2} m_0 + \dot{q}^l \phi_l + \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}\, \phi_4 - \psi,$$

$$\bar{L}_1 = -m_0 \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2} + \phi_4.$$

In coordinates $(t, q^l, v^l)$, adapted to the fibration $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, i.e. such that

$$\dot{q}^l = v^l \dot{q}^4, \quad \dot{q}^4 = \frac{1}{\sqrt{1 - v^2}}\,,$$

and with the notation $(\phi_l) = \vec{A}$, $\phi_4 = -V$ we obtain

$$\bar{L} = -\frac{1}{2} m_0 + \frac{1}{\sqrt{1 - v^2}} (\vec{v} \vec{A} - V) - \psi\,, \quad \bar{L}_1 = -\frac{m_0}{\sqrt{1 - v^2}} - V.$$

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# Projective metrizability in Finsler geometry

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**Abstract.** The projective Finsler metrizability problem deals with the question whether a projective-equivalence class of sprays is the geodesic class of a (locally or globally defined) Finsler function. This paper describes an approach to the problem using an analogue of the multiplier approach to the inverse problem in Lagrangian mechanics.

## 1 Introduction

Let $M$ be a manifold of class $C^\infty$ which is Hausdorff, second-countable and connected; let $\tau\colon T^\circ M \to M$ denote its slit tangent bundle; let $(x^i)$ be local coordinates coresponding to some chart on $M$, and let $(x^i, y^i)$ be the corresponding fibred coordinates on $T^\circ M$.

A Finsler function [1] is a smooth map $F\colon T^\circ M \to \mathbb{R}$ which is positive, *positively homogeneous* so that $F(kv) = kF(v)$ for $v \in T^\circ M$ whenever $k \in \mathbb{R}$, $k > 0$, and *strongly convex* so that at each point of $T^\circ M$ the matrix

$$g_{ij} = \tfrac{1}{2}\frac{\partial^2(F^2)}{\partial y^i\,\partial y^j}$$

is positive definite. Each Finsler function $F$ gives rise to a variational problem on $M$ of a special kind, where if $\gamma\colon (a,b) \to M$ is an extremal (in other words, a geodesic) then so is $\gamma \circ \phi$ where $\phi\colon (a,b) \to (a,b)$ with $\phi'(t) > 0$.

On the other hand, a spray [5] is a vector field $\Gamma$ on $T^\circ M$ which is *second-order*, so that $S(\Gamma) = \Delta$ where $S$ is the almost tangent structure on $T^\circ M$, and which is also *homogeneous*, so that $[\Delta, \Gamma] = \Gamma$ where $\Delta$ is the vector field on $T^\circ M$ given by the restriction of the dilation field on the tangent manifold $TM$. Locally

$$\Gamma = y^i\frac{\partial}{\partial x^i} - 2\Gamma^i\frac{\partial}{\partial y^i}$$

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for some local functions $\Gamma^i$ which are positively homogeneous of degree 2. Two sprays $\Gamma_1, \Gamma_2$ are said to be projectively related if $\Gamma_1 - \Gamma_2 = \alpha\Delta$ for some function $\alpha$.

Every Finsler function $F$ gives rise to a projective class of sprays in the following way. The *Hilbert form* of $F$ is the 1-form $\theta_F = S(\mathrm{d}F)$ given locally by

$$\theta_F = \frac{\partial F}{\partial y^i}\mathrm{d}x^i$$

and having the property that if $\gamma\colon (a,b) \to M$ is a geodesic of $F$ then $\gamma'\colon (a,b) \to T^\circ M$ is an integral curve of a spray $\Gamma \in \ker \mathrm{d}\theta_F$. Furthermore, if $\gamma \circ \phi$ is a reparametrized geodesic then $(\gamma \circ \phi)'$ is an integral curve of a projectively related spray $\Gamma - \alpha\Delta \in \ker \mathrm{d}\theta_F$, and indeed

$$\ker \mathrm{d}\theta_F = \langle \Gamma, \Delta \rangle \,.$$

The *projective metrizability problem* is about the converse question. Given a projective class $\{\Gamma\}$ of sprays on $T^\circ M$, when are these sprays derived from a Finsler function $F$ on $T^\circ M$, either locally or globally? Here, 'locally' means on $T^\circ U$ where $v \in T^\circ M$ and $U$ is an open neighbourhood of $\tau(v)$. There are several approaches to this problem; we consider only the *multiplier approach* as an analogue of a similarly-named approach to the inverse problem in Lagrangian mechanics (see [4] for a recent survey of this latter problem). We also restrict attention to $\dim M \geq 3$.

