VOLUME
19/2011
No. 2
ISSN 1804-1388

COMMUNICATIONS IN MATHEMATICS

Geometrical aspects of variational calculus on manifolds

Guest Editor: László Kozma

Available online at
http://cm.osu.cz

UNIVERSITAS OSTRAVIENSIS

Subscriptions

This journal is available for exchange. Information on exchange and/or subscription is available at: *Library of The University of Ostrava, Bráfova 7, 701 03 Ostrava, Czech Republic*
Website: http://cm.osu.cz

The journal is reviewed in Zentralblatt für Mathematik and Mathematical Reviews.

ISSN 1804-1388

Communications in Mathematics 19 (2011) 89–90

Editorial

László Kozma

I am pleased to introduce this issue of *Communications in Mathematics* which is devoted to research and survey papers on topics in differential geometry and variational calculus. As well as two regular papers it contains some contributions from workshops on differential geometry held in Ostrava in May and October 2011.

The cooperation of the Czech and the Hungarian research groups in differential geometry has a long tradition. This year we held the 7th Bilateral Joint Workshop on Differential Geometry, previously were held in Opava, Ostrava, Olomouc in the Czech Republic and Debrecen, Síkfőkút in Hungary. The core interest of the two groups is common: geometrization of differential equations and variational calculus on manifolds. While the Czech team reached remarkable results on the variational aspects, the Hungarian group is strong in Finsler geometry. So the aspects fruitfully complete each other.

This issue contains the written version of the minicourse of David Saunders, five research papers and a survey paper, and a book review. Almost all were read at the latest workshops.

- Saunders' work presents the material of the minicourse which introduces a version of the geometrical background of the problem where the extremals are submanifolds, but where the action function still depends upon no more than the first derivatives of the submanifold.

- The paper of Fatibene, Francaviglia and Mercadante shows that when in a higher order variational principle one fixes fields at the boundary leaving the field derivatives unconstrained, then the variational principle is not invariant with respect to the addition of the boundary terms to the action.

- In their paper Muzsnay and Nagy aim at finding the largest Lie algebra of vector fields on the indicatrix such that all its elements are tangent to the holonomy group of a Finsler manifold.

- Szilasi and Tóth apply the apparatus of the calculus along the tangent bundle projection, and give a series of characterizations of affine and conformal vector fields on Finsler manifolds.

- Havelková studies the dynamics of singular Lagrangian systems described by implicit differential equations from a geometrical point of view using the approach of exterior differential systems.
- In the field of variational principles, Tulczyjew's new notes are based on the definition of equilibrium related to the response of a system to virtual displacements rather than the minima of the internal energy.

I am sure that these works will stimulate further studies in their respective subjects, and that international collaboration and joint meetings such as ours will accelerate scientific advance in fields of common interest.

László Kozma
Guest Editor

Editor's address:
Institute of Mathematics, University of Debrecen, H-4010 Debrecen, P.O.B. 12, Hungary

E-mail: kozma@math.klte.hu

Communications in Mathematics 19 (2011) 91–128

Homogeneous variational problems: a minicourse

David J. Saunders

Abstract. A Finsler geometry may be understood as a homogeneous variational problem, where the Finsler function is the Lagrangian. The extremals in Finsler geometry are curves, but in more general variational problems we might consider extremal submanifolds of dimension m. In this minicourse we discuss these problems from a geometric point of view.

1 Introduction

This paper is a written-up version of the major part of a minicourse given at the sixth Bilateral Workshop on Differential Geometry and its Applications, held in Ostrava in May 2011. Much of the discussion at these workshops is on Finsler geometry, where the interest is in variational problems defined on tangent manifolds by a 'Finsler function', a smooth function defined on the slit tangent manifold (excluding the zero section) and satisfying certain homogeneity and nondegeneracy properties. The extremals of such problems are geometric curves in the original (base) manifold, without any particular parametrization but with an orientation.

For this particular workshop it was felt that it might be worthwhile to describe slightly more general problems, looking at variational problems where the extremals were submanifolds of dimension m, but where the action function still depended upon no more than the first derivatives of the submanifold [2], [4]; for example, minimal surface problems would be included in this description. This minicourse introduces a version of the geometric background needed to express such problems, in terms of velocity manifolds. There is an alternative approach to such problems involving manifolds of contact elements (quotients of velocity manifolds); we refer to this only briefly, when we consider the action of the jet group.

Although we consider only first order variational problems, we nevertheless need to use second order velocities: for instance, the Euler-Lagrange equations for first order variational problems are second-order differential equations. We do this in

2010 MSC: 35A15, 58A10, 58A20
Key words: calculus of variations, parametric problems

a slightly unusual way, looking at a particular submanifold of the double velocity manifold. Having done this, we look at some geometrical and cohomological constructions, before obtaining a version of the first variation formula for variational problems with fixed boundary conditions. The final part of the minicourse, which considered various concepts of regularity, has been omitted from this paper for reasons of space; the concepts described may be found in a recent paper [1]. We give only a few other references: [3] provides extensive background material on various types of jet manifold and the actions of the jet groups; [5] introduces in a more general context the type of cohomological approach we use these types of variational problem; and [6], with a philosophy similar to that of the present paper, compares these problems with those defined on jets of sections of fibrations.

I should like to thank the organisers of the Workshop for inviting me to give this course. I acknowledge the support of grant no. 201/09/0981 for Global Analysis and its Applications from the Czech Science Foundation; grant no. MEB 041005 for Finsler structures and the Calculus of Variations; and also the joint IRSES project GEOMECH (EU FP7, nr 246981).

2 Velocities

In this section we see how to construct manifolds of first order and second order velocities, and also how certain groups, the jet groups, act on these manifolds.

2.1 First order velocities

Let E be a connected, paracompact, Hausdorff manifold of class C^∞ and of finite dimension n; let $O \subset \mathbb{R}^m$ (with $m < n$) be open and connected, with $0 \in O$. A map $\gamma : O \to E$ will be called an *m-curve in E*. The 1-jet $j^1_0\gamma$ of γ at zero will be called a *velocity* (or *m-velocity*), and the set $T_m E = \{j^1_0\gamma\}$ of velocities of all m-curves in E will be called the *velocity (or m-velocity) manifold of E*. We map $T_m E$ to E by

$$\tau_{mE} : T_m E \to E\,, \qquad \tau_{mE}(j^1_0\gamma) = \gamma(0)\,.$$

We shall show that $T_m E$ really is a manifold (and is connected, paracompact and Hausdorff, and indeed is a vector bundle over E) by identifying it with the Whitney sum over E of m copies of the tangent manifold TE.

Lemma 1. *There is a canonical identification $T_m E \cong \bigoplus^m TE$.*

Proof. Let $i_k : \mathbb{R} \to \mathbb{R}^m$ be the inclusion $i_k(s) = (0, \ldots, 0, s, 0, \ldots, 0)$. Then each $\gamma \circ i_k$ is a curve in E, and the map

$$j^1_0\gamma \mapsto \big(j^1_0(\gamma \circ i_1), \ldots, j^1_0(\gamma \circ i_m)\big)$$

is a bijection $T_m E \to \bigoplus^m TE$ preserving the fibration over E. □

Corollary 1. *Let $\{dt^i\}$ be the canonical basis of $\mathbb{R}^{m*}$; then*

$$T_m E \to TE \otimes \mathbb{R}^{m*}\,, \qquad (\xi_1, \ldots, \xi_m) \mapsto \xi_i \otimes dt^i$$

is a vector bundle isomorphism. □

If $(U;u^a)$ is a chart on E then $(U^1;u^a,u^a_i)$ is a chart on T_mE, where

$$U^1=\tau_{mE}^{-1}(U)\,,\qquad u^a_i(j^1_0\gamma)=D_i\gamma^a(0)=D_i(u^a\circ\gamma)(0)\,.$$

If $j^1_0\gamma=(\xi_1,\dots,\xi_m)$ then it is clear that $u^a_i(j^1_0\gamma)=\dot u^a(\xi_i)$. The rule for changing coordinates on T_mE is therefore

$$v^b_i(j^1_0\gamma)=\left.\frac{\partial v^b}{\partial u^a}\right|_{\gamma(0)}u^a_i(j^1_0\gamma)\,.$$

We can see from this that the superscript a labeling the coordinate function u^a_i depends on the original choice of chart u^a on E, whereas the subscript i is independent of this choice and so is the index of a component of the velocity (namely, the tangent vector ξ_i). We call indices of this latter type *counting indices* rather than *coordinate indices*.

We shall be particularly interested in the subsets of T_mE containing those velocities $j^1_0\gamma$ where the m-curve γ has certain properties. Write $\mathring{T}_mE$ for the subset

$$\{j^1_0\gamma\in T_mE:\gamma\text{ is an immersion near zero}\}\,;$$

if $j^1_0\gamma=(\xi_1,\dots,\xi_m)$ and $j^1_0\gamma\in\mathring{T}_mE$ then $\{\xi_1,\dots,\xi_m\}$ will be linearly independent. An element of $\mathring{T}_mE\subset T_mE$ will be called a *regular velocity*.

Proposition 1. *The regular velocities form an open-dense submanifold.*

Proof. To show that $\mathring{T}_mE$ is open in T_mE, define the map $\wedge:T_mE\to\bigwedge^mTE$ by $(\xi_1,\dots,\xi_m)\mapsto\xi_1\wedge\cdots\wedge\xi_m$. Then

- The map $\wedge$ is fibred over the identity on E and is continuous (it is polynomial in the fibre coordinates u^a_i);
- $j^1_0\gamma\in\mathring{T}_mE$ exactly when $\wedge(j^1_0\gamma)\neq0$;
- the zero section of $\bigwedge^mTE$ is closed.

To show that $\mathring{T}_mE$ is dense in T_mE, define the map $f:U^1\to\mathbb{R}$ by $f(j^1_0\gamma)=\det\bigl(u^j_i(j^1_0\gamma)\bigr)$, where (u^j_i) is the $m\times m$ submatrix containing the first m rows of the $n\times m$ matrix u^a_i. If $j^1_0\gamma\in O\subset U^1$ where O is open and $O\cap\mathring{T}_mE=\emptyset$ then f vanishes on O. But

$$\left.\frac{\partial^mf}{\partial u^1_1\,\partial u^2_2\cdots\partial u^m_m}\right|_{j^1_0\gamma}=1\,.\qquad\square$$

2.2 Second order velocities

We define a *second-order m-velocity* in the same way as a 2-jet at zero of an m-curve, and write

$$T^2_mE=\{j^2_0\gamma\}\,,\qquad\mathring{T}^2_mE=\{j^2_0\gamma:\gamma\text{ is an immersion near zero}\}\,.$$

We also let $\tau^2_{mE} : T^2_m E \to E$, $\tau^{2,1}_{mE} : T^2_m E \to T_m E$ be the projections

$$\tau^2_{mE}(j^2_0\gamma) = \gamma(0)\,, \qquad \tau^{2,1}_{mE}(j^2_0\gamma) = j^1_0\gamma\,.$$

We take charts on $T^2_m E$ to be $(U^2; u^a, u^a_i, u^a_{ij})$ where $U^2 = (\tau^2_{mE})^{-1}(U)$ and

$$u^a_i(j^2_0\gamma) = D_i\gamma^a(0)\,, \qquad u^a_{ij}(j^2_0\gamma) = D_i D_j \gamma^a(0)$$

so that $u^a_{ij} = u^a_{ji}$ (this constraint will cause complications in certain coordinate formulæ). These charts form an atlas such that $T^2_m E$ becomes a manifold with the standard properties. We shall not demonstrate this directly; we shall show instead that it may be identified with a closed submanifold of a larger manifold, the manifold of double velocities.

2.3 Double velocities

We know that $T_m E$ is a manifold, so it has its own velocity manifold

$$T_{m'} T_m E = \{j^1_0\tilde{\gamma}\}$$

where $\tilde{\gamma}$ is an m'-curve in $T_m E$. This is the (m', m) *double velocity manifold*. Charts on $T_{m'} T_m E$ are therefore

$$\big((U^1)^1; u^a, u^a_i, u^a_{;j}, u^a_{i;j}\big)\,,$$

where $1 \le i \le m$ and $1 \le j \le m'$, corresponding to the charts $(U^1; u^a, u^a_i)$ on $T_m E$. In most applications we have either $m' = m$ or $m' = 1$. We shall be interested in a particular submanifold of double velocities, known as holonomic double velocities.

2.4 Holonomic double velocities

If γ is an m-curve in E then its *prolongation* is the m-curve $\bar{\jmath}^1\gamma$ in $T_m E$ where

$$\bar{\jmath}^1\gamma(t) = j^1_0(\gamma \circ \mathrm{T}_t)$$

and $\mathrm{T}_t : \mathbb{R}^m \to \mathbb{R}^m$ is the translation map $\mathrm{T}_t(s) = t + s$. Thus $j^1_0\bar{\jmath}^1\gamma \in T_m T_m E$. We use the notation $\bar{\jmath}^1\gamma$ rather than $j^1\gamma$; the latter would be a map satisfying $j^1\gamma(t) = j^1_t\gamma$ whose codomain would be a set containing jets at arbitrary points of $\mathbb{R}^m$ rather than just at zero.

Proposition 2. *The map*

$$\iota : T^2_m E \to T_m T_m E\,, \qquad \iota(j^2_0\gamma) = j^1_0\bar{\jmath}^1\gamma$$

is an injection. Its image is the submanifold described in coordinates by

$$u^a_i = u^a_{;i}\,, \qquad u^a_{i;j} = u^a_{j;i}\,.$$

The image of the chart $(U^2; u^a, u^a_i, u^a_{ij})$ under the injection is the restriction of the chart $\big((U^1)^1; u^a, u^a_i, u^a_{;j}, u^a_{i;j}\big)$ to the submanifold.

Proof. Suppose γ_1, γ_2 are two m-curves in E such that $j_0^1\bar{\jmath}^1\gamma_1 = j_0^1\bar{\jmath}^1\gamma_2$. Then for γ_1

$$\begin{aligned} u^a(j_0^2\gamma_1) &= u^a(\gamma_1(0)) = u^a(\bar{\jmath}^1\gamma_1(0)) = u^a(j_0^1\bar{\jmath}^1\gamma_1)\,; \\ u_i^a(j_0^2\gamma_1) &= D_i(u^a\circ\gamma_1)(0) = u_i^a(\bar{\jmath}^1\gamma_1(0)) = u_i^a(j_0^1\bar{\jmath}^1\gamma_1)\,; \\ u_{ij}^a(j_0^2\gamma_1) &= D_iD_j(u^a\circ\gamma_1)(0) = D_i(u_j^a\circ\bar{\jmath}^1\gamma_1)(0) = u_{j;i}^a(j_0^1\bar{\jmath}^1\gamma_1) \end{aligned}$$

and similarly for γ_2, so that $j_0^2\gamma_1 = j_0^2\gamma_2$ and the map is an injection.

For any m-curve γ in E

$$u_{;i}^a(j_0^1\bar{\jmath}^1\gamma) = D_i(u^a\circ\bar{\jmath}^1\gamma)(0) = D_i(u^a\circ\gamma)(0) = u_i^a(\bar{\jmath}^1\gamma(0)) = u_i^a(j_0^1\bar{\jmath}^1\gamma)$$

and

$$u_{j;i}^a(j_0^1\bar{\jmath}^1\gamma) = D_i(u_j^a\circ\bar{\jmath}^1\gamma)(0) = D_i(D_j(u^a\circ\gamma))(0)$$

so that $u_i^a = u_{;i}^a$ and $u_{i;j}^a = u_{j;i}^a$ when restricted to the image of the injection.

Furthermore, if $\tilde\gamma$ is an m-curve in T_mE satisfying

$$u_i^a(j_0^1\tilde\gamma) = u_{;i}^a(j_0^1\tilde\gamma)\,, \qquad u_{i;j}^a(j_0^1\tilde\gamma) = u_{j;i}^a(j_0^1\tilde\gamma)$$

then the m-curve γ in E given in coordinates near $\tau_{mE}(\tilde\gamma(0))$ by

$$\gamma^a(t) = u^a(j_0^1\tilde\gamma) + u_i^a(j_0^1\tilde\gamma)t^i + \tfrac{1}{2}u_{i;j}^a(j_0^1\tilde\gamma)t^it^j$$

so that $j_0^1\bar{\jmath}^1\gamma = j_0^1\tilde\gamma$; thus the image of the injection is described locally by the equations $u_i^a = u_{;i}^a$, $u_{i;j}^a = u_{j;i}^a$ and is therefore a submanifold of T_mT_mE.

The relationship between the charts $(U^2; u^a, u_i^a, u_{ij}^a)$ and $\big((U^1)^1; u^a, u_i^a, u_{;j}^a, u_{i;j}^a\big)$ is immediate. □

The image of T_m^2E in T_mT_mE is called the submanifold of *holonomic* double velocities. There is no canonical projection $T_mT_mE \to T_m^2E$; we may, however, consider a tubular neighbourhood $\nu : N \to T_m^2E$ of T_m^2E in T_mT_mE, and then the condition $\nu\circ\iota = \mathrm{id}_{T_m^2E}$ (where $\iota : T_m^2E \to T_mT_mE$ is the injection) gives rise to the constraints

$$\frac{\partial\nu^a}{\partial u^c} = \delta_c^a\,, \qquad \frac{\partial\nu^a}{\partial u_p^c} + \frac{\partial\nu^a}{\partial u_{;p}^c} = 0\,, \qquad \frac{\partial\nu^a}{\partial u_{p;q}^c} + \frac{\partial\nu^a}{\partial u_{q;p}^c} = 0\,,$$

$$\frac{\partial\nu_i^a}{\partial u^c} = 0\,, \qquad \frac{\partial\nu_i^a}{\partial u_p^c} + \frac{\partial\nu_i^a}{\partial u_{;p}^c} = \delta_c^a\delta_i^p\,, \qquad \frac{\partial\nu_i^a}{\partial u_{p;q}^b} + \frac{\partial\nu_i^a}{\partial u_{q;p}^b} = 0\,,$$

$$\frac{\partial\nu_{ij}^a}{\partial u^c} = 0\,, \qquad \frac{\partial\nu_{ij}^a}{\partial u_p^c} + \frac{\partial\nu_{ij}^a}{\partial u_{;p}^c} = 0\,, \qquad \frac{\partial\nu_{ij}^a}{\partial u_{p;q}^c} + \frac{\partial\nu_{ij}^a}{\partial u_{q;p}^c} = \delta_c^a(\delta_i^p\delta_j^q + \delta_j^p\delta_i^q)\,.$$

for the coordinates of ν, and hence to the conditions

$$d\nu^a = du^a + \frac{\partial\nu^a}{\partial u^c_p}(du^c_p - du^c_{;p}) + \tfrac{1}{2}\frac{\partial\nu^a}{\partial u^c_{p;q}}(du^c_{p;q} - du^c_{q;p})$$
$$d\nu^a_i = \tfrac{1}{2}(du^a_i + du^a_{;i}) + \tfrac{1}{2}\left(\frac{\partial\nu^a_i}{\partial u^c_p} - \frac{\partial\nu^a_i}{\partial u^c_{;p}}\right)(du^c_p - du^c_{;p}) + \tfrac{1}{2}\frac{\partial\nu^a_i}{\partial u^c_{p;q}}(du^c_{p;q} - du^c_{q;p})$$
$$d\nu^a_{ij} = \tfrac{1}{2}(du^a_{i;j} + du^a_{j;i}) + \frac{\partial\nu^a_{ij}}{\partial u^c_p}(du^c_p - du^c_{;p}) + \tfrac{1}{2}\frac{\partial\nu^a_{ij}}{\partial u^c_{p;q}}(du^c_{p;q} - du^c_{q;p})\,.$$

We shall use these conditions later on.

2.5 The exchange map

There is another way of describing the submanifold of holonomic velocities.

A map $\psi : O' \times O \to E$, where $O \subset \mathbb{R}^m$, $O' \subset \mathbb{R}^{m'}$ are open and connected, and where $0_{\mathbb{R}^m} \in O$ and $0_{\mathbb{R}^{m'}} \in O'$, is called a *double* (m', m)*-curve*. For each $s \in O'$

$$\psi_s : O \to E\,, \qquad \psi_s(t) = \psi(s,t)$$

is then an m-curve in E, so that $j^1_0\psi_s \in T_mE$. Thus

$$j^1_0(s \mapsto j^1_0\psi_s) \in T_{m'}T_mE\,.$$

Lemma 2. *The* exchange map $e : T_{m'}T_mE \to T_mT_{m'}E$ *is well-defined by* $\psi \mapsto \hat{\psi}$ *where* $\hat{\psi}(t,s) = \psi(s,t)$ *and is a smooth bijection.*

Proof. The element of T_mT_mE defined by ψ satisfies

$$\begin{aligned}
u^a(j^1_0(s \mapsto j^1_0\psi_s)) &= u^a(j^1_0\psi_0) = \psi^a_0(0) = \psi^a(0,0)\,,\\
u^a_i(j^1_0(s \mapsto j^1_0\psi_s)) &= u^a_i(j^1_0\psi_0) = D_i(u^a \circ \psi_0)(0) = D_{2:i}\psi^a(0,0)\,,\\
u^a_{;j}(j^1_0(s \mapsto j^1_0\psi_s)) &= D_j(u^a \circ (s \mapsto j^1_0\psi_s))(0) = D_j(s \mapsto \psi^a_s)(0) = D_{1:j}\psi^a(0,0)\,,\\
u^a_{i;j}(j^1_0(s \mapsto j^1_0\psi_s)) &= D_j(u^a_i \circ (s \mapsto j^1_0\psi_s))(0)\\
&= D_j(s \mapsto D_i\psi^a_s(0))(0) = D_{1:j}D_{2:i}\psi^a(0,0)\,,
\end{aligned}$$

and carrying out the same calculation for $\hat{\psi}$ shows that e is a well-defined injection. It is clearly an involution, and hence is a bijection. The coordinate formulæ

$$u^a \circ e = u^a\,, \qquad u^a_i \circ e = u^a_{;i}\,, \qquad u^a_{;j} \circ e = u^a_j\,, \qquad u^a_{i;j} \circ e = u^a_{j;i}$$

show that it is smooth. □

Proposition 3. *The holonomic submanifold of* T_mT_mE *is the fixed point set of the exchange map.*

Proof. This is immediate from the coordinate formulæ for e. □

2.6 Jet groups

If we consider m-curves in $\mathbb{R}^m$ rather than in some other manifold, then we have the possibility of composing two such m-curves. If we insist that the origin must map to itself then the composition will always exist, although possibly with a smaller domain then the domains of the two original m-curves. We shall want the jets of these m-curves to have inverses, so that the curves themselves will need to be immersions near zero; it is convenient to assume that they are, in fact, diffeomorphisms onto their images.

So let $O \subset \mathbb{R}^m$ be open and connected with $0 \in O$, and let $\phi : O \to \phi(O) \subset \mathbb{R}^m$ be a diffeomorphism with $\phi(0) = 0$. The *first and second order jet groups* are

$$L^1_m = \{j^1_0\phi\}\,, \qquad L^2_m = \{j^2_0\phi\}\,.$$

The products for L^1_m and L^2_m are given by

$$j^1_0\phi_1 \cdot j^1_0\phi_2 = j^1_0(\phi_1 \circ \phi_2)\,, \qquad j^2_0\phi_1 \cdot j^2_0\phi_2 = j^2_0(\phi_1 \circ \phi_2)\,.$$

Lemma 3. *The product rules define group structures on L^1_m and L^2_m.*

Proof. The products are well-defined because the first (or second) derivatives of a composite depend only upon the first (or second) derivatives of the individual maps, by the first (or second) order chain rule; sssociativity of the products is inherited from that of composition. The diffeomorphism $\mathrm{id}_{\mathbb{R}^m}$ satisfies

$$j^1_0(\mathrm{id}_{\mathbb{R}^m}) \cdot j^1_0\phi = j^1_0(\mathrm{id}_{\mathbb{R}^m} \circ \phi) = j^1_0\phi\,;$$

the map $\bar\phi : \phi(O) \to O$ given by $\bar\phi = \phi^{-1}$ satisfies $\bar\phi(0) = 0$, and

$$j^1_0\bar\phi \cdot j^1_0\phi = j^1_0(\bar\phi \circ \phi) = j^1_0(\mathrm{id}_O) = j^1_0(\mathrm{id}_{\mathbb{R}^m})\,.$$

Similar formulæ hold for second-order jets. □

The map $L^1_m \to \mathbb{R}^{m^2}$, $j^1_0\phi \mapsto \big(D_j\phi^i(0)\big)$ defines global coordinates on L^1_m, and identifies it with $\mathrm{GL}(m, \mathbb{R})$. The map $L^2_m \to \mathbb{R}^{m^2(m+3)/2}$,

$$j^2_0\phi \mapsto \big(D_j\phi^i(0), D_jD_k\phi^i(0)\big)$$

defines global coordinates on L^2_m. Writing

$$A^i_j = D_j\phi^i(0)\,, \qquad B^i_{jk} = D_jD_k\phi_i(0)$$

where $\det A^i_j \neq 0$ because ϕ is a diffeomorphism, the product rule in L^1_m is

$$(A\hat A)^i_j = A^i_h \hat A^h_j$$

and the product rule in L^2_m is

$$\begin{aligned}
\big((A,B)(\hat A,\hat B)\big)^i_j &= A^i_h \hat A^h_j\,,\\
\big((A,B)(\hat A,\hat B)\big)^i_{jk} &= A^i_l \hat B^l_{jk} + B^i_{hl}\hat A^h_j \hat A^l_k\,,
\end{aligned}$$

the latter formula arising from the second order chain rule

$$\begin{aligned} D_j D_k(\phi\hat{\phi})^i(0) &= D_j(D_l\phi^i \circ \hat{\phi})D_k\hat{\phi}^l)(0) \\ &= D_l\phi^i(0)D_jD_k\hat{\phi}^l(0) + D_hD_l\phi^i(0)D_j\hat{\phi}^h(0)D_k\hat{\phi}^l(0) \end{aligned}$$

using $\phi(0) = \hat{\phi}(0) = 0$.

Corollary 2. *The groups L^1_m and L^2_m are Lie groups.* □

Lemma 4. *The* oriented *subgroups L^{1+}_m and L^{2+}_m, where ϕ preserves orientation, are connected.*

Proof. As L^1_m may be identified with $\mathrm{GL}(m,\mathbb{R})$, the subgroup L^1_m where ϕ preserves orientation may be identified with $\mathrm{GL}^+(m,\mathbb{R})$, the subgroup of matrices satisfying $\det A^i_j > 0$, which is connected.

The map $L^1_m \to L^2_m$ given by $j^1_0\phi \mapsto j^2_0\hat{\phi}$, where $\hat{\phi}$ is the linear map $\hat{\phi}^i(t) = A^i_j t^j$ with (A^i_j) being the matrix corresponding to $j^1_0\phi$, is continuous; the coordinates of the image are $(A^i_j, 0)$. The image of the subgroup L^{1+}_m under this map is therefore connected. But every element of L^{2+}_m may be joined to an element of this image by a path given in coordinates by

$$s \mapsto (A^i_j, sB^i_{jk}), \qquad s \in [0,1]$$ □

2.7 Group actions

The jet groups L^1_m and L^2_m act on the velocity manifolds T_mE and T^2_mE by

$$(j^1_0\phi, j^1_0\gamma) \mapsto j^1_0(\gamma\circ\phi), \qquad (j^2_0\phi, j^2_0\gamma) \mapsto j^2_0(\gamma\circ\phi).$$

These are right actions, and in coordinates they are

$$\begin{aligned} u^a &\mapsto u^a \\ u^a_i &\mapsto u^a_h A^h_i \\ u^a_{ij} &\mapsto u^a_{hk} A^h_i A^k_j + u^a_h B^h_{jk} \end{aligned}$$

where A^i_j and B^i_{jk} are the global coordinates of $j^2_0\phi$.

Lemma 5. *The action of L^1_m on T_mE restricts to $\mathring{T}_mE$, and the restricted action is free. The action of L^2_m on T^2_mE restricts to $\mathring{T}^2_mE$, and the restricted action is free.*

Proof. The map ϕ is a diffeomorphism onto its image, so if γ is an immersion near zero then so is $\gamma\circ\phi$.

We use coordinates to show that the restricted actions are free. Suppose first that $j^1_0(\gamma\circ\phi) = j^1_0\gamma$, so that

$$u^a_j(j^1_0\gamma) = u^a_i(j^1_0\gamma)A^i_j;$$

as γ is an immersion near zero and $u^a_i(j^1_0\gamma) = D_i\gamma^a(0)$, it follows that the $m \times n$ matrix $u^a_i(j^1_0\gamma)$ must have rank m, so that $A^i_j = \delta^i_j$ and hence $j^1_0\phi = j^1_0(\mathrm{id}_{\mathbb{R}^m})$.

Now suppose that $j^2_0(\gamma \circ \phi) = j^2_0\gamma$, so that $u^a_j(j^2_0\gamma) = u^a_i(j^2_0\gamma)A^i_j$ and now also

$$u^a_{hk}(j^2_0\gamma) = u^a_{ij}(j^2_0\gamma)A^i_h A^j_k + u^a_i(j^2_0\gamma)B^i_{hk}\,.$$

As before we see that $A^i_j = \delta^i_j$, so that

$$u^a_{hk}(j^2_0\gamma) = u^a_{hk}(j^2_0\gamma) + u^a_i(j^2_0\gamma)B^i_{hk}$$

and therefore that $u^a_i(j^2_0\gamma)B^i_{hk} = 0$; the rank condition on $u^a_i(j^2_0\gamma)$ now tells us that $B^i_{hk} = 0$. □

2.8 Infinitesimal actions

Let (a^i_j) be an element of the Lie algebra of L^1_m; the identification of the group with $\mathrm{GL}(m, r)$ means that its Lie algebra may be identified with $\mathfrak{gl}(m, \mathbb{R})$ so that (a^i_j) is an arbitrary $m \times m$ matrix.

Lemma 6. *The vector field on T_mE corresponding to (a^i_j) is*

$$a^i_j u^a_i \frac{\partial}{\partial u^a_j}\,.$$

Proof. The map $\sigma : (-\varepsilon, \varepsilon) \to \mathrm{GL}(m, \mathbb{R})$, defined for sufficiently small ε by $\sigma(s) = (\delta^i_j + s a^i_j)$, is a curve in $\mathrm{GL}(m, \mathbb{R})$ whose tangent vector at the identity is (a^i_j). If $j^1_0\gamma \in T_mE$ then the corresponding curve through $j^1_0\gamma$ is given in coordinates by

$$s \mapsto \left(u^b(j^1_0\gamma), (\delta^i_j + s a^i_j)u^b_i(j^1_0\gamma)\right)\,.$$

The resulting tangent vector $\xi \in T_{j^1_0\gamma}T_mE$ satisfies

$$\dot{u}^b(\xi) = 0\,, \qquad \dot{u}^b_j(\xi) = a^i_j u^b_i(j^1_0\gamma)$$

so that the vector field on T_mE defined by the Lie algebra element (a^i_j) is

$$a^i_j u^b_i \frac{\partial}{\partial u^b_j}\,.$$ □

We write d^j_i for the Lie derivative operation of the basis vector field $\Delta^j_i = u^a_i \partial/\partial u^a_j$.

2.9 Second order infinitesimal actions

There is a similar result for the action of the Lie algebra of L^2_m.

Lemma 7. *Let (a^i_j, b^i_{jk}) be an element of the Lie algebra of L^2_m. The corresponding vector field on T^2_mE is*

$$a^i_j u^a_i \frac{\partial}{\partial u^a_j} + \frac{1}{\#(jk)}\left(2a^i_j u^a_{ik} + b^i_{jk}u^a_i\right)\frac{\partial}{\partial u^a_{jk}}\,.$$

where $\#(jk)$ equals 1 if $j = k$ and equals 2 otherwise.

Proof. Let γ be the curve in L^2_m through the identity $j^2_0(\mathrm{id})$ given in coordinates by

$$s \mapsto \left(\delta^i_j + s a^i_j, s b^i_{jk}\right).$$

If $j^2_0\gamma \in T^2_m E$ then the corresponding curve through $j^2_0\gamma$ is given in coordinates by

$$s \mapsto \left(u^a(j^2_0\gamma), u^a_i(j^2_0\gamma)(\delta^i_j + s a^i_j), u^a_{hi}(j^2_0\gamma)(\delta^h_j + s a^h_j)(\delta^i_k + s a^i_k) + s u^a_i(j^2_0\gamma) b^i_{jk}\right).$$

The resulting tangent vector $\xi \in T_{j^2_0\gamma} T^2_m E$ satisfies

$$\begin{aligned}
\dot u^a(\xi) &= 0 \\
\dot u^a_j(\xi) &= a^i_j u^a_i(j^2_0\gamma) \\
\dot u^a_{jk}(\xi) &= a^i_k u^a_{ij}(j^2_0\gamma) + a^i_j u^a_{ik}(j^2_0\gamma) + b^i_{jk} u^a_i(j^2_0\gamma)
\end{aligned}$$

so that the vector field on $T^2_m E$ defined by the Lie algebra element corresponding to (a^i_j, b^i_{jk}) is

$$a^i_j u^a_i \frac{\partial}{\partial u^a_j} + \frac{1}{\#(jk)}\left(2 a^i_j u^a_{ik} + u^i_{jk} u^a_i\right) \frac{\partial}{\partial u^a_{jk}}. \qquad \square$$

We write d^j_i and d^{jk}_i for the Lie derivative operation of the basis vector fields

$$\Delta^j_i = u^a_i \frac{\partial}{\partial u^a_j} + \frac{2}{\#(jk)} u^a_{ik} \frac{\partial}{\partial u^a_{jk}}, \qquad \Delta^{jk}_i = \frac{1}{\#(jk)} u^a_i \frac{\partial}{\partial u^a_{jk}}.$$

Note the use of the symbol $\#(jk)$ to compensate for the fact that the coordinate functions u^a_{jk} and u^a_{kj} are equal, so that summing over j and k could result in double-counting.

3 Geometric structures

The special structure of velocity manifolds manifests itself in the existence of certain differential operators ('total derivatives') and differential forms ('contact forms') which capture certain aspects of the structure. The total derivatives and contact forms may also be used to identify those maps between velocity manifolds, and vector fields on velocity manifolds, which have been constructed by a process known as prolongation. Finally, there is an algebraic method of lifting tangent vectors from a manifold to its velocity manifold called the vertical lift, and this gives rise to vertical endomorphisms.

3.1 Total derivatives

The identity map $T_m E \to T_m E$ defines a section of the pull-back bundle

$$\tau^*_{mE} T_m E \to T_m E.$$

Its components d_i are the *total derivatives*, vector fields along τ_{mE}. At a point $j^1_0\gamma$, the identification $T_m E \cong \bigoplus^m TE$ from Lemma 1 gives the k-th component of $j^1_0\gamma$ as

$$d_k|_{j^1_0\gamma} = j^1_0(\gamma \circ i_k) = T\gamma(j^1_0 i_k) = T\gamma\left(\left.\frac{\partial}{\partial t^k}\right|_0\right).$$

Note that the subscript k is a counting index, not a coordinate index. In coordinates, if f is a function on E then

$$\begin{aligned} d_k f|_{j_0^1\gamma} = d_k|_{j_0^1\gamma} f = T\gamma\left(\left.\frac{\partial}{\partial t^k}\right|_0\right) f = \left.\frac{\partial (f\circ\gamma)}{\partial t^k}\right|_0 \\ = \left.\frac{\partial f}{\partial u^a}\right|_{\gamma(0)} D_k\gamma^a(0) = u_k^a(j_0^1\gamma)\left.\frac{\partial f}{\partial u^a}\right|_{\gamma(0)} \end{aligned}$$

so that

$$d_k = u_k^a \frac{\partial}{\partial u^a}\,.$$

It is clear from this coordinate formula that the image of $(d_1,\ldots,d_m)$, a subspace of $T_{\gamma(0)}E$ corresponding to each point $j_0^1\gamma \in T_m E$, does not have constant rank on $T_m E$. But its restriction to $\mathring{T}_m E$, where the $m\times n$ matrix u_i^a has maximal rank, *does* have constant rank m.

3.2 Second order total derivatives

We take a similar approach to second order total derivatives. The inclusion map $T_m^2 E \to T_m T_m E$ defines a section of the pull-back bundle

$$\tau_{mE}^{2,1\,*} T_m T_m E \to T_m^2 E\,;$$

its components d_i are the *second order total derivatives*, vector fields along $\tau_{mE}^{2,1}$. At a point $j_0^2\gamma$,

$$d_k|_{j_0^2\gamma} = T(j^1\gamma)\left(\left.\frac{\partial}{\partial t^k}\right|_0\right);$$

in coordinates

$$d_k = u_k^a \frac{\partial}{\partial u^a} + u_{kj}^a \frac{\partial}{\partial u_j^a}\,.$$

Once again the image of $(d_1,\ldots,d_m)$, a subspace of $T_{j_0^1\gamma} T^m E$ corresponding to each point $j_0^2\gamma \in T_m^2 E$, does not have constant rank on $T_m^2 E$, but its restriction to $\mathring{T}_m^2 E$ *does* have constant rank m.

3.3 Contact 1-forms

Contact 1-forms on $T_m E$ or on $T_m^2 E$ are the horizontal 1-forms which annihilate total derivatives, so that θ is a contact 1-form exactly when

$$\langle \theta, d_k\rangle = 0\,.$$

Here, 'horizontal' means horizontal over E for a 1-form on $T_m E$, and it means horizontal over $T_m E$ for a 1-form on $T_m^2 E$, so that it makes sense to evaluate such forms on total derivatives; indeed, the modules of such horizontal 1-forms are dual to the modules of vector fields along $T_m E \to E$ or along $T_m^2 E \to T_m E$.