This paper is based on a talk given by the author at the satellite thematic session 'Geometric Methods in Calculus of Variations' of the 6th European Congress of Mathematics in Kraków, July 2012, and reports on joint work with Mike Crampin and Tom Mestdag [2][3].

## 2 The comparison with Lagrangian mechanics

Lagrangian mechanics, in the time-independent case, considers a function $L$ on the tangent manifold $TM$, and the corresponding local Euler-Lagrange equations

$$\frac{\partial L}{\partial x^j} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial y^j}\,;$$

by writing $z^i = \dot{y}^i = \ddot{x}^i$ the total derivative $\mathrm{d}/\mathrm{d}t$ on the right-hand side may be replaced to give the explicit formulation

$$\frac{\partial L}{\partial x^j} = z^i \frac{\partial^2 L}{\partial y^i\,\partial y^j}\,.$$

If the Hessian matrix $h_{ij} = \partial L/\partial y^i\,\partial y^j$ is regular then this equation may be solved locally for the second derivatives $z^i$, and there is a unique vector field $\Gamma$ on $TM$ satisfying $S(\Gamma) = \Delta$ and with the property that if $\gamma$ is a solution of the Euler-Lagrange equations (an extremal of the variational problem defined by $L$) then $\gamma'$ is an integral curve of $\Gamma$.

The inverse problem of Lagrangian mechanics is to start with a vector field $\Gamma$ satisfying $S(\Gamma) = \Delta$, and to determine whether $\Gamma$ arises from a Lagrangian in this way. Any such vector field may again be written locally as

$$\Gamma = y^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial y^i}$$

(without, of course, any homogeneity condition on the functions $\Gamma^i$), and any integral curve of $\Gamma$ will be the derivative of a curve in $M$ satisfying the second-order equation $\ddot{z}^i + 2\Gamma^i = 0$. Comparing this with the Euler-Lagrange equations $\ddot{z}^i h_{ij} = \partial L/\partial x^j$ for a possible Lagrangian $L$ shows the importance of the regularity of the multiplier matrix $h_{ij}$ in the study of this problem.

## 3 Positivity and strong convexity

The projective metrizability problem for Finsler geometry is, on the face of it, quite similar to the inverse problem of Lagrangian mechanics. A spray is a vector field on $T^\circ M \subset TM$ of the required form, and a Finsler function may be regarded as a Lagrangian. The difference is that a Finsler function is required to be positively homogeneous, and so its Hessian matrix can never be regular; indeed

$$y^j \frac{\partial^2 F}{\partial y^i \, \partial y^j} = 0\,.$$

We shall, though, need some kind of regularity, and we can see how to approach this by writing

$$h_{ij} = \frac{\partial^2 F}{\partial y^i \, \partial y^j}\,, \qquad g_{ij} = \tfrac{1}{2}\frac{\partial^2 (F^2)}{\partial y^i \, \partial y^j} = h_{ij}F + \frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j}\,.$$

Define $h_{ij}$ to be *positive quasidefinite* if $h_{ij}(y)v^i v^j \geq 0$, with equality only when $v = \lambda y$; say that a function $F$ on $T^\circ M$ is a *pseudo-Finsler function* if it is positively homogeneous and if its Hessian $h_{ij}$ is positive quasidefinite. The following result is essentially Theorem 1 of [2].

**Theorem 1.** *If $F$ is a pseudo-Finsler function on $T^\circ M$ then locally there is a Finsler function $\tilde{F}$ such that $F - \tilde{F}$ is a total derivative, so that $F$ and $\tilde{F}$ satisfy the same Euler-Lagrange equations and therefore have the same geodesics. If in addition $F$ is positive then $g_{ij}$ is positive definite, so that $F$ is itself a Finsler function. If $F$ is absolutely homogeneous, so that $F(kv) = |k|F(v)$ for any $k \neq 0$ rather than only for $k > 0$, then $F$ is necessarily positive, so that again it is a Finsler function.*

## 4 Projective classes of sprays

The projective metrizability problem considers a projective class $\{\Gamma\}$ of sprays, and asks whether there is a corresponding Finsler function $F$. (Given $F$, one may select a distinguished spray from the class by requiring $\Gamma(F) = 0$; this gives rise to a different inverse problem, starting with a single spray, which we do not consider here.)