In fact we shall consider contact 1-forms, not on the whole of $T_m E$ or $T_m^2 E$, but on the submanifolds of regular velocities $\mathring{T}_m E$ and $\mathring{T}_m^2 E$. The reason is that, as mentioned previously, the image of the map $(d_1,\ldots,d_m)$ has constant rank m

only on the regular submanifolds; it is, for example, zero on the zero section of $T_m E$, and so every horizontal cotangent vector on that zero section is annihilated by all the total derivatives. If we were to include non-regular velocities then there would be 'contact' cotangent vectors which were not the values of any (smooth, and hence continuous) contact 1-form.

The important property of contact 1-forms is that they always pull back to zero under prolongations.

Lemma 8. *If θ is a contact 1-form on $\mathring{T}_m E$ then $(\bar{\jmath}^1\gamma)^*\theta = 0$. If it is a contact 1-form on $\mathring{T}^2_m E$ then $(\bar{\jmath}^2\gamma)^*\theta = 0$, where the prolonged m-curve $\bar{\jmath}^2\gamma$ is defined by $\bar{\jmath}^2\gamma(t) = j^2_0(\gamma \circ \mathrm{T}_t)$.*

Proof. If θ is a contact 1-form on $\mathring{T}_m E$ then

$$\begin{aligned}
\left\langle (\bar{\jmath}^1\gamma)^*\theta\big|_t \,,\, \frac{\partial}{\partial t^k}\bigg|_t \right\rangle &= \left\langle (j^1(\gamma \circ \mathrm{T}_t))^*\theta\big|_t \,,\, \frac{\partial}{\partial t^k}\bigg|_t \right\rangle \\
&= \left\langle (j^1\gamma)^*\theta\big|_0 \,,\, \frac{\partial}{\partial t^k}\bigg|_0 \right\rangle \\
&= \left\langle \theta\big|_{j^1_0\gamma} \,, T\gamma\bigg(\frac{\partial}{\partial t^k}\bigg|_0 \bigg) \right\rangle \\
&= \langle \theta|_{j^1_0\gamma} \,,\, d_k|_{j^1_0\gamma} \rangle = 0\,.
\end{aligned}$$

The proof for a contact 1-form on $\mathring{T}^2_m E$ is similar. □

Proposition 4. *If θ is a 1-form on $\mathring{T}_m E$ satisfying $(\bar{\jmath}^1\gamma)^*\theta = 0$ for every prolonged m-curve $\bar{\jmath}^1\gamma$ in $\mathring{T}_m E$ then θ is horizontal over E, and is a contact 1-form. A similar result holds for contact 1-forms on $\mathring{T}^2_m E$.*

Proof. We show first that θ is horizontal over E, by showing that it is horizontal at each point $j^1_0\gamma \in \mathring{T}_m E$. Write θ in coordinates around such a point as

$$\theta = \theta_a du^a + \theta^i_a du^a_i\,;$$

then if γ is a representative m-curve for the velocity $j^1_0\gamma$ we have

$$(\bar{\jmath}^1\gamma)^*\theta = (\theta_a \circ \bar{\jmath}^1\gamma)\big((\bar{\jmath}^1\gamma)^* du^a\big) + (\theta^i_a \circ \bar{\jmath}^1\gamma)\big((\bar{\jmath}^1\gamma)^* du^a_i\big)\,.$$

But

$$\begin{aligned}
(\bar{\jmath}^1\gamma)^* du^a\big|_0 &= d(u^a \circ \bar{\jmath}^1\gamma)\big|_0 = d\gamma^a\big|_0 = \frac{\partial \gamma^a}{\partial t^j}\bigg|_0 dt^j\big|_0 \\
(\bar{\jmath}^1\gamma)^* du^a_i\big|_0 &= d(u^a_i \circ \bar{\jmath}^1\gamma)\big|_0 = d\bigg(\frac{\partial \gamma^a}{\partial t^i}\bigg)\bigg|_0 = \frac{\partial \gamma^a}{\partial t^i\,\partial t^j}\bigg|_0 dt^j\big|_0
\end{aligned}$$

so that

$$0 = (\bar{\jmath}^1\gamma)^*\theta\big|_0 = \left((\theta_a \circ \bar{\jmath}^1\gamma)(0)\,\frac{\partial\gamma^a}{\partial t^j}\bigg|_0 + (\theta_a^i \circ \bar{\jmath}^1\gamma)(0)\,\frac{\partial\gamma^a}{\partial t^i\,\partial t^j}\bigg|_0\right) dt^j\big|_0$$

and hence

$$\theta_a(j_0^1\gamma)\,\frac{\partial\gamma^a}{\partial t^j}\bigg|_0 + \theta_a^i(j_0^1\gamma)\,\frac{\partial\gamma^a}{\partial t^i\,\partial t^j}\bigg|_0 = 0\,.$$

Choosing a different representative m-curve $\hat{\gamma}$ of $j_0^1\gamma$ which differs in its second derivatives from γ (although necessarily having the same first derivatives) allows us to conclude that $\theta_a^i(j_0^1\gamma) = 0$, so that θ is horizontal at $j_0^1\gamma$ and hence is a horizontal 1-form. We also see from this argument that

$$\theta_a(j_0^1\gamma)\,\frac{\partial\gamma^a}{\partial t^j}\bigg|_0 = 0\,.$$

Finally we observe that

$$\langle\theta\,, d_k\rangle = \left\langle\theta_a du^a\,, u_k^b\frac{\partial}{\partial u^b}\right\rangle = \theta_a u_k^a$$

so that

$$\langle\theta\,, d_k\rangle|_{j_0^1\gamma} = \theta_a(j_0^1\gamma)\,\frac{\partial\gamma^a}{\partial t^k}\bigg|_0 = 0$$

for each point $j_0^1\gamma \in T_m E$, showing that $\langle\theta\,, d_k\rangle = 0$ and hence that θ is a contact 1-form.

The proof for forms on $\mathring{T}_m^2 E$ is similar in principle but involves more complicated calculations. □

The coordinate expressions for contact 1-forms on velocity manifolds are quite different from those on jet manifolds, and involve determinants: indeed, contact 1-forms on $\mathring{T}_m E$ are sums of scalar multiples of $(m+1)\times(m+1)$ determinants

$$\theta^{a_1 a_2\cdots a_{m+1}} = \begin{vmatrix} u_1^{a_1} & u_1^{a_2} & \cdots & u_1^{a_{m+1}} \\ u_2^{a_1} & u_2^{a_2} & \cdots & u_2^{a_{m+1}} \\ \vdots & \vdots & & \vdots \\ u_m^{a_1} & u_m^{a_2} & \cdots & u_m^{a_{m+1}} \\ du^{a_1} & du^{a_2} & \cdots & du^{a_{m+1}} \end{vmatrix}\,.$$

To see that such a determinant is indeed a contact 1-form, evaluate it on the total derivative $d_k = u_k^b\partial/\partial u^b$ to give

$$\langle\theta^{a_1 a_2\cdots a_{m+1}}\,, d_k\rangle = \begin{vmatrix} u_1^{a_1} & u_1^{a_2} & \cdots & u_1^{a_{m+1}} \\ u_2^{a_1} & u_2^{a_2} & \cdots & u_2^{a_{m+1}} \\ \vdots & \vdots & & \vdots \\ u_m^{a_1} & u_m^{a_2} & \cdots & u_m^{a_{m+1}} \\ u_k^{a_1} & u_k^{a_2} & \cdots & u_k^{a_{m+1}} \end{vmatrix} = 0\,.$$

To show that these forms span the local contact 1-forms, we show that their values at each point span the contact cotangent vectors at that point. Let the coordinate functions on the fibres of $T^*\mathring{T}_m E$ corresponding to the coordinates (u^a, u^a_i) on $\mathring{T}_m E$ be (p_a, p^i_a); then horizontal cotangent vectors satisfy the equations $p^i_a = 0$, and we have seen that the condition $\langle \theta\,, d_k \rangle = 0$ corresponds to a coordinate condition which may now be written as $u^a_k p_a = 0$.

Now observe that at each point $j^1_0\gamma$ there is at least one set of m coordinates $(u^{a_1}_1, u^{a_2}_2, \ldots, u^{a_m}_m)$ such that the determinant $\det u^{a_i}_j$ does not vanish at $j^1_0\gamma$; suppose, without loss of generality, that this set is $(u^1_1, u^2_2, \ldots, u^m_m)$, for we may always rearrange the order of the base coordinates u^a if necessary. It is clear that the cotangent vectors

$$\theta^{12\cdots m,m+1}_{j^1_0\gamma}, \theta^{12\cdots m,m+2}_{j^1_0\gamma}, \ldots, \theta^{12\cdots m,n}_{j^1_0\gamma}$$

are linearly independent, so that the subspace of the space of contact cotangent vectors at $j^1_0\gamma$ spanned by them has dimension $n-m$. But $\dim \tau^*_{mE}(T^*_{j^1_0\gamma}E) = n$ and the m equations $u^a_k p_a$ characterising contact 1-forms are linearly independent for regular velocities, so that the dimension of the space of contact cotangent vectors at $j^1_0\gamma$ is $n - m$.

3.4 Contact r-forms

We define contact r-forms using the pull-back condition, so that an r-form ω on $\mathring{T}_m E$ is a contact r-form if $(\bar{\jmath}^1\gamma)^*\omega = 0$, and an r-form ω on $\mathring{T}^2_m E$ is a contact r-form if $(\bar{\jmath}^2\gamma)^*\omega = 0$. Note that contact r-forms need not be horizontal if $r > 1$.

We now see another important difference between contact forms on velocity manifolds and contact forms on jet manifolds. In the latter context, the contact r-forms are generated by the contact 1-forms and their exterior derivatives; but this is not the case on velocity manifolds. For example, on $\mathring{T}_2\mathbb{R}^3$ the contact 1-forms are generated by the single 1-form

$$\theta = \begin{vmatrix} u^1_1 & u^2_1 & u^3_1 \\ u^1_2 & u^2_2 & u^3_2 \\ du^1 & du^2 & du^3 \end{vmatrix} \; ;$$

but $(u^1_1 du^2 - u^2_1 du^1) \wedge du^3_2 - (u^1_2 du^2 - u^2_2 du^1) \wedge du^3_1$ is a contact 2-form which cannot be written in terms of θ and $d\theta$.

3.5 Prolongations of maps

Let E_1, E_2 be manifolds, and let $f : E_1 \to E_2$ a map. The *prolongation* of f to $T_m E_1$ is the map

$$T_m f : T_m E_1 \to T_m E_2$$

defined by

$$T_m f(j^1_0\gamma) = j^1_0(f \circ \gamma)\,.$$

It is immediate from this definition that $T_m(f \circ g) = T_m f \circ T_m g$ and that $T_m(\mathrm{id}_E) = \mathrm{id}_{T_m E}$, so that T_m is a covariant functor. In coordinates,

$$u^a \circ T_m f = f^a \,, \qquad u_i^a \circ T_m f = d_i f^a \,.$$

It is important to note that $T_m f$ might not restrict to a map $\mathring{T}_m E_1 \to \mathring{T}_m E_2$, because $f \circ \gamma$ might not be an immersion, even though γ is an immersion.

3.6 Prolongations and the exchange map

As a particular example, the prolongation of the vector bundle projection $\tau_{mE} : T_m E \to E$ to $T_{m'} T_m E$ is

$$T_{m'} \tau_{mE} : T_{m'} T_m E \to T_{m'} E \,.$$

Lemma 9. *The exchange map $e : T_{m'} T_m E \to T_m T_{m'} E$ satisfies*

$$T_{m'} \tau_{mE} \circ e = \tau_{m(T_{m'} E)} \,.$$

Proof. From Lemma 2, e may be expressed in coordinates as

$$u^a \circ e = u^a \,, \qquad u_i^a \circ e = u_{;i}^a \,, \qquad u_{;j}^a \circ e = u_j^a \,, \qquad u_{i;j}^a \circ e = u_{j;i}^a \,.$$

Thus

$$u^a \circ \tau_{m(T_{m'} E)} = u^a \,, \qquad u_i^a \circ \tau_{m(T_{m'} E)} = u_i^a$$

whereas

$$u^a \circ T_{m'} \tau_{mE} \circ e = u^a \circ e = u^a \,, \qquad u_i^a \circ T_{m'} \tau_{mE} \circ e = u_{;i}^a \circ e = u_i^a \,. \qquad \square$$

In other words, the exchange map interchanges these two diagrams.

$$\begin{array}{ccc} T_{m'} T_m E & \xrightarrow{T_{m'} \tau_{mE}} & T_{m'} E \\ \downarrow{\scriptstyle \tau_{m'(T_m E)}} & & \downarrow{\scriptstyle \tau_{m'E}} \\ T_m E & \xrightarrow[\tau_{mE}]{} & E \end{array} \quad \overset{e}{\longleftrightarrow} \quad \begin{array}{ccc} T_m T_{m'} E & \xrightarrow{\tau_{m(T_{m'} E)}} & T_{m'} E \\ \downarrow{\scriptstyle T_m \tau_{m'E}} & & \downarrow{\scriptstyle \tau_{m'E}} \\ T_m E & \xrightarrow[\tau_{mE}]{} & E \end{array}$$

3.7 Prolongations of vector fields

A vector field X on E is a map $E \to TE$, and so its prolongation (as a map) is $T_m X : T_m E \to T_m TE$.

Lemma 10. *The composition $X_m^1 = e \circ T_m X$, where $e : T_m TE \to TT_m E$ is the exchange map, is a vector field on $T_m E$*

Proof. From Lemma 9,

$$\begin{aligned}\tau_{T_mE} \circ e \circ T_m X &= T_m \tau_E \circ T_m X \\ &= T_m(\tau_E \circ X) \\ &= T_m(\mathrm{id}_E) \\ &= \mathrm{id}_{T_m E} \,.\end{aligned}$$

□

The vector field X^1_m is called the *prolongation of X to $T_m E$.*

Proposition 5. *If ψ_s is the flow of X then $T_m\psi_s$ is the flow of X^1_m.*

Proof. We first compute a coordinate formula for the vector field whose flow is $T_m\psi_s$.

Choose a point $j^1_0\gamma \in T_m E$ and let φ be the flow of X in a neighbourhood of $\gamma(0)$. Let (U, y) be a chart around $\gamma(0)$ so that, if

$$X = X^a \frac{\partial}{\partial u^a} \,,$$

φ satisfies

$$\left.\frac{\partial \varphi^a}{\partial s}\right|_{(0,\cdot)} = X^a \,.$$

Let $\tilde\varphi$ denote the map $(s, q) \mapsto T_m\varphi_s(q)$, so that

$$\tilde\varphi^a = \varphi^a \,, \qquad \tilde\varphi^a_i = d_i\varphi^a$$

where we define $(d_i\varphi^a)(s, q) = (d_i\varphi^a_s)(q)$. Then

$$\begin{aligned}\left.\frac{\partial \tilde\varphi^a_i}{\partial s}\right|_{(0,\cdot)} &= \left.\frac{\partial (d_i\varphi^a)}{\partial s}\right|_{(0,\cdot)} = \left.\frac{\partial}{\partial s}\right|_{(0,\cdot)} \left(u^b_i \frac{\partial \varphi^a}{\partial u^b}\right) \\ &= u^b_i \left.\frac{\partial \varphi^a}{\partial u^b\,\partial s}\right|_{(0,\cdot)} = d_i\left(\frac{\partial \varphi^a}{\partial s}(0,\cdot)\right)\bigg|_{(0,\cdot)} = d_i X^a \,,\end{aligned}$$

so that, in coordinates, the vector field whose flow is $T_m\psi_s$ is

$$X^a \frac{\partial}{\partial u^a} + (d_i X^a)\frac{\partial}{\partial u^a_i} \,.$$

On the other hand, regarding X as a map $E \to TE$, and writing $\dot u^a$ as u^a_1,

$$u^a \circ X = u^a \,, \qquad u^a_1 \circ X = X^a$$

so that

$$\begin{aligned} u^a \circ T_m X &= u^a \,, & u^a_1 \circ T_m X &= X^a \,, \\ u^a_{;i} \circ T_m X &= u^a_i \,, & u^a_{1i} \circ T_m X &= d_i X^a \,;\end{aligned}$$

thus

$$u^a \circ e \circ T_m X = u^a\,, \qquad u^a_{;1} \circ e \circ T_m X = X^a\,,$$
$$u^a_i \circ e \circ T_m X = u^a_i\,, \qquad u^a_{i1} \circ e \circ T_m X = d_i X^a$$

so that

$$X^1_m = e \circ T_m X = X^a \frac{\partial}{\partial u^a} + (d_i X^a) \frac{\partial}{\partial u^a_i}\,. \qquad \square$$

Unlike prolongations of maps, prolongations of vector fields *do* restrict to $\mathring{T}_m E$.

3.8 Second prolongations

By extending the first order approach, maps $f : E_1 \to E_2$ may be prolonged to maps $T^2_m f : T^2_m E_1 \to T^2_m E_2$, and vector fields X on E may be prolonged to vector fields X^2_m on $T^2_m E$. In coordinates,

$$u^a \circ T^2_m f = f^a\,, \qquad u^a_i \circ T^2_m f = d_i f^a\,, \qquad u^a_{ij} \circ T^2_m f = d_i d_j f^a$$

and if $X = X^a \partial/\partial u^a$ then

$$X^2_m = X^a \frac{\partial}{\partial u^a} + (d_i X^a) \frac{\partial}{\partial u^a_i} + \frac{1}{\#(ij)} (d_i d_j X^a) \frac{\partial}{\partial u^a_{ij}}\,.$$

The calculations are similar in principle to those given for the first order case, but more complicated in detail. Again $T^2_m f$ might not restrict to a map $\mathring{T}^2_m E_1 \to \mathring{T}^2_m E_2$, whereas X^2_m does restrict to $\mathring{T}^2_m E$.

3.9 Prolongations, contact forms, and total derivatives

Let $f : E_1 \to E_2$ be a map. If θ is a contact form on $\mathring{T}_m E_2$ and if $T_m f$ restricts to $\mathring{T}_m E_1$ then $(T_m f)^* \theta$ is a contact form on $\mathring{T}_m E_1$, because

$$(\bar{\jmath}^1 \gamma)^* (T_m f)^* \theta = (T_m f \circ \bar{\jmath}^1 \gamma)^* \theta = \bigl(\bar{\jmath}^1 (f \circ \gamma)\bigr)^* \theta = 0\,.$$

If X is a vector field on E and θ is a contact form on $\mathring{T}_m E$ then the Lie derivative $\mathcal{L}_{X^1_m} \theta$ by the prolongation of X is also a contact form, because the flow of X^1_m is the prolongation of the flow of X. These results, using the characterisation of a contact form by vanishing pullback, apply to both 1-forms and to r-forms with $r > 1$. They also hold for contact forms on $\mathring{T}^2_m E$.

The corresponding result for total derivatives is more complicated, as these operators are vector fields along a map rather than on a manifold.

Lemma 11. *Prolongations and basis total derivatives commute, so that*

$$d_i \circ \mathcal{L}_X = \mathcal{L}_{X^1_m} \circ d_i\,, \qquad d_i \circ \mathcal{L}_{X^1_m} = \mathcal{L}_{X^2_m} \circ d_i\,.$$

Proof. We check this using coordinates. In the first order case, if f is a function on E then

$$d_i(\mathcal{L}_X f) = d_i\left(X^a \frac{\partial f}{\partial u^a}\right) = u_i^b\left(\frac{\partial X^a}{\partial u^b}\frac{\partial f}{\partial u^a} + X^a \frac{\partial f}{\partial u^b\,\partial u^a}\right)$$

whereas

$$\mathcal{L}_{X_m^1}(d_i f) = \mathcal{L}_{X_m^1}\left(u_i^b \frac{\partial f}{\partial u^b}\right) = (d_i X^b)\frac{\partial f}{\partial u^b} + u_i^b X^a \frac{\partial f}{\partial u^b\,\partial u^a}\,.$$

A similar but slightly more lengthy calculation is used in the second order case. □

3.10 Vertical endomorphisms

We have seen that $T_m E \to E$ is a vector bundle and so, as with every vector bundle, it has a canonical vertical lift operator. Denote the vertical lift to $(\eta_i) \in \bigoplus^m TE \cong T_m E$ by

$$T_{m|\tau_m(\eta_i)}E \to T_{(\eta_i)}T_m E\,, \qquad (\xi_k) \mapsto (\xi_k)^{\uparrow(\eta_i)}\,;$$

in coordinates this is

$$(\xi_k)^{\uparrow(\eta_i)} = u_j^a(\xi_k)\left.\frac{\partial}{\partial u_j^a}\right|_{(\eta_i)}\,.$$

For each vector $\zeta \in T_{(\eta_i)}T_m E$ define the vector $S^j\zeta \in T_{(\eta_i)}T_m E$ by

$$S^j\zeta = (0, \ldots, 0, T\tau_m(\zeta), 0, \ldots, 0)^{\uparrow(\eta_i)}$$

where the non-zero vector $T\tau_m(\zeta)$ is in the j-th position. It is evident that S^j is a vector bundle map $TT_m E \to TT_m E$, or alternatively a type $(1,1)$ tensor field on $T_m E$, called a *vertical endomorphism*. Note that the superscript j is a counting index, not a coordinate index. In coordinates

$$S^j = du^a \otimes \frac{\partial}{\partial u_j^a}\,.$$

There is a close relationship between vertical endomorphisms and total derivatives.

Lemma 12. *If ω is an r-form on E then*

$$S^j d_k\omega = r\delta_k^j(\tau_{mE}^*\omega)\,.$$

Proof. Suppose first that θ is a 1-form; we shall give a proof in coordinates, omitting explicit mention of the pullback map. If $\theta = \theta_a du^a$ then

$$S^j d_k\theta = S^j\big((d_k\theta_a)du^a + \theta_a du_k^a\big) = \delta_k^j\theta_a du^a = \delta_k^j\theta\,.$$

We now use induction on r. Suppose ω is an r-form and that $S^j d_k\omega = r\delta^j_k(\tau^*_{mE}\omega)$; then

$$\begin{aligned} S^j d_k(\theta\wedge\omega) &= S^j\big(d_k\theta\wedge\tau^*_{mE}\omega + \tau^*_{mE}\theta\wedge d_k\omega\big)\\ &= S^j d_k\theta\wedge\tau^*_{mE}\omega + \tau^*_{mE}\theta\wedge S^j d_k\omega\\ &= \delta^j_k(\tau^*_{mE}\theta\wedge\tau^*_{mE}\omega) + r\delta^j_k(\tau^*_{mE}\theta\wedge\tau^*_{mE}\omega)\\ &= (r+1)\delta^j_k\,\tau^*_{mE}(\theta\wedge\omega)\end{aligned}$$

using the fact that $\tau^*_{mE}\theta$ and $\tau^*_{mE}\omega$ are horizontal over E. The result now follows by linearity. □

3.11 Second order vertical endomorphisms

There is also a version of the vertical endomorphism defined on second order velocity manifolds. This cannot be constructed in the same way as the first order vertical endomorphism, as $T^2_mE \to T_mE$ is not a vector bundle but is instead an affine sub-bundle of $T_mT_mE \to T_mE$. We shall establish our construction by modifying the first-order vertical endomorphism on T_mT_mE. There is an alternative method, based on the construction of vertical lifts using double $(1,m)$-curves, which may be used in both first and second order cases, but we shall not describe that here.

So let $\nu : T^2_mE \to T_mE$ be some tubular neighbourhood of T^2_mE in T_mT_mE, and let $\iota : T^2_mE \to T_mT_mE$ be the inclusion from Proposition 2. As before, let $e : T_mT_mE \to T_mT_mE$ be the exchange map.

Proposition 6. *Let θ be a 1-form on T^2_mE; then the operation*

$$\theta \mapsto \iota^*\big(S^k \lrcorner (\nu^*\theta + e^*\nu^*\theta)\big)\,,$$

where S^k is the vertical endomorphism on T_mT_mE), does not depend on the choice of tubular neighbourhood map ν and hence defines a vertical endomorphism on T^2_mE.

Proof. We use coordinates to show that the result is independent of ν. Let $\theta = \theta_a du^a + \theta^i_a du^a_i + \theta^{ij}_a du^a_{ij}$; then

$$\begin{aligned}\nu^*\theta &= (\nu^*\theta_a)d\nu^a + (\nu^*\theta^i_a)d\nu^a_i + (\nu^*\theta^{ij}_a)d\nu^a_{ij}\\ &= (\nu^*\theta_a)\left(du^a + \frac{\partial\nu^a}{\partial u^c_p}(du^c_p - du^c_{;p}) + \tfrac12\frac{\partial\nu^a}{\partial u^c_{p;q}}(du^c_{p;q} - du^c_{q;p})\right)\\ &\qquad + (\nu^*\theta^i_a)\left(\tfrac12(du^a_i + du^a_{;i}) + \tfrac12\left(\frac{\partial\nu^a_i}{\partial u^c_p} - \frac{\partial\nu^a_i}{\partial u^c_{;p}}\right)(du^c_p - du^c_{;p})\right.\\ &\qquad\quad \left. + \tfrac12\frac{\partial\nu^a_i}{\partial u^c_{p;q}}(du^c_{p;q} - du^c_{q;p})\right)\\ &\qquad + (\nu^*\theta^{ij}_a)\left(\tfrac12(du^a_{i;j} + du^a_{j;i}) + \frac{\partial\nu^a_{ij}}{\partial u^c_p}(du^c_p - du^c_{;p}) + \tfrac12\frac{\partial\nu^a_{ij}}{\partial u^c_{p;q}}(du^c_{p;q} - du^c_{q;p})\right)\end{aligned}$$

using the coordinate formulæ for the tubular neighbourhood map given in Section 2. Thus

$$
\begin{aligned}
S^k \lrcorner\, \nu^*\theta = (\nu^*\theta_a)\Big(&-\frac{\partial\nu^a}{\partial u^c_k}du^c + \tfrac{1}{2}\Big(\frac{\partial\nu^a}{\partial u^c_{p;k}} - \frac{\partial\nu^a}{\partial u^c_{k;p}}\Big)du^c_p\Big)\\
&+ (\nu^*\theta^i_a)\Big(\delta^k_i\tfrac{1}{2}du^a - \tfrac{1}{2}\Big(\frac{\partial\nu^a_i}{\partial u^c_k} - \frac{\partial\nu^a_i}{\partial u^c_{;k}}\Big)du^c\\
&\qquad + \tfrac{1}{2}\Big(\frac{\partial\nu^a_i}{\partial u^c_{p;k}} - \frac{\partial\nu^a_i}{\partial u^c_{k;p}}\Big)du^c_p\Big)\\
&+ (\nu^*\theta^{ij}_a)\Big(\tfrac{1}{2}(\delta^k_j du^a_i + \delta^k_i du^a_j) - \frac{\partial\nu^a_{ij}}{\partial u^c_k}du^c\\
&\qquad + \tfrac{1}{2}\Big(\frac{\partial\nu^a_{ij}}{\partial u^c_{p;k}} - \frac{\partial\nu^a_{ij}}{\partial u^c_{k;p}}\Big)du^c_p\Big)
\end{aligned}
$$

so that

$$
\begin{aligned}
\iota^*(S^k \lrcorner\, \nu^*\theta) = \theta_a\Big(&-\iota^*\Big(\frac{\partial\nu^a}{\partial u^c_k}\Big)du^c + \tfrac{1}{2}\iota^*\Big(\frac{\partial\nu^a}{\partial u^c_{p;k}} - \frac{\partial\nu^a}{\partial u^c_{k;p}}\Big)du^c_p\Big)\\
&+ \theta^i_a\Big(\delta^k_i\tfrac{1}{2}du^a - \tfrac{1}{2}\iota^*\Big(\frac{\partial\nu^a_i}{\partial u^c_k} - \frac{\partial\nu^a_i}{\partial u^c_{;k}}\Big)du^c\\
&\qquad + \tfrac{1}{2}\iota^*\Big(\frac{\partial\nu^a_i}{\partial u^c_{p;k}} - \frac{\partial\nu^a_i}{\partial u^c_{k;p}}\Big)du^c_p\Big)\\
&+ \theta^{ij}_a\Big(\tfrac{1}{2}(\delta^k_j du^a_i + \delta^k_i du^a_j) - \iota^*\Big(\frac{\partial\nu^a_{ij}}{\partial u^c_k}\Big)du^c\\
&\qquad + \tfrac{1}{2}\iota^*\Big(\frac{\partial\nu^a_{ij}}{\partial u^c_{p;k}} - \frac{\partial\nu^a_{ij}}{\partial u^c_{k;p}}\Big)du^c_p\Big)\,;
\end{aligned}
$$

and similarly

$$
\begin{aligned}
S^k \lrcorner\, e^*\nu^*\theta = (e^*\nu^*\theta_a)\Big(&e^*\Big(\frac{\partial\nu^a}{\partial u^c_k}\Big)du^c + \tfrac{1}{2}e^*\Big(\frac{\partial\nu^a}{\partial u^c_{k;p}} - \frac{\partial\nu^a}{\partial u^c_{p;k}}\Big)du^c_p\Big)\\
&+ (e^*\nu^*\theta^i_a)\Big(\tfrac{1}{2}\delta^k_i du^a + \tfrac{1}{2}e^*\Big(\frac{\partial\nu^a_i}{\partial u^c_k} - \frac{\partial\nu^a_i}{\partial u^c_{;k}}\Big)du^c\\
&\qquad + \tfrac{1}{2}e^*\Big(\frac{\partial\nu^a_i}{\partial u^c_{k;p}} - \frac{\partial\nu^a_i}{\partial u^c_{p;k}}\Big)du^c_p\Big)\\
&+ (e^*\nu^*\theta^{ij}_a)\Big(\tfrac{1}{2}(\delta^k_j du^a_i + \delta^k_i du^a_j) + e^*\Big(\frac{\partial\nu^a_{ij}}{\partial u^c_k}\Big)du^c\\
&\qquad + \tfrac{1}{2}e^*\Big(\frac{\partial\nu^a_{ij}}{\partial u^c_{k;p}} - \frac{\partial\nu^a_{ij}}{\partial u^c_{p;k}}\Big)du^c_p\Big)
\end{aligned}
$$

so that, using $\iota^* e^* = \iota^*$,

$$\begin{aligned}\iota^*(S^k \lrcorner\, e^*\nu^*\theta) = \theta_a\biggl(\iota^*\biggl(\frac{\partial \nu^a}{\partial u^c_k}\biggr)du^c + \tfrac{1}{2}\iota^*\biggl(\frac{\partial \nu^a}{\partial u^c_{k;p}} - \frac{\partial \nu^a}{\partial u^c_{p;k}}\biggr)du^c_p\biggr) \\ + \theta^i_a\biggl(\tfrac{1}{2}\delta^k_i du^a + \tfrac{1}{2}\iota^*\biggl(\frac{\partial \nu^a_i}{\partial u^c_k} - \frac{\partial \nu^a_i}{\partial u^c_{;k}}\biggr)du^c \\ + \tfrac{1}{2}\iota^*\biggl(\frac{\partial \nu^a_i}{\partial u^c_{k;p}} - \frac{\partial \nu^a_i}{\partial u^c_{p;k}}\biggr)du^c_p\biggr) \\ + \theta^{ij}_a\biggl(\tfrac{1}{2}(\delta^k_j du^a_i + \delta^k_i du^a_j) + \iota^*\biggl(\frac{\partial \nu^a_{ij}}{\partial u^c_k}\biggr)du^c \\ + \tfrac{1}{2}\iota^*\biggl(\frac{\partial \nu^a_{ij}}{\partial u^c_{k;p}} - \frac{\partial \nu^a_{ij}}{\partial u^c_{p;k}}\biggr)du^c_p\biggr).\end{aligned}$$

Thus, adding, we obtain

$$\iota^*(S^k \lrcorner\, (\nu^*\theta + e^*\nu^*\theta)) = \theta^k_a du^a + 2\theta^{ik}_a du^a_i$$

using $\theta^{ki}_a = \theta^{ik}_a$. □

In coordinates, therefore, the second order vertical endomorphisms may be written as tensor fields

$$S^k = du^a \otimes \frac{\partial}{\partial u^a_k} + \frac{2}{\#(ik)} du^a_i \otimes \frac{\partial}{\partial u^a_{ik}};$$

the factor $1/\#(ik)$ arises here because the contraction of $\partial/\partial u^a_{ik}$ with du^c_{pq} equals $\frac{1}{2}\#(ik)\delta^c_a(\delta^i_p\delta^k_q + \delta^i_q\delta^k_p)$, so that

$$\frac{\partial}{\partial u^a_{ik}} \lrcorner\, (\theta^{pq}_c du^c_{pq}) = \frac{\#(ik)}{2}\delta^c_a(\delta^i_p\delta^k_q + \delta^i_q\delta^k_p)\theta^{pq}_c = \#(ik)\,\theta^{ik}_a\,.$$

The relationship given in Lemma 12 between vertical endomorphisms and total derivatives may now be extended to a kind of homotopy formula.

Lemma 13. *If ω is an r-form on $T_m E$ then*

$$S^j d_k\omega - d_k S^j\omega = r\,\delta^j_k(\tau^{2,1\,*}_{mE}\omega)\,.$$

Proof. Suppose first that θ is a 1-form; we shall give a proof in coordinates, omitting explicit mention of the pullback map. If $\theta = \theta_a du^a + \theta^i_a du^a_i$ then

$$d_k\theta = (d_k\theta_a)du^a + \theta_a du^a_k + (d_k\theta^i_a)du^a_i + \theta^i_a du^a_{ik}$$

so that

$$S^j d_k\theta = \bigl(\delta^j_k\theta_a + (d_k\theta^j_a)\bigr)du^a + \delta^j_k\theta^i_a du^a_i + \theta^j_a du^a_k\,.$$

On the other hand, $S^j\theta = \theta^j_a du^a$, so that

$$d_k S^j\theta = (d_k\theta^j_a)du^a + \theta^j_a du^a_k$$

and hence

$$S^j d_k\theta - d_k S^j\theta = \delta^j_k \theta_a du^a + \delta^j_k \theta^i_a du^a_i = \delta^j_k \theta\,.$$

We now use induction on r. Suppose ω is an r-form and that $S^j d_k\omega - d_k S^j\omega = r\delta^j_k(\tau^{2,1\,*}_{mE}\omega)$; then, as both S^j and d_k are derivations of degree zero, their commutator is a derivation of degree zero, and so

$$\begin{aligned}(S^j d_k - d_k S^j)(\theta\wedge\omega) &= (S^j d_k - d_k S^j)\theta \wedge \tau^{2,1\,*}_{mE}\omega + \tau^{2,1\,*}_{mE}\theta\wedge(S^j d_k - d_k S^j)\omega\\ &= r\delta^j_k\,\tau^{2,1\,*}_{mE}\theta\wedge\tau^{2,1\,*}_{mE}\omega + \delta^j_k\tau^{2,1\,*}_{mE}\theta\wedge\tau^{2,1\,*}_{mE}\omega\\ &= (r+1)\delta^j_k\tau^{2,1\,*}_{mE}(\theta\wedge\omega)\,.\end{aligned}$$

The result now follows by linearity. □

4 Vector forms

We often use *vectors* of operators, tensors, forms, and so on. For instance, we have defined the total derivatives d_k and the vertical endomorphisms S^j, where j and k are counting indices rather than coordinate indices. These operators fit into a framework of *vector forms*, to which we can associate a cohomology theory. Although the full cohomology theory requires the use of higher-order velocity manifolds, we can see some aspects of the theory in the first and second order cases.

4.1 Vector forms

We consider differential forms on E, $\mathring{T}_m E$ and $\mathring{T}^2_m E$ taking values in the vector space $\mathbb{R}^{m*}$ and its exterior powers. Write $\mathring{T}^k_m E$ with $k = 0, 1, 2$ and put

$$\Omega^{r,s}_k = \left(\Omega^r \mathring{T}^k_m E\right) \otimes \left(\bigwedge\nolimits^s \mathbb{R}^{m*}\right)\,.$$

Then a typical element of $\Omega^{r,s}_k$ is

$$\Xi = \chi_{i_1\cdots i_s} \otimes dt^{i_1}\wedge\ldots\wedge dt^{i_s} \in \Omega^{r,s}_k$$

where the scalar forms $\chi_{i_1\cdots i_s}$ are skew-symmetric in their indices, and where, as in Corollary 1, $\{dt^i\}$ is the canonical basis of $\mathbb{R}^{m*}$. It is clear that $\Omega^{r,s}_k$ is a module over the algebra of functions on $\mathring{T}^k_m E$.

4.2 Operations on vector forms

Define the operators d and d_{T} on the modules of vector forms by their actions on decomposable forms,

$$\begin{aligned} d : \Omega^{r,s}_k &\to \Omega^{r+1,s}_k\,, & d(\chi\otimes\omega) &= d\chi\otimes\omega\\ d_{\mathrm{T}} : \Omega^{r,s}_k &\to \Omega^{r,s+1}_{k+1}\,, & d_{\mathrm{T}}(\chi\otimes\omega) &= d_i\chi\otimes(dt^i\wedge\omega)\,,\end{aligned}$$

so that

$$\begin{aligned} dd_{\mathrm{T}}(\chi\otimes\omega) &= d\big(d_i\chi\otimes(dt^i\wedge\omega)\big) = dd_i\chi\otimes(dt^i\wedge\omega)\\ &= d_i d\chi\otimes(dt^i\wedge\omega) = d_{\mathrm{T}}(d\chi\otimes\omega) = d_{\mathrm{T}} d(\chi\otimes\omega)\end{aligned}$$

and

$$d_{\mathrm{T}}^2(\chi \otimes \omega) = d_j d_i \chi \otimes (dt^j \wedge dt^i \wedge \omega) = 0\,,$$

showing that $dd_{\mathrm{T}} = d_{\mathrm{T}}d$ and $d_{\mathrm{T}}^2 = 0$. We say that $d\Xi$ is the *differential* of the vector form Ξ, and that $d_{\mathrm{T}}\Xi$ is its *total derivative*.

The total derivative of a vector form is a type of Lie derivative, and so we can also define the corresponding contraction operation. Put

$$i_{\mathrm{T}} : \Omega_k^{r,s} \to \Omega_{k+1}^{r-1,s+1}\,, \qquad i_{\mathrm{T}}(\chi \otimes \omega) = (d_i \lrcorner \chi) \otimes dt^i \wedge \omega$$

where $d_i \lrcorner \chi$ denotes the contraction of the 'vector field along a map' d_i with the scalar form χ, so that

$$d_{\mathrm{T}} = di_{\mathrm{T}} + i_{\mathrm{T}}d\,.$$

4.3 Equivariant vector forms

Let $\alpha_{j_0^1\phi} : \mathring{T}_m E \to \mathring{T}_m E$ denote the right action of $j_0^1\phi \in L_m^{1+}$ on $\mathring{T}_m E$ by

$$\alpha_{j_0^1\phi}(j_0^1\gamma) = j_0^1(\gamma \circ \phi)\,;$$

also, let $A_{j_0^1\phi} : \mathbb{R}^{m*} \to \mathbb{R}^{m*}$ denote the linear map

$$A_{j_0^1\phi}(dt^i) = \big(D_j\phi^i(0)\big)dt^j\,,$$

and extend this by multilinearity to $A_{j_0^1\phi} : \bigwedge^s\mathbb{R}^{m*} \to \bigwedge^s\mathbb{R}^{m*}$. The vector form $\chi_{i_1\cdots i_s} \otimes (dt^{i_1} \wedge \cdots \wedge dt^{i_s}) \in \Omega_1^{r,s}$ is said to be *equivariant* if, for every $j_0^1\phi$,

$$\alpha_{j_0^1\phi}^*(\chi_{i_1\cdots i_s}) \otimes (dt^{i_1} \wedge \cdots \wedge dt^{i_s}) = \chi_{i_1\cdots i_s} \otimes A_{j_0^1\phi}(dt^{i_1} \wedge \cdots \wedge dt^{i_s})\,.$$

Thus an equivariant form, regarded as a map from objects defined on a velocity manifold to elements of a vector space, commutes with the action of the jet group on the manifold and the vector space. We use the *oriented* jet group in our definition, as our application will be to problems in the calculus of variations where we need to integrate the forms.

We shall be particularly interested in equivariant elements of $\Omega_1^{0,m}$, namely 0-forms (functions) taking their values in the one-dimensional vector space $\bigwedge^m\mathbb{R}^{m*}$. Then

$$A_{j_0^1\phi}(dt^1 \wedge \cdots \wedge dt^m) = \mathcal{J}\phi(0)(dt^1 \wedge \cdots \wedge dt^m)$$

where $\mathcal{J}\phi = \det(D_j\phi^i)$ is the Jacobian of ϕ, and so, writing $d^m t$ for $dt^1 \wedge \cdots \wedge dt^m$, an element $\Lambda = L\,d^m t$ is equivariant when

$$(L \circ \alpha_{j_0^1\phi})d^m t = \det\big(D_j\phi^i(0)\big)L\,d^m t\,.$$

Thus, writing an element of $T_m E \cong \bigoplus^m TE$ as $(\xi_1, \ldots, \xi_m)$, Λ is equivariant when for each matrix $A \in \mathrm{GL}^+(m, \mathbb{R})$,

$$L(\xi_i A_j^i) = (\det A)L(\xi_j)\,.$$

As the oriented jet group L_m^{1+} is connected, there is an infinitesimal condition for equivariance. For a vector form $\chi_{i_1\cdots i_s} \otimes (dt^{i_1} \wedge \cdots \wedge dt^{i_s}) \in \Omega_1^{r,s}$, we require

$$d_i^j(\chi_{i_1\cdots i_s}) \otimes (dt^{i_1} \wedge \cdots \wedge dt^{i_s}) = \chi_{i_1\cdots i_s} \otimes \mathcal{L}_{t^j \partial/\partial t^i}(dt^{i_1} \wedge \cdots \wedge dt^{i_s})$$

In the particular case where $s = m$ we have $\mathcal{L}_{t^j \partial/\partial t^i} d^m t = \delta_i^j d^m t$, so the condition simplifies to

$$d_i^j \chi = \delta_i^j \chi \,.$$

4.4 The bicomplex

It is clear that for $-1 \leq s \leq m-2$ we can use the operators d and d_{T} to construct a bicomplex:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \overline{\Omega}_0^{0,s} & \xrightarrow{d} & \Omega_0^{1,s} & \xrightarrow{d} & \Omega_0^{2,s} & \xrightarrow{d} & \Omega_0^{3,s} & \longrightarrow & \\
 & & \downarrow{\scriptstyle d_{\mathrm{T}}} & & \downarrow{\scriptstyle d_{\mathrm{T}}} & & \downarrow{\scriptstyle d_{\mathrm{T}}} & & \downarrow{\scriptstyle d_{\mathrm{T}}} & & \\
0 & \longrightarrow & \overline{\Omega}_1^{0,s+1} & \xrightarrow{d} & \Omega_1^{1,s+1} & \xrightarrow{d} & \Omega_1^{2,s+1} & \xrightarrow{d} & \Omega_1^{3,s+1} & \longrightarrow & \\
 & & \downarrow{\scriptstyle d_{\mathrm{T}}} & & \downarrow{\scriptstyle d_{\mathrm{T}}} & & \downarrow{\scriptstyle d_{\mathrm{T}}} & & \downarrow{\scriptstyle d_{\mathrm{T}}} & & \\
0 & \longrightarrow & \overline{\Omega}_2^{0,s+2} & \xrightarrow{d} & \Omega_2^{1,s+2} & \xrightarrow{d} & \Omega_2^{2,s+2} & \xrightarrow{d} & \Omega_2^{3,s+2} & \longrightarrow &
\end{array}$$

where if $s = -1$ then $\Omega_*^{*,s} = 0$. In this bicomplex $\overline{\Omega}_*^{0,*}$ means 'modulo constant functions', and is used instead of the usual beginning $0 \to \mathbb{R} \to \Omega^0 \to \ldots$ of the de Rham sequence.

An important property of the bicomplex is that all columns (apart from the first) are globally exact, we show this by obtaining a homotopy formula for d_{T}. Strictly speaking the homotopy formula involves *third order* forms which are horizontal over E, because the operator P_2 defined in the statement of the theorem involves applying a total derivative to (scalar) second-order forms which are horizontal over E; but if $d_{\mathrm{T}}\Xi = 0$ then the operator P_2 is not involved and the formula is genuinely second order. We feel, nevertheless, that it is worthwhile giving the more general statement, on the understanding that the definition of the total derivative of a second order form, and the consequent generalisation of Lemma 13, follow exactly the same pattern as before. We also use the operator P_2 when studying equivalents of first-order Lagrangians, although in that context the image of P_2 is always second-order rather than third-order.

Theorem 1. *If $\Xi \in \Omega_1^{r,s+1}$ with $r > 0$ then, to within a pullback,*

$$P_2 d_{\mathrm{T}}\Xi + d_{\mathrm{T}} P_1 \Xi = \Xi \,,$$

where

$$P_1\big(\chi_{i_1\cdots i_{s+1}} \otimes dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}\big) = \frac{1}{r(m-s)} S^j \chi_{i_1\cdots i_{s+1}} \otimes \left(\frac{\partial}{\partial t^j} \lrcorner\, dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}\right)$$

for first-order r-forms $\chi_{i_1\cdots i_{s+1}}$, and

$$\begin{aligned}
&P_2\big(\eta_{i_1\cdots i_{s+2}} \otimes dt^{i_1} \wedge \cdots \wedge dt^{i_{s+2}}\big) \\
&= \left(\frac{1}{r(m-s-1)} S^j \eta_{i_1\cdots i_{s+2}} - \frac{1}{r^2(m-s)(m-s-1)} d_l S^l S^j \eta_{i_1\cdots i_{s+2}}\right) \otimes \\
&\quad \otimes \left(\frac{\partial}{\partial t^j} \lrcorner\, dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}\right)
\end{aligned}$$

for second-order r-forms $\eta_{i_1\cdots i_{s+2}}$.

Proof. This is a consequence of Lemma 13. Put

$$\begin{aligned}
P_1^j &= \frac{1}{r(m-s)} S^j \\
P_2^j &= \frac{1}{r(m-s-1)} S^j - \frac{1}{r^2(m-s)(m-s-1)} d_l S^l S^j\,;
\end{aligned}$$

then

$$\begin{aligned}
P_2 d_{\mathrm{T}} \Xi &= P_2\big(d_k \chi_{i_1\cdots i_{s+1}} \otimes dt^k \wedge dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}\big) \\
&= P_2^j d_k \chi_{i_1\cdots i_{s+1}} \otimes \left(\frac{\partial}{\partial t^j} \lrcorner\, dt^k \wedge dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}\right) \\
&= P_2^j d_k \chi_{i_1\cdots i_{s+1}} \otimes (\delta^k_j\, dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}) \\
&\qquad - P_2^j d_k \chi_{i_1\cdots i_{s+1}} \otimes dt^k \wedge \left(\frac{\partial}{\partial t^j} \lrcorner\, dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}\right) \\
&= P_2^k d_k \chi_{i_1\cdots i_{s+1}} \otimes dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}} \\
&\qquad - (s+1) P_2^j d_{i_1} \chi_{j i_2\cdots i_{s+1}} \otimes dt^{i_1} \wedge dt^{i_2} \wedge \cdots \wedge dt^{i_{s+1}}
\end{aligned}$$

whereas

$$\begin{aligned}
d_{\mathrm{T}} P_1 \Xi &= d_{\mathrm{T}}\left(P_1^j \chi_{i_1\cdots i_{s+1}} \otimes \left(\frac{\partial}{\partial t^j} \lrcorner\, dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}\right)\right) \\
&= d_k P_1^j \chi_{i_1\cdots i_{s+1}} \otimes dt^k \wedge \left(\frac{\partial}{\partial t^j} \lrcorner\, dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}\right) \\
&= (s+1) d_{i_1} P_1^j \chi_{j i_2\cdots i_{s+1}} \otimes dt^{i_1} \wedge dt^{i_2} \wedge \cdots \wedge dt^{i_{s+1}}
\end{aligned}$$

so that

$$\begin{aligned}
P_2 d_{\mathrm{T}} \Xi + d_{\mathrm{T}} P_1 \Xi = P_2^k d_k \chi_{i_1\cdots i_{s+1}} &\otimes dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}} \\
- (s+1) P_2^j d_{i_1} \chi_{j i_2\cdots i_{s+1}} &\otimes dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}} \\
+ (s+1) d_{i_1} P_1^j \chi_{j i_2\cdots i_{s+1}} &\otimes dt^{i_1} \wedge \cdots \wedge dt^{i_{s+1}}\,.
\end{aligned}$$

But, using Lemma 13, the operators acting on $\chi_{ji_2\cdots i_{s+1}}$ satisfy

$$
\begin{aligned}
\delta^j_{i_1} P^k_2 d_k &= \frac{1}{r(m-s-1)} \delta^j_{i_1} S^k d_k - \frac{1}{r^2(m-s)(m-s-1)} \delta^j_{i_1} d_l S^l S^k d_k \\
&= \frac{1}{r(m-s-1)} \delta^j_{i_1} (d_k S^k + mr) \\
&\quad - \frac{1}{r^2(m-s)(m-s-1)} \delta^j_{i_1} (d_l d_k S^l S^k + (m+1) r d_l S^l) \\
&= \frac{m}{m-s-1} \delta^j_{i_1} - \frac{s+1}{r(m-s)(m-s-1)} \delta^j_{i_1} d_k S^k \,,
\end{aligned}
$$

using the fact that $S^l S^k \chi_{i_1\cdots i_{s+1}} = 0$ because the $\chi_{i_1\cdots i_{s+1}}$ are first-order forms. Similarly

$$
\begin{aligned}
-(s+1) P^j_2 d_{i_1} &= -\frac{s+1}{r(m-s-1)} S^j d_{i_1} + \frac{s+1}{r^2(m-s)(m-s-1)} d_l S^l S^j d_{i_1} \\
&= -\frac{s+1}{r(m-s-1)} (d_{i_1} S^j + r \delta^j_{i_1}) \\
&\quad + \frac{s+1}{r^2(m-s)(m-s-1)} (d_l d_{i_1} S^l S^j + r d_{i_1} S^j + r \delta^j_{i_1} d_l S^l) \\
&= -\frac{s+1}{(m-s-1)} \delta^j_{i_1} - \frac{(s+1)}{r(m-s)} d_{i_1} S^j \\
&\quad + \frac{s+1}{r(m-s)(m-s-1)} \delta^j_{i_1} d_k S^k
\end{aligned}
$$

and

$$
(s+1) d_{i_1} P^j_1 = \frac{s+1}{r(m-s)} d_{i_1} S^j \,,
$$

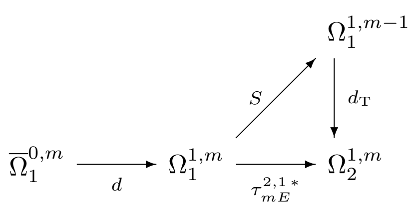
from which we see that

$$
\delta^j_{i_1} P^k_2 d_k - (s+1) P^j_2 d_{i_1} + (s+1) d_{i_1} P^j_1 = \delta^j_{i_1}
$$

and the result follows. □

4.5 The bottom left corner

The part of the bicomplex which holds the major interest for the calculus of variations is in the bottom left-hand corner; we shall repeat it, with a pull-back map shown explicitly where appropriate.



Take $[\Lambda] \in \overline{\Omega}_1^{0,m}$, so that, for some function L on $\mathring{T}_m E$, we have for any representative

$$\Lambda = L\, dt^1 \wedge \cdots \wedge dt^m = L\, d^m t\,.$$

Here, L will play the role of a (first order) Lagrangian function in the calculus of variations, and the vector-valued function Λ will have the capability of being integrated along m-curves in $\mathring{T}_m E$ (and, in particular, along prolongations to $\mathring{T}_m E$ of m-curves in E). So, given the equivalence class $[\Lambda]$, define

$$\Theta_1 = S d\Lambda\,, \qquad \mathcal{E}_0 = \tau_{mE}^{2,1\,*} d\Lambda - d_{\mathrm{T}}\Theta_1\,,$$

where the choice of representative in the equivalence class is immaterial as we consider only $d\Lambda$ in the definition. We may compute Θ_1 and $\mathcal{E}_0$ in coordinates; they are

$$\begin{aligned} \Theta_1 &= S^j\left(\frac{\partial L}{\partial u^a} du^a + \frac{\partial L}{\partial u^a_i} du^a_i\right) \otimes \left(\frac{\partial}{\partial t^j} \lrcorner\, (dt^1 \wedge \cdots \wedge dt^m)\right) \\ &= \left(\frac{\partial L}{\partial u^a_j} du^a\right) \otimes \left(\frac{\partial}{\partial t^j} \lrcorner\, (dt^1 \wedge \cdots \wedge dt^m)\right) \end{aligned}$$

and

$$\begin{aligned} \mathcal{E}_0 = &\left(\frac{\partial L}{\partial u^a} du^a + \frac{\partial L}{\partial u^a_i} du^a_i\right) \otimes (dt^1 \wedge \cdots \wedge dt^m) \\ &- d_k\left(\frac{\partial L}{\partial u^a_j} du^a\right) \otimes dt^k \wedge \left(\frac{\partial}{\partial t^j} \lrcorner\, (dt^1 \wedge \cdots \wedge dt^m)\right) \\ = &\left(\frac{\partial L}{\partial u^a} - d_k\left(\frac{\partial L}{\partial u^a_k}\right)\right) du^a \otimes (dt^1 \wedge \cdots \wedge dt^m)\,. \end{aligned}$$

5 Variational problems

Our main application of the theory of vector forms, and their associated cohomology, will be to problems in the calculus of variations. These will be *parametric problems*: that is, problems where the solutions are submanifolds without a given parametrization (although with a particular orientation). In the one-dimensional case, as exemplified by Finsler geometry, all the vector forms are essentially scalar forms, and so this theory only provides further insight in the case where the submanifolds have dimension two or more.

5.1 Homogeneous variational problems

We now study m-dimensional variational problems on E, with fixed boundary conditions. As before, a vector function $\Lambda = L\, d^m t \in \Omega_1^{0,m}$ will be called a *Lagrangian* for a variational problem. It will be called *homogeneous* if it is equivariant with respect to the action of the oriented jet group L_m^{1+}. Thus Λ is homogeneous when the scalar function L satisfies the infinitesimal condition

$$d^i_j L = \delta^i_j L$$

or, equivalently, the finite condition

$$L \circ \alpha_{j_0^1\phi} = (\det D_j \phi^i(0))L$$

for every every $j_0^1\phi \in L_m^{1+}$.

We now consider submanifolds of E of the form $\gamma(C)$ where $\gamma : \mathbb{R}^m \to E$ is an immersion and $C \subset \mathbb{R}^m$ is a connected compact m-dimensional submanifold with boundary ∂C. The *fixed-boundary variational problem defined by* Λ is the search for extremal submanifolds $\gamma(C) \subset E$ satisfying

$$\int_C ((\bar{\jmath}^1\gamma)^* \mathcal{L}_{X_m^1} L) d^m t = 0$$

for every variation field X on E satisfying $X|_{\gamma(\partial C)} = 0$.

Theorem 2. *If Λ is homogeneous and $\gamma(C)$ is an extremal submanifold then $\gamma \circ \phi$ is also an extremal submanifold, for any orientation-preserving reparametrization ϕ whose image contains C.*

Proof. We shall show that if Λ is homogeneous then, for any immersion γ,

$$\int_{\phi^{-1}(C)} \big(L \circ \bar{\jmath}^1(\gamma \circ \phi)\big) d^m t = \int_C \big(L \circ \bar{\jmath}^1\gamma\big) d^m t$$

so that the integral itself is invariant under reparametrization; hence extremals will be invariant under reparametrization. As

$$\begin{aligned} \int_{\phi^{-1}(C)} \big(L \circ \bar{\jmath}^1(\gamma \circ \phi)\big) d^m t &= \int_C (\phi^{-1})^* \big(\big(L \circ \bar{\jmath}^1(\gamma \circ \phi)\big) d^m t\big) \\ &= \int_C \big(L \circ \bar{\jmath}^1(\gamma \circ \phi) \circ \phi^{-1}\big) \, (\phi^{-1})^* d^m t \,, \end{aligned}$$

it will be sufficient to show that

$$\big(L \circ \bar{\jmath}^1(\gamma \circ \phi) \circ \phi^{-1}\big) \, (\phi^{-1})^* d^m t = \big(L \circ \bar{\jmath}^1\gamma\big) d^m t \,.$$

Now for any $s \in \mathbb{R}^m$

$$d^m t\big|_s = (\mathcal{J}\phi \circ \phi^{-1})(s) \; (\phi^{-1})^* d^m t\big|_s \;,$$

and so it will be sufficient to show that, for each s,

$$\big(L \circ \bar{\jmath}^1(\gamma \circ \phi) \circ \phi^{-1}\big)(s) = \big(L \circ \bar{\jmath}^1\gamma\big)(s)(\mathcal{J}\phi \circ \phi^{-1})(s) \,.$$

Note that we do not require the diffeomorphism ϕ to satisfy the condition $\phi(0) = 0$.

To see how this can be obtained from the homogeneity condition, write the latter as

$$L \circ \alpha_{j_0^1\varphi} = (\mathcal{J}\varphi)(0)L$$

where φ is a diffeomorphism which *does* satisfy $\varphi(0) = 0$; then, for any immersion $\gamma : \mathbb{R}^m \to E$,

$$\begin{aligned}(\mathcal{J}\varphi)(0)L\big(j_0^1(\gamma \circ \mathrm{T}_s)\big) &= L\big(\alpha_{j_0^1\varphi}\big(j_0^1(\gamma \circ \mathrm{T}_s)\big)\big) \\ &= L\big(j_0^1(\gamma \circ \mathrm{T}_s \circ \varphi)\big)\,.\end{aligned}$$

Now put $\varphi = \mathrm{T}_{-s} \circ \phi \circ \mathrm{T}_{\phi^{-1}(s)}$, and note that $\varphi(0) = 0$; also

$$(\gamma \circ \mathrm{T}_s) \circ \varphi = \gamma \circ \phi \circ \mathrm{T}_{\phi^{-1}(s)}$$

and

$$(\mathcal{J}\varphi)(0) = (\mathcal{J}\phi)(\phi^{-1}(s))\,,$$

so that

$$(\mathcal{J}\phi)(\phi^{-1}(s))L\big(j_0^1(\gamma \circ \mathrm{T}_s)\big) = L\big(j_0^1(\gamma \circ \phi \circ \mathrm{T}_{\phi^{-1}(s)})\big)$$

and hence

$$(\mathcal{J}\phi)(\phi^{-1}(s))L\big(\bar{j}^1\gamma(s)\big) = L\big(\bar{j}^1(\gamma \circ \phi) \circ \phi^{-1}(s)\big)\,. \qquad \square$$

5.2 Equivalents of Lagrangians

Let $\Lambda \in \Omega_1^{0,m}$ be a homogeneous Lagrangian. Any scalar m-form $\Theta_m \in \Omega_1^{m,0}$ which is horizontal over E will be called an *integral equivalent* of Λ if

$$\Lambda = \left(\frac{(-1)^{m(m-1)/2}}{m!}\right) i_{\mathrm{T}}^m \Theta_m\,;$$

any vector r-form $\Theta_r \in \Omega_1^{r,m-r}$ which is horizontal over E will be called an *intermediate equivalent* if

$$\Lambda = \frac{(-1)^{r(r-1)/2}(m-r)!}{m!}\, i_{\mathrm{T}}^r \Theta_r \qquad 0 \le r \le m-1\,.$$

Lemma 14. *If Θ_{r+1} is an equivalent of Λ then*

$$\Theta_r = \frac{(-1)^r}{m-r}\, i_{\mathrm{T}} \Theta_{r+1}$$

is also an equivalent.

Proof. If Θ_{r+1} is an equivalent of Λ then by definition

$$\Lambda = \frac{(-1)^{r(r+1)/2}(m-r-1)!}{m!}\, i_{\mathrm{T}}^{r+1} \Theta_{r+1}\,,$$

so that

$$\frac{(-1)^{r(r-1)/2}(m-r)!}{m!}\, i_{\mathrm{T}}^r \Theta_r = \frac{(-1)^{r(r-1)/2}(m-r)!}{m!}\, i_{\mathrm{T}}^r \left(\frac{(-1)^r}{m-r}\, i_{\mathrm{T}} \Theta_{r+1}\right) = \Lambda\,. \quad \square$$

In the case $r = m$ we use the term 'integral equivalent' for the following reason.

Lemma 15. *If γ is an m-curve in E then $(\bar{\jmath}^1\gamma)^*\Lambda = (\bar{\jmath}^1\gamma)^*\Theta_m$, so that*

$$\int_C (\bar{\jmath}^1\gamma)^*\Lambda = \int_C (\bar{\jmath}^1\gamma)^*\Theta_m \,.$$

It follows that $\Lambda = \Theta_0$ and Θ_m have the same extremals.

Proof. Suppose $\Theta \in \Omega^{r,m-r}$ may be written in coordinates in the particular form

$$\Theta = \Theta_{a_1\cdots a_m} u^{a_{r+1}}_{k_{r+1}} \cdots u^{a_m}_{k^m} du^{a_1} \wedge \cdots \wedge du^{a_r} \otimes dt^{k_{r+1}} \wedge \cdots \wedge dt^{k_m}$$

where the functions $\Theta_{a_1\cdots a_m}$ are skew-symmetric in their indices; then

$$\begin{aligned} i_{\mathrm{T}}\Theta &= \Theta_{a_1\cdots a_m} u^{a_{r+1}}_{k_{r+1}} \cdots u^{a_m}_{k^m} \left(u^b_{k_r} \frac{\partial}{\partial u^b} \lrcorner\, du^{a_1} \wedge \cdots \wedge du^{a_r} \right) \otimes \\ &\qquad \otimes dt^{k_r} \wedge dt^{k_{r+1}} \wedge \cdots \wedge dt^{k_m} \\ &= \sum_{p=1}^{r} (-1)^{p-1} \Theta_{a_1\cdots a_m} u^{a_{r+1}}_{k_{r+1}} \cdots u^{a_m}_{k^m} \left(u^{a_p}_{k_r} du^{a_1} \wedge \cdots \widehat{du^{a_p}} \cdots \wedge du^{a_r} \right) \otimes \\ &\qquad \otimes dt^{k_r} \wedge dt^{k_{r+1}} \wedge \cdots \wedge dt^{k_m} \\ &= r(-1)^{r-1} \Theta_{a_1\cdots a_m} u^{a_r}_{k_r} u^{a_{r+1}}_{k_{r+1}} \cdots u^{a_m}_{k^m} du^{a_1} \wedge \cdots \wedge du^{a_{r-1}} \otimes \\ &\qquad \otimes dt^{k_r} \wedge dt^{k_{r+1}} \wedge \cdots \wedge dt^{k_m} \,. \end{aligned}$$

Thus if $\Theta \in \Omega^{m,0}$ we see that

$$\begin{aligned} i^m_{\mathrm{T}}\Theta &= m!(-1)^{m(m-1)/2} \Theta_{a_1\cdots a_m} u^{a_1}_{k_1} \cdots u^{a_m}_{k^m} dt^{k_1} \wedge \cdots \wedge dt^{k_m} \\ &= m!(-1)^{m(m-1)/2} \Theta_{a_1\cdots a_m} \det\bigl(u^{a_i}_{k_j}\bigr) dt^1 \wedge \cdots \wedge dt^m \end{aligned}$$

so that

$$(\bar{\jmath}^1\gamma)^* i^m_{\mathrm{T}}\Theta = m!(-1)^{m(m-1)/2} \bigl(\Theta_{a_1\cdots a_m} \circ \bar{\jmath}^1\gamma\bigr) \det\left(\frac{\partial \gamma^{a_i}}{\partial t^{k_j}}\right) dt^1 \wedge \cdots \wedge dt^m \,.$$

On the other hand,

$$\begin{aligned} (\bar{\jmath}^1\gamma)^*\Theta &= \bigl(\Theta_{a_1\cdots a_m} \circ \bar{\jmath}^1\gamma\bigr)(\bar{\jmath}^1\gamma)^*\bigl(du^{a_1} \wedge \cdots \wedge du^{a_m}\bigr) \\ &= \bigl(\Theta_{a_1\cdots a_m} \circ \bar{\jmath}^1\gamma\bigr) \det\left(\frac{\partial \gamma^{a_i}}{\partial t^{k_j}}\right) dt^1 \wedge \cdots \wedge dt^m \,. \end{aligned}$$

□

5.3 Euler forms

Let Θ_m be an integral equivalent of Λ. Define the scalar $(m+1)$-form $\mathcal{E}_m \in \Omega^{m+1,0}_1$ by

$$\mathcal{E}_m = d\Theta_m$$

and the vector forms $\mathcal{E}_r \in \Omega^{r+1,m-r}_2$ by

$$\mathcal{E}_r = \tau^{2,1\,*}_{mE} d\Theta_r - (-1)^r d_{\mathrm{T}}\Theta_{r+1} \qquad 0 \le r \le m-1 \,.$$

The forms $\mathcal{E}_r$ are called the *Euler forms* of Θ_m.

Lemma 16. *The Euler forms satisfy the recurrence relation*

$$\mathcal{E}_r = \frac{(-1)^{r+1}}{m-r}\, i_{\mathrm{T}}\mathcal{E}_{r+1} \qquad 0 \le r \le m-1\,;$$

consequently if $d\Theta_m = \mathcal{E}_m = 0$ *then* $\mathcal{E} = 0$.

Proof. This follows from the definition and Lemma 14. We have, omitting the pull-back maps,

$$\begin{aligned} i_{\mathrm{T}}\mathcal{E}_{r+1} &= i_{\mathrm{T}} d\Theta_{r+1} (-1)^{r+1} i_{\mathrm{T}} d_{\mathrm{T}}\Theta_{r+2} \\ &= d_{\mathrm{T}}\Theta_{r+1} - d i_{\mathrm{T}}\Theta_{r+1} - (-1)^r d_{\mathrm{T}} i_{\mathrm{T}}\Theta_{r+2} \\ &= d_{\mathrm{T}}\Theta_{r+1} - (-1)^r d\Theta_r + (m-r-1) d_{\mathrm{T}}\Theta_{r+1} \\ &= (m-r)\big(d_{\mathrm{T}}\Theta_{r+1} - (-1)^r d\Theta_r\big) \end{aligned}$$

when $r+1 < m$, so that

$$\frac{(-1)^{r+1}}{m-r} i_{\mathrm{T}}\mathcal{E}_{r+1} = (-1)^{r+1} d_{\mathrm{T}}\Theta_{r+1} + d\Theta^r = \mathcal{E}_r\,.$$

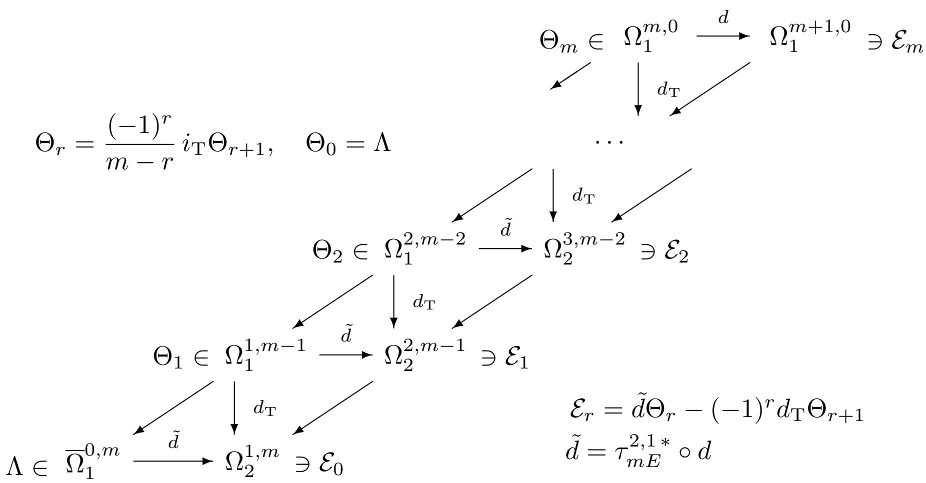
Similarly,

$$\begin{aligned} i_{\mathrm{T}}\mathcal{E}_m &= i_{\mathrm{T}} d\Theta_m \\ &= d_{\mathrm{T}}\Theta_m - d i_{\mathrm{T}}\Theta_m \\ &= d_{\mathrm{T}}\Theta_m - (-1)^{m-1} d\Theta_{m-1} \end{aligned}$$

so that

$$(-1)^m i_{\mathrm{T}}\mathcal{E}_m = (-1)^m d_{\mathrm{T}}\Theta_m + d\Theta_{m-1} = \mathcal{E}_{m-1}\,. \qquad \square$$

The different spaces containing the various equivalents and Euler forms may be seen in this diagonal part of the bicomplex.



5.4 Lepagian forms

Let Λ be a homogeneous Lagrangian, and let Θ_r be an equivalent of Λ $(1 \leq r \leq m)$. We shall say that Θ_r is *Lepagian* if the corresponding Euler form $\mathcal{E}_0 \in \Omega_2^{1,m}$ satisfies

$$S\mathcal{E}_0 = 0\,,$$

so that $\mathcal{E}_0$ is horizontal over E.

Theorem 3. *The vector 1-form*

$$\Theta_1 = Sd\Lambda$$

is an integral equivalent of Λ $(m = 1)$ or an intermediate equivalent $(m \geq 2)$, and is Lepagian. It is called the Hilbert equivalent *of $\Lambda = L\,d^m t$.*

Proof. From the definition of S,

$$S\Xi = S^j \chi \otimes d^{m-1}t_j\,,$$

so that

$$\begin{aligned} i_{\mathrm{T}} Sd\Lambda &= i_{\mathrm{T}} S(dL \otimes d^m t) \\ &= i_{\mathrm{T}}(S^j dL \otimes d^{m-1}t_j) \\ &= i_k S^j dL \otimes dt^k \wedge d^{m-1}t_j \\ &= i_j S^j dL \otimes d^m t\,. \end{aligned}$$

But for any 1-form θ on $\mathring{T}_m E$, if in coordinates $\theta = \theta_a du^a + \theta_a^i du_i^a$ then

$$i_j S^j \theta = i_j(\theta_a^j du^a) = u_j^a \theta_a^j\,,$$

so that

$$i_j S^j dL = u_j^a \frac{\partial L}{\partial u_j^a} = d_j^j L = mL$$

using the homogeneity of the Lagrangian.

To show that Θ_1 is Lepagian, note that

$$\begin{aligned} Sd_{\mathrm{T}}\Theta_1 &= Sd_{\mathrm{T}} Sd\Lambda \\ &= Sd_{\mathrm{T}}(S^j dL \otimes d^{m-1}t_j) \\ &= S\big(d_i S^j dL \otimes (dt^i \wedge d^{m-1}t_j)\big) \\ &= S\big(d_j S^j dL \otimes d^m t\big) \\ &= S^i d_j S^j dL \otimes d^{m-1}t_i \\ &= (d_j S^i + \delta_j^i) S^j dL \otimes d^{m-1}t_i \\ &= S^i dL \otimes d^{m-1}t_i \\ &= S(dL \otimes d^m t) = Sd\Lambda \end{aligned}$$

using Lemma 13 and the fact that L is defined on $\mathring{T}_m E$ so that $S^i S^j dL = 0$; thus $S\mathcal{E}_0 = 0$, as required. □

Theorem 4. *If $\widetilde{\Theta}_1$ is another Lepagian vector 1-form equivalent to Λ, with corresponding Euler form $\widetilde{\mathcal{E}}_0$, then*

$$\widetilde{\mathcal{E}}_0 = \mathcal{E}_0 \,, \qquad \widetilde{\Theta}_1 - \Theta_1 = d_{\mathrm{T}}\Phi \qquad (\Phi \in \Omega_0^{1,m-2})\,.$$

Proof. It follows straightforwardly from the Lepagian condition $S\widetilde{\mathcal{E}}_0 = 0$ that $P_2\widetilde{\mathcal{E}}_0 = 0$, so that we may use the homotopy condition of Theorem 1 to see that

$$0 = P_2\widetilde{\mathcal{E}}_0 = P_2(d\Lambda - d_{\mathrm{T}}\widetilde{\Theta}_1) = \Theta_1 - P_2 d_{\mathrm{T}}\widetilde{\Theta}_1 = \Theta_1 - (1 - d_{\mathrm{T}}P_1)\widetilde{\Theta}_1 \,,$$

giving $\widetilde{\Theta}_1 - \Theta_1 = d_{\mathrm{T}}P_1\widetilde{\Theta}_1$ (or $\widetilde{\Theta}_1 = \Theta_1$ if $m = 1$). Thus

$$\widetilde{\mathcal{E}}_0 - \mathcal{E}_0 = (d\Lambda - d_{\mathrm{T}}\widetilde{\Theta}_1) - (d\Lambda - d_{\mathrm{T}}\Theta_1) = -d_{\mathrm{T}}^2 P_1\widetilde{\Theta}_1 = 0\,.$$

(Note that, as $d\Lambda$ is a first-order vector 1-form, $P_2 d\Lambda = Sd\Lambda = \Theta_1$.) □

5.5 The First Variation Formula

Theorem 5. *Let C be a compact connected m-dimensional submanifold of $\mathbb{R}^m$ with boundary ∂C, let γ be an m-curve in E whose domain contains C, and let X be a variation field on E vanishing on $\gamma(\partial C)$ with prolongation X_m^1 on $\mathring{T}_m E$. Then*

$$\int_C (\bar{\jmath}^1\gamma)^* \mathcal{L}_{X_m^1}\Lambda = \int_C (\bar{\jmath}^2\gamma)^* i_X \mathcal{E}_0\,;$$

consequently γ is an extremal of Λ precisely when $\mathcal{E}_0$ vanishes along the image of $\bar{\jmath}^2\gamma$.

Proof. We note first that

$$\begin{aligned}
\int_C (\bar{\jmath}^1\gamma)^* \mathcal{L}_{X_m^1}\Lambda &= \int_C (\bar{\jmath}^1\gamma)^* i_{X_m^1} d\Lambda \\
&= \int_C (\bar{\jmath}^2\gamma)^* i_{X_m^2} \tau_{mE}^{2,1\,*} d\Lambda \\
&= \int_C (\bar{\jmath}^2\gamma)^* i_{X_m^2} \mathcal{E}_0 + \int_C (\bar{\jmath}^2\gamma)^* i_{X_m^2} d_{\mathrm{T}}\Theta_1 \,,
\end{aligned}$$

using the definition of the Euler form $\mathcal{E}_0$. But prolongations commute with basis total derivatives and Θ_1 is horizontal over E, so that

$$\int_C (\bar{\jmath}^2\gamma)^* i_{X_m^2} d_{\mathrm{T}}\Theta_1 = \int_C (\bar{\jmath}^2\gamma)^* d_{\mathrm{T}} i_{X_m^1}\Theta_1 = \int_C d(\bar{\jmath}^1\gamma)^* i_X\Theta_1 = 0$$

and we see that the second integral vanishes; thus

$$\int_C (\bar{\jmath}^1\gamma)^* \mathcal{L}_{X_m^1}\Lambda = \int_C (\bar{\jmath}^2\gamma)^* i_{X_m^2}\mathcal{E}_0 = \int_C (\bar{\jmath}^2\gamma)^* i_X\mathcal{E}_0$$

because $\mathcal{E}_0$ is horizontal over E.