We approach this problem by adapting a technique which has been used to study the inverse problem in Lagrangian mechanics. Every spray on $T^\circ M$ gives rise to a nonlinear connection on $\tau$ with horizontal projector

$$H_\Gamma = \tfrac{1}{2}(I - \mathcal{L}_\Gamma S) = \mathrm{d}x^i \otimes \left( \frac{\partial}{\partial x^i} - \frac{\partial \Gamma^j}{\partial y^i}\frac{\partial}{\partial y^j} \right);$$

the connection allows us to define the horizontal lift $X^{\mathrm{h}} = H_\Gamma(X)$ of a vector field $X$ along $\tau$ (that is, of a section of the pull-back bundle $\tau^* TM \to T^\circ M$). We may also use the almost tangent structure to define the vertical lift $X^{\mathrm{v}} = S(X)$; in coordinates, if $X = X^i\partial/\partial x^i$ where $X^i$ are locally defined functions on $T^\circ M$ then

$$X^{\mathrm{h}} = X^i\left(\frac{\partial}{\partial x^i} - \frac{\partial \Gamma^j}{\partial y^i}\frac{\partial}{\partial y^j}\right), \qquad X^{\mathrm{v}} = X^i\frac{\partial}{\partial y^i}\,.$$

We now define the *dynamical covariant derivative* $\nabla$ and the *Jacobi endomorphism* $\Phi$ acting on a vector field $X$ along $\tau$ by

$$[\Gamma, X^{\mathrm{h}}] = (\nabla X)^{\mathrm{h}} + (\Phi X)^{\mathrm{v}}\,, \qquad [\Gamma, X^{\mathrm{v}}] = -X^{\mathrm{h}} + (\nabla X)^{\mathrm{v}}\,.$$

With these tools at hand, we can now state a result which is essentially Theorem 2 of [2].

**Theorem 2.** *Suppose given a projective class of sprays. If, in a contractible chart, a positive quasidefinite matrix of functions $h_{ij}$ satisfies the Helmholtz conditions*

$$h_{ji} = h_{ij}\,, \qquad \frac{\partial h_{ij}}{\partial y^k} = \frac{\partial h_{ik}}{\partial y^j}\,, \qquad h_{ij}y^j = 0$$

*and*

$$(\nabla h)_{ij} = 0\,, \qquad h_{ij}\Phi^k_j = h_{kj}\Phi^k_i\,,$$

*where $\nabla h$ and $\Phi^k_j$ are the dynamical covariant derivative and Jacobi endomorphism of any spray in the class, then there is a local pseudo-Finsler function $F$ with Euler-Lagrange equations satisfied by the geodesics of the sprays.*

It follows from Theorem 1 that, when these conditions are satisfied, there is a local Finsler function with Euler-Lagrange equations satisfied by the geodesics of the sprays.

## 5 Global aspects

The result of Theorem 2 has been given in coordinates and is essentially local, although it is valid for complete fibres (it is '$y$-global' in the terminology of Finsler geometry). To consider the existence of a pseudo-Finsler function globally on $T^\circ M$, we use the techniques of Čech cohomology.

If $\{U_\lambda\}$ is an open cover of $M$, then we say that $\{U_\lambda\}$ is a *good cover* if all nonempty finite intersections of the sets $U_\lambda$ are contractible. It may be shown that if there is a spray defined on $M$ then $M$ admits a good cover by the domains of coordinate charts ([2], Appendix B); the proof uses Whitehead's result on the existence of geodesically convex sets [6][7].

Let $\{U_\lambda\}$ be such a cover. Given a projective class of sprays and a (0,2) tensor field $h$ along $\tau$ whose components in each chart satisfy the conditions of Theorem 2, there is a pseudo-Finsler function $F_\lambda$ defined on each $U_\lambda$. If $U_\lambda \cap U_\mu$ is nonempty then

$$F_\lambda - F_\mu = y^i\frac{\partial \phi_{\lambda\mu}}{\partial x^i}$$

for some function $\phi_{\lambda\mu}$ defined on $T^\circ(U_\lambda \cap U_\mu)$ which is unique to within a constant. Also, if $U_\lambda \cap U_\mu \cap U_\nu$ is nonempty then

$$\phi_{\mu\nu} - \phi_{\lambda\nu} + \phi_{\lambda\mu} = k_{\lambda\mu\nu}$$

is constant on the connected set $T^\circ(U_\lambda \cap U_\mu \cap U_\nu)$, and if $U_\kappa \cap U_\lambda \cap U_\mu \cap U_\nu$ is nonempty then

$$k_{\lambda\mu\nu} - k_{\kappa\mu\nu} + k_{k\lambda\nu} - k_{\kappa\lambda\mu} = 0$$

on $T^\circ(U_\kappa \cap U_\lambda \cap U_\mu \cap U_\nu)$. We see from this that the obstruction to the construction of a global pseudo-Finsler function lies in the second Čech cohomology group of the cover, and as we have taken a good cover this is isomorphic to the de Rham cohomology group $H^2(M)$. The following result is essentially the second part of Theorem 3 of [2].