Now let γ be an immersion. If $\mathcal{E}_0 = 0$ at every point in the image of $\bar{\jmath}^2\gamma$, then for any vector field X on E and any $t \in C$ we will have $(\bar{\jmath}^2\gamma)^* i_X\mathcal{E}_0\big|_t = 0$, so that the integral over C will vanish and γ will be an extremal.

If, instead, $q = j_0^2(\gamma \circ \mathrm{T}_t)$ is some point in the image of $\bar{\jmath}^2\gamma$ where $\mathcal{E}_0|_q$ is non-zero, then there must be a vector field X on E such that the vector-valued function $i_X\mathcal{E}_0$ gives a strictly positive multiple of $d^m t$ when evaluated at q, and hence when evaluated in some neighbourhood U of q. Let b be a positive bump function on E whose support lies in the interior of U and which satisfies $b(q) = 1$. Then

$$\int_C (\bar{\jmath}^1\gamma)^* \mathcal{L}_{(bX)^1_m} \Lambda = \int_C (\bar{\jmath}^2\gamma)^* i_{bX}\mathcal{E}_0 > 0\,,$$

so that γ cannot be an extremal. □

5.6 Integral equivalents for $m \geq 2$

Let $\Lambda = L\, d^m t$ be a homogeneous Lagrangian with $m \geq 2$, and write its Hilbert equivalent Θ_1 as

$$\Theta_1 = \vartheta^i \otimes d^{m-1}t_i\,;$$

the scalar 1-forms ϑ_i are called the *Hilbert forms* of Λ. If Λ never vanishes, define the *Carathéodory equivalent* $\widetilde{\Theta}_m \in \Omega_1^{m,0}$ by

$$\widetilde{\Theta}_m = \frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \vartheta^i\,.$$

Theorem 6. *The Carathéodory equivalent $\widetilde{\Theta}_m$ is an integral equivalent of Λ.*

Proof. We must show that $i_{\mathrm{T}}^m \Theta_m = (-1)^{m(m-1)/2}\, m!\Lambda$, so rewrite Θ_m as

$$\Theta_m = \frac{1}{m! L^{m-1}} \sum_{\sigma\in\mathfrak{S}_m} (-1)^\sigma \vartheta^{\sigma(1)} \wedge \cdots \wedge \vartheta^{\sigma(m)}\,,$$

where $\mathfrak{S}_m$ is the permutation group, and use induction. The calculation uses $d_j \lrcorner\, \vartheta^i = \delta^i_j L$, the proof of which is similar to that used to show that $i_{\mathrm{T}}\Theta_1 = m\Lambda$; we also define $\tau_{r,s} \in \mathfrak{S}_m$ by

$$\tau_{r,s}(i) = \begin{cases} m-s & (i = r) \\ i-1 & (r+1 \leq i \leq m-s) \\ i & \text{otherwise}\,. \end{cases}$$

Now

$$
\begin{aligned}
& i_{\mathrm{T}}\Big(\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\otimes dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\Big)\\
&= \sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma d_j \lrcorner \big(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\big)\otimes dt^j\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\\
&= \sum_{r=1}^{m-s}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma(-1)^{r-1}\big(\vartheta^{\sigma(1)}\wedge\cdots\wedge(d_j\lrcorner\vartheta^{\sigma(r)})\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\big)\otimes\\
&\qquad\qquad \otimes dt^j\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\\
&= L\sum_{r=1}^{m-s}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma(-1)^{r-1}\big(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(r-1)}\wedge\vartheta^{\sigma(r+1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\big)\otimes\\
&\qquad\qquad \otimes dt^{\sigma(r)}\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\\
&= L\sum_{r=1}^{m-s}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma(-1)^{r-1}(-1)^{m-r-s}\Big\{\\
&\quad \big(\vartheta^{\sigma\tau_{r,s}(1)}\wedge\cdots\wedge\vartheta^{\sigma\tau_{r,s}(r-1)}\wedge\vartheta^{\sigma\tau_{r,s}(r+1)}\wedge\cdots\wedge\vartheta^{\sigma\tau_{r,s}(m-s)}\big)\otimes\\
&\qquad\qquad \otimes dt^{\sigma\tau_{r,s}(r)}\wedge dt^{\sigma\tau_{r,s}(m-s+1)}\wedge\cdots\wedge dt^{\sigma\tau_{r,s}(m)}\Big\}\\
&= (-1)^{m-s-1}L\sum_{r=1}^{m-s}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\big(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s-1)}\big)\otimes\\
&\qquad\qquad \otimes dt^{\sigma(m-s)}\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\\
&= (-1)^{m-s-1}(m-s)L\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\big(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s-1)}\big)\otimes\\
&\qquad\qquad \otimes dt^{\sigma(m-s)}\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)},
\end{aligned}
$$

so if

$$
\begin{aligned}
i_{\mathrm{T}}^s\Theta_m = \frac{(-1)^{s(2m-s-1)/2}}{(m-s)!L^{m-s-1}}\Big\{&\\
&\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\otimes dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\Big\}
\end{aligned}
$$

then

$$
\begin{aligned}
i_{\mathrm{T}}^{s+1}\Theta_m = \frac{(-1)^{s(2m-s-1)/2}}{(m-s)!L^{m-s-1}}\Big\{&\\
&(-1)^{m-s-1}(m-s)L\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\big(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s-1)}\big)\otimes\\
&\qquad \otimes dt^{\sigma(m-s)}\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\Big\}
\end{aligned}
$$

$$= \frac{(-1)^{(s+1)(2m-s-2)/2}}{(m-s-1)!L^{m-s-2}} \sum_{\sigma\in\mathfrak{S}_m} (-1)^\sigma \left(\vartheta^{\sigma(1)} \wedge \cdots \wedge \vartheta^{\sigma(m-s-1)}\right) \otimes$$
$$\otimes\, dt^{\sigma(m-s)} \wedge dt^{\sigma(m-s+1)} \wedge \cdots \wedge dt^{\sigma(m)}$$

as required. Hence

$$\begin{aligned} i_{\mathrm{T}}^m \Theta_m &= \frac{(-1)^{m(m-1)/2}}{L^{-1}} \sum_{\sigma\in\mathfrak{S}_m} (-1)^\sigma dt^{\sigma(1)} \wedge \cdots \wedge dt^{\sigma(m)} \\ &= (-1)^{m(m-1)/2}\, m!\, L\, dt^1 \wedge \cdots \wedge dt^m \\ &= (-1)^{m(m-1)/2}\, m!\, \Lambda\,. \end{aligned}$$ □

We see also from the induction formula that

$$i_{\mathrm{T}}^{m-1}\Theta_m = (-1)^{m(m-1)/2}\,(m-1)!\Theta_1$$

where Θ_1 is the Hilbert equivalent; consequently Θ_m is Lepagian. Then, as $d\Theta_m = \mathcal{E}_m$,

$$\begin{aligned} \int_C (j^1\gamma)^* \mathcal{L}_{X^1_m} \Theta_m &= \int_C (j^1\gamma)^* i_{X^1_m} \mathcal{E}_m \\ &= (-1)^{m(m-1)/2} m! \int_C (j^2\gamma)^* i_X \mathcal{E}_0 \end{aligned}$$

for any vector field X on E vanishing on $\gamma(\partial C)$, because contractions by vector fields anticommute, so that $i_{\mathrm{T}}^m i_{X^1_m} \mathcal{E}_m = (-1)^m i_{X^1_m} i_{\mathrm{T}}^m \mathcal{E}_m$.

5.7 Another integral equivalent

When $m = 1$ then the only Lepagian integral equivalent of a Lagrangian is the Hilbert equivalent. But when $m > 1$ there may be other integral equivalents. Put

$$\Theta_{r+1} = \frac{(-1)^r}{(r+1)^2} S d\Theta_r \qquad (1 \le r < m)$$

where, as usual, $\Theta_0 = \Lambda$.

Lemma 17. *Each Θ_r is a first-order vector form, an element of $\Omega_1^{r,m-r}$, horizontal over E.*

Proof. Each Θ_r is first-order because neither S nor d increases the order of a vector form. By definition Θ_0 is horizontal over E, and if Θ_r is horizontal over E then the contraction of $d\Theta_r$ with any vector field on $\mathring{T}_m E$ vertical over E will again be horizontal over E; thus Θ_{r+1} will also be horizontal over E. □

Theorem 7. *The scalar m-form Θ_m is a Lepagian integral equivalent of Λ called the* fundamental equivalent *of Λ.*

Proof. We first show that, in coordinates,

$$\Theta_r = \frac{1}{(r!)^2} \frac{\partial^r L}{\partial u^{a_1}_{i_1} \cdots \partial u^{a_r}_{i_r}} du^{a_1} \wedge \cdots \wedge du^{a_r} \otimes \left(\frac{\partial}{\partial t^{i_r}} \lrcorner \cdots \lrcorner \frac{\partial}{\partial t^1} \lrcorner\, d^m t \right).$$

This formula clearly holds for $r = 1$ (and, indeed, for $r = 0$); so suppose that it holds for a given value of r. Then

$$\begin{aligned}
\Theta_{r+1} &= \frac{(-1)^r}{(r+1)^2} S d\Theta_r \\
&= \frac{(-1)^r}{(r+1)^2} \frac{1}{(r!)^2} S^j \left(\frac{\partial^{r+1} L}{\partial u^{a_1}_{i_1} \cdots \partial u^{a_r}_{i_r} \partial u^{a_{r+1}}_{i_{r+1}}} du^{a_{r+1}}_{i_{r+1}} + \cdots \right) \wedge \\
&\qquad \wedge du^{a_1} \wedge \cdots \wedge du^{a_r} \otimes \left(\frac{\partial}{\partial t^j} \lrcorner \frac{\partial}{\partial t^{i_r}} \lrcorner \cdots \lrcorner \frac{\partial}{\partial t^1} \lrcorner\, d^m t \right) \\
&= \frac{1}{((r+1)!)^2} \frac{\partial^{r+1} L}{\partial u^{a_1}_{i_1} \cdots \partial u^{a_{r+1}}_{i_{r+1}}} du^{a_1} \wedge \cdots \wedge du^{a_{r+1}} \otimes \\
&\qquad \otimes \left(\frac{\partial}{\partial t^{i_{r+1}}} \lrcorner \cdots \lrcorner \frac{\partial}{\partial t^1} \lrcorner\, d^m t \right)
\end{aligned}$$

so that the formula also holds for the case $r + 1$. In particular, therefore, we have

$$\begin{aligned}
\Theta_m &= \frac{1}{(m!)^2} \frac{\partial^m L}{\partial u^{a_1}_{i_1} \cdots \partial u^{a_m}_{i_m}} du^{a_1} \wedge \cdots \wedge du^{a_m} \times \left(\frac{\partial}{\partial t^{i_m}} \lrcorner \cdots \lrcorner \frac{\partial}{\partial t^1} \lrcorner\, d^m t \right) \\
&= \frac{1}{(m!)^2} \frac{\partial^m L}{\partial u^{a_1}_{i_1} \cdots \partial u^{a_m}_{i_m}} du^{a_1} \wedge \cdots \wedge du^{a_m} \times \begin{vmatrix} \delta^1_{i_1} & \cdots & \delta^1_{i_m} \\ \vdots & & \vdots \\ \delta^m_{i_1} & \cdots & \delta^m_{i_m} \end{vmatrix} \\
&= \frac{1}{m!} \frac{\partial^m L}{\partial u^{a_1}_1 \cdots \partial u^{a_m}_m} du^{a_1} \wedge \cdots \wedge du^{a_m} .
\end{aligned}$$

Thus, using the calculation in the proof of Lemma 15,

$$\begin{aligned}
i^m_{\mathrm{T}} \Theta_m &= m!(-1)^{m(m-1)/2} \left(\frac{1}{m!} \frac{\partial^m L}{\partial u^{a_1}_1 \cdots \partial u^{a_m}_m} \det\bigl(u^{a_i}_{k_j}\bigr) \right) dt^1 \wedge \cdots \wedge dt^m \\
&= (-1)^{m(m-1)/2} \frac{\partial^m L}{\partial u^{a_1}_1 \cdots \partial u^{a_m}_m} \det\bigl(u^{a_i}_{k_j}\bigr) dt^1 \wedge \cdots \wedge dt^m \\
&= (-1)^{m(m-1)/2} m!\, L\, dt^1 \wedge \cdots \wedge dt^m \\
&= (-1)^{m(m-1)/2} m!\, \Lambda .
\end{aligned}$$

□

Theorem 8. *The fundamental equivalent Θ_m of a homogeneous Lagrangian Λ has the property that $d\Theta_m = \mathcal{E}_m = 0$ if, and only if, $\mathcal{E}_0 = 0$.*

Proof. If $\mathcal{E}_m = 0$ then $\mathcal{E}_0 = 0$ by the recurrence relation of Lemma 16. So show the converse, we use the definition

$$\Theta_{r+1} = \frac{(-1)^r}{(r+1)^2} S d\Theta_r$$

and the fact that $d\Theta_r \in \Omega_1^{r+1,m-r}$ to see that the homotopy operator P_1 from Theorem 1 takes the form

$$P_1\big(\chi_{i_1\cdots i_{m-r}}\otimes dt^{i_1}\wedge\cdots\wedge dt^{i_{m-r}}\big) = \frac{1}{(r+1)^2}S^j\chi_{i_1\cdots i_{m-r}}\otimes\left(\frac{\partial}{\partial t^j}\lrcorner\, dt^{i_1}\wedge\cdots\wedge dt^{i_{m-r}}\right)$$

(the formula in the proof of Theorem 1 was for an element of $\Omega_1^{r,s+1}$); thus we may rewrite the definition of Θ_{r+1} as

$$\Theta_{r+1} = (-1)^r P d\Theta_r\,.$$

Now from

$$\mathcal{E}_r = d\Theta_r - (-1)^r d_{\mathrm{T}}\Theta_{r+1}$$

we obtain

$$P_2 d\mathcal{E}_r = -(-1)^r P_2 d_{\mathrm{T}} d\Theta_{r+1} = (-1)^r\big(d_{\mathrm{T}}P_1 d\Theta_{r+1} - d\Theta_{r+1}\big)$$

so that

$$(-1)^{r+1}P_2 d\mathcal{E}_r = d\Theta_{r+1} - d_{\mathrm{T}}P_1 d\Theta_{r+1}$$

using the homotopy formula of Theorem 1; but

$$\mathcal{E}_{r+1} = d\Theta_{r+1} - (-1)^{r+1}d_{\mathrm{T}}\Theta_{r+2} = d\Theta_{r+1} - d_{\mathrm{T}}P_1 d\Theta_{r+1}$$

so that

$$\mathcal{E}_{r+1} = (-1)^{r+1}P_2 d\mathcal{E}_r\,.$$

Similarly,

$$P_2 d\mathcal{E}_{m-1} = -(-1)^{m-1}P_2 d_{\mathrm{T}} d\Theta_m = (-1)^m d\Theta_m = (-1)^m\mathcal{E}_m\,.$$

It follows that if $\mathcal{E}_0 = 0$ then $\mathcal{E}_m = 0$. □

References

[1] M. Crampin, D.J. Saunders: Some concepts of regularity for parametric multiple-integral problems in the calculus of variations. *Czech Math. J.* 59 (3) (2009) 741–758.

[2] M. Giaquinta, S. Hildebrandt: *Calculus of Variations II*. Springer (1996).

[3] I. Kolář, P.W. Michor, J. Slovák: *Natural Operations in Differential Geometry*. Springer (1993).

[4] H. Rund: *The Hamilton-Jacobi Equation in the Calculus of Variations*. Krieger (1973).

[5] D.J. Saunders: Homogeneous variational complexes and bicomplexes. *J. Geom. Phys.* 59 (2009) 727–739.

[6] D.J. Saunders: Some geometric aspects of the calculus of variations in several independent variables. *Comm. Math.* 18 (1) (2010) 3–19.

Author's address:
Department of Mathematics, Faculty of Science, The University of Ostrava, 30. dubna 22, 701 03 Ostrava, Czech Republic

E-mail: david@symplectic.demon.co.uk

Received: 18 August, 2011
Accepted for publication: 28 August, 2011

Communications in Mathematics 19 (2011) 129–136

About Boundary Terms in Higher Order Theories

Lorenzo Fatibene, Mauro Francaviglia, Silvio Mercadante

Abstract. It is shown that when in a higher order variational principle one fixes fields at the boundary leaving the field derivatives unconstrained, then the variational principle (in particular the solution space) is not invariant with respect to the addition of boundary terms to the action, as it happens instead when the correct procedure is applied. Examples are considered to show how leaving derivatives of fields unconstrained affects the physical interpretation of the model. This is justified in particular by the need of clarifying the issue for the purpose of applications to relativistic gravitational theories, where a bit of confusion still exists. On the contrary, as it is well known for variational principles of order k, if one fixes variables together with their derivatives (up to order $k-1$) on the boundary then boundary terms leave solution space invariant.

1 Introduction

Recently the interest in higher order Lagrangian theories has been renewed within the framework of covariant field theories in various contexts, aiming to suitably extend standard (Hilbert-Einstein) General Relativity in order to model, at least partially, dark energy/matter effects (see [1], [9] and references quoted therein) via the use of gravitational Lagrangians depending non-linearly on the curvature.

In gravitational literature different attitudes towards boundary conditions in GR and in alternative gravitational theories are presented (see [10] for a detailed review). We shall here stress that mathematical consequences of different attitudes must be considered *before* any physical interpretation is attempted and that of course one is not free to ignore these consequences, that might be (and usually are) rather crucial for a number of physically relevant issues, e.g. the definition of conservation laws and their correct physical interpretation.

From the mathematical viewpoint, any attitude towards boundary conditions should be dictated by Hamilton's least action principle. This principle is a *definition* of the critical sections which have to be understood as physical configurations.

2010 MSC: 35A15, 37K05, 49S99
Key words: higher order field theories, boundary terms

Being it a *definition* one is logically free to choose the formulation which is more suitable to the situation. However, there are physical and mathematical consequences of this choice which must be in any case taken into account. Moreover, it would be appreciated if a general guiding principle would avoid to treat each model on its own on the base of physical considerations which in some cases (e.g. when dealing with exotic physics or non-trivial generalizations of the models already considered) could be unclear.

In particular, we shall hereafter show that if one assumes that only the value of fields must be fixed while (higher order) derivatives are left unconstrained at the boundary, then one cannot keep that pure divergences in the action leave the solution space invariant, as it happens in the standard applications of Calculus of Variations. This is particularly relevant for Gravity, since in the literature (see e.g. [12]) it is often claimed that in standard GR one is free to choose not to fix first derivatives of the metric at the boundary, since the boundary terms of the Hilbert action can be written as a total variation and hence can be compensated in various non-unique ways by adding suitable boundary terms to the action. Even if this is mathematically correct in GR it is in any case rather misleading since such a procedure fails to hold if one considers Lagrangians that are non-degenerate and non-linear in curvature. Accordingly we believe that whenever such a choice is adopted one should clearly state that this is done at the expense of *changing* the space of solutions and affecting conservation laws which is unfortunately physically disturbing; see also [7].

As a motivation for such an uncanonical choice it is often claimed that fixing higher order variations of the fields may affect their physical interpretation so that this standard attitude should not be embraced without considering these effects. This is of course true and we fully agree that detailed discussions on the role that different boundary conditions have in GR is extremely important. However, it is also true the other way around, i.e. when leaving variations of field derivatives free at the boundary one should always be careful about the change of solution space, the interpretation of boundary fluxes as well as the further spurious boundary equations that appear besides the (bulk) field equations.

Of course there are also other issues to be considered when fixing boundary terms of variational principles. For example boundary terms affect also conservation laws and their effect should be considered as further criteria to choose among boundary terms that leave the solution space unchanged; see [4], [8].

Hereafter, we shall present explicit examples in Mechanics and Field Theory. From these examples it is clearly shown that if one artificially wants to describe a system by a higher order Lagrangian adding pure divergences to the Lagrangian itself, then in order to maintain the standard interpretation of the physical system one is forced to fix variations *and their derivatives* at the boundary. The examples will in fact show, *en passant*, how the solution space may drastically change and even reduce to empty if the standard procedures of Calculus of Variations are not used.

2 The Relation between Higher Order Variations and Boundary Terms

Let us consider the following Lagrangian

$$L'(q, \dot{q}, \ddot{q}) = \dot{q}\ddot{q} + \frac{1}{2}\left(\dot{q}^2 - \omega^2 q^2\right) + \omega^2 q\dot{q} \tag{1}$$

which is easily found to be equivalent to the Lagrangian of an harmonic oscillator and to give rise to the same dynamics via Euler-Lagrange equations (of order 2). Varying it we have

$$\begin{aligned}
\delta L' &= \delta\dot{q}\ddot{q} + \dot{q}\delta\ddot{q} + \dot{q}\delta\dot{q} - \omega^2 q\delta q + \omega^2\delta q\dot{q} + \omega^2 q\delta\dot{q} = \\
&= \frac{d}{dt}\left(\delta q\ddot{q}\right) - \delta q\frac{d^3 q}{dt^3} + \frac{d}{dt}\left(\dot{q}\delta\dot{q}\right) - \frac{d}{dt}\left(\ddot{q}\delta q\right) + \frac{d^3 q}{dt^3}\delta q + \\
&\quad + \frac{d}{dt}\left(\dot{q}\delta q\right) - \ddot{q}\delta q - \omega^2 q\delta q + \omega^2\delta q\dot{q} + \frac{d}{dt}\left(\omega^2 q\delta q\right) - \omega^2\dot{q}\delta q = \\
&= \frac{d}{dt}\left(\dot{q}\delta\dot{q} + \left(\dot{q} + \omega^2 q\right)\delta q\right) - \left(\ddot{q} + \omega^2 q\right)\delta q
\end{aligned} \tag{2}$$

If following the standard prescriptions of Calculus of Variations we assume $\delta q = 0$ and $\delta\dot{q} = 0$ on the boundary of an interval $[t_0, t_1]$ then we obtain in fact the equation of motion of the $1d$-harmonic oscillator

$$\ddot{q} + \omega^2 q = 0 \tag{3}$$

This is no mystery since the Lagrangian (1) can be easily recasted as follows

$$L'(q, \dot{q}, \ddot{q}) = \frac{1}{2}\left(\dot{q}^2 - \omega^2 q^2\right) + \frac{d}{dt}\left(\frac{1}{2}\left(\dot{q}^2 + \omega^2 q^2\right)\right)$$

so that it manifestly differs from the harmonic oscillator Lagrangian $L(q, \dot{q}) = \frac{1}{2}\left(\dot{q}^2 - \omega^2 q^2\right)$ by a total time derivative (which is the mechanical equivalent of a pure divergence term in field theory). Hence, in this case, we know that the pure-divergence-term $\frac{d}{dt}\left(\frac{1}{2}\left(\dot{q}^2 + \omega^2 q^2\right)\right)$ in the Lagrangian L' is totally unessential with respect to the equation of motion. Let us stress that in this case the pure divengence term is even zero on-shell because of the conservation of total energy, since the boundary term is nothing but the total derivative of the Hamiltonian.

If one decides instead to fix only $\delta q = 0$ on the boundary, leaving $\delta\dot{q}$ unfixed, then extra boundary field equations are added in order to kill the extra boundary contribution to the action. The equations of motion that follow form (2) in this case are

$$\begin{cases} \ddot{q} + \omega^2 q = 0 \\ \dot{q}_0 = 0 \end{cases}$$

which in fact admit less solutions than Eq. (3). Notice that solutions to this problem are in fact just a zero-measure set in the solution space of the $1d$-harmonic oscillator!

If one decides not to keep the first derivatives fixed, by adding pure divergences one can even invent nastier and nastier examples. For instance, by considering the following 1-parameter family of Lagrangians

$$L''(q,\dot q,\ddot q;\Lambda) = \frac{1}{2}\left(\dot q^2 - \omega^2 q^2\right) + \frac{d}{dt}\left(\frac{1}{6}\,\dot q^3 + \left(\frac{\omega^2}{2}\,q^2 + \Lambda^2\right)\dot q\right)$$

with Λ real, which produce equations of motion in the form

$$\begin{cases} \ddot q + \omega^2 q = 0 \\ \dot q_0^2 + \omega^2 q_0^2 = -\Lambda^2 \end{cases}$$

wee see that, for any $\Lambda \neq 0$, one has no solution at all, since there are no initial conditions satisfying the boundary equation. And even for $\Lambda = 0$ the solution space is much smaller than the solution space of the harmonic oscillator, since it reduces again to quiet.

3 Examples in General Relativity

Of course one could argue that field theory is not Mechanics and that in Field Theory there is more space to play with. Such an assumption is of course true, but still one has to pay a lot of attention when playing...! Let us then present similar situations in GR.

Let M be a 4-dimensional manifold with boundary Ω and let us consider the metric Lagrangian

$$\begin{aligned} L &= \sqrt{g}R - \nabla_\alpha\left(\sqrt{g}g^{\mu\nu}\left(u^\alpha_{\mu\nu} - \bar u^\alpha_{\mu\nu}\right)\right) = \\ &= \left[\sqrt{g}g^{\alpha\beta}(\Gamma^\rho_{\alpha\sigma}\Gamma^\sigma_{\rho\beta} - \Gamma^\sigma_{\sigma\rho}\Gamma^\rho_{\alpha\beta}) + d_\sigma(\sqrt{g}g^{\alpha\beta}\bar u^\sigma_{\alpha\beta})\right] ds \end{aligned} \tag{4}$$

where: ds is the standard local volume element induced by the coordinates; here and below, $\Gamma^\alpha_{\beta\mu}$ are the coefficients of the Levi-Civita connection of the metric g; we set $u^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \delta^\lambda_{(\mu}\Gamma^\alpha_{\nu)\alpha}$ and $\bar u^\lambda_{\mu\nu} = \bar\Gamma^\lambda_{\mu\nu} - \delta^\lambda_{(\mu}\bar\Gamma^\alpha_{\nu)\alpha}$ for any connection $\bar\Gamma^\lambda_{\mu\nu}$ chosen at will on M. $\Gamma^\alpha_{\beta\mu}$ as well as $u^\lambda_{\mu\nu}$ are functions of the first derivatives of the field $g_{\mu\nu}$, while $\bar\Gamma^\lambda_{\mu\nu}$ is just a "fixed parametrization" i.e. a non-dynamical background (as one could easily see by realizing that the Euler-Lagrange equations of (4) with respect to $\bar\Gamma^\lambda_{\mu\nu}$ are identities). As long as the background connection $\bar\Gamma^\alpha_{\beta\mu}$ is considered, one is free to fix it at will: it can be a generic connection or the Levi-Civita connection of a background metric $\bar g$ (which could even have in principle a different signature) depending on the situation.

The Lagrangian (4) is covariant and first order in $g_{\mu\nu}$; the connection $\bar\Gamma^\alpha_{\beta\mu}$ is not subjected to any field equations so that it can be any connection both *a priori* and *a posteriori* (we stress that connections exist globally on any manifold); bulk field equations for g are vacuum Einstein field equations.

The background $\bar u^\lambda_{\mu\nu}$ is here added to preserve covariance. One could fix coordinates so that $\bar u^\lambda_{\mu\nu} = 0$ (usually at a point), or consider a fixed $\bar u^\lambda_{\mu\nu}(x)$ as a point dependence (we stress that it is relegated into a divergence). Our procedure is analogous to the one used by Hawking and Ellis (see [6]) to study the Cauchy problem in Relativity; there a background (metric) is used at the level of field equations, to show essential hyperbolicity, while here it is used at the level of the action. The two approaches are equivalent since the background is non-dynamical and its fixing commutes with the derivation of field equations; see also [5].

The variation of this Lagrangian is given by

$$\delta L = \sqrt{g} G_{\mu\nu} \delta g^{\mu\nu} - \nabla_\lambda \left(\sqrt{g} (\delta^\mu_{(\alpha} \delta^\nu_{\beta)} - \frac{1}{2} g_{\alpha\beta})(u^\lambda_{\mu\nu} - \bar{u}^\lambda_{\mu\nu}) \delta g^{\alpha\beta} - \sqrt{g} g^{\mu\nu} \delta \bar{u}^\lambda_{\mu\nu} \right)$$

with $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$. Applying standard techniques of Calculus of Variation one obtains only the bulk standard field equations $G_{\mu\nu} = 0$. If, instead, one fixes only $\delta g^{\mu\nu} = 0$ on the boundary, then a new boundary equation (associated to $\delta \bar{u}^\lambda_{\mu\nu}$) is added

$$\sqrt{g} g^{\mu\nu}|_\Omega = 0 \quad \Rightarrow \quad g_{\mu\nu}|_\Omega = 0$$

This boundary condition is not only incompatible with the bulk field equations, but with kinematics in the first place (metrics are assumed in fact to be non-degenerate so that they are everywhere forbidden to vanish). Hence if one considers the Lagrangian (4), that differs from standard GR by a divergence, and fixes the metric only, then the solution space is *empty*!

One could argue that the background $\bar{u}^\lambda_{\mu\nu}$ is unphysical since it has no dynamics and that therefore there is no need to consider its variations. That is certainly reasonable though the argument can be reversed: since the field $\bar{u}^\lambda_{\mu\nu}$ is unphysical, then physics should be independent of how one decides to treat it: keeping it fixed or varying it, possibly varying an underlying metric $\bar{g}_{\mu\nu}$ that fixes it on the boundary, alone or together with its first derivative. The above example shows instead how the physical predictions of the theory (in particular the solution space) do depend on which unphysical degree of freedom is kept fixed on the boundary. Moreover, conservation laws would result to be affected by terms ensuing form the divergence (they can be easily calculated as in [5]).

Similar (but nastier) examples can be considered: e.g. the Lagrangian

$$L' = \sqrt{g} R - \frac{1}{\Lambda} \nabla_\alpha \left(\sqrt{g} g^{\mu\nu} R \left(u^\alpha_{\mu\nu} - \bar{u}^\alpha_{\mu\nu} \right) \right)$$

that is again classically equivalent to the Hilbert Lagrangian. The variation is now

$$\delta L' = \sqrt{g} G_{\mu\nu} \delta g^{\mu\nu} - \nabla_\lambda \left(\frac{1}{\Lambda} \delta(\sqrt{g} g^{\mu\nu}) R \left(u^\lambda_{\mu\nu} - \bar{u}^\lambda_{\mu\nu} \right) + \frac{1}{\Lambda} \delta R \sqrt{g} g^{\mu\nu} \left(u^\lambda_{\mu\nu} - \bar{u}^\lambda_{\mu\nu} \right) \right) + \\ - \nabla_\lambda \left(\frac{1}{\Lambda} \sqrt{g} g^{\mu\nu} (R - \Lambda) \delta u^\lambda_{\mu\nu} - \frac{1}{\Lambda} \sqrt{g} g^{\mu\nu} R \delta \bar{u}^\lambda_{\mu\nu} \right)$$

Here, if we fix $\delta g^{\mu\nu} = 0$ leaving δR, $\delta u^\lambda_{\mu\nu}$ and $\delta \bar{u}^\lambda_{\mu\nu}$ unconstrained on the boundary, we have three boundary field equations

$$\begin{cases} \sqrt{g} g^{\mu\nu} \left(u^\alpha_{\mu\nu} - \bar{u}^\alpha_{\mu\nu} \right) \delta R|_\Omega = 0 & \Rightarrow \quad u^\alpha_{\mu\nu}|_\Omega = \bar{u}^\alpha_{\mu\nu}|_\Omega \\ \sqrt{g} g^{\mu\nu} (R - \Lambda) \delta u^\lambda_{\mu\nu}|_\Omega = 0 & \Rightarrow \quad R|_\Omega = \Lambda \\ g^{\mu\nu} R \delta \bar{u}^\lambda_{\mu\nu}|_\Omega = 0 & \Rightarrow \quad R|_\Omega = 0 \end{cases} \tag{5}$$

As in the previous example, these three conditions are incompatible and the resulting solution space is again empty. Unlike the previous example, however, if in this case one decides not to vary the background the first two equations in (5) are still obtained along with Einstein equation; they (in particular, the second one) are enough to force the solution space to be empty. Here the troubles are generated exactly from not fixing $\delta u^\lambda_{\mu\nu}$ at the boundary. If now one adds to the Lagrangian a divergence that suitably counterbalance the first constraint, then this is enough, for any $\Lambda \neq 0$, to prevent Minkowski spacetime from being a solution of the theory, with a devastating effect on Newtonian limit and the physical interpretation of the whole theory.) The first condition imposes in fact to $g_{\mu\nu}$ an arbitrary asymptotic; if $\bar{u}^\lambda_{\mu\nu}$ is suitably chosen, then one could impose to $g_{\mu\nu}$ to be asymptotically anti-de Sitter, de Sitter or anything else. In any case, the solution space is again *empty*!

Other even more complicated examples can be studied under the form

$$L_f = \sqrt{g}R - \nabla_\alpha \left(\sqrt{g} g^{\mu\nu} f(R; \Lambda, \dots) \left(u^\alpha_{\mu\nu} - \bar{u}^\alpha_{\mu\nu}\right)\right)$$

We stress that of course there are reasonable boundary terms which do not force the solution space to be empty, but there is no guiding principle helping one in distinguishing good boundary terms from bad ones, so that such a procedure should be better avoided (being misleading) or, if really necessary, treated with the correct mathematical instruments. All this in the case that the "real" Lagrangian we start deforming is the Hilbert Lagrangian, that is known to be the only non-trivial second order Lagrangian linear in the curvature of a metric field. Linearity implies Hamiltonian degeneracy, so that the second order theory is essentially equivalent to a first order theory with second order field equations. It is exactly this degeneracy and the existence of a family of covariant first order (see [5]) that allows one to play with a certain success with the addition of divergences. One should be aware that such a method *cannot* hold any longer in more general families of gravitational theories, such as e.g. all $f(R)$, Gauss-Bonnet, Lovelock, Chern-Simons Lagrangians and so on, including all effective Lagrangians that ensue from low limits of spacetime and/or quantum requirements.

4 Conclusions

We have here considered two attitudes in a variational principle of order k. Let us summarize our point. A *weakly critical configuration* is a configuration that extremizes the action for any deformation which vanishes along the boundary (while the field derivatives are left unconstrained).

A *critical configuration* is instead a configuration which extremizes the action for any deformation which vanishes together with its derivatives (up to order $k-1$) along the boundary.

Of course a weakly critical configuration is also critical, while the converse is false in general. From these simple examples we may easily conclude that, in a theory of order k, pure-divergence-terms may be considered unessential with respect to the field equations *only if* one considers critical configurations. On the contrary, by adding boundary terms to the action one can easily force the space of weakly critical configurations to be smaller or even empty.

Of course one is free to abandon the invariance of the action with respect to boundary contributions (as in a sense is done in the Hamiltonian formalism).

Unfortunately, such an attitude strongly impacts on conservation laws which are an essential part of the physical interpretation of the theory as well.

Weakly critical configurations are considered in [12] (against the standard results in Variational Calculus and other important monographs in GR that more correctly consider only critical configurations; see [2], [6], [11]). In our opinion there is no real reason to impose an often artificial boundary term to a covariant action, breaking general covariance, in order to allow more general deformations of fields. Deformations in Lagrangian formalism have indeed no physical meaning. In Mechanics they are called in fact *virtual* dispacements also to stress the fact that they are not physical and they just need to be generically independent.

Any procedure that fixes fields and no derivatives at the boundary is certainly very similar (if not technically identical) to a gauge fixing. Gauge fixing are useful in practice in special situations but there is no reason to break gauge covariance by fixing a gauge when a gauge covariant procedure allows to obtain the same result from a more fundamentally satisfactory point of view.

Another way of considering these examples is from control theory in the Hamiltonian framework. Boundary terms of the action are exactly the way of mimiking control theory at the Lagrangian level. In such a framework one is not concerned with computing physical configurations (namely, solutions of field equations) but how (and whether) physical configurations can respond to some constraint imposed at the boundary. For example, computing the electric field in a space with a conductor, knowing that the boundary, i.e. the surface of the conductor, is equipotential.

In this context the extra boundary equations are exactly interpreted as the condition one wishes to impose at the boundary. Here (and only here) one should guarantee that the boundary conditions imposed can be physically realized. It is no surprise that in certain cases there exist no configuration obeying those boundary conditions, meaning that one cannot physically impose those particular boundary conditions.

We have to stress that in gravitational experiments we are now technologically unable to impose *any* boundary conditions. It is therefore interesting to know that some requirements are forbidden in principle.

We have also to stress that the framework of control theory is by no means related to the determination of solutions of field equations, where by definition one wants to obtain *all* possible field configurations. Moreover, if we *unnecessarily* rely on boundary terms to obtain field equations, then this freedom cannot be exploited to deal with conservation laws (see [3]). In fact, it is well-known that, although divergences leave invariant critical configurations, they affect conservation laws and conserved quantities that are an important part of the physical interpretation of the model. If boundary terms are fixed for field equations one could only hope conservation laws to turn out to make sense.

Acknowledgements

This work is partially supported by the contribution of INFN (Iniziativa Specifica NA12) and the local research founds of Dipartimento di Matematica of Torino University.

References

[1] S. Capozziello, M. De Laurentis, V. Faraoni: *A bird's eye view of $f(R)$-gravity.* (2009), arXiv:0909.4672

[2] S.M. Carrol: *Lecture Notes on General Relativity.* arXiv:gr-qc/9712019v1

[3] L. Fatibene, M. Ferraris, M. Francaviglia: Augmented Variational Principles and Relative Conservation Laws in Classical Field Theory. *Int. J. Geom. Methods Mod. Phys.* 2 (3) (2005) 373–392.

[4] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri: Noether charges, Brown-York quasilocal energy, and related topics. *J. Math. Phys.* 42 (3) (2001) 1173–1195.

[5] M. Ferraris, M. Francaviglia: Covariant first-order Lagrangians, energy-density and superpotentials in general relativity. *GRG* 22 (9) (1990) 965–985.

[6] S.W. Hawking, G.F.R. Ellis, P.V. Landshoff, D.R. Nelson, D.W. Sciama, S. Weinberg: *The Large Scale Structure of Space-Time.* Cambridge University Press, New York (1973).

[7] S.W. Hawking, C.J. Hunter: Gravitational Entropy and Global Structure. *Phys. Rev.* D59 (1999) 044025. arXiv:hep-th/9808085v2

[8] K. Hinterbichler: Boundary Terms, Variational Principles and Higher Derivative Modified Gravity. *Phys. Rev.* D79 (2009) 024028. arXiv:0809.4033

[9] S. Nojiri, S.D. Odintsov: *Unified cosmic history in modified gravity: from $F(R)$ theory to Lorentz non-invariant models.* arXiv:1011.0544v4 [gr-qc]

[10] T.P. Sotiriou, V. Faraoni: *$f(R)$ theories of gravity.* arXiv:0805.1726

[11] G.'t Hooft: *Introduction to General Relativity.* Rinton Press (2001).

[12] R.M. Wald: *General Relativity.* University of Chicago Press (1984).

Authors' addresses:
Lorenzo Fatibene: Dipartimento di Matematica, University of Torino, via Carlo Alberto 10, I-10123 Torino, Italy
and INFN – Sez. Torino: Iniziativa Specifica NA12

E-mail: lorenzo.fatibene@unito.it

Mauro Francaviglia: Dipartimento di Matematica, University of Torino, via Carlo Alberto 10, I-10123 Torino, Italy
and INFN – Sez. Torino: Iniziativa Specifica NA12
and LCS, University of Calabria

E-mail: mauro.francaviglia@unito.it

Silvio Mercadante: Dipartimento di Matematica, University of Torino, via Carlo Alberto 10, I-10123 Torino, Italy
and INFN – Sez. Torino: Iniziativa Specifica NA12

E-mail: silvio.mercadante@unito.it

Received: 18 June, 2011
Accepted for publication: 21 July, 2011

Communications in Mathematics 19 (2011) 137–147

Tangent Lie algebras to the holonomy group of a Finsler manifold

Zoltán Muzsnay, Péter T. Nagy

Abstract. Our goal in this paper is to make an attempt to find the largest Lie algebra of vector fields on the indicatrix such that all its elements are tangent to the holonomy group of a Finsler manifold. First, we introduce the notion of the *curvature algebra*, generated by curvature vector fields, then we define the *infinitesimal holonomy algebra* by the smallest Lie algebra of vector fields on an indicatrix, containing the curvature vector fields and their horizontal covariant derivatives with respect to the Berwald connection. At the end we introduce conjugates of infinitesimal holonomy algebras by parallel translations with respect to the Berwald connection. We prove that this holonomy algebra is tangent to the holonomy group.

1 Introduction

The notion of the holonomy group of Riemannian manifolds can be generalized very naturally for Finsler manifolds: it is the group generated by canonical homogeneous (nonlinear) parallel translations along closed loops. Until now the holonomy groups of non-Riemannian Finsler manifolds were described only in special cases: the Berwald manifolds have the same holonomy group as some Riemannian manifolds (cf. Z.I. Szabó, [8]) and the holonomy groups of Landsberg manifolds are compact Lie groups (cf. L. Kozma, [3]). A thorough study of the holonomy algebras of homogeneous (nonlinear) connections was initiated by W. Barthel [1], he gave a successive extension by Berwald's covariant derivation of the Lie algebras generated by the curvature vector fields. A general setting for the study of infinite dimensional holonomy groups and holonomy algebras of nonlinear connections was initiated by P. Michor in [5], but the tangential properties of the holonomy algebras to the holonomy group were not clarified.

Recently, the authors introduced in [6] the notion of tangent Lie algebras to the holonomy group and proved that the curvature algebra (the Lie algebra generated

2010 MSC: 53B40, 53C29, 22E65
Key words: Finsler geometry, holonomy, infinite-dimensional Lie groups.

by curvature vector fields) is a tangent algebra to the holonomy group. With this technique we have constructed a Finsler manifold (with singular metric) with infinite dimensional curvature algebra, which implies that the holonomy group can not be a finite dimensional Lie group in this case. We suspect that for most of non-Riemannian Finsler manifolds, the holonomy group is not a finite dimensional Lie group.

In a recent paper [2] M. Crampin, D.J. Saunders carried on a deep analysis of the holonomy structures of bundles with fibre metrics, and in particular the holonomy structures of Landsbergian type Finsler manifolds. In these cases, the holonomy groups are finite dimensional Lie groups. They introduced the notion of holonomy algebra and proved a version of Ambrose-Singer Theorem for such spaces. Reflecting to our results, they noticed that in the general Finslerian framework the holonomy algebra should contain the parallel translated curvature algebras. They showed that in this case the topological closure of this holonomy algebra contains the covariant derivatives of curvature vector fields, but the tangent properties of the successive covariant derivatives of curvature vector fields are not obvious from this approach in the cases, when the holonomy group is not a finite dimensional Lie group. The difficulty comes from the fact, that a topologically non-closed infinite dimensional Lie algebra of vector fields may expand, if we add the covariant derivatives of its elements.

Our goal in this paper is to make an attempt to find the right notion of the holonomy algebra of Finsler spaces. The holonomy algebra should be the largest Lie algebra such that all its elements are tangent to the holonomy group. In our attempt we are building successively Lie algebras having the tangent properties. First, we introduce the notion of the *curvature algebra* (the Lie algebra generated by curvature vector fields) which is a tangent Lie algebra to the holonomy group (cf. [6]). Then we define the *infinitesimal holonomy algebra* by the smallest Lie algebra of vector fields on an indicatrix, containing the curvature vector fields and their horizontal covariant derivatives with respect to the Berwald connection and prove the tangential property of this Lie algebra to the holonomy group. At the end we introduce the notion of the *holonomy algebra* of a Finsler manifold by all conjugates of infinitesimal holonomy algebras by parallel translations with respect to the Berwald connection. We prove that this holonomy algebra is tangent to the holonomy group. The question of whether the holonomy algebra introduced in this way is the largest Lie algebra, which is tangent to the holonomy group, is still open.

2 Preliminaries

Let M be an n-dimensional C^∞ manifold and let $\mathfrak{X}^\infty(M)$ denote the vector space of smooth vector fields on M. For a local coordinate system $(x^1,\dots,x^n)$ on M we denote by $(x^1,\dots,x^n;y^1,\dots,y^n)$ the induced local coordinate system on the tangent bundle TM.

Finsler manifold, canonical connection, parallelism

A *Finsler manifold* is a pair $(M,\mathcal{F})$, where the Finsler function $\mathcal{F}\colon TM\to\mathbb{R}$ is continuous, smooth on $\hat{T}M := TM\setminus\{0\}$, its restriction $\mathcal{F}_x=\mathcal{F}|_{T_xM}$ is a positively homogeneous function of degree 1 and the symmetric bilinear form (the Finsler metric)

$$g_{x,y}\colon (u,v)\mapsto g_{ij}(x,y)u^iv^j = \frac{1}{2}\frac{\partial^2\mathcal{F}^2_x(y+su+tv)}{\partial s\,\partial t}\Big|_{t=s=0}$$

is positive definite at every $y\in\hat{T}_xM$.

Geodesics of Finsler manifolds are determined by a system of second order ordinary differential equation $\ddot{x}^i + 2G^i(x,\dot{x}) = 0$, $i=1,\dots,n$, where $G^i(x,\dot{x})$ are locally given by

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big)y^jy^k. \tag{1}$$

The associated *homogeneous (nonlinear) parallel translation* $\tau_c\colon T_{c(0)}M\to T_{c(1)}M$ along a curve $c\colon[0,1]\to\mathcal{R}$ is defined by vector fields $X(t) = X^i(t)\frac{\partial}{\partial x^i}$ along $c(t)$ which are solutions of the differential equation

$$D_{\dot{c}}X(t) := \Big(\frac{dX^i(t)}{dt} + G^i_j(c(t),X(t))\dot{c}^j(t)\Big)\frac{\partial}{\partial x^i} = 0, \quad\text{where}\quad G^i_j = \frac{\partial G^i}{\partial y^j}. \tag{2}$$

Horizontal distribution, Berwald connection, curvature

Let (TM,π,M) and (TTM,ρ,TM) denote the first and the second tangent bundle of the manifold M, respectively. The horizontal distribution $\mathcal{H}TM\subset TTM$ associated with the Finsler manifold $(M,\mathcal{F})$ can be defined as the image of the horizontal lift which is an isomorphism $X\to X^h$ between T_xM and $\mathcal{H}_yTM$ at $y\in T_xM$ defined by

$$\Big(X^i\frac{\partial}{\partial x^i}\Big)^h := X^i\left(\frac{\partial}{\partial x^i} - G^k_i(x,y)\frac{\partial}{\partial y^k}\right). \tag{3}$$

If $\mathcal{V}TM := \operatorname{Ker}\pi_*\subset TTM$ denotes the vertical distribution on TM, then for any $y\in TM$ we have $T_yTM = \mathcal{H}_yTM\oplus\mathcal{V}_yTM$. The projectors corresponding to this decomposition will be denoted by $h\colon TTM\to\mathcal{H}TM$ and $v\colon TTM\to\mathcal{V}TM$. We note that the vertical distribution is integrable.

Let $(\hat{\mathcal{V}}TM,\rho,\hat{T}M)$ be the vertical bundle over $\hat{T}M := TM\setminus\{0\}$. We denote by $\mathfrak{X}^\infty(M)$, respectively by $\hat{\mathfrak{X}}^\infty(TM)$ the vector space of smooth vector fields on M and of smooth sections of the bundle $(\hat{\mathcal{V}}TM,\tau,\hat{T}M)$, respectively. The *horizontal Berwald covariant derivative* of a section $\xi\in\hat{\mathfrak{X}}^\infty(TM)$ by a vector field $X\in\mathfrak{X}^\infty(M)$ is $\nabla_X\xi := [X^h,\xi]$.

In an induced local coordinate system (x^i,y^i) on TM for vector fields $\xi(x,y) = \xi^i(x,y)\frac{\partial}{\partial y^i}$ and $X(x) = X^i(x)\frac{\partial}{\partial x^i}$ we have (3) and hence

$$\nabla_X\xi = \left(\frac{\partial\xi^i(x,y)}{\partial x^j} - G^k_j(x,y)\frac{\partial\xi^i(x,y)}{\partial y^k} + \frac{\partial G^i_j(x,y)}{\partial y^k}(x,y)\xi^k(x,y)\right)X^j\frac{\partial}{\partial y^i}. \tag{4}$$

Let $(\pi^*TM,\bar{\pi},\hat{T}M)$ be the pull-back bundle of $(\hat{T}M,\pi,M)$ by the map $\pi\colon TM\to M$. Clearly, the mapping

$$\Big(x,y,\xi^i\frac{\partial}{\partial y^i}\Big)\mapsto\Big(x,y,\xi^i\frac{\partial}{\partial x^i}\Big)\colon\hat{\mathcal{V}}TM\to\pi^*TM \tag{5}$$

is a canonical bundle isomorphism. In the following we will use the isomorphism (5) for the identification of these bundles.

The *Riemannian curvature tensor* field $R_{(x,y)}(X,Y) := v\big[X^h, Y^h\big]$, $X, Y \in T_xM$, $(x,y) \in \hat{T}M$ characterizes the integrability of the horizontal distribution. Namely, if the horizontal distribution $\mathcal{H}\hat{T}M$ is integrable, then the Riemannian curvature is identically zero. The expression of the Riemannian curvature tensor

$$R_{(x,y)} = R^i_{jk}(x,y)dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$$

on the pull-back bundle $(\pi^*TM, \bar{\pi}, \hat{T}M)$ is

$$R^i_{jk}(x,y) = \frac{\partial G^i_j(x,y)}{\partial x^k} - \frac{\partial G^i_k(x,y)}{\partial x^j} + G^m_j(x,y)G^i_{km}(x,y) - G^m_k(x,y)G^i_{jm}(x,y).$$

Indicatrix bundle

The *indicatrix* $\mathcal{I}_pM$ of an n-dimensional Finsler manifold $(M, \mathcal{F})$ at a point $p \in M$ is the compact hypersurface $\mathcal{I}_pM := \{y \in T_pM; \mathcal{F}(y) = 1\}$ in T_pM, diffeomorphic to the standard $(n-1)$-sphere. The *indicatrix bundle* $(\mathcal{I}M, \pi, M)$ of $(M, \mathcal{F})$ is a smooth subbundle of the tangent bundle (TM, π, M). The group $\mathsf{Diff}^\infty(\mathcal{I}_pM)$ of all smooth diffeomorphisms of an indicatrix $\mathcal{I}_pM$ is a regular infinite dimensional Lie group modeled on the vector space $\mathfrak{X}^\infty(\mathcal{I}_pM)$ of smooth vector fields on $\mathcal{I}_pM$. The Lie algebra of the infinite dimensional Lie group $\mathsf{Diff}^\infty(\mathcal{I}_pM)$ is the vector space $\mathfrak{X}^\infty(\mathcal{I}_pM)$, equipped with the negative of the usual Lie bracket, (cf. A. Kriegl and P.W. Michor [4], Section 43).

Let $c(t)$, $0 \le t \le a$ be a smooth curve joining the points $p = c(0)$ and $q = c(a)$ in the Finsler manifold $(M, \mathcal{F})$. Since the parallel translation $\tau_c \colon T_pM \to T_qM$ along the curve $c\colon [0,a] \to M$ is a differentiable map between $\hat{T}_pM$ and $\hat{T}_qM$ preserving the value of the Finsler function, it induces a parallel translation $\tau_c \colon \mathcal{I}_pM \to \mathcal{I}_qM$ in the indicatrix bundle.

Holonomy group

The notion of the holonomy group of Riemannian manifolds can be generalized very naturally for Finsler manifolds:

Definition 1. The *holonomy group* $\mathsf{Hol}(p)$ of a Finsler space $(M, \mathcal{F})$ at $p \in M$ is the subgroup of the group of diffeomorphisms $\mathsf{Diff}^\infty(\mathcal{I}_pM)$ of the indicatrix $\mathcal{I}_pM$ determined by parallel translation of $\mathcal{I}_pM$ along piece-wise differentiable closed curves initiated at the point $p \in M$.

Clearly, the holonomy groups at different points of M are isomorphic. We note that the holonomy group $\mathsf{Hol}(p)$ is a topological subgroup of the regular infinite dimensional Lie group $\mathsf{Diff}^\infty(\mathcal{I}_pM)$, but its differentiable structure is not known in general.

3 Tangent Lie algebras to diffeomorphism groups

Here we discuss the tangential properties of Lie algebras of vector fields to an abstract subgroup of the diffeomorphism group of a manifold. The results of this section will be applied in the following to the investigation of tangent Lie algebras of the holonomy subgroup of the diffeomorphism group of an indicatrix $\mathcal{I}_xM$ and to the fibred holonomy subgroup of the diffeomorphism group of the indicatrix bundle $\mathcal{I}(M)$.

Let P be a C^∞ manifold, let H be a (not necessarily differentiable) subgroup of the diffeomorphism group $\mathsf{Diff}^\infty(P)$ and let $\mathfrak{X}^\infty(P)$ be the Lie algebra of smooth vector fields on P.

Definition 2. A vector field $X \in \mathfrak{X}^\infty(P)$ is called *tangent* to the subgroup H of $\mathsf{Diff}^\infty(P)$, if there exists a $\mathcal{C}^1$-differentiable 1-parameter family $\{\phi_t \in H\}_{t\in(-\varepsilon,\varepsilon)}$ of diffeomorphisms of M such that $\phi_0 = \mathsf{Id}$ and $\frac{\partial \phi_t}{\partial t}\big|_{t=0} = X$. A Lie subalgebra $\mathfrak{g}$ of $\mathfrak{X}^\infty(P)$ is called *tangent* to H, if all elements of $\mathfrak{g}$ are tangent vector fields to H.

Unfortunately, it is not true, that tangent vector fields to the group H generate a tangent Lie algebra to H. This is why we have to introduce a stronger tangency property in Definition 4.

Definition 3. A $\mathcal{C}^\infty$-differentiable k-parameter family

$$\{\phi_{(t_1,\dots,t_k)} \in \mathsf{Diff}^\infty(P)\}_{t_i\in(-\varepsilon,\varepsilon)}$$

of diffeomorphisms of P is called a *commutator-like family* if it satisfies the equations

$$\phi_{(t_1,\dots,t_k)} = \mathsf{Id}, \quad \text{whenever} \quad t_j = 0 \quad \text{for some} \quad 1 \le j \le k.$$

We remark, that the commutators of commutator-like families are commutator-like, and the inverse of commutator-like families are commutator-like.

Definition 4. A vector field $X \in \mathfrak{X}^\infty(P)$ is called *strongly tangent* to the subgroup H of $\mathsf{Diff}^\infty(P)$, if there exists a commutator-like family

$$\{\phi_{(t_1,\dots,t_k)} \in \mathsf{Diff}^\infty(P)\}_{t_i\in(-\varepsilon,\varepsilon)}$$

of diffeomorphisms satisfying the conditions

(A) $\phi_{(t_1,\dots,t_k)} \in H \quad \text{for all} \quad t_i \in (-\varepsilon,\varepsilon), \quad 1 \le i \le k,$

(B) $\frac{\partial^k \phi_{(t_1,\dots,t_k)}}{\partial t_1 \cdots \partial t_k}\big|_{(0,\dots,0)} = X.$

It follows from the commutator-like property that $\frac{\partial^k \phi_{(t_1,\dots,t_k)}}{\partial t_1 \dots \partial t_k}\big|_{(0,\dots,0)}$ is the first non-necessarily vanishing derivative of the diffeomorphism family $\{\phi_{(t_1,\dots,t_k)}\}$ at any point $x \in P$, and therefore it determines a vector field. On the other hand, by reparametrizing the commutator like family of diffeomorphism, it can be shown that if a vector field is strongly tangent to a group H, then it is also tangent to H. Moreover, we have the following

Theorem 1. *Let $\mathcal{V}$ be a set of vector fields strongly tangent to the group $H \subset \mathsf{Diff}^\infty(P)$. The Lie subalgebra $\mathfrak{v}$ of $\mathfrak{X}^\infty(P)$ generated by $\mathcal{V}$ is tangent to H.*

The proof of the theorem is based on two important observations. The first is a generalization of the well-known relation between the commutator of vector fields and the commutator of their induced flows. Namely, if $\{\phi_{(s_1,\dots,s_k)}\}$ and $\{\psi_{(t_1,\dots,t_l)}\}$ are commutator-like k-parameter, respectively l-parameter families of local diffeomorphisms, then the family of (local) diffeomorphisms $[\phi_{(s_1,\dots,s_k)},\psi_{(t_1,\dots,t_l)}]$ defined by the commutator of the group $\mathsf{Diff}^\infty(U)$ is a commutator-like $(k+l)$-parameter family and

$$\frac{\partial^{k+l}[\phi_{(s_1,\dots,s_k)},\psi_{(t_1,\dots,t_l)}]}{\partial s_1\dots\partial s_k\,\partial t_1\dots\partial t_l}\Big|_{(0,\dots,0;0,\dots,0)}(x) = -\left[\frac{\partial^k\phi_{(s_1,\dots,s_k)}}{\partial s_1\dots\partial s_k}\Big|_{(0,\dots,0)},\frac{\partial^l\psi_{(t_1,\dots,t_l)}}{\partial t_1\dots\partial t_l}\Big|_{(0,\dots,0)}\right](x)$$

at any point $x \in U$. The second important fact to prove the theorem is that the linear combinations of vector fields tangent to H are also tangent to H. The detailed computations can be found in [6].

4 The curvature algebra at a point

Now, se summarize our results on the tangent Lie algebras of the holonomy group $\mathsf{Hol}(p)$ at a point $p \in M$, their proofs can be found in [6].

Definition 5. A vector field $\xi \in \mathfrak{X}(\mathcal{I}_pM)$ on the indicatrix $\mathcal{I}_pM$ of the Finsler manifold $(M,\mathcal{F})$ is called a *curvature vector field at the point* $p \in M$, if it is in the image of the curvature tensor, i.e. if there exist $X,Y \in T_pM$ such that $\xi = r_p(X,Y)$, where

$$r_p(X,Y)(y) := R_{(p,y)}(X^h,Y^h) \tag{6}$$

The Lie subalgebra $\mathfrak{R}_p := \big\langle r_p(X,Y); X,Y \in T_pM\big\rangle$ of $\mathfrak{X}(\mathcal{I}_pM)$ generated by the curvature vector fields at the point $p \in M$ is called the *curvature algebra at the point* $p \in M$.

Since the Finsler function is preserved by parallel translations, its derivatives with respect to horizontal vector fields are identically zero. According to [7], eq. (10.9), the derivative of the Finsler metric with respect to $R_{(p,y)}(X^h,Y^h)$ vanishes, i.e.

$$g_{(p,y)}\big(y,R_{(p,y)}(X^h,Y^h)\big) = 0, \quad \text{for any} \quad y,X,Y \in T_xM.$$

This means that the curvature vector fields $\xi = r_p(X,Y)$ are tangent to the indicatrix. In the sequel we investigate the tangential properties of the curvature algebra to the holonomy group of the canonical connection ∇ of a Finsler manifold.

Proposition 1. *Any curvature vector field at a point $p \in M$ is strongly tangent to the holonomy group $\mathsf{Hol}(p)$.*

Proposition 2. *The curvature algebra $\mathfrak{R}_p$ at any point $p \in M$ of a Riemannian manifold (M, g) is isomorphic to the linear Lie algebra on the tangent space T_pM generated by the curvature operators of (M, g) at $p \in M$.*

Remark 1. *The dimension of the curvature algebra at any point $p \in M$ of a Finsler surface is ≤ 1.*

5 Fibred holonomy group and fibred holonomy algebra

Now, we introduce the notion of the fibred holonomy group of a Finsler manifold $(M, \mathcal{F})$ as a subgroup of the diffeomorphism group of the total manifold $\mathcal{I}M$ of the bundle $(\mathcal{I}M, \pi, M)$ and apply our results on tangent vector fields to an abstract subgroup of the diffeomorphism group to the study of tangent Lie algebras to the fibred holonomy group.

Definition 6. The *fibred holonomy group* $\mathsf{Hol_f}(M)$ of $(M, \mathcal{F})$ consists of fibre preserving diffeomorphisms $\Phi \in \mathsf{Diff}^\infty(\mathcal{I}M)$ of the indicatrix bundle $(\mathcal{I}M, \pi, M)$ such that for any $p \in M$ the restriction $\Phi_p = \Phi|_{\mathcal{I}_pM} \in \mathsf{Diff}^\infty(\mathcal{I}_pM)$ belongs to the holonomy group $\mathsf{Hol}(p)$.

We note that the holonomy group $\mathsf{Hol}(p)$ and the fibred holonomy group $\mathsf{Hol_f}(M)$ are topological subgroups of the infinite dimensional Lie groups $\mathsf{Diff}^\infty(\mathcal{I}_pM)$ and $\mathsf{Diff}^\infty(\mathcal{I}M)$ respectively.

The definition of strongly tangent vector fields yields

Remark 2. *A vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ is strongly tangent to the fibred holonomy group $\mathsf{Hol_f}(M)$ if and only if there exists a family $\{\Phi_{(t_1,\dots,t_k)}|_{\mathcal{I}M}\}_{t_i\in(-\varepsilon,\varepsilon)}$ of fibre preserving diffeomorphisms of the bundle $(\mathcal{I}M, \pi, M)$ such that for any indicatrix $\mathcal{I}_p$ the induced family $\{\Phi_{(t_1,\dots,t_k)}|_{\mathcal{I}_pM}\}_{t_i\in(-\varepsilon,\varepsilon)}$ of diffeomorphisms is contained in the holonomy group $\mathsf{Hol}(p)$ and $\xi\big|_{\mathcal{I}_pM}$ is strongly tangent to $\mathsf{Hol}(p)$.*

Since $\pi\big(\Phi_{(t_1,\dots,t_k)}(p)\big) \equiv p$ and $\pi_*(\xi) = 0$ for every $p \in U$, we get the

Corollary 1. *Strongly tangent vector fields to the fibred holonomy group $\mathsf{Hol_f}(M)$ are vertical vector fields. If $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ is strongly tangent to $\mathsf{Hol_f}(M)$ then its restriction $\xi_p := \xi\big|_{\mathcal{I}_p}$ to any indicatrix $\mathcal{I}_p$ is strongly tangent to the holonomy group $\mathsf{Hol}(p)$.*

The curvature vector fields and the curvature algebra at a point has been defined on an indicatrix of the manifold M. Now we extend the domain of their definition to the total manifold of the indicatrix bundle.

Definition 7. A vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ on the indicatrix bundle $\mathcal{I}M$ is a *curvature vector field* of the Finsler manifold $(M, \mathcal{F})$, if there exist $X, Y \in \mathfrak{X}^\infty(M)$ such that $\xi = r(X, Y)$, where $r(X, Y)(x, y) := R_{(x,y)}(X_x, Y_x)$ for $x \in M$ and $y \in \mathcal{I}_xM$. The Lie algebra $\mathfrak{R}(M)$ generated by the curvature vector fields of $(M, \mathcal{F})$ is called the *curvature algebra* of the Finsler manifold $(M, \mathcal{F})$.

Proposition 3. *If the Finsler manifold $(M,\mathcal{F})$ is diffeomorphic to $\mathbb{R}^n$ then any curvature vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I}M)$ of $(M,\mathcal{F})$ is strongly tangent to the fibred holonomy group $\mathsf{Hol_f}(M)$.*

Proof. Since M is diffeomorphic to $\mathbb{R}^n$ we can identify the manifold M with the vector space $\mathbb{R}^n$. Let $\xi = r(X,Y) \in \mathfrak{X}^\infty(\mathcal{I}\mathbb{R}^n)$ be a curvature vector field with $X,Y \in \mathfrak{X}^\infty(\mathbb{R}^n)$. According to Proposition 1 its restriction $\xi\big|_{\mathcal{I}_p\mathbb{R}^n}$ to any indicatrix $\mathcal{I}_p\mathbb{R}^n$ is strongly tangent to the holonomy groups $\mathsf{Hol}(p)$. We have to prove that there exists a family $\{\Phi_{(t_1,\dots,t_k)}\big|_{\mathcal{I}\mathbb{R}^n}\}_{t_i\in(-\varepsilon,\varepsilon)}$ of fibre preserving diffeomorphisms of the indicatrix bundle $(\mathcal{I}\mathbb{R}^n,\pi,\mathbb{R}^n)$ such that for any $p\in\mathbb{R}^n$ the family of diffeomorphisms induced on the indicatrix $\mathcal{I}_p$ is contained in $\mathsf{Hol}(p)$ and $\xi\big|_{\mathcal{I}_p\mathbb{R}^n}$ is strongly tangent to $\mathsf{Hol}(p)$.

For any $p\in\mathbb{R}^n$ and $-1<s,t<1$ let $\Pi(sX_p,tY_p)$ be the parallelogram in $\mathbb{R}^n$ determined by the vertexes $p, p+sX_p, p+sX_p+tY_p, p+tY_p \in \mathbb{R}^n$ and let $\tau_{\Pi(sX_p,tY_p)}\colon \mathcal{I}_p \to \mathcal{I}_p$ denote the (nonlinear) parallel translation of the indicatrix $\mathcal{I}_p$ along the parallelogram $\Pi(sX_p,tY_p)$ with respect to the associated homogeneous (nonlinear) parallel translation of the Finsler manifold $(\mathbb{R}^n,\mathcal{F})$. Clearly we have $\tau_{\Pi(sX_p,tY_p)} = \mathsf{Id}_{\mathcal{I}\mathbb{R}^n}$, if $s=0$ or $t=0$ and

$$\frac{\partial^2\tau_{\Pi(sX_p,tY_p)}}{\partial s\,\partial t}\bigg|_{(s,t)=(0,0)} = \xi_p \quad \text{for every} \quad p\in\mathbb{R}^n.$$

Since the mapping $(p,s,t)\mapsto \Pi(sX_p,tY_p)$ is a differentiable field of parallelograms in $\mathbb{R}^n$, the maps $\tau_{\Pi(sX_p,tY_p)}$, $p\in\mathbb{R}^n$, $0<s,t<1$, define a 2-parameter family of fibre preserving diffeomorphisms of the indicatrix bundle $\mathcal{I}\mathbb{R}^n$. The diffeomorphisms induced by the family $\{\tau_{\Pi(sX_p,tY_p)}\}_{s,t\in(-1,1)}$ on any indicatrix $\mathcal{I}_p$ are contained in $\mathsf{Hol}(p)$. Hence the vector field $\xi\in\mathfrak{X}^\infty(\mathbb{R}^n)$ is strongly tangent to the fibred holonomy group $\mathsf{Hol_f}(M)$, hence the assertion is proved. □

Corollary 2. *If M is diffeomorphic to $\mathbb{R}^n$ then the curvature algebra $\mathfrak{R}(M)$ of $(M,\mathcal{F})$ is tangent to the fibred holonomy group $\mathsf{Hol_f}(M)$.*

The following assertion shows that similarly to the Riemannian case, the curvature algebra can be extended to a larger tangent Lie algebra containing all horizontal covariant derivatives of the curvature algebra vector fields.

Proposition 4. *If $\xi\in\mathfrak{X}^\infty(\mathcal{I}M)$ is strongly tangent to the fibred holonomy group $\mathsf{Hol_f}(M)$ of $(M,\mathcal{F})$, then its horizontal covariant derivative $\nabla_X\xi$ along any vector field $X\in\mathfrak{X}^\infty(M)$ is also strongly tangent to $\mathsf{Hol_f}(M)$.*

Proof. Let τ be the (nonlinear) parallel translation along the flow φ of the vector field X, i.e. for every $p\in M$ and $t\in(-\varepsilon_p,\varepsilon_p)$ the map $\tau_t(p)\colon \mathcal{I}_pM \to \mathcal{I}_{\varphi_t(p)}M$ is the (nonlinear) parallel translation along the integral curve of X. If $\{\Phi_{(t_1,\dots,t_k)}\}_{t_i\in(-\varepsilon,\varepsilon)}$ is a $\mathcal{C}^\infty$-differentiable k-parameter family $\{\Phi_{(t_1,\dots,t_k)}\}_{t_i\in(-\varepsilon,\varepsilon)}$ of fibre preserving diffeomorphisms of the indicatrix bundle $(\mathcal{I}M,\pi|_M,M)$ satisfying the conditions of Definition 1 then the commutator

$$[\Phi_{(t_1,\dots,t_k)},\tau_{t_{k+1}}] := \Phi^{-1}_{(t_1,\dots,t_k)}\circ(\tau_{t_{k+1}})^{-1}\circ\Phi_{(t_1,\dots,t_k)}\circ\tau_{t_{k+1}}$$

of the group $\mathsf{Diff}^\infty(\mathcal{I}M)$ fulfills $[\Phi_{(t_1,\dots,t_k)},\tau_{t_{k+1}}] = \mathsf{Id}$, if some of its variables equals 0. Moreover

$$\left.\frac{\partial^{k+1}[\Phi_{(t_1,\dots,t_k)},\tau_{(t_{k+1})}]}{\partial t_1\dots\partial t_{k+1}}\right|_{(0,\dots,0)} = -\big[\xi, X^h\big] \tag{7}$$

at any point of M, which shows that the vector field $\big[\xi, X^h\big]$ is strongly tangent to $\mathsf{Hol}_{\mathsf{f}}(M)$. Moreover, since the vector field ξ is vertical, we have $h[X^h,\xi] = 0$, and using $\nabla_X\xi := [X^h,\xi]$ we obtain

$$-[\xi, X^h] = [X^h,\xi] = v[X^h,\xi] = \nabla_X\xi$$

which yields the assertion. □

Definition 8. Let $\mathfrak{hol}_{\mathfrak{f}}(M)$ be the smallest Lie algebra of vector fields on the indicatrix bundle $\mathcal{I}M$ satisfying the properties

(i) any curvature vector field ξ belongs to $\mathfrak{hol}_{\mathfrak{f}}(M)$,

(ii) if $\xi \in \mathfrak{hol}_{\mathfrak{f}}(M)$ and $X \in \mathfrak{X}^\infty(M)$, then the covariant derivative $\nabla_X\xi$ also belongs to $\mathfrak{hol}_{\mathfrak{f}}(M)$.

The Lie algebra $\mathfrak{hol}_{\mathfrak{f}}(M) \subset \mathfrak{X}^\infty(\mathcal{I}M)$ is called the *fibred holonomy algebra* of the Finsler manifold $(M,\mathcal{F})$.

Remark 3. *The fibred holonomy algebra $\mathfrak{hol}_{\mathfrak{f}}(M)$ is invariant with respect to the horizontal covariant derivation with respect to the Berwald connection, i.e.*

$$\xi \in \mathfrak{hol}_{\mathfrak{f}}(M) \quad \text{and} \quad X \in \mathfrak{X}^\infty(M) \quad \Rightarrow \quad \nabla_X\xi \in \mathfrak{hol}_{\mathfrak{f}}(M). \tag{8}$$

The results of this sections yield the following

Theorem 2. *If M is diffeomorphic to $\mathbb{R}^n$ then the fibred holonomy algebra $\mathfrak{hol}_{\mathfrak{f}}(M)$ is tangent to the fibred holonomy group $\mathsf{Hol}_{\mathsf{f}}(M)$.*

6 Infinitesimal holonomy algebra

Let $\mathfrak{hol}_{\mathfrak{f}}(M) \subset \mathfrak{X}^\infty(\mathcal{I}M)$ be the fibred holonomy algebra of the Finsler manifold $(M,\mathcal{F})$ and let p be a a given point in M.

Definition 9. The Lie algebra $\mathfrak{hol}^*(p) := \big\{\xi_p; \xi \in \mathfrak{hol}_{\mathfrak{f}}(M)\big\} \subset \mathfrak{X}^\infty(\mathcal{I}_pM)$ of vector fields on the indicatrix $\mathcal{I}_pM$ is called the *infinitesimal holonomy algebra at the point $p \in M$.*

Clearly, $\mathfrak{R}_p \subset \mathfrak{hol}^*(p)$ for any $p \in M$.

The following assertion is a direct consequence of the definition. It shows that the infinitesimal holonomy algebra at a point p of $(M,\mathcal{F})$ can be calculated in a neighbourhood of p.

Remark 4. *Let $(U,\mathcal{F}|_U)$ be an open submanifold of $(M,\mathcal{F})$ such that $U \subset M$ is diffeomorphic to $\mathbb{R}^n$ and let $p \in U$. The infinitesimal holonomy algebras at p of the Finsler manifolds $(M,\mathcal{F})$ and $(U,\mathcal{F}|_U)$ coincide.*

Now, we can prove the following

Theorem 3. *The infinitesimal holonomy algebra* $\mathfrak{hol}^*(p)$ *at any point* p *of the Finsler manifold* $(M, \mathcal{F})$ *is tangent to the holonomy group* $\mathsf{Hol}(p)$.

Proof. Let $U \subset M$ be an open submanifold of M, diffeomorphic to $\mathbb{R}^n$ and containing $p \in M$. According to the previous remark we have $\mathfrak{hol}^*(p) := \{\xi_p; \xi \in \mathfrak{hol}_{\mathfrak{f}}(U)\}$. Since the fibred holonomy algebra $\mathfrak{hol}_{\mathfrak{f}}(U)$ is tangent to the fibred holonomy group $\mathsf{Hol}_{\mathfrak{f}}(U)$ we obtain that $\mathfrak{hol}^*(p)$ is tangent to the holonomy group $\mathsf{Hol}(p)$. □

7 Holonomy algebra

Let $x(t)$, $0 \leq t \leq a$ be a smooth curve joining the points $q = x(0)$ and $p = x(a)$ in the Finsler manifold $(M, \mathcal{F})$. If $y(t) = \tau_t y(0) \in \mathcal{I}_{x(t)}M$ is a parallel vector field along $x(t)$, $0 \leq t \leq a$, where $\tau_t \colon \mathcal{I}_q M \to \mathcal{I}_{x(t)}M$ denotes the homogeneous (nonlinear) parallel translation, then we have $D_{\dot{x}}y(t) := \Big(\frac{dy^i(t)}{dt} + G^i_j(x(t), y(t))\dot{x}^j(t)\Big)\frac{\partial}{\partial x^i} = 0$. Considering a vector field ξ on the indicatrix $\mathcal{I}_q M$, the map $\tau_{a*}\xi \circ \tau_a^{-1} \colon (p, y) \mapsto \tau_{a*}\xi(y(a))$ gives a vector field on the indicatrix $\mathcal{I}_p M$. Hence we can formulate

Lemma 1. *For any vector field* $\xi \in \mathfrak{hol}^*(q) \subset \mathfrak{X}^\infty(\mathcal{I}_q M)$ *in the infinitesimal holonomy algebra at* q *the vector field* $\tau_{a*}\xi \circ \tau_a^{-1} \in \mathfrak{X}^\infty(\mathcal{I}_p M)$ *is tangent to the holonomy group* $\mathsf{Hol}(p)$.