**Theorem 3.** *Suppose given a projective class of sprays. If there is a (0,2) tensor field $h$ along $\tau$ such that*

- *in each chart of a good atlas the components $h_{ij}$ satisfy the Helmholtz conditions and are positive quasidefinite, and*
- *$H^2(M) = 0$,*

*then there is a global pseudo-Finsler function $F$ with Euler-Lagrange equations satisfied by the geodesics of the sprays, and each point of $T^\circ M$ has a neighbourhood on which there is a corresponding local Finsler function.*

The example of the spray

$$\Gamma = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} + \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}\left( y^1 \frac{\partial}{\partial y^2} - y^2 \frac{\partial}{\partial y^1} \right)$$

defined on $T^\circ\mathbb{R}^3$, which is in the projective class of sprays arising from the global pseudo-Finsler function

$$F = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} + \tfrac{1}{2}(x^2 y^1 - x^1 y^2)\,,$$

shows that there need not be a global Finsler function giving rise to the projective class.

## 6 Multiplier tensors and 2-forms

In a global formulation, the multiplier matrix $h_{ij}$ is the coordinate representation of a symmetric (0,2) tensor field $h$ along the projection $T^\circ M \to M$ (that is, locally $h = h_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$). This tensor field is closely related to a 2-form on $T^\circ M$ which, given the existence of a Finsler function $F$, will be the differential $d\theta_F$ of its Hilbert form. We can therefore translate the conditions on $h$ given above into conditions on the 2-form; these results are essentially Theorems 5 and 6 of [3].

**Theorem 4.** *Suppose given a spray $\Gamma$ and a 2-form $\omega$ on $T^\circ M$, and let $\{\mathrm{d}x^i\,, \phi^i = H_\Gamma(\mathrm{d}y^i)\}$ be a local basis of 1-forms on $T^\circ M$. If*

- $\langle \Gamma, \Delta \rangle \subset \ker\omega$ *and* $\mathcal{L}_\Gamma \omega = 0$,
- $\omega(V_1, V_2) = 0$ *if $V_1$, $V_2$ are vertical, and*
- $\mathrm{d}\omega(H, V_1, V_2) = 0$ *if $V_1$, $V_2$ are vertical and $H$ horizontal*

*then in any chart we may write*

$$\omega = h_{ij}\mathrm{d}x^i \wedge \phi^j$$

*where $h_{ij}$ satisfies the Helmholtz conditions. It is also the case that a 2-form $\omega$ satisfying the stated conditions must be closed.*

It follows that if the matrix $h_{ij}$ obtained above is positive quasidefinite on a contractible chart then there will be a local pseudo-Finsler function for $\Gamma$.

**Theorem 5.** *Suppose given a projective class of sprays. If there is a 2-form $\omega$ satisfying the conditions of Theorem 4 for any spray in the class, and if the functions $h_{ij}$ are positive quasidefinite, and if furthermore $H^2(M) = 0$, then there is a global pseudo-Finsler function $F$ with Euler-Lagrange equations satisfied by the geodesics of the sprays.*

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# Conference announcements

**The 33rd Winter School on Geometry and Physics**

January 12–19, 2013, Srní, Czech Republic.

http://www.math.muni.cz/~srni

**28th Journées Arithmétiques**

July 1–5, 2013, Grenoble, France

http://www-fourier.ujf-grenoble.fr/ja2013/index-en.html

**Differential Geometry and its Applications**

August 19–23, 2013, Brno, Czech Republic.

http://inserv.math.muni.cz/dga2013

**21st Czech and Slovak International Conference on Number Theory**

September 2–6, 2013, Czech Republic.

http://ntc.osu.cz/2013

**4th International Conference on Uniform Distribution Theory**

July 1–5, 2014, Czech Republic.

http://ntc.osu.cz/UDT2014

**International Congress on Mathematical Physics**

August 3–8, 2015, Santiago de Chile

http://www.iamp.org/bulletins/Bulletin-Oct12.pdf

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