Proof. Let $\{\phi_t \in \mathsf{Hol}(q)\}_{t \in (-\varepsilon, \varepsilon)}$ be a $\mathcal{C}^1$-differentiable 1-parameter family of diffeomorphisms of $\mathcal{I}_q M$ belonging to the holonomy group $\mathsf{Hol}(q)$ and satisfying the conditions $\phi_0 = \mathsf{Id}$, $\frac{\partial \phi_t}{\partial t}\big|_{t=0} = \xi$. Since the 1-parameter family

$$\tau_a \circ \phi_t \circ \tau_a^{-1} \in \mathsf{Diff}^\infty(\mathcal{I}_p M)\}_{t \in (-\varepsilon, \varepsilon)}$$

of diffeomorphisms consists of elements of the holonomy group $\mathsf{Hol}(p)$ and satisfies the conditions

$$\tau_a \circ \phi_0 \circ \tau_a^{-1} = \mathsf{Id}, \qquad \frac{\partial\big(\tau_a \circ \phi_t \circ \tau_a^{-1}\big)}{\partial t}\bigg|_{t=0} = \tau_{a*}\xi \circ \tau_a^{-1},$$

the assertion follows. □

Definition 10. A vector field $\mathbf{B}_\gamma \xi \in \mathfrak{X}^\infty(\mathcal{I}_p M)$ on the indicatrix $\mathcal{I}_p M$ will be called *the Berwald translate* of the vector field $\xi \in \mathfrak{X}^\infty(\mathcal{I}_q M)$ along the curve $\gamma = x(t)$ if

$$\mathbf{B}_\gamma \xi = \tau_{a*}\xi \circ (\tau_a)^{-1}.$$

Remark 5. *Let* $y(t) = \tau_t y(0) \in \mathcal{I}_{x(t)}M$ *be a parallel vector field along* $\gamma = x(t)$, $0 \leq t \leq a$, *started at* $y(0) \in \mathcal{I}_{x(0)}M$. *Then, the vertical vector field* $\xi_t = \xi(x(t), y(t))$ *along* $(x(t), y(t))$ *is the Berwald translate* $\xi_t = \tau_{t*}\xi_0 \circ {\tau_t}^{-1}$ *if and only if*

$$\nabla_{\dot{x}}\xi = \left(\frac{\partial \xi^i(x,y)}{\partial x^j} - G^k_j(x,y)\frac{\partial \xi^i(x,y)}{\partial y^k} + G^i_{jk}(x,y)\xi^k(x,y)\right)\dot{x}^j \frac{\partial}{\partial y^i} = 0.$$

Now, lemma 1 yields the following

Corollary 3. *If* $\xi \in \mathfrak{hol}^*(q)$ *then its Berwald translate* $\mathbf{B}_\gamma\xi \in \mathfrak{X}^\infty(\mathcal{I}_pM)$ *along any curve* $\gamma = x(t)$, $0 \le t \le a$, *joining* $q = x(0)$ *with* $p = x(a)$ *is tangent to the holonomy group* $\mathsf{Hol}(p)$.

This last statement motivates the following

Definition 11. The *holonomy algebra* $\mathfrak{hol}_p(M)$ of the Finsler manifold $(M, \mathcal{F})$ at the point $p \in M$ is defined by the smallest Lie algebra of vector fields on the indicatrix $\mathcal{I}_pM$, containing the Berwald translates of all infinitesimal holonomy algebras along arbitrary curves $x(t)$, $0 \le t \le a$ joining any points $q = x(0)$ with the point $p = x(a)$.

Clearly, the holonomy algebras at different points of the Finsler manifold $(M, \mathcal{F})$ are isomorphic. The previous lemma and corollary yield the following

Theorem 4. *The holonomy algebra* $\mathfrak{hol}_p(M)$ *at* $p \in M$ *of a Finsler manifold* $(M, \mathcal{F})$ *is tangent to the holonomy group* $\mathsf{Hol}(p)$.

Acknowledgement

The paper was supported by the Hungarian Scientific Research Fund (OTKA) grant K67617 and by grant no. MEB 041005 for Finsler structures and the Calculus of Variations.

References

[1] W. Barthel: Nichtlineare Zusammenhänge und deren Holonomiegruppen. *J. Reine Angew. Math.* 212 (1963) 120–149.

[2] M. Crampin, D.J. Saunders: *Holonomy of a class of bundles with fibre metrics.* arXiv: 1005.5478v1.

[3] L. Kozma: Holonomy structures in Finsler geometry, Part 5. In: P.L. Antonelli (ed.): *Handbook of Finsler Geometry.* Kluwer Academic Publishers, Dordrecht (2003) 445–490.

[4] A. Kriegl, P.W. Michor: *The Convenient Setting for Global Analysis.* AMS, Providence, Surveys and Monographs 53 (1997).

[5] P.W. Michor: *Gauge Theory for Fiber Bundles.* Monographs and Textbooks in Physical Sciences, Lecture Notes 19, Bibliopolis, Napoli (1991).

[6] Z. Muzsnay, P.T. Nagy: Finsler manifolds with non-Riemannian holonomy. *Houston J. Math.*

[7] Z. Shen: *Differential Geometry of Spray and Finsler Spaces.* Kluwer Academic Publishers, Dordrecht (2001).

[8] Z.I. Szabó: Positive definite Berwald spaces. *Tensor*, New Ser., 35 (1981) 25–39.

Authors' address:
Institute of Mathematics, University of Debrecen, H-4010 Debrecen, P.O.B. 12, Hungary

E-mail: `muzsnay@science.unideb.hu`

E-mail: `petert.nagy@science.unideb.hu`

Received: 19 September, 2011
Accepted for publication: 29 October, 2011

Communications in Mathematics 19 (2011) 149–168

Conformal vector fields on Finsler manifolds

József Szilasi, Anna Tóth

Abstract. Applying concepts and tools from classical tangent bundle geometry and using the apparatus of the calculus along the tangent bundle projection ('pull-back formalism'), first we enrich the known lists of the characterizations of affine vector fields on a spray manifold and conformal vector fields on a Finsler manifold. Second, we deduce consequences on vector fields on the underlying manifold of a Finsler structure having one or two of the mentioned geometric properties.

Introduction

The theory of 'geometrical' – projective, affine, conformal, isometric – vector fields on a Finsler manifold has a vast literature, mainly from the period dominated technically by the classical tensor calculus, visually, 'the debauch of indices'. Chapter VIII of K. Yano's book 'The theory of Lie derivatives and its applications' presents a survey of the main achievements from the beginning of the 20th century to 1957. A good overview of the developments of the next decades can be found in R. B. Misra's paper [15], written in 1981, revised and updated in 1993. It is important to note that in a 2-part paper, see [13], [14], M. Matsumoto clarified and improved some results of Yano in the framework of his theory of Finsler connections.

From the (relatively) modern, but partly tensor calculus based literature the works of H. Akbar-Zadeh [2], [3], J. Grifone [9], [10] and R. L. Lovas [12] are worth mentioning. Grifone applies systematically the '$\tau_{TM}\colon TTM \to TM$ formalism', combining with the Frölicher-Nijenhuis calculus of vector-valued forms; Lovas formulates and proves his results in terms of the 'pull-back formalism

$$\overset{\circ}{\pi}\colon \overset{\circ}{T}M \times_M TM \to \overset{\circ}{T}M\text{'}.$$

Our paper is a continuation of both Grifone's and Lovas's works. Although we are going to develop the greater part of the theory in terms of the pull-back bundle,

2010 MSC: 53C60, 53A30
Key words: spray manifold, Finsler manifold, projective vector field, affine vector field, conformal vector field

the concepts and techniques of the tangent bundle geometry, including the vertical calculus on TM, also play an eminent role in our considerations. To make the paper more readable, in section 1 we summarize in a coherent way the various concepts and tools which will be indispensable in the following.

We apply two types of a Lie derivative operator: besides the classical Lie derivative operator $\mathcal{L}_\xi$ on TM ($\xi \in \mathfrak{X}(TM)$) we need a further operator, denoted by $\widetilde{\mathcal{L}}_\xi$, which acts on the tensor algebra of the $C^\infty(TM)$-module of the sections of the vector bundle $\pi\colon TM \times_M TM \to TM$ (or of the bundle $\overset{\circ}{\pi}$). To assure the validity of the crucial identity $[\widetilde{\mathcal{L}}_\xi, \widetilde{\mathcal{L}}_\eta] = \widetilde{\mathcal{L}}_{[\xi,\eta]}$ in case of the 'new' operator, we are forced to differentiate with respect to *projectable* vector fields on TM. In section 2 some basic properties of the operator $\widetilde{\mathcal{L}}_\xi$ are established.

The affine and projective properties of a Finsler manifold depend only on its canonical spray, so it is natural to examine affine and projective vector fields in the (virtual) generality of spray manifolds. A vector field X on a manifold M is said to be an affine vector field or a Lie symmetry for a spray $S\colon TM \to TTM$ if S is invariant under the flow of the complete lift X^{c} of X, that is, if $\mathcal{L}_{X^{\mathsf{c}}}S = [X^{\mathsf{c}}, S] = 0$. In Lovas's paper [12] various equivalents of this property are established. In section 3 we enrich his list with some new items, which will be technically useful in the next section.

By a conformal vector field on a Finsler manifold (M, F) we mean a vector field X on M satisfying

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g = \varphi g,$$

where g is the metric tensor of the Finsler manifold (the vertical Hessian of the energy function $E = \frac{1}{2}F^2$) and φ is a function, defined and continuous on TM, smooth on the deleted bundle $\overset{\circ}{T}M$. It turns out at once that φ has to be fibrewise constant, i.e., of the form $\varphi = f \circ \tau$, where f is a smooth function on M and τ is the tangent bundle projection. Homothetic and isometric (or Killing) vector fields are the particular cases for which φ is a constant function, resp. identically zero. In section 4 we present further characterizations of conformal vector fields on a Finsler manifold (Proposition 2), one of them has already been proposed by Grifone in [10]. We show that if a vector field $X \in \mathfrak{X}(M)$ is both affine and conformal on a Finsler manifold (M, F), then X^{c} is a conformal vector field for the Sasaki extension of the metric tensor of (M, F) (Proposition 3).

At this stage, the following 'expectable', but non-trivial conclusions may be deduced fairly easily:

(a) Homothetic vector fields on a Finsler manifold are affine vector fields (Proposition 4).

(b) If a vector field on a Finsler manifold is both projective and conformal, then it is a homothetic vector field (Proposition 5).

(c) If a vector field preserves the Dazord volume form of a Finsler manifold and it is also projective, then it is an affine vector field (Proposition 6, (i)).

(d) If a vector field is both volume-preserving (in the above sense) and conformal, then it is a Killing field (Proposition 6, (ii)).

1 Basic setup

1.1 Generalities

Most of our basic notations and conventions will be the same as in [4], see also [16]. However, for the reader's convenience, we present here a short review on the most essential things.

(**a**) By a manifold we mean a finite dimensional smooth manifold whose underlying topological space is Hausdorff, second countable and connected. In what follows, M will be an n-dimensional manifold, where $n \geq 2$. Let $k \in \mathbb{N} \cup \{\infty\}$. We denote by $C^k(M)$ the set of k-times continuously differentiable real-valued functions on M, with the convention that $C^0(M)$ is the set of continuous functions on M. In particular, $C^\infty(M)$ is the real algebra of smooth functions on M.

(**b**) The tangent space of M at a point $p \in M$ is denoted by T_pM;

$$TM := \bigcup_{p \in M} T_pM\,.$$

The tangent bundle of M is the triplet (TM, τ, M), where the tangent bundle projection τ is defined by $\tau(v) := p$ if $v \in T_pM$. Instead of (TM, τ, M) we usually write $\tau\colon TM \to M$ or simply τ. Similarly, the tangent bundle of TM is (TTM, τ_{TM}, TM) or $\tau_{TM}\colon TTM \to TM$ or τ_{TM}. In general, we prefer to denote a bundle by the same symbol as we use for its projection.

A *vector field* on M is a smooth section of the tangent bundle $\tau\colon TM \to M$. The vector fields on M form a $C^\infty(M)$-module which will be denoted by $\mathfrak{X}(M)$. The *zero vector field* o on M is defined by

$$p \in M \mapsto o(p) := 0_p := \text{ the zero vector in } T_pM.$$

The *deleted bundle* for τ is the fibre bundle $\overset{\circ}{\tau}\colon \overset{\circ}{T}M \to M$, where $\overset{\circ}{T}M := TM \backslash o(M)$, $\overset{\circ}{\tau} := \tau \upharpoonright \overset{\circ}{T}M$.

(**c**) If $\varphi\colon M \to N$ is a smooth mapping between smooth manifolds, then we denote its derivative by φ_*, which is a fibrewise linear smooth mapping of TM into TN. Two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are *φ-related* if $\varphi_* \circ X = Y \circ \varphi$; then we write $X \underset{\varphi}{\sim} Y$. A vector field ξ on TM is said to be *projectable* if there exists a vector field X on M such that $\xi \underset{\tau}{\sim} X$.

(**d**) The classical graded derivations of the graded algebra $\Omega(M) := \bigoplus_{k=0}^n \Omega^k(M)$ of differential forms on M are

the Lie derivative $\mathcal{L}_X$ $(X \in \mathfrak{X}(M))$,
the substitution operator i_X $(X \in \mathfrak{X}(M))$,
the exterior derivative d,

related by H. Cartan's 'magic' formula

$$\mathcal{L}_X = i_X \circ d + d \circ i_X. \tag{1}$$

1.2 Canonical constructions and objects

(**a**) By the *vertical lift* of a smooth function f on M we mean the function

$$f^{\mathsf{v}} := f \circ \tau \in C^\infty(TM);$$

the *complete lift* of f is the function $f^{\mathsf{c}} \in C^\infty(TM)$ given by

$$f^{\mathsf{c}}(v) := v(f), v \in TM.$$

(**b**) A vector field ξ on TM is *vertical* if $\xi \underset{\tau}{\sim} o$. The vertical vector fields form a $C^\infty(TM)$-module $\mathfrak{X}^{\mathsf{v}}(TM)$, which is also a subalgebra of the Lie algebra $\mathfrak{X}(TM)$. The *Liouville vector field* on TM is the unique vertical vector field $C \in \mathfrak{X}^{\mathsf{v}}(TM)$ such that

$$Cf^{\mathsf{c}} = f^{\mathsf{c}} \textit{ for all } f \in C^\infty(M). \tag{2}$$

The *vertical lift of a vector field* X on M is the unique vertical vector field $X^{\mathsf{v}} \in \mathfrak{X}^{\mathsf{v}}(TM)$ satisfying

$$X^{\mathsf{v}} f^{\mathsf{c}} = (Xf)^{\mathsf{v}} \textit{ for all } f \in C^\infty(M); \tag{3}$$

the *complete lift* $X^{\mathsf{c}} \in \mathfrak{X}(TM)$ of X is characterized by

$$X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}}, \quad f \in C^\infty(M) \tag{4}$$

(see [19], Ch. I.3). Then we have

$$X^{\mathsf{c}} f^{\mathsf{v}} = (Xf)^{\mathsf{v}}, \quad f \in C^\infty(M). \tag{5}$$

Both X^{v} and X^{c} are projectable: $X^{\mathsf{v}} \underset{\tau}{\sim} o$, $X^{\mathsf{c}} \underset{\tau}{\sim} X$. Lie brackets involving vertical and complete lifts satisfy the rules

$$[X^{\mathsf{v}}, Y^{\mathsf{v}}] = 0, \quad [X^{\mathsf{c}}, Y^{\mathsf{v}}] = [X, Y]^{\mathsf{v}}, \quad [X^{\mathsf{c}}, Y^{\mathsf{c}}] = [X, Y]^{\mathsf{c}}, \tag{6a–c}$$

$$[C, X^{\mathsf{v}}] = -X^{\mathsf{v}}, \quad [C, X^{\mathsf{c}}] = 0. \tag{7a–b}$$

(**c**) Let

$$TM \times_M TM := \big\{(u, v) \in TM \times TM \mid \tau(u) = \tau(v)\big\},$$
$$\overset{\circ}{T}M \times_M TM := \big\{(u, v) \in \overset{\circ}{T}M \times TM \mid \overset{\circ}{\tau}(u) = \tau(v)\big\}.$$

If

$$\pi := \mathrm{pr}_1 \restriction TM \times_M TM, \quad \overset{\circ}{\pi} := \mathrm{pr}_1 \restriction \overset{\circ}{T}M \times_M TM,$$

then both π and $\overset{\circ}{\pi}$ are vector bundles over TM and $\overset{\circ}{T}M$, resp., with fibres

$$\{u\} \times T_{\tau(u)}M \cong T_{\tau(u)}M; \quad u \in TM, \text{ resp. } u \in \overset{\circ}{T}M.$$

We denote by $\mathrm{Sec}(\pi)$ and $\mathrm{Sec}(\overset{\circ}{\pi})$ the $C^\infty(TM)$-, resp. $C^\infty(\overset{\circ}{T}M)$-module of the sections of these bundles. A typical section in $\mathrm{Sec}(\pi)$ is of the form

$$\widetilde{X} \colon v \in TM \longmapsto (v, \underline{X}(v)) \in TM \times_M TM,$$

where $\underline{X}\colon TM \to TM$ is a smooth mapping such that $\tau \circ \underline{X} = \tau$. $\underline{X}$ is called the *principal part* of $\widetilde{X}$. We have a *canonical section* in $\mathrm{Sec}(\pi)$, denoted by δ, whose principal part is the identity mapping of TM. Every vector field X on M yields a section $\widehat{X}$ in $\mathrm{Sec}(\pi)$, called a *basic section*, whose principal part is $X \circ \tau$. Locally, the $C^\infty(TM)$-module $\mathrm{Sec}(\pi)$ is generated by the basic sections.

We denote by $\mathcal{T}^k_l(\pi)$ the $C^\infty(TM)$-module of the type (k,l) tensors over the module $\mathrm{Sec}(\pi)$; the meaning of $\mathcal{T}^k_l(\overset{\circ}{\pi})$ is analogous.

(**d**) We have a canonical $C^\infty(TM)$-linear injection $\mathbf{i}\colon \mathrm{Sec}(\pi) \to \mathfrak{X}(TM)$ given on the basic sections by

$$\mathbf{i}(\widehat{X}) := X^{\mathsf{v}}, \quad X \in \mathfrak{X}(M), \tag{8}$$

and a canonical $C^\infty(TM)$-linear surjection $\mathbf{j}\colon \mathfrak{X}(TM) \to \mathrm{Sec}(\pi)$ such that

$$\mathbf{j}(X^{\mathsf{v}}) := 0, \quad \mathbf{j}(X^{\mathsf{c}}) := \widehat{X}. \tag{9}$$

Then $\mathrm{Im}(\mathbf{i}) = \mathrm{Ker}(\mathbf{j}) = \mathfrak{X}^{\mathsf{v}}(TM)$. The mapping $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ is said to be the *vertical endomorphism* of $\mathfrak{X}(TM)$. It follows immediately that

$$\mathrm{Im}(\mathbf{J}) = \mathrm{Ker}(\mathbf{J}) = \mathfrak{X}^{\mathsf{v}}(TM), \mathbf{J}^2 = 0.$$

Due to their $C^\infty(TM)$-linearity, $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{J}$ have a natural pointwise interpretation.

1.3 Some vertical calculus

(**a**) We define the *vertical differential* $\nabla^{\mathsf{v}} F$ of a function $F \in C^\infty(TM)$ as a 1-form in $\mathcal{T}^0_1(\pi)$ given by

$$\nabla^{\mathsf{v}} F(\widetilde{X}) := \nabla^{\mathsf{v}}_{\widetilde{X}} F := (\mathbf{i}\widetilde{X})F, \widetilde{X} \in \mathrm{Sec}(\pi). \tag{10}$$

The vertical differential $\nabla^{\mathsf{v}}\widetilde{Y}$ of a section $\widetilde{Y} \in \mathrm{Sec}(\pi)$ is the type $(1,1)$ tensor in $\mathcal{T}^1_1(\pi)$ defined by

$$\begin{cases} \nabla^{\mathsf{v}}\widetilde{Y}(\widetilde{X}) := \nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y} := \mathbf{j}[\mathbf{i}\widetilde{X}, \eta], \\ \eta \in \mathfrak{X}(TM), \quad \mathbf{j}(\eta) = \widetilde{Y}. \end{cases} \tag{11}$$

(It is easy to check that $\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}$ does not depend on the choice of η satisfying $\mathbf{j}(\eta) = \widetilde{Y}$.)

By the standard technique, to make sure that Leibniz's rule holds, the operators $\nabla^{\mathsf{v}}_{\widetilde{X}}$ may be extended to tensor derivations of the full tensor algebra of $\mathrm{Sec}(\pi)$.

(**b**) Next we consider the graded algebra $\Omega(TM)$ of differential forms on TM, and we define an operator

$$d_{\mathbf{J}}\colon \Omega(TM) \longrightarrow \Omega(TM)$$

by the rules

$$d_{\mathbf{J}}F := dF \circ \mathbf{J}, \quad d_{\mathbf{J}}dF := -dd_{\mathbf{J}}F; \quad F \in C^\infty(TM). \tag{12}$$

Then $d_{\mathbf{J}}$ is a graded derivation of degree 1 of $\Omega(TM)$, called the *vertical differentiation on TM*. We have (and we shall need) the following important relation:

$$d_{\mathbf{J}} \circ \mathcal{L}_C - \mathcal{L}_C \circ d_{\mathbf{J}} = d_{\mathbf{J}}. \tag{13}$$

For details, we refer to the book [6]. We mention that ∇^{v} and $d_{\mathbf{J}}$, at the level of functions, are related by

$$d_{\mathbf{J}}F = \nabla^{\mathsf{v}}F \circ \mathbf{j}, \quad F \in C^\infty(TM).$$

(**c**) Let K be a type $(1,1)$ tensor on TM, interpreted as an endomorphism of the $C^\infty(TM)$-module $\mathfrak{X}(TM)$. It will be convenient to denote the Lie derivative $-\mathcal{L}_\eta K$ ($\eta \in \mathfrak{X}(TM)$) by $[K,\eta]$. Then, for any vector field ξ on TM,

$$[K,\eta]\xi = [K\xi,\eta] - K[\xi,\eta].$$

We have, in particular,

$$[\mathbf{J},C] = \mathbf{J}; \quad [\mathbf{J},X^{\mathsf{v}}] = 0, \quad [\mathbf{J},X^{\mathsf{c}}] = 0 \quad (X \in \mathfrak{X}(M)). \tag{14a–c}$$

In what follows, for simplicity, we shall denote also by $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{J}$ the restrictions of these mappings to $\mathrm{Sec}(\overset{\circ}{\pi})$ and $\mathfrak{X}(\overset{\circ}{T}M)$.

1.4 Ehresmann connections

(**a**) By an *Ehresmann connection* in $\overset{\circ}{T}M$ we mean a $C^\infty(\overset{\circ}{T}M)$-linear mapping

$$\mathcal{H}\colon \mathrm{Sec}(\overset{\circ}{\pi}) \longrightarrow \mathfrak{X}(\overset{\circ}{T}M)$$

such that

$$\mathbf{j} \circ \mathcal{H} = 1_{\mathrm{Sec}(\overset{\circ}{\pi})}.$$

We emphasize (cf. 1.2(d)) that the $C^\infty(\overset{\circ}{T}M)$-linearity of $\mathcal{H}$ makes it possible to interpret an Ehresmann connection as a strong bundle map.

(**b**) Let $\mathcal{H}\colon \mathrm{Sec}(\overset{\circ}{\pi}) \to \mathfrak{X}(\overset{\circ}{T}M)$ be an Ehresmann connection in $\overset{\circ}{T}M$. Then $\mathfrak{X}^{\mathsf{h}}(\overset{\circ}{T}M) := \mathrm{Im}(\mathcal{H})$ is a submodule of $\mathfrak{X}(\overset{\circ}{T}M)$, and we have the direct decomposition $\mathfrak{X}(\overset{\circ}{T}M) = \mathfrak{X}^{\mathsf{v}}(\overset{\circ}{T}M) \oplus \mathfrak{X}^{\mathsf{h}}(\overset{\circ}{T}M)$. Vector fields on $\overset{\circ}{T}M$ belonging to $\mathfrak{X}^{\mathsf{h}}(\overset{\circ}{T}M)$ are called *horizontal*. Notice that they do not form, in general, a subalgebra of the Lie algebra $\mathfrak{X}(\overset{\circ}{T}M)$. The mappings

$$\mathbf{h} := \mathcal{H} \circ \mathbf{j}, \quad \mathbf{v} := 1_{\mathfrak{X}(\overset{\circ}{T}M)} - \mathbf{h},$$

$$\mathcal{V} := \mathbf{i}^{-1} \circ \mathbf{v}\colon \mathfrak{X}(\overset{\circ}{T}M) \longrightarrow \mathrm{Sec}(\overset{\circ}{\pi})$$

are called the *horizontal projection*, the *vertical projection* and the *vertical mapping* associated to $\mathcal{H}$, respectively. $\mathbf{h}$ and $\mathbf{v}$ are indeed projection operators in $\mathfrak{X}(\overset{\circ}{T}M)$, while the mapping $\mathcal{V}$ has the properties

$$\mathcal{V} \circ \mathbf{i} = 1_{\mathrm{Sec}(\overset{\circ}{\pi})}, \quad \mathrm{Ker}(\mathcal{V}) = \mathrm{Im}(\mathcal{H}).$$

The *horizontal lift* of a vector field X on M (with respect to $\mathcal{H}$) is

$$X^{\mathsf{h}} := \mathcal{H}(\widehat{X}) = \mathbf{h}(X^{\mathsf{c}}).$$

($\widehat{X}$ and X^{c} are regarded here as a section in $\mathrm{Sec}(\overset{\circ}{\pi})$ and a vector field on $\overset{\circ}{T}M$, resp.; for simplicity, we make no notational distinction.)

(**c**) An Ehresmann connection $\mathcal{H}$ is said to be *homogeneous* if

$$[C, X^{\mathsf{h}}] = 0 \textit{ for all } X \in \mathfrak{X}(M).$$

Then $\mathcal{H}$, as a strong bundle map of $\overset{\circ}{T}M \times_M TM$ to $T\overset{\circ}{T}M$, may be extended continuously to a mapping $TM \times_M TM \to TTM$ such that

$$\mathcal{H}(0_p, v) = (o_*)_p(v) \textit{ for all } p \in M, v \in T_pM.$$

Thus, in what follows, we shall always assume that a homogeneous Ehresmann connection is defined on the entire $TM \times_M TM$ (or on $\mathrm{Sec}(\pi)$).

(**d**) If $\mathcal{H}$ is an Ehresmann connection in $\overset{\circ}{T}M$, then the mapping

$$\nabla \colon \mathfrak{X}(\overset{\circ}{T}M) \times \mathrm{Sec}(\overset{\circ}{\pi}) \longrightarrow \mathrm{Sec}(\overset{\circ}{\pi}), \quad (\xi, \widetilde{Y}) \longmapsto \nabla_\xi \widetilde{Y}$$

given by

$$\nabla_{\mathbf{v}\xi}\widetilde{Y} := \nabla^{\mathsf{v}}_{\mathcal{V}\xi}\widetilde{Y} \overset{(11)}{=} \mathbf{j}[\mathbf{v}\xi, \mathcal{H}\widetilde{Y}] \tag{15a}$$

$$\nabla_{\mathbf{h}\xi}\widetilde{Y} := \nabla^{\mathsf{h}}_{\mathbf{j}\xi}\widetilde{Y} := \mathcal{V}[\mathbf{h}\xi, \mathbf{i}\widetilde{Y}] \tag{15b}$$

is a covariant derivative operator in the vector bundle $\overset{\circ}{\pi}$, called the *Berwald derivative* induced by $\mathcal{H}$.

By the *tension* of $\mathcal{H}$ we mean the ∇^{h}-differential $\mathbf{t} := \nabla^{\mathsf{h}}\delta$ of the canonical section. Then, for any section $\widetilde{X} \in \mathrm{Sec}(\overset{\circ}{\pi})$,

$$\mathbf{t}(\widetilde{X}) := (\nabla^{\mathsf{h}}\delta)(\widetilde{X}) := \nabla^{\mathsf{h}}_{\widetilde{X}}\delta = \mathcal{V}[\mathcal{H}\widetilde{X}, C]. \tag{16}$$

In particular,

$$\mathbf{it}(\widehat{X}) = [X^{\mathsf{h}}, C], \quad X \in \mathfrak{X}(M);$$

therefore $\mathcal{H}$ *is homogeneous if, and only if, its tension vanishes.*

With the help of the induced Berwald derivative we define the *torsion* $\mathbf{T}$ of an Ehresmann connection $\mathcal{H}$ by

$$\mathbf{T}(\widetilde{X}, \widetilde{Y}) := \nabla_{\mathcal{H}\widetilde{X}}\widetilde{Y} - \nabla_{\mathcal{H}\widetilde{Y}}\widetilde{X} - \mathbf{j}[\mathcal{H}\widetilde{X}, \mathcal{H}\widetilde{Y}]; \quad \widetilde{X}, \widetilde{Y} \in \mathrm{Sec}(\overset{\circ}{\pi}).$$

Evaluating on basic sections, we obtain the more expressive formula

$$\mathbf{iT}(\widehat{X}, \widehat{Y}) = [X^{\mathsf{h}}, Y^{\mathsf{v}}] - [Y^{\mathsf{h}}, X^{\mathsf{v}}] - [X, Y]^{\mathsf{v}}; \quad X, Y \in \mathfrak{X}(M).$$

2 Lie derivative along the tangent bundle projection

Let ξ be a projectable vector field on TM (1.1(c)). We define a Lie derivative operator $\widetilde{\mathcal{L}}_\xi$ on the tensor algebra of the $C^\infty(TM)$-module $\mathrm{Sec}(\pi)$ by the rules

$$\widetilde{\mathcal{L}}_\xi \varphi := \xi\varphi, \text{ if } \varphi \in C^\infty(TM); \tag{17a}$$

$$\widetilde{\mathcal{L}}_\xi \widetilde{Y} := \mathbf{i}^{-1}[\xi, \mathbf{i}\widetilde{Y}], \text{ if } \widetilde{Y} \in \mathrm{Sec}(\pi), \tag{17b}$$

and by extending it to the whole tensor algebra in such a way that $\widetilde{\mathcal{L}}_\xi$ satisfies the product rule of tensor derivations. Since ξ is a projectable and $\mathbf{i}\widetilde{Y}$ is a vertical vector field, it follows that the vector field $[\xi, \mathbf{i}\widetilde{Y}]$ is vertical, so $\widetilde{\mathcal{L}}_\xi \widetilde{Y}$ is well-defined. If $\mathbf{v} = \mathbf{i} \circ \mathcal{V}$ is the vertical projection associated to an Ehresmann connection $\mathcal{H}$ in TM, then $\mathbf{i}^{-1}[\xi, \mathbf{i}\widetilde{Y}] = \mathcal{V}[\xi, \mathbf{i}\widetilde{Y}]$, so we get the useful formula

$$\widetilde{\mathcal{L}}_\xi \widetilde{Y} = \mathcal{V}[\xi, \mathbf{i}\widetilde{Y}]. \tag{18}$$

Notice, however, that the Lie derivative operator $\widetilde{\mathcal{L}}_\xi$ does not depend on any Ehresmann connection in TM.

If, in particular, $\xi := X^{\mathsf{c}}$ or $\xi := X^{\mathsf{h}}$, where X is a vector field on M, then (18) takes the form

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widetilde{Y} = \mathcal{V}[X^{\mathsf{c}}, \mathbf{i}\widetilde{Y}], \tag{19}$$

resp.

$$\widetilde{\mathcal{L}}_{X^{\mathsf{h}}} \widetilde{Y} = \mathcal{V}[X^{\mathsf{h}}, \mathbf{i}\widetilde{Y}] \overset{(15b)}{=} \nabla^{\mathsf{h}}_{\widehat{X}} \widetilde{Y}. \tag{20}$$

Since $[X^{\mathsf{c}}, \mathbf{i}\delta] = [X^{\mathsf{c}}, C] \overset{(7b)}{=} 0$, it follows that

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \delta = 0. \tag{21}$$

The Lie derivative of a basic section with respect to a complete lift leads essentially to the ordinary Lie derivative. Namely, for any vector fields X, Y on M we have

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \widehat{Y} \overset{(19)}{=} \mathcal{V}[X^{\mathsf{c}}, Y^{\mathsf{v}}] \overset{(6b)}{=} \mathcal{V}[X, Y]^{\mathsf{v}} = \mathcal{V} \circ \mathbf{i}\widehat{[X, Y]} = \widehat{[X, Y]} = \widehat{\mathcal{L}_X Y}.$$

This relation indicates that our Lie derivative operator $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}$ is a natural extension of the classical Lie derivative $\mathcal{L}_X$ on M.

Lemma 1. *For any projectable vector fields ξ, η on TM,*

$$[\widetilde{\mathcal{L}}_\xi, \widetilde{\mathcal{L}}_\eta] = \widetilde{\mathcal{L}}_{[\xi, \eta]}. \tag{22}$$

Proof. Obviously, both sides of (22) act in the same way on smooth functions on TM. If $\widetilde{Y}$ is a section of π, then, applying (18) repeatedly,

$$\begin{aligned}
[\widetilde{\mathcal{L}}_\xi, \widetilde{\mathcal{L}}_\eta]\widetilde{Y} &= \widetilde{\mathcal{L}}_\xi \mathcal{V}[\eta, \mathbf{i}\widetilde{Y}] - \widetilde{\mathcal{L}}_\eta \mathcal{V}[\xi, \mathbf{i}\widetilde{Y}] = \mathcal{V}([\xi, \mathbf{i}\mathcal{V}[\eta, \mathbf{i}\widetilde{Y}]] - [\eta, \mathbf{i}\mathcal{V}[\xi, \mathbf{i}\widetilde{Y}]]) \\
&= \mathcal{V}([\xi, [\eta, \mathbf{i}\widetilde{Y}]] + [\eta, [\mathbf{i}\widetilde{Y}, \xi]]) = -\mathcal{V}[\mathbf{i}\widetilde{Y}, [\xi, \eta]] = \mathcal{V}[[\xi, \eta], \mathbf{i}\widetilde{Y}] \\
&= \widetilde{\mathcal{L}}_{[\xi, \eta]} \widetilde{Y}.
\end{aligned}$$

□

Lemma 2. *Let $X \in \mathfrak{X}(M)$, $\eta \in \mathfrak{X}(TM)$. Then*

$$\widetilde{\mathcal{L}}_{X^c}\mathbf{j}\eta = \mathbf{j}\mathcal{L}_{X^c}\eta. \tag{23}$$

Proof. Since

$$0 \overset{(14c)}{=} [\mathbf{J}, X^c]\eta = [\mathbf{J}\eta, X^c] - \mathbf{J}[\eta, X^c],$$

we find

$$\mathbf{i}\widetilde{\mathcal{L}}_{X^c}\mathbf{j}\eta = [X^c, \mathbf{J}\eta] = \mathbf{J}[X^c, \eta] = \mathbf{i}(\mathbf{j}\mathcal{L}_{X^c}\eta),$$

which implies (23). □

We end this section with the definition of the Lie derivative $\widetilde{\mathcal{L}}_\xi D$ of a covariant derivative $D\colon \mathfrak{X}(TM) \times \mathrm{Sec}(\pi) \to \mathrm{Sec}(\pi)$: it is given by the rule

$$(\widetilde{\mathcal{L}}_\xi D)(\eta, \widetilde{Z}) := \widetilde{\mathcal{L}}_\xi(D_\eta \widetilde{Z}) - D_\eta(\widetilde{\mathcal{L}}_\xi \widetilde{Z}) - D_{[\xi,\eta]}\widetilde{Z},$$

where $\eta \in \mathfrak{X}(TM)$, $\widetilde{Z} \in \mathrm{Sec}(\pi)$.

Notice finally that the theory of Lie derivatives 'along the tangent bundle projection' sketched here works without any change also on the bundle

$$\mathring{\pi}\colon \mathring{T}M \times_M TM \to \mathring{T}M\,.$$

3 Affine vector fields on a spray manifold

3.1

By a *spray* for M we mean a C^1 mapping $S\colon TM \to TTM$, smooth on $\mathring{T}M$, such that

$$\tau_{TM} \circ S = 1_{TM}; \tag{24}$$

$$\mathbf{J}S = C; \tag{25}$$

$$[C, S] = S. \tag{26}$$

Condition (25) is equivalent to the requirement $\tau_* \circ S = 1_{TM}$, so a spray for M is a section also of the secondary vector bundle $\tau_*\colon TTM \to TM$. In view of (26), a spray is a *homogeneous* vector field (of class C^1) *of degree 2*. We say that a manifold endowed with a spray is a *spray manifold*.

3.2

If $\mathcal{H}$ is a homogeneous Ehresmann connection in TM, then $S := \mathcal{H} \circ \delta$ is a spray for M, called the *spray associated to* $\mathcal{H}$. Indeed, for any vector w in TM, $S(w) = \mathcal{H}(w, w) \in T_wTM$, therefore $\tau_{TM}(S(w)) = w$, so (24) is valid. Since

$$\mathbf{J} \circ S = \mathbf{i} \circ \mathbf{j} \circ \mathcal{H} \circ \delta = \mathbf{i} \circ \delta = C,$$

condition (25) also holds. To check (26), observe first that the vector field $[C, S] - S$ is vertical, and hence $\mathbf{h}[C, S] = \mathbf{h}S$. However, $\mathbf{h}S = \mathcal{H} \circ \mathbf{j} \circ \mathcal{H} \circ \delta = \mathcal{H} \circ \delta =: S$, so we get $\mathbf{h}[C, S] = S$. On the other hand, by the homogeneity of $\mathcal{H}$,

$$0 = -\mathbf{it}(\delta) = -\mathbf{v}[\mathcal{H} \circ \delta, C] = \mathbf{v}[C, S],$$

therefore $\mathbf{h}[C,S]=[C,S]$ and $[C,S]=S$. Finally, the C^1 differentiability of S can be shown using the 'Observation' in 3.11 (p. 1378) of [16].

Thus sprays exist in abundance for a manifold. Conversely, if S is a spray for M, then there exists a unique torsion-free homogeneous Ehresmann connection $\mathcal{H}$ in TM such that the horizontal lifts with respect to $\mathcal{H}$ are given by

$$X^{\mathsf{h}} := \mathcal{H}(\widehat{X}) = \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}}, S]), \quad X \in \mathfrak{X}(M). \tag{27}$$

For a proof of this fundamental fact we refer to [16], 3.3, or to the original source [5]. The Ehresmann connection specified by (27) is said to be the *Ehresmann connection induced by the spray S*.

3.3

Let (M,S) be a spray manifold. We say that a vector field X on M is a *projective vector field* for (M,S) (or for the spray S) if there is a continuous function φ on TM, smooth on $\overset{\circ}{T}M$, such that

$$[X^{\mathsf{c}}, S] = \varphi C. \tag{28}$$

If, in particular, φ is the zero function, then we say that X is an *affine vector field* for (M,S), or a *Lie symmetry* of S.

Proposition 1. *Suppose (M,S) is a spray manifold. Let $\mathcal{H}$ be the Ehresmann connection induced by S, and let ∇ be the Berwald derivative arising from $\mathcal{H}$. For a vector field X on M, the following conditions are equivalent:*

(i) X *is a Lie symmetry of* S*;*

(ii) $[\mathbf{h}, X^{\mathsf{c}}] = 0$*;*

(iii) $[\mathbf{v}, X^{\mathsf{c}}] = 0$*;*

(iv) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\nabla = 0$*;*

(v) $[X^{\mathsf{c}}, Y^{\mathsf{h}}] = [X,Y]^{\mathsf{h}}$, *for any vector field* Y *on* M*;*

(vi) $[\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}, \widetilde{\mathcal{L}}_{Y^{\mathsf{h}}}] = \widetilde{\mathcal{L}}_{[X,Y]^{\mathsf{h}}}$, $Y \in \mathfrak{X}(M)$*;*

(vii) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} \circ \mathcal{V} = \mathcal{V} \circ \mathcal{L}_{X^{\mathsf{c}}}$.

Proof. The equivalence of conditions *(i)*, *(ii)* and *(iv)* has already been proved in [12].
(ii) $\Longleftrightarrow$ *(iii)* This is evident, since $\mathbf{v} = \mathbf{1} - \mathbf{h}$ $(\mathbf{1} := 1_{\mathfrak{X}(TM)})$ and $[\mathbf{1}, \xi] = 0$ for all $\xi \in \mathfrak{X}(TM)$.
(ii) $\Longleftrightarrow$ *(v)* For any vector field Y on M,

$$[\mathbf{h}, X^{\mathsf{c}}]Y^{\mathsf{c}} = [\mathbf{h}Y^{\mathsf{c}}, X^{\mathsf{c}}] - \mathbf{h}[Y^{\mathsf{c}}, X^{\mathsf{c}}] = [Y^{\mathsf{h}}, X^{\mathsf{c}}] - \mathbf{h}[Y,X]^{\mathsf{c}} = [Y^{\mathsf{h}}, X^{\mathsf{c}}] - [Y,X]^{\mathsf{h}},$$

so the vanishing of $[\mathbf{h}, X^{\mathsf{c}}]$ implies that $[X^{\mathsf{c}}, Y^{\mathsf{h}}] = [X, Y]^{\mathsf{h}}$. The converse is also true, since $[\mathbf{h}, X^{\mathsf{c}}]$ annihilates the module of vector fields: for any vector field ξ on TM we have

$$[\mathbf{h}, X^{\mathsf{c}}]\mathbf{J}\xi = [\mathbf{h} \circ \mathbf{J}(\xi), X^{\mathsf{c}}] - \mathbf{h}[\mathbf{J}\xi, X^{\mathsf{c}}] = 0.$$

(v) $\iff$ *(vi)* This is an immediate consequence of the identity

$$[\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}, \widetilde{\mathcal{L}}_{Y^{\mathsf{h}}}] = \widetilde{\mathcal{L}}_{[X^{\mathsf{c}}, Y^{\mathsf{h}}]}$$

(see Lemma 1).
(iii) $\iff$ *(vii)* For any vector field ξ on TM,

$$\mathbf{i}\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathcal{V}\xi) = [X^{\mathsf{c}}, \mathbf{v}\xi], \quad \mathbf{i}\mathcal{V}(\mathcal{L}_{X^{\mathsf{c}}}\xi) = \mathbf{v}[X^{\mathsf{c}}, \xi],$$

hence $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}(\mathcal{V}\xi) = \mathcal{V}(\mathcal{L}_{X^{\mathsf{c}}}\xi)$ if, and only if,

$$0 = [\mathbf{v}\xi, X^{\mathsf{c}}] - \mathbf{v}[\xi, X^{\mathsf{c}}] = [\mathbf{v}, X^{\mathsf{c}}]\xi.$$ □

4 Conformal vector fields on a Finsler manifold

4.1

Let (M, F) be a *Finsler manifold.* We recall that the *Finsler function* $F\colon TM \to \mathbb{R}$ here is assumed to be *smooth* on $\overset{\circ}{T}M$, *positive* ($F(v) > 0$, if $v \in \overset{\circ}{T}M$), *positive-homogeneous of degree* 1 ($F(\lambda v) = \lambda F(v)$ for all $v \in TM$ and positive real number λ), and it is also required that the *metric tensor*

$$g := \frac{1}{2}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F^2$$

is *fibrewise non-degenerate.* The function $E := \frac{1}{2}F^2$ is the *energy function* of (M, F). The homogeneity of F implies that over $\overset{\circ}{T}M$ we have

$$CF = F, \quad CE = 2E.$$

The *Hilbert 1-form* of (M, F) is

$$\widetilde{\theta} := \nabla^{\mathsf{v}}E = F\nabla^{\mathsf{v}}F \text{ – in the pull-back formalism,}$$
$$\theta := d_{\mathbf{J}}E \text{ – in the } \tau_{TM} \text{ formalism.}$$

It is easy to check that

$$\widetilde{\theta}(\widetilde{X}) = g(\widetilde{X}, \delta) \text{ for each } \widetilde{X} \in \mathrm{Sec}(\overset{\circ}{\pi}).$$

$\widetilde{\theta}$ and θ are related by

$$\theta = \widetilde{\theta} \circ \mathbf{j}. \tag{29}$$

The 2-form

$$\omega := d\theta = dd_{\mathbf{J}}E$$

on $\overset{\circ}{T}M$ is said to be the *fundamental 2-form* of (M, F). Its relation to the metric tensor is given by

$$\omega(\mathbf{J}\xi, \eta) = g(\mathbf{j}\xi, \mathbf{j}\eta); \quad \xi, \eta \in \mathfrak{X}(\overset{\circ}{T}M). \tag{30}$$

The non-degeneracy of g implies the non-degeneracy of ω – and vice versa.

Lemma 3. *With the notations introduced above, let (M, F) be a Finsler manifold, and let X be a vector field on M. Then*

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widetilde{\theta}) \circ \mathbf{j} = \mathcal{L}_{X^{\mathsf{c}}}\theta; \tag{31}$$

$$(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\mathbf{j}\xi, \mathbf{j}\eta) = (\mathcal{L}_{X^{\mathsf{c}}}\omega)(\mathbf{J}\xi, \eta); \quad \xi, \eta \in \mathfrak{X}(\overset{\circ}{T}M). \tag{32}$$

Proof. We check only the less trivial second relation:

$$\begin{aligned}(\mathcal{L}_{X^{\mathsf{c}}}\omega)(\mathbf{J}\xi, \eta) &= X^{\mathsf{c}}\omega(\mathbf{J}\xi, \eta) - \omega(\mathcal{L}_{X^{\mathsf{c}}}\mathbf{J}\xi, \eta) - \omega(\mathbf{J}\xi, \mathcal{L}_{X^{\mathsf{c}}}\eta) \\ &\overset{(23),(30)}{=} X^{\mathsf{c}}g(\mathbf{j}\xi, \mathbf{j}\eta) - \omega(\mathcal{L}_{X^{\mathsf{c}}}\mathbf{J}\xi, \eta) - g(\mathbf{j}\xi, \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{j}\eta).\end{aligned}$$

Since $\mathcal{L}_{X^{\mathsf{c}}}\mathbf{J}\xi = [X^{\mathsf{c}}, \mathbf{J}\xi] = -[\mathbf{J}, X^{\mathsf{c}}]\xi + \mathbf{J}[X^{\mathsf{c}}, \xi] = \mathbf{J}\mathcal{L}_{X^{\mathsf{c}}}\xi$, the second term at the right-hand side of the above relation takes the form

$$\omega(\mathcal{L}_{X^{\mathsf{c}}}\mathbf{J}\xi, \eta) = \omega(\mathbf{J}\mathcal{L}_{X^{\mathsf{c}}}\xi, \eta) \overset{(30)}{=} g(\mathbf{j}\mathcal{L}_{X^{\mathsf{c}}}\xi, \mathbf{j}\eta) \overset{(23)}{=} g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{j}\xi, \mathbf{j}\eta).$$

So we obtain

$$(\mathcal{L}_{X^{\mathsf{c}}}\omega)(\mathbf{J}\xi, \eta) = X^{\mathsf{c}}g(\mathbf{j}\xi, \mathbf{j}\eta) - g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{j}\xi, \mathbf{j}\eta) - g(\mathbf{j}\xi, \widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\mathbf{j}\eta) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}g)(\mathbf{j}\xi, \mathbf{j}\eta). \quad \square$$

4.2

We continue to assume that (M, F) is a Finsler manifold. The $2n$-form

$$\sigma := \frac{(-1)^{\frac{n(n-1)}{2}}}{n!}\omega^n,$$

where $\omega^n = \omega \wedge \cdots \wedge \omega$ (n factors) is a volume form on $\overset{\circ}{T}M$, called the *Dazord volume form* of (M, F). By the *divergence* of a vector field ξ on $\overset{\circ}{T}M$ (with respect to σ) we mean the unique function $\operatorname{div}\xi \in C^\infty(\overset{\circ}{T}M)$ such that

$$\mathcal{L}_\xi\sigma = (\operatorname{div}\xi)\sigma.$$

Lemma 4. *If (M, F) is a Finsler manifold, then the divergence of the Liouville vector field C on $\overset{\circ}{T}M$ with respect to the Dazord volume form is $n = \dim M$.*

Proof. $\mathcal{L}_C\omega = \mathcal{L}_C dd_{\mathbf{J}}E = d\mathcal{L}_C d_{\mathbf{J}}E \overset{(13)}{=} dd_{\mathbf{J}}\mathcal{L}_C E - dd_{\mathbf{J}}E = 2dd_{\mathbf{J}}E - dd_{\mathbf{J}}E = \omega$. From this it follows by induction that $\mathcal{L}_C\omega^n = n\omega^n$, whence our claim. $\square$

4.3

If (M, F) is a Finsler manifold, then there exists a unique spray S for M such that

$$i_S dd_{\mathbf{J}}E = -dE \quad \text{over } \overset{\circ}{T}M, \text{ and } S \restriction o(M) = 0. \tag{33}$$

We say that S is the *canonical spray* of (M, F); the Ehresmann connection induced by S according to (27) is said to be the *canonical connection* of (M, F). It may be characterized as *the unique torsion-free homogeneous Ehresmann connection* $\mathcal{H}$ *for* M *which is compatible with the Finsler function* in the sense that $dF \circ \mathcal{H} = 0$, or, equivalently,

$$X^{\mathsf{h}}F = 0 \quad \text{for all} \quad X \in \mathfrak{X}(M).$$

With the help of the canonical connection, we define the *Sasaki extension* G of the metric tensor g of (M, F) by the rule

$$G(\xi, \eta) := g(\mathbf{j}\xi, \mathbf{j}\eta) + g(\mathcal{V}\xi, \mathcal{V}\eta); \quad \xi, \eta \in \mathfrak{X}(\overset{\circ}{T}M), \tag{34}$$

where $\mathcal{V}$ is the vertical mapping associated to $\mathcal{H}$. Then G is a Riemannian metric tensor on $\overset{\circ}{T}M$.

For subsequent applications, we collect here some further technical results.

Lemma 5. *For any section* $\widetilde{X}$ *in* $\mathrm{Sec}(\pi)$, *we have*

$$\nabla^{\mathsf{v}}_{\widetilde{X}}\delta = \widetilde{X}. \tag{35}$$

Proof. Let $\mathcal{H}$ be a homogeneous Ehresmann connection for M and let $S := \mathcal{H} \circ \delta$ be the spray associated to $\mathcal{H}$ (**3.2**). Then, applying the so-called Grifone identity ([8], Prop. I.7) in the last step, we find that

$$\nabla^{\mathsf{v}}_{\widetilde{X}}\delta := \mathbf{j}[\mathbf{i}\widetilde{X}, \mathcal{H}\delta] = \mathbf{j}[\mathbf{i}\widetilde{X}, S] = \widetilde{X}. \qquad \square$$

Lemma 6. *The energy function of a Finsler manifold can be obtained from the metric tensor by*

$$g(\delta, \delta) = 2E; \tag{36}$$

from the fundamental 2*-form by*

$$\omega(C, S) = 2E, \tag{37}$$

where S *is a spray for the base manifold.*

Proof.

$$\begin{aligned}
g(\delta,\delta) &= \nabla^{\mathsf{v}}(\nabla^{\mathsf{v}}E)(\delta,\delta) = \nabla^{\mathsf{v}}_{\delta}(\nabla^{\mathsf{v}}E)(\delta) = \nabla^{\mathsf{v}}_{\delta}(\nabla^{\mathsf{v}}E(\delta)) - \nabla^{\mathsf{v}}E(\nabla^{\mathsf{v}}_{\delta}\delta)\\
&\overset{(35)}{=} \nabla^{\mathsf{v}}_{\delta}(CE) - \nabla^{\mathsf{v}}E(\delta) = C(CE) - CE = 4E - 2E = 2E;\\
\omega(C,S) &= dd_{\mathbf{J}}E(C,S) = Cd_{\mathbf{J}}E(S) - S(d_{\mathbf{J}}E(C)) - d_{\mathbf{J}}E([C,S])\\
&= C(CE) - d_{\mathbf{J}}E(S) = 4E - 2E = 2E. \qquad \square
\end{aligned}$$

Lemma 7. *The divergence of the canonical spray of a Finsler manifold vanishes.*

Proof. $\mathcal{L}_S\omega = \mathcal{L}_S dd_{\mathbf{J}}E \overset{(1)}{=} i_S ddd_{\mathbf{J}}E + di_S dd_{\mathbf{J}}E \overset{(33)}{=} -ddE = 0$, which implies our claim. $\square$

4.4

Let (M,F) be a Finsler manifold. We say that a vector field X on M is a *projective*, resp. an *affine vector field* of (M,F), if it is a projective vector field, resp. a Lie symmetry for the canonical spray of (M,F). A vector field X on M is said to be a *conformal vector field*, if the Lie derivative of the metric tensor of (M,F) with respect to the complete lift of X satisfies the relation

$$\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} g = \varphi g \tag{38}$$

for a continuous function $\varphi\colon TM \to \mathbb{R}$, of class C^1 on $\overset{\circ}{T}M$, called the *conformal factor* of X. Particular cases of conformal vector fields are *homothetic vector fields* for which the conformal factor is a constant function and *isometric vector fields*, also called *Killing vector fields*, for which the conformal factor is the zero function on TM.

Lemma 8. *If X is a conformal vector field on a Finsler manifold (M,F) with conformal factor φ, then $X^{\mathsf{c}}E = \varphi E$.*

Proof.

$$\begin{aligned} 2X^{\mathsf{c}}E &\overset{(36)}{=} X^{\mathsf{c}}(g(\delta,\delta)) = (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} g)(\delta,\delta) + 2g(\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\delta,\delta) \overset{(21)}{=} (\widetilde{\mathcal{L}}_{X^{\mathsf{c}}} g)(\delta,\delta) \\ &\overset{(38)}{=} \varphi g(\delta,\delta) \overset{(36)}{=} 2\varphi E. \end{aligned}$$ □

Lemma 9. *If X is a conformal vector field on a Finsler manifold (M,F), then the conformal factor of X is the vertical lift of a smooth function on M.*

Proof. In view of the previous lemma, $X^{\mathsf{c}}E = \varphi E$, where $\varphi \in C^0(TM) \cap C^1(\overset{\circ}{T}M)$. Acting on both sides of this relation by the Liouville vector field, we get on the one hand

$$C(X^{\mathsf{c}}E) = C(\varphi E) = (C\varphi)E + 2\varphi E,$$

on the other hand

$$C(X^{\mathsf{c}}E) = [C, X^{\mathsf{c}}]E + X^{\mathsf{c}}(CE) = 2X^{\mathsf{c}}E = 2\varphi E,$$

so it follows that $(C\varphi)E = 0$, and hence $C\varphi = 0$. This means that φ is positive--homogeneous of degree 0, which implies (see, e.g., [16], 2.6, Lemma 2) that φ is of the form $\varphi = f \circ \tau$, $f \in C^\infty(M)$. □

Proposition 2. *Let (M,F) be a Finsler manifold. For a vector field X on M, the following conditions are equivalent:*

(i) *X is a conformal vector field with conformal factor φ;*

(ii) $X^{\mathsf{c}}E = \varphi E$;

(iii) $\mathcal{L}_{X^{\mathsf{c}}}\theta = \varphi\theta$;

(iv) $\widetilde{\mathcal{L}}_{X^{\mathsf{c}}}\widetilde{\theta} = \varphi\widetilde{\theta}$;

(v) $\mathcal{L}_{X^c}\omega = \varphi\omega + d\varphi \wedge d_{\mathbf{J}}E; \quad \varphi = f \circ \tau, \quad f \in C^\infty(M)$.

In conditions (ii)–(iv), $\varphi \in C^0(TM) \cap C^1(\overset{\circ}{T}M)$.

Proof. The arrangement of our reasoning follows the scheme

$$\begin{array}{ccccc} (i) & \Longrightarrow & (ii) & & \\ \Uparrow & & \Downarrow & & \\ (v) & \Longleftarrow & (iii) & \Longleftrightarrow & (iv). \end{array}$$

(i) $\Longrightarrow$ *(ii)* This is just a restatement of Lemma 8.
(ii) $\Longrightarrow$ *(iii)* Let Y be a vector field on M. We have on the one hand

$$\begin{aligned}(\mathcal{L}_{X^c}\theta)(Y^v) &= X^c(\theta(Y^v)) - \theta([X^c, Y^v]) \overset{(6b)}{=} X^c(\theta(Y^v)) - \theta([X,Y]^v) = 0 \\ &= (\varphi\theta)(Y^v),\end{aligned}$$

since the vertical vector fields are annullated by the 1-form $\theta = d_{\mathbf{J}}E$. On the other hand,

$$\begin{aligned}(\mathcal{L}_{X^c}\theta)(Y^c) &= X^c(d_{\mathbf{J}}E(Y^c)) - d_{\mathbf{J}}E([X^c, Y^c]) \overset{(6c)}{=} X^c(Y^vE) - [X,Y]^vE \\ &\overset{(6b)}{=} X^c(Y^vE) - [X^c, Y^v]E = Y^v(X^cE) \overset{(ii)}{=} Y^v(\varphi E) \overset{(*)}{=} \varphi(Y^vE) \\ &= (\varphi d_{\mathbf{J}}E)(Y^c) = (\varphi\theta)(Y^c).\end{aligned}$$

At step $(*)$ we used the fact that our condition $X^cE = \varphi E$ implies, as it turns out from the proof of Lemma 9, that φ is a vertical lift. Thus $\mathcal{L}_{X^c}\theta = \varphi\theta$, as we claimed.

(iii) $\Longrightarrow$ *(v)* By our condition,

$$\mathcal{L}_{X^c}\omega = \mathcal{L}_{X^c}d\theta = d\mathcal{L}_{X^c}\theta \overset{(iii)}{=} d(\varphi\theta) = d\varphi \wedge \theta + \varphi d\theta = \varphi\omega + d\varphi \wedge d_{\mathbf{J}}E.$$

To check that the function φ here is a vertical lift, we evaluate both sides of *(iii)* at a spray S. Then $\theta(S) = d_{\mathbf{J}}E(S) = dE(C) = 2E$, while

$$(\mathcal{L}_{X^c}\theta)(S) = X^c(d_{\mathbf{J}}E(S)) - d_{\mathbf{J}}E([X^c, S]) = 2X^cE - \mathbf{J}[X^c, S]E = 2X^cE,$$

since $[X^c, S]$ is vertical (see, e.g., [16], p. 1350). Thus we obtain that $X^cE = \varphi E$, which implies, as we have just remarked, that $\varphi = f \circ \tau$, $f \in C^\infty(M)$.

(v) $\Longrightarrow$ *(i)* For any vector fields ξ, η on $\overset{\circ}{T}M$,

$$\begin{aligned}(\widetilde{\mathcal{L}}_{X^c}g)(\mathbf{j}\xi, \mathbf{j}\eta) &\overset{(32)}{=} (\mathcal{L}_{X^c}\omega)(\mathbf{J}\xi, \eta) \overset{(v)}{=} (\varphi\omega + d\varphi \wedge d_{\mathbf{J}}E)(\mathbf{J}\xi, \eta) \\ &= \varphi\omega(\mathbf{J}\xi, \eta) + d_{\mathbf{J}}\varphi(\xi)d_{\mathbf{J}}E(\eta) - d\varphi(\eta)d_{\mathbf{J}}E(\mathbf{J}\xi) \\ &\overset{d_{\mathbf{J}}\varphi=0}{=} \varphi\omega(\mathbf{J}\xi, \eta) \overset{(30)}{=} (\varphi g)(\mathbf{j}\xi, \mathbf{j}\eta),\end{aligned}$$

hence $\widetilde{\mathcal{L}}_{X^c}g = \varphi g$.

(iii) $\Longleftrightarrow$ *(iv)* If $\mathcal{L}_{X^c}\theta = \varphi\theta$, then for any vector field ξ on $\overset{\circ}{T}M$,

$$(\widetilde{\mathcal{L}}_{X^c}\widetilde{\theta})(\mathbf{j}\xi) \overset{(31)}{=} (\mathcal{L}_{X^c}\theta)(\xi) \overset{(iii)}{=} (\varphi\theta)(\xi) \overset{(29)}{=} \varphi\widetilde{\theta}(\mathbf{j}\xi),$$

whence $\widetilde{\mathcal{L}}_{X^c}\widetilde{\theta} = \varphi\widetilde{\theta}$. The converse may be checked in the same way. □

We note that relation *(v)*, as a characterization of conformal vector fields on a Finsler manifold, was announced first by J. Grifone [10].

Corollary 1. *Let (M,F) be a Finsler manifold. For a vector field X on M, the following conditions are equivalent:*

(i) X *is a homothetic vector field, i.e.,* $\widetilde{\mathcal{L}}_{X^c} g = \alpha g$, *where* α *is a real number;*

(ii) the energy function is an eigenfunction of X^c *with eigenvalue* α, *i.e.,* $X^c E = \alpha E$;

(iii) $\mathcal{L}_{X^c}\theta = \alpha\theta$;

(iv) $\widetilde{\mathcal{L}}_{X^c}\widetilde{\theta} = \alpha\widetilde{\theta}$;

(v) $\mathcal{L}_{X^c}\omega = \alpha\omega$.

In conditions *(iii)–(v)* α is a real number. With the choice $\alpha := 0$ we obtain criteria that a vector field X on M be a Killing vector field of (M,F).

Proposition 3. *Let (M,F) be a Finsler manifold. If a vector field X on M is both affine and conformal, then X^c is a conformal vector field on the Riemannian manifold $(\overset{\circ}{T}M, G)$, i.e., $\mathcal{L}_{X^c} G = \varphi G$, where $\varphi \in C^0(TM) \cap C^1(\overset{\circ}{T}M)$ and G is the Sasaki extension of the metric tensor of (M,F).*

Conversely, if X^c is a conformal vector field of $(\overset{\circ}{T}M, G)$, then X is a conformal vector field on the Finsler manifold (M,F).

Proof. Suppose first that X is both an affine and a conformal vector field on (M,F). Applying (34), (23) and Proposition 1/*(vii)*, for any vector fields ξ,η on $\overset{\circ}{T}M$ we have

$$\begin{aligned}
(\mathcal{L}_{X^c} G)(\xi,\eta) &= \mathcal{L}_{X^c}(G(\xi,\eta)) - G(\mathcal{L}_{X^c}\xi,\eta) - G(\xi,\mathcal{L}_{X^c}\eta) \\
&= \mathcal{L}_{X^c}(g(\mathbf{j}\xi,\mathbf{j}\eta)) + \mathcal{L}_{X^c}(g(\mathcal{V}\xi,\mathcal{V}\eta)) - g(\mathbf{j}\mathcal{L}_{X^c}\xi,\mathbf{j}\eta) \\
&\quad - g(\mathcal{V}\mathcal{L}_{X^c}\xi,\mathcal{V}\eta) - g(\mathbf{j}\xi,\mathbf{j}\mathcal{L}_{X^c}\eta) - g(\mathcal{V}\xi,\mathcal{V}\mathcal{L}_{X^c}\eta) \\
&= \widetilde{\mathcal{L}}_{X^c}(g(\mathbf{j}\xi,\mathbf{j}\eta)) + \widetilde{\mathcal{L}}_{X^c}(g(\mathcal{V}\xi,\mathcal{V}\eta)) - g(\widetilde{\mathcal{L}}_{X^c}(\mathbf{j}\xi),\mathbf{j}\eta) \\
&\quad - g(\widetilde{\mathcal{L}}_{X^c}(\mathcal{V}\xi),\mathcal{V}\eta) - g(\mathbf{j}\xi,\widetilde{\mathcal{L}}_{X^c}(\mathbf{j}\eta)) - g(\mathcal{V}\xi,\widetilde{\mathcal{L}}_{X^c}(\mathcal{V}\eta)) \\
&= (\widetilde{\mathcal{L}}_{X^c} g)(\mathbf{j}\xi,\mathbf{j}\eta) + (\widetilde{\mathcal{L}}_{X^c} g)(\mathcal{V}\xi,\mathcal{V}\eta) \\
&= \varphi g(\mathbf{j}\xi,\mathbf{j}\eta) + \varphi g(\mathcal{V}\xi,\mathcal{V}\eta) = \varphi G(\xi,\eta).
\end{aligned}$$

This proves that X^c is a conformal vector field on $(\overset{\circ}{T}M, G)$. Conversely, under this condition we find that

$$\begin{aligned}
2\varphi E &= \varphi g(\delta,\delta) = \varphi g(\mathcal{V}C,\mathcal{V}C) = \varphi G(C,C) = (\mathcal{L}_{X^c} G)(C,C) \\
&= X^c(G(C,C)) - G([X^c,C],C) - G(C,[X^c,C]) = X^c(G(C,C)) \\
&= X^c g(\delta,\delta) = 2X^c E,
\end{aligned}$$

so, by Proposition 2, X is a conformal vector field on (M,F). □

Proposition 4. *Any homothetic vector field on a Finsler manifold is an affine vector field.*

Proof. Let (M,F) be a Finsler manifold, and let S be the canonical spray for (M,F). Suppose that X is a homothetic vector field of (M,F). Then, by Corollary 1, there is a real number α such that $X^{\mathsf{c}}E = \alpha E$, or, equivalently, $\mathcal{L}_{X^{\mathsf{c}}}\omega = \alpha\omega$. So we have

$$\begin{aligned}\mathcal{L}_{X^{\mathsf{c}}}dE &= d(X^{\mathsf{c}}E) = \alpha dE \overset{(33)}{=} -\alpha i_S\omega = -i_S(\alpha\omega) = -i_S(\mathcal{L}_{X^{\mathsf{c}}}\omega)\\ &= -\mathcal{L}_{X^{\mathsf{c}}}i_S\omega + i_{[X^{\mathsf{c}},S]}\omega = \mathcal{L}_{X^{\mathsf{c}}}dE + i_{[X^{\mathsf{c}},S]}\omega.\end{aligned}$$

Thus $i_{[X^{\mathsf{c}},S]}\omega = 0$, and hence – by the non-degeneracy of ω – $[X^{\mathsf{c}},S] = 0$. This means that X is a Lie symmetry of the canonical spray of (M,F). □

Lemma 10. *If X is a conformal vector field on an n-dimensional Finsler manifold, then (with respect to the Dazord volume form)* $\operatorname{div} X^{\mathsf{c}} = n\varphi$, *where φ is the conformal factor of X.*

Proof. Choose a local frame $(X_i)_{i=1}^n$ for TM over an open subset U of M. Then the family $(X_i^{\mathsf{v}}, X_i^{\mathsf{c}})_{i=1}^n$ is a local frame for TTM over $\tau^{-1}(U)$. It may be shown by a little lengthy inductive argument that

$$(\mathcal{L}_{X^{\mathsf{c}}}\omega)(X_1^{\mathsf{v}}, X_1^{\mathsf{c}}, \dots, X_n^{\mathsf{v}}, X_n^{\mathsf{c}}) = n\varphi\omega(X_1^{\mathsf{v}}, X_1^{\mathsf{c}}, \dots, X_n^{\mathsf{v}}, X_n^{\mathsf{c}}),$$

which implies our claim. □

Proposition 5. *If a vector field is both a projective and a conformal vector field on a Finsler manifold, then it is a homothetic vector field.*

Proof. Let (M,F) be an n-dimensional Finsler manifold. Suppose that a vector field X on M is both projective and conformal. Then, on the one hand,

$$[X^{\mathsf{c}}, S] = \psi C, \quad \psi \in C^0(TM) \cap C^1(\mathring{T}M),$$

where S is the canonical spray of (M,F). On the other hand, by Proposition 2,

$$X^{\mathsf{c}}E = f^{\mathsf{v}}E, \quad f \in C^\infty(M).$$

Thus we get

$$\begin{aligned}2\psi E &= \psi(CE) = [X^{\mathsf{c}}, S]E = X^{\mathsf{c}}(SE) - S(X^{\mathsf{c}}E) = -S(f^{\mathsf{v}}E)\\ &= -(Sf^{\mathsf{v}})E - f^{\mathsf{v}}(SE) = -f^{\mathsf{c}}E,\end{aligned}$$

taking into account that S is horizontal with respect to the canonical connection of (M,F) and hence $SE = \frac{1}{2}SF^2 = F(SF) = 0$ (see 4.3), applying furthermore the relation $Sf^{\mathsf{v}} = f^{\mathsf{c}} (f \in C^\infty(M))$, whose verification is routine. It follows that

$$\psi = -\frac{1}{2}f^{\mathsf{c}}.$$

Now we determine the divergence (with respect to the Dazord volume form) of both sides of the relation $[X^{\mathsf{c}}, S] = -\frac{1}{2} f^{\mathsf{c}} C$. Applying the well-known rules for calculation (see, e.g., [1], §6.5 or [11], XV, §1) we find that

$$\operatorname{div}[X^{\mathsf{c}}, S] = X^{\mathsf{c}} \operatorname{div} S - S \operatorname{div} X^{\mathsf{c}} \overset{\text{Lemmas 7, 10}}{=} -S(nf^{\mathsf{v}}) = -nf^{\mathsf{c}}$$

and

$$\operatorname{div}\Big(-\frac{1}{2} f^{\mathsf{c}} C\Big) = -\frac{1}{2}(Cf^{\mathsf{c}} + f^{\mathsf{c}} \operatorname{div} C) \overset{\text{Lemma 4}}{=} -\frac{1}{2}(n+1) f^{\mathsf{c}}.$$

So $(n-1)f^{\mathsf{c}} = 0$, where $n \geq 2$ (1.1 (a)), whence $f^{\mathsf{c}} = 0$. This implies by the connectedness of M that f is a constant function, and therefore the conformal factor of X is constant. □

We note that this result is an infinitesimal version of Theorem 2 in [17].

Proposition 6. *Let (M, F) be a Finsler manifold. Suppose that a vector field X on M preserves the Dazord volume form of (M, F), i.e., $\mathcal{L}_{X^{\mathsf{c}}} \sigma = 0$. If, in addition,*

(i) X is a projective vector field, then X is affine;

(ii) X is a conformal vector field, then X is isometric.

Proof. First we note that our condition $\mathcal{L}_{X^{\mathsf{c}}} \sigma = 0$ implies that $\operatorname{div} X^{\mathsf{c}} = 0$.
(i) Suppose that X is also a projective vector field, i.e.,

$$[X^{\mathsf{c}}, S] = \psi C, \quad \psi \in C^0(TM) \cap C^1(\overset{\circ}{T}M).$$

Observe that over $\overset{\circ}{T}M$ the function ψ satisfies the relation $C\psi = \psi$. Indeed, by the Jacobi identity

$$0 = [C, [X^{\mathsf{c}}, S]] + [X^{\mathsf{c}}, [S, C]] + [S, [C, X^{\mathsf{c}}]] = [C, [X^{\mathsf{c}}, S]] - [X^{\mathsf{c}}, S],$$

hence

$$[X^{\mathsf{c}}, S] = [C, [X^{\mathsf{c}}, S]] = [C, \psi C] = (C\psi)C,$$

therefore $(C\psi)C = \psi C$, and so $C\psi = \psi$.

Now, as in the previous proof, we calculate the divergence of both sides of the relation $[X^{\mathsf{c}}, S] = \psi C$. Since $\operatorname{div} X^{\mathsf{c}} = \operatorname{div} S = 0$, we have

$$\operatorname{div}[X^{\mathsf{c}}, S] = X^{\mathsf{c}} \operatorname{div} S - S \operatorname{div} X^{\mathsf{c}} = 0.$$

On the other hand, by our above remark,

$$\operatorname{div}(\psi C) = \psi \operatorname{div} C + C\psi = (n+1)\psi.$$

So it follows that $\psi = 0$, hence $[X^{\mathsf{c}}, S] = 0$. Thus X is an affine vector field on (M, F).
(ii) Now suppose that ($\operatorname{div} X^{\mathsf{c}} = 0$ and) X is also a conformal vector field. Then, by Proposition 2, $X^{\mathsf{c}} E = f^{\mathsf{v}} E$, $f \in C^\infty(M)$. Since

$$nf^{\mathsf{v}} \overset{\text{Lemma 10}}{=} \operatorname{div} X^{\mathsf{c}} \overset{\text{cond.}}{=} 0,$$

it follows that $X^{\mathsf{c}} E = 0$. Thus, by Corollary 1, X is an isometric vector field on (M, F). □

Acknowledgements

This research was carried out in the framework of the Cooperation of the Czech and Hungarian Government (MEB 041005). The first author was supported also by Hungarian Scientific Research Fund OTKA No. NK 81402. The authors wish to express their gratitude to Bernadett Aradi, Dávid Cs. Kertész and Rezső L. Lovas for their useful comments and technical help during the preparation of the manuscript.

References

[1] R. Abraham, J. E. Marsden, T. Ratiu: *Manifolds, Tensor Analysis, and Applications.* 2nd edition, Springer-Verlag, New York and Berlin (1988).

[2] H. Akbar-Zadeh: Transformations infinitésimales conformes des variétés finsleriennes compactes. *Annales Polonici Mathematici* XXXVI (1979) 213–229.

[3] H. Akbar-Zadeh: Champs de vecteurs projectifs sur le fibré unitaire. *J. Math. pures et appl.* 65 (1986) 47–79.

[4] S. Bácsó, Z. Szilasi: On the projective theory of sprays. *Acta Math. Acad. Paed. Nyíregyháziensis* 26 (2010) 171–207.

[5] M. Crampin: On horizontal distributions on the tangent bundle of a differentiable manifold. *J. London Math. Soc* (2) 3 (1971) 178–182.

[6] M. de León, P.R. Rodrigues: *Methods of Differential Geometry in Analytical Mechanics.* North-Holland, Amsterdam (1989).

[7] W. Greub, S. Halperin, R. Vanstone: *Connections, Curvature, and Cohomology.* Vol. I, Academic Press, New York and London (1972).

[8] J. Grifone: Structure presque-tangente et connexions, I. *Ann. Inst. Fourier (Grenoble)* 22 (1972) 287–334.

[9] J. Grifone: Transformations infinitésimales conformes d'une variété finslerienne. *C.R. Acad. Sc. Paris* 280, Série A (1975) 519–522.

[10] J. Grifone: Sur les transformations infinitésimales conformes d'une variété finslérienne. *C.R. Acad. Sc. Paris* 280, Série A (1975) 583–585.

[11] S. Lang: *Fundamentals of Differential Geometry.* Springer-Verlag, New York (1999).

[12] R.L. Lovas: Affine and projective vector fields on spray manifolds. *Periodica Mathematica Hungarica* 48 (2004) 165–179.

[13] M. Matsumoto: Theory of extended point transformations of Finsler spaces I. *Tensor N.S.* 45 (1987) 109–115.

[14] M. Matsumoto: Theory of extended point transformations of Finsler spaces II. *Tensor N.S.* 47 (1988) 203–214.

[15] R.B. Misra: *Groups of transformations in Finslerian spaces.* Internal Reports of the ICTP, Trieste (1993).

[16] J. Szilasi: A Setting for Spray and Finsler Geometry. In: *Handbook of Finsler Geometry.* Kluwer Academic Publishers, Dordrecht (2003) 1183–1426.

[17] J. Szilasi, Cs. Vincze: On conformal equivalence of Riemann-Finsler metrics. *Publ. Math. Debrecen* 52 (1998) 167–185.

[18] K. Yano: *The Theory of Lie Derivatives and its Applications.* North-Holland, Amsterdam (1957).

[19] K. Yano and S. Ishihara: *Tangent and Cotangent Bundles.* Marcel Dekker Inc., New York (1973).

Authors' address:
Institute of Mathematics, University of Debrecen, Debrecen, Hungary

E-mail: `szilasi@science.unideb.hu`

E-mail: `tothanna@science.unideb.hu`

Received: 3 September, 2011
Accepted for publication: 29 October, 2011

Communications in Mathematics 19 (2011) 169–178

A geometric analysis of dynamical systems with singular Lagrangians

Monika Havelková

Abstract. We study dynamics of singular Lagrangian systems described by implicit differential equations from a geometric point of view using the exterior differential systems approach. We analyze a concrete Lagrangian previously studied by other authors by methods of Dirac's constraint theory, and find its complete dynamics.[1]

1 Introduction

Singular (or degenerate) Lagrangian systems were first systematically considered by Dirac [2]. He was probably the first who noticed that the classical Hamilton equations make sense only for Lagrangians $L(t, q^\sigma, \dot{q}^\sigma)$ satisfying the regularity condition

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}\right) \neq 0,$$

and proposed a generalization to describe and understand dynamics of singular Lagrangians. Unfortunately, his approach was more heuristic than rigorous from the mathematical point of view, with an unpleasant consequence: study of the dynamics of concrete Lagrangian systems provided by different authors using Dirac's procedure can lead to different results.

A mathematically correct approach has been achieved later, with help of differential geometry. The dynamics of degenerate Lagrangian systems can be investigated in two geometrically distinct ways:

Indirect (image) approach concerns the well-known Hamiltonian formalism in symplectic geometry mapping a Lagrangian system from the tangent to the cotangent bundle: Hamiltonian dynamics then appears as image dynamics via Legendre

2010 MSC: 51P05
Key words: singular Lagrangian systems, geometric constraint algorithm, extended dynamics, proper dynamics, final constraint submanifold

[1]This work awarded prize at the 12th Czech-Slovak Students' Research Conference (SVOČ), Ústí nad Labem, Czech Republic, May 2011

map which is degenerate. An explicit study of the image (hamiltonian) dynamics is possible if the Legendre map has a constant rank (the image space is a submanifold in the cotangent space). Applying a procedure called "constraint algorithm" one can obtain, under some other assumptions on the Lagrangian, a *final constraint submanifold* where the image motion proceeds (among many references see e.g. [1], [5], [6], [7]).

Direct approach originally due to O. Krupková [8] concerns study of Hamiltonian exterior differential systems in jet bundles. This approach develops the idea of Goldschmidt and Sternberg [4] understanding Hamilton equations as equations for integral sections of an exterior differential system in the first jet bundle over a fibred manifold and is not restricted to some particular kind of Lagrangian systems (regarding rank, or order). Whatever is the Legendre map, in this approach there is a direct geometric relation to extremals (solutions of the Euler-Lagrange equations) as those integral sections of the Hamiltonian exterior differential system which are *holonomic* (i.e. take the form of prolongations). As proposed in [8], within the "direct" setting one can study in a unified way both the Hamiltonian (extended) and the Lagrangian (proper) dynamics of any Lagrangian system (including highly singular), and to obtain a *geometrically exact description of the dynamics*. From the point of view of mathematics, this is a *method of analysing the structure of solutions of implicit second (or higher) order differential equations*. It should be pointed out (also shown on examples in [8], [9]) that the resulting Hamiltonian (and Lagrangian) dynamics need *not* be bounded to a "final constraint submanifold", and may proceed rather in the whole phase space in a way which can be understood as a "controlled chaotic motion". Moreover, in cases when a "final constraint submanifold" exists, the motion typically is not described by a vector field along this submanifold but rather by a more complicated *family of vector fields* (vector distribution).

In this article we investigate a singular mechanical system given by the Lagrangian

$$L = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3.$$

This Lagrangian system has been studied by several authors (e.g. [3] and references therein) by means of the Dirac constraint algorithm, however, its dynamical properties were not clarified: the obtained results are incomplete and conclusions on the dynamical properties of this Lagrangian system made by different authors are not in agreement. It should be pointed out that the main result – to obtain Hamiltonian dynamics for this Lagrangian system, has not been achieved.

We show that problems of this kind can be rigorously solved by application of the above mentioned Hamiltonian exterior differential systems method. For the given Lagrangian, we obtain the corresponding dynamical distribution in the first jet bundle, and show that this distribution is not completely integrable and has a nonconstant rank. This means, however, that to obtain the dynamics one has to apply a general integration method developed in [8], called a "geometric constraint algorithm". With help of the geometric constraint algorithm we solve the problem completely: we compute the Euler-Lagrange equations and Hamilton equations in terms of the corresponding distributions and find the complete structure of solutions (so-called *proper dynamics* and *extended dynamics*, respectively). In particular, the Hamilton and Euler-Lagrange equations are not equivalent, and the dynamics are *not* representable by a *vector field*, they are even not representable by a *vector field* along a certain "final constraint submanifold" of the evolution space. It turns out that the dynamics *are* restricted to (the same) final constraint submanifold, however, along this submanifold the extended (Hamiltonian) and the proper (Lagrangian) motion are governed by distinct *nonintegrable* distributions of *rank greater than one*, the Lagrangian dynamics being characterized by the rank 2 semispray subdistribution of the distribution describing the Hamiltonian dynamics. Among others this means that neither the Hamiltonian nor the Lagrangian "solution pattern" follows within a foliation in the evolution space (or its submanifold). It can be shown, however, that the nonintegrable semispray subdistribution of rank 2 inherits an intrinsic structure such that, within the final constraint submanifold, (prolonged) extremals are constrained to a *family of submanifolds* parametrized by functions on the evolution space. Moreover, along each of these submanifolds the motion is regular, i.e. extremals are integral sections of a *semispray vector field*.

2 Singular Lagrangian systems

We shall consider a fibred manifold $\pi\colon Y \to X$; $Y = R \times M$ where M is a smooth manifold of dimension m, and its first jet prolongation J^1Y. Local fibred coordinates are denoted by (t, q^σ), where $1 \le \sigma \le m$, and the corresponding coordinates on J^1Y are denoted by $(t, q^\sigma, \dot{q}^\sigma)$. The manifold J^1Y is called evolution space.

We shall use the following setting due to [8], [9]:

- A geometric description of the dynamics using a vector distribution on J^1Y.
- Formulation of Hamilton theory as a problem of finding all solutions of this distribution.
- Formulation of Lagrange theory as a problem of finding holonomic solutions of this distribution.

Equations of motion of a Lagrangian system defined by a Lagrangian $\lambda = L dt$, $L = L(t, q^\sigma, \dot{q}^\sigma)$, are represented by the Euler-Lagrange form

$$E_\lambda = E_\sigma \omega^\sigma \wedge dt, \quad E_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma},$$

$1 \le \sigma \le m$, where $\omega^\sigma = dq^\sigma - \dot{q}^\sigma dt$. In what follows we assume that the Euler-Lagrange equations are nontrivially of order two, and denote

$$E_\sigma = A_\sigma + B_{\sigma\nu}\ddot{q}^\nu.$$

It is known that there exists a unique 2-contact form F on J^2Y such that the 2-form $\alpha = E_\lambda + F$ is closed and projectable onto J^1Y [9].

The form α gives rise to the following two distributions on J^1Y which are in general distinct but their holonomic sections are the same and coincide with prolongations of extremals:

- The *characteristic distribution* of α

$$D = \mathrm{span}\{i_\xi \alpha\}, \quad \text{where } \xi \text{ runs over all vector fields on } J^1Y$$

- The *Euler-Lagrange distribution* of α

$$\Delta = \mathrm{span}\{i_\xi \alpha\}, \quad \text{where } \xi \text{ runs over all vertical vector fields on } J^1Y$$

Note that $D \subset \Delta$.

In terms of a Lagrangian

$$\alpha = d\theta_\lambda,$$

where

$$\theta_\lambda = Ldt + \frac{\partial L}{\partial \dot{q}^\sigma}\omega^\sigma$$

is the Cartan form.

Definition 1. Equations for integral sections of Δ are called *Hamilton equations*, solutions of the Hamilton equations are called *Hamilton extremals*.

Definition 2. λ is called *regular* if $\mathrm{rank}\,\Delta = 1$. λ is called *semiregular* if Δ is weakly horizontal (i.e. at each point $p \in J^1Y$ the vector space $\Delta(x)$ is not vertical) and $\mathrm{rank}\,\Delta$ is constant.

Dynamical properties of a Lagrangian system are determined by properties of its related distributions:

Theorem 1. *Δ is weakly horizontal at $x \in J^1Y$ if and only if $D(x) = \Delta(x)$.*

Definition 3. The set $\tilde{P} = \{x \in J^1Y | D(x) = \Delta(x)\}$ is called *the primary constraint set*.

$\widetilde{P} \subset J^1Y$ need not be a submanifold. This set has the meaning of "possibly admissible" initial conditions for the Hamilton equations – more precisely, $J^1Y - \widetilde{P}$ is a primary obstruction set for the hamiltonian initial conditions (outside $\widetilde{P}$ there passes no solution of the Hamilton equations, and consequently, no solution of the Euler-Lagrange equations).

Theorem 2. *The following conditions are equivalent:*

1. *λ is regular*
2. $\Delta = \mathrm{span}\{\frac{\partial}{\partial t} + \dot{q}^\sigma \frac{\partial}{\partial q^\sigma} - B^{\sigma\nu}A_\nu \frac{\partial}{\partial \dot{q}^\sigma}\} = \mathrm{annih}\{A_\sigma dt + B_{\sigma\nu} d\dot{q}^\nu, \omega^\sigma\}$
3. $\det B_{\sigma\nu} = \det\left(\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}\right) \neq 0.$

2.1 The geometric constraint algorithm

The dynamics of a smooth singular Lagrangian system cannot be characterized by a vector field, or even by a system of continuous vector fields in the evolution space. In this section we recall a general procedure which enables one to solve the Euler-Lagrange distribution explicitly [8].

Since in general the extended dynamics and proper dynamics do not coincide, we have to distinguish two levels of the integration problem:

(1) to find the *extended dynamics*, i.e., all integral sections of the Euler-Lagrange distribution (Hamilton extremals)

(2) to find the *proper dynamics*, i.e., holonomic integral sections.

2.1.1 Extended dynamics

We shall describe an algorithm for finding the structure of solutions of Hamilton equations (the dynamics of a Hamiltonian system).

Let us denote

$$F = \begin{pmatrix} \frac{1}{2}\left(\frac{\partial A_\sigma}{\partial \dot{q}^\nu} - \frac{\partial A_\nu}{\partial \dot{q}^\sigma}\right) & B_{\sigma\nu} \\ -B_{\sigma\nu} & 0 \end{pmatrix}$$

and $(F|A)$ the matrix F extended by the column $A_\sigma, 1 \leq \sigma \leq m$, where:

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t \partial \dot{q}^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot{q}^\sigma}\dot{q}^\nu \quad \text{and} \quad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot{q}^\sigma \partial \dot{q}^\nu}.$$

Step 1: Find the primary constraint set $\widetilde{P}$. As proved in [8],

$$\widetilde{P} = \{x \in J^1Y \,|\, \operatorname{rank} F = \operatorname{rank}(F|A)\}.$$

If $\widetilde{P} = \emptyset$, there is no extended dynamics, hence no dynamics at all. If $\widetilde{P} \neq \emptyset$, choose a point $x \in \widetilde{P}$, and proceed to the next step.

Step 2: Denote $M_{(1)} \subset \widetilde{P}$ a submanifold of maximal dimension around x and calculate the Euler-Lagrange distribution $\Delta_{(1)}$ along $M_{(1)}$.

Step 3: Exclude from $M_{(1)}$ the points where the restriction of $\Delta_{(1)}$ to the tangent bundle of $M_{(1)}$ is not weakly horizontal and denote the resulting set by $\widetilde{P}'$. Repeat Step 2 with $\widetilde{P}'$ instead of $\widetilde{P}$.

Continue until the procedure is finalized. Then take another (distinct) submanifold $M_{(2)}$ in $\widetilde{P}$ around x, repeat the procedure.

After sufficiently many steps one obtains either a bunch of final constraint submanifolds at x, or finds that there is no final constraint submanifold passing through x.

Considering then the collection of final constraint submanifolds together with to them constrained Euler-Lagrange distributions, we get the dynamical picture corresponding to the solutions of the Hamilton equations.

2.1.2 Proper dynamics

We have to exclude solutions of Hamilton equations which are not holonomic. First we find the set

$$P = \{x \in J^1Y \mid \operatorname{rank} B = \operatorname{rank}(B|A)\},$$

called *primary semispray constraint set*. Again, it need not be a submanifold in J^1Y. Outside this set, there exist no prolonged extremals, hence there is no motion. If $P \neq \emptyset$, we choose a point $x \in P$ and proceed in a similar way as described above in searching for the extended dynamics: however, in this case we consider as admissible only those submanifolds and vector fields belonging to Δ which along the submanifold can be identified with a semispray.

3 A singular Lagrangian dynamics

Let us consider the following singular Lagrangian

$$L = \dot{q}^1\dot{q}^3 - q^2\dot{q}^3 + q^1q^3. \tag{1}$$

Its Euler-Lagrange equations are implicit second order differential equations

$$q^3 - \ddot{q}^3 = 0, \quad \dot{q}^3 = 0, \quad q^1 + \dot{q}^2 - \ddot{q}^1 = 0.$$

Momenta take the form

$$p_1 = \dot{q}^3, \quad p_2 = 0, \quad p_3 = \dot{q}^1 - q^2$$

and the Hamiltonian reads

$$H = \dot{q}^1\dot{q}^3 - q^1q^3. \tag{2}$$

We can use momenta as a part of new coordinates on J^1Y by considering a local coordinate transformation as follows

$$(t, q^1, q^2, q^3, \dot{q}^1, \dot{q}^2, \dot{q}^3) \to (t, q^1, q^2, q^3, p_3, \dot{q}^2, p_1), \tag{3}$$

and get for the Hamiltonian the expression

$$H = (p_3 + q^2)p_1 - q^1q^3. \tag{4}$$

Computing the Hessian matrix of L we get

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

hence the Legendre map $R \times TM \to R \times T^*M$ has constant rank equal to 2 and defines a submanifold of dimension 5.

Let us turn to the analysis of the dynamics on $R \times TM$ with help of the corresponding distributions. To this end we need the Cartan form θ_λ and its exterior derivative $d\theta_\lambda$:

$$\theta_\lambda = (q^1q^3 - \dot{q}^1\dot{q}^3)dt + \dot{q}^3dq^1 + (\dot{q}^1 - q^2)dq^3,$$

$$d\theta_\lambda = (q^3dq^1 + q^1dq^3 - \dot{q}^3d\dot{q}^1 - \dot{q}^1d\dot{q}^3) \wedge dt + (d\dot{q}^1 - dq^2) \wedge dq^3 + d\dot{q}^3 \wedge dq^1.$$

Computing the distributions D and Δ we get:

$$D = \operatorname{annih}\{q^3dq^1 + q^1dq^3 - \dot{q}^3d\dot{q}^1 - \dot{q}^1d\dot{q}^3,\ q^3dt - d\dot{q}^3,\ dq^3,\ q^1dt + dq^2 - d\dot{q}^1,\ \omega^3,\ \omega^1\}$$

and

$$\Delta = \operatorname{annih}\{q^3dt - d\dot{q}^3,\ dq^3,\ q^1dt + dq^2 - d\dot{q}^1,\ \omega^3,\ \omega^1\}.$$

We can see that $D \subset \Delta$ and $D \neq \Delta$.

3.1 Extended dynamics

The Lagrangian system possesses primary dynamical constraints (primary obstructions to initial conditions for the Hamilton equations). The primary constraint set $\widetilde{P}$ is the set of points in the evolution space where $\operatorname{rank} F = \operatorname{rank}(F|A)$, hence

$$\widetilde{P} = \{x \in J^1Y |\ \dot{q}^3 = 0\},$$

and it is a closed submanifold in J^1Y of codimension 1.

Note that, indeed, outside the submanifold $\widetilde{P}$ the Euler-Lagrange distribution is spanned by two vector fields

$$\frac{\partial}{\partial q^2} + \frac{\partial}{\partial \dot{q}^1}, \quad \frac{\partial}{\partial \dot{q}^2}$$

which are vertical over the base R, hence among the integral curves there are no *sections* (the integral curves describe no evolution).

To get the Hamiltonian dynamics we have first to restrict our considerations to the admissible submanifold $\widetilde{P}$. Along this submanifold the Euler-Lagrange distribution Δ and the dynamical distribution D coincide and are spanned by the following three vector fields:

$$D = \Delta = \mathrm{span}\left\{ \frac{\partial}{\partial t} + \dot{q}^1 \frac{\partial}{\partial q^1} - q^1 \frac{\partial}{\partial q^2} + q^3 \frac{\partial}{\partial \dot{q}^3};\ \frac{\partial}{\partial q^2} + \frac{\partial}{\partial \dot{q}^1};\ \frac{\partial}{\partial \dot{q}^2} \right\}.$$

This distribution is weakly horizontal, but we have to exclude points where it is not tangent to $\widetilde{P}$, that is, the points where $q^3 \neq 0$. Indeed, at these points restriction of $D = \Delta$ to the tangent bundle of $\widetilde{P}$ is a vertical distribution. We obtain a submanifold

$$M = \{x \in J^1Y |\ q^3 = 0,\ \dot{q}^3 = 0\} \tag{5}$$

of $\widetilde{P}$, and along M the distribution

$$\begin{aligned} D_M = \Delta_M &= \mathrm{span}\left\{ \frac{\partial}{\partial t} + \dot{q}^1 \frac{\partial}{\partial q^1} - q^1 \frac{\partial}{\partial q^2};\ \frac{\partial}{\partial q^2} + \frac{\partial}{\partial \dot{q}^1};\ \frac{\partial}{\partial \dot{q}^2} \right\} \\ &= \mathrm{span}\left\{ f_1 \left(\frac{\partial}{\partial t} + \dot{q}^1 \frac{\partial}{\partial q^1} \right) + (f_2 - f_1 q^1) \frac{\partial}{\partial q^2} + f_2 \frac{\partial}{\partial \dot{q}^1} + f_3 \frac{\partial}{\partial \dot{q}^2} \right\}, \end{aligned} \tag{6}$$

where f_1, f_2, f_3 are arbitrary functions on M. This distribution is tangent to M, and weakly horizontal at each point of M, as required. Note that its annihilator is spanned by the following two 1-forms: $q^1 dt + dq^2 - d\dot{q}^1$ and $\omega^1 = dq^1 - \dot{q}^1 dt$. We can see that $\mathrm{rank}\,\Delta_M$ is constant and equal to 3, however, Δ_M is not completely integrable. Summarizing, we have obtained the following structure of solutions of the Hamilton equations for our Lagrangian:

Theorem 3. *Hamilton equations of L are equations for integral sections of the not completely integrable rank 3 distribution Δ_M on the closed 5-dimensional manifold $M \subset J^1Y$ above.*

In fibred coordinates, the Hamilton equations are equations for sections $\delta(t) = (t, x^\sigma(t), y^\sigma(t))$ of J^1Y, where we have denoted $x^\sigma(t) = q^\sigma \circ \delta$ and $y(t) = \dot{q}^\sigma \circ \delta$, and take the following form:

$$\begin{aligned} &\frac{dx^1}{dt} = y^1, \quad &&\frac{dx^2}{dt} = g(t) - x^1, \quad &&\frac{dx^3}{dt} = 0, \\ &\frac{dy^1}{dt} = g(t), \quad &&\frac{dy^2}{dt} = h(t), \quad &&\frac{dy^3}{dt} = 0, \end{aligned}$$

where g, h are arbitrary functions on M, $g(t) = g \circ \delta$ and $h(t) = h \circ \delta$.

In a more conventional way, in "partial Legendre coordinates" defined by (3), and in terms of Hamiltonian (4) we can write

$$\frac{dq^1}{dt} = \frac{\partial H}{\partial p_1}, \quad \frac{dq^2}{dt} = g + \frac{\partial H}{\partial q^3}, \quad \frac{dq^3}{dt} = 0,$$
$$\frac{dp_1}{dt} = 0, \quad \frac{d\dot{q}^2}{dt} = h, \quad \frac{dp_3}{dt} = -\frac{\partial H}{\partial q^3}.$$

For this Hamiltonian system M is the *final constraint submanifold* (having the meaning of a genuine evolution space, or phase space). Extended motion is constraint to this submanifold, is chaotic, not uniquely determined by initial conditions.

3.2 Proper dynamics

We are looking for *holonomic* Hamilton extremals = prolongations of extremals.

Computing the primary semispray-constraint set we get the following closed submanifold in the evolution space

$$P = \{x \in J^1Y \mid \dot{q}^3 = 0\}.$$

Outside this submanifold there are no (prolonged) extremals.

Since $P = \widetilde{P}$, the procedure of restricting the Euler-Lagrange distribution to P ends with the same submanifold M and the restricted distribution $D_M = \Delta_M$ as in the Hamiltonian case above. Now, however, this is not yet the end of the story, since we are interested in *holonomic* solutions, and the distribution Δ_M still has solutions which are not holonomic, hence do not correspond to extremals. We have to continue to another step in the geometric constraint algorithm in order to obtain a maximal submanifold of M with a distribution whose nonvertical vector fields are *semisprays*. It is easily seen that this is achieved by taking the manifold M itself and the rank 2 subdistribution of Δ_M which is obtained by choosing $f_1 = 1$ and $f_2 = \dot{q}^2 + q^1$, i.e. takes the form

$$\mathcal{D} = \operatorname{span}\left\{\frac{\partial}{\partial t} + \dot{q}^1\frac{\partial}{\partial q^1} + \dot{q}^2\frac{\partial}{\partial q^2} + (\dot{q}^2 + q^1)\frac{\partial}{\partial \dot{q}^1} + f\frac{\partial}{\partial \dot{q}^2}\right\}, \tag{7}$$

where f is an arbitrary function on M. Hence we have obtained the following result:

Theorem 4. *The Euler-Lagrange equations of L are equations for integral sections of the not completely integrable rank 2 distribution $\mathcal{D}$ on the closed 5-dimensional manifold $M \subset J^1Y$ above.*

Theorem 4 gives a geometric solution to the extremal problem and a complete geometric "dynamical picture" for the proper dynamics of the given singular Lagrangian system (1). We can see that the motion is restricted to a final constraint submanifold of dimension 5, and is chaotic and indeterministic there (cannot be uniquely determined by initial conditions). Compared with the Hamiltonian dynamics obtained in the previous section, the final constraint submanifold is the same, and the Lagrangian dynamics is given by a rank 2 semispray subdistribution of the distribution describing the Hamiltonian dynamics.

The distribution $\mathcal{D}$ is not completely integrable which means that the (prolonged) extremals do not proceed within leaves of a foliation of M. Nevertheless, as shown below, the geometric picture can be further refined to give us a more precise and fine description of the Lagrangian dynamics within the final constraint submanifold M.

Let us turn back to the distribution Δ_M (6) and note that it has the following rank 2 weakly horizontal subdistribution

$$\operatorname{span}\left\{\frac{\partial}{\partial t}+\dot{q}^1\frac{\partial}{\partial q^1}+(g-q^1)\frac{\partial}{\partial q^2}+g\frac{\partial}{\partial \dot{q}^1}+h\frac{\partial}{\partial \dot{q}^2}\right\}, \tag{8}$$

where g, h are arbitrary functions on M. Now, for every fixed $g(t, q^1, q^2, \dot{q}^1, \dot{q}^2)$ consider a manifold

$$M_g = \{x \in J^1Y \mid q^3 = \dot{q}^3 = 0,\ \dot{q}^2 = g - q^1\} \subset M. \tag{9}$$

If

$$\phi \equiv \frac{\partial g}{\partial \dot{q}^2} - 1 \neq 0, \tag{10}$$

then along M_g distribution (8) takes the form of a rank 2 *semispray* distribution spanned by the following vector fields:

$$\frac{\partial}{\partial t}+\dot{q}^1\frac{\partial}{\partial q^1}+\dot{q}^2\frac{\partial}{\partial q^2}+(\dot{q}^2+q^1)\frac{\partial}{\partial \dot{q}^1}+h\frac{\partial}{\partial \dot{q}^2}, \tag{11}$$

i.e., it is the distribution $\mathcal{D}$ restricted to the submanifold M_g. We have to find its subdistribution *tangent* to M_g. To this end let us consider local coordinates $\bar{t} = t$, $\bar{q}^1 = q^1$, $\bar{q}^2 = q^2$, $\dot{\bar{q}}^1 = \dot{q}^1$, $z = \dot{q}^2 - g + q^1$, adapted to the submanifold M_g. Note that regularity of the transformation means that at each point condition (10) holds true. Transforming (11) to the new coordinates we can see that there is a unique (up to a multiplier) vector field tangent to M_g, with

$$h = \frac{1}{\phi}(\dot{q}^1 - X(g)), \tag{12}$$

where we have denoted

$$X = \frac{\partial}{\partial t}+\dot{q}^1\frac{\partial}{\partial q^1}+\dot{q}^2\frac{\partial}{\partial q^2}+(\dot{q}^2+q^1)\frac{\partial}{\partial \dot{q}^1}.$$

Theorem 5. *Euler-Lagrange equations of L are equations for integral sections of the following family of rank one (hence completely integrable) constraint semispray distributions:*

$$\mathcal{S}_g = \operatorname{span}\left\{\frac{\partial}{\partial t}+\dot{q}^1\frac{\partial}{\partial q^1}+\dot{q}^2\frac{\partial}{\partial q^2}+(\dot{q}^2+q^1)\frac{\partial}{\partial \dot{q}^1}+\frac{1}{\phi}(\dot{q}^1 - X(g))\frac{\partial}{\partial \dot{q}^2}\right\}$$

each defined on the closed 4*-dimensional manifold*

$$M_g = \{x \in J^1Y \mid q^3 = \dot{q}^3 = 0,\ \dot{q}^2 - g + q^1 = 0\} \subset M \subset J^1Y,$$

where $g(t, q^1, q^2, \dot{q}^1, \dot{q}^2)$ *is an arbitrary function satisfying condition* (10).

Hence, the structure of extremals of the considered singular Lagrangian is completely described by a *family* of 4-dimensional submanifolds M_g of the 5-dimensional "final constraint submanifold" M, endowed with *semispray distributions of rank* 1. This means that every manifold M_g is foliated by one-dimensional foliation, and the family of these "constraint foliations" in M represents the structure of integral sections of the non-integrable rank 2 distribution $\mathcal{D}$ (7) on the final constraint submanifold in the evolution space.

Acknowledgments

I would like to thank to Professor Olga Krupková for supervision, advice and help, and to Professor David Saunders for valuable comments. Also I wish to acknowledge support of grant GAČR 201/09/0981 of the Czech Science Foundation, grant SGS9/PrF/2011 of the University of Ostrava and grant MEB 041005.

References

[1] D. Chinea, M. de León, J.C. Marrero: The constraint algorithm for time-dependent Lagrangians. *J. Math. Phys.* 35 (1994) 3410–3447.

[2] P.A.M. Dirac: Generalized Hamiltonian dynamics. *Canad. J. Math.* II (1950) 129–148.

[3] H.A. El-Zalan, S.I. Muslih, F.M.F. Elsabaa: The Hamiltonian-Jacobi analysis of dynamical system with singular higher order Lagrangians. *Hadronic Journal* 30 (2007) 209–220.

[4] H. Goldschmidt, S. Sternberg: The Hamilton-Cartan formalism in the calculus of variations. *Ann. Inst. Fourier, Grenoble* 23 (1973) 203–267.

[5] M.J. Gotay, J.M. Nester: Presymplectic Lagrangian systems I: the constraint algorithm and the equivalence theorem. *Ann. Inst. H. Poincaré* Sect. A, 30 (1979) 129–142.

[6] M.J. Gotay, J.M. Nester: Presymplectic Lagrangian systems II: the second order equation problem. *Ann. Inst. H. Poincaré* Sect. A, 32 (1980) 1–13.

[7] M.J. Gotay, J.M. Nester, G. Hinds: Presymplectic manifolds and the Dirac-Bergmann theory of constraints. *J. Math. Phys.* 19 (1978) 2388–2399.

[8] O. Krupková: A geometric setting for higher-order Dirac-Bergmann theory of constraints. *J. Math. Phys.* 35 (1994) 6557–6576.

[9] O. Krupková: *The Geometry of Ordinary Variational Equations.* Springer (1997).

[10] D.J. Saunders: *The Geometry of Jet Bundles.* Cambridge Univ. Press, Cambridge (2nd Ed. 2004).

Author's address:
Department of Mathematics, Faculty of Science, The University of Ostrava, 30. dubna 22, 701 03 Ostrava, Czech Republic

E-mail: moninka17@seznam.cz

Received: 30 September, 2011
Accepted for publication: 15 November, 2011

Communications in Mathematics 19 (2011) 179–206

Variational formulations I: Statics of mechanical systems.

Włodzimierz M. Tulczyjew

Abstract. Two improvements of variational formulations of mechanics are proposed. The first consists in a modification of the definition of equilibrium. The second consists in adding elements of control by external devices. In the present note the proposed improvements are applied to variational principles of statics. Numerous examples are given.

Introduction

The fundamental concept in variational formulations of physical theories is that of equilibrium. In the current literature on mechanics an equilibrium configuration is a configuration at which a function such as internal energy or action assumes a local minimum. This definition is too narrow. It excludes the treatment of dissipative systems. A definition of equilibrium based on the response to virtual displacements is proposed. This proposal does net affect the treatment of potential unconstrained systems. It allows the treatment of dissipative systems. Applying constraints to virtual displacements and not to configurations is a natural consequence of this proposal. A different interpretation of non holonomic constraints is obtained as one of the results. This modified version of non holonomic constraints applies to statics as well as dynamics.

The study of motions of an isolated object in a configuration space is the subject of geometric formulations of mechanics. Let Q be an affine configuration space modelled on a vector space V. For a potential unconstrained system a motion

$$\mathfrak{q}\colon \mathbb{R} \to Q$$

is required to satisfy the Hamilton principle

$$\delta \int_{-\infty}^{\infty} L \circ (\mathfrak{q}, \dot{\mathfrak{q}}) = 0\,.$$

2010 MSC: 70C20
Key words: variational principles, constraints, Legendre-Fenchel transformation

Here

$$L\colon Q \times V \to \mathbb{R}$$

is the Lagrangian and

$$\dot{\mathfrak{q}}\colon \mathbb{R} \to V$$

is the velocity. The Hamilton principle must be satisfied for all variations

$$\delta\mathfrak{q}\colon \mathbb{R} \to V$$

with compact support. Variations with compact support are used in order to make the integration meaningful. The Euler-Lagrange equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right) \circ (\mathfrak{q}, \dot{\mathfrak{q}}, \ddot{\mathfrak{q}}) = 0$$

follow from the variational principle.

The formulation of mechanics based on the Hamilton principle is suitable for studying motions of isolated systems such as planets. Modern formulations of mechanics should treat boundary value problems and should include elements of control theory. A motion is typically observed in a precise time interval $[t_0, t_1]$. The observed object is not created at the initial moment t_0 and does not disappear at the terminal moment t_1. The past motion of the object interacts with the motion in the time interval $[t_0, t_1]$ by supplying the initial momentum p_0 and the terminal momentum p_1 is passed onto the future motion. This type of interaction is well described by the variational principle

$$\delta \int_{t_0}^{t_1} L \circ (\mathfrak{q}, \dot{\mathfrak{q}}) = \langle p_1, \delta\mathfrak{q}(t_1)\rangle - \langle p_0, \delta\mathfrak{q}(t_0)\rangle \tag{1}$$

with free variations of the boundary configurations. This principle leads to the equations

$$p_0 = \frac{\partial L}{\partial \dot{q}}(\mathfrak{q}(t_0), \dot{\mathfrak{q}}(t_0))$$

and

$$p_1 = \frac{\partial L}{\partial \dot{q}}(\mathfrak{q}(t_1), \dot{\mathfrak{q}}(t_1))$$

in addition to the Euler-Lagrange equations satisfied inside the interval $[t_0, t_1]$.

The variational principle (1) provides a theoretical background for ballistics. It is not general enough for treatig guided missiles and not even cars or planes. External forcess applied to the object during the interval $[t_0, t_1]$ must be included. An external force represented by

$$\mathfrak{f}\colon \mathbb{R} \to V^*.$$

appears in the variational principle

$$\delta \int_{t_0}^{t_1} L \circ (\mathfrak{q}, \dot{\mathfrak{q}}) = -\int_{t_0}^{t_1} \langle \mathfrak{f}, \delta\mathfrak{q}\rangle + \langle p_1, \delta\mathfrak{q}(t_1)\rangle - \langle p_0, \delta\mathfrak{q}(t_0)\rangle.$$

Equations

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q}\right) \circ (\mathfrak{q}, \dot{\mathfrak{q}}, \ddot{\mathfrak{q}}) = \mathfrak{f}, \tag{2}$$

$$p_0 = \frac{\partial L}{\partial \dot q}(\mathfrak{q}(t_0), \dot{\mathfrak{q}}(t_0)),$$

and

$$p_1 = \frac{\partial L}{\partial \dot q}(\mathfrak{q}(t_1), \dot{\mathfrak{q}}(t_1))$$

follow from the principle. The equation (2) is to be satisfied in the interval $[t_0, t_1]$.

Control by external forces and boundary momenta is not the only form of control. We suggest that at least this form of control be explicitly included in modern formulations of mechanics.

This note is a part of a series of notes on variational formulations of physical theories. Static mechanical systems are considered. Formulations of dynamics of mechanical systems and field theories will follow.

Statics of mechanical systems is hardly present in modern literature. Static systems appeared in catastrophe theory. Equilibrium configurations of isolated systems defined as minima of internal energy functions were studied. Some elements of control were present. All proposed improvements are fully implemented in the present note.

1 Equilibria

1.1 Two simple examples

Example 1. Let Q be an affine space modelled on a vector space V with a Euclidean metric $g\colon V \to V^*$. A material point with configuration $q \in Q$ is connected with a spring of spring constant k to a fixed point $q_0 \in Q$. The configuration $q = q_0$ is the only stable configuration of the material point.

Example 2. The material point with configuration $q \in Q$ in Example 1 is subject to friction. The friction is measured by the coefficient ρ. The set

$$\{q \in Q; \|q - q_0\| \leqslant \rho/k\}$$

is the set of equilibrium configurations.

Definitions of equilibrium:

A) A stable equilibrium configuration is a configuration at which the internal energy of the system assumes its minimum value.

B) A configuration q is a stable equilibrium configuration if the work of each process starting at q and not ending at q is positive.

Definition A) applies to the first example. The internal energy is the function

$$U\colon Q \to \mathbb{R}\colon q \mapsto \frac{k}{2}\|q - q_0\|^2.$$

It assumes its minimum value at the configuration $q = q_0$. Definition A) does not apply to the second example.

Definition B) applies to both examples. In the first example the work of a process starting at q_1 and ending at q_2 equals $U(q_2) - U(q_1)$. This work is always positive unless $q_1 = q_0$. In the second example the work of a process from q_1 to q_2 equals

$$U(q_2) - U(q_1) + \rho \times [\text{length of process}]. \tag{3}$$

If $q_1 = q$, $q_2 = q + \Delta q \neq q$, then

$$\begin{aligned} &U(q_2) - U(q_1) + \rho \times [\text{length of process}] \\ &= \frac{k}{2}\|q - q_0 + \Delta q\|^2 - \frac{k}{2}\|q - q_0\|^2 + \rho\|\Delta q\| \\ &= k\langle g(q - q_0), \Delta q\rangle + \frac{k}{2}\|\Delta q\|^2 + \rho\|\Delta q\|\,. \end{aligned}$$

Let

$$\|q - q_0\| > \rho/k.$$

Choose Δq in the direction opposite to $(q - q_0)$ and assume that the process is the straight segment from q to $q + \Delta q$. We have

$$\begin{aligned} &U(q_2) - U(q_1) + \rho \times [\text{length of process}] \\ &= -k\|q - q_0\|\|\Delta q\| + \rho\|\Delta q\| + \frac{k}{2}\|\Delta q\|^2. \end{aligned}$$

This quantity is negative if $\|\Delta q\|$ is small enough since

$$-k\|q - q_0\|\|\Delta q\| + \rho\|\Delta q\| < 0.$$

It follows that q is not a configuration of equilibrium.

Let

$$\|q - q_0\| \leqslant \rho/k.$$

The quantity

$$\rho \times [\text{length of process}]$$

is always positive. It assumes its lowest value

$$\rho \times [\text{length of process}] = \rho\|\Delta$$

for given q and Δq if the process is a segment of a straight line. The lowest value of the term

$$U(q_2) - U(q_1) = U(q + \Delta q) - U(q)$$

with a given $\|\Delta q\|$ is obtained when Δq points in the direction opposite to $(q - q_0)$. In this case

$$k\langle g(q - q_0), \Delta q\rangle = -k\|q - q_0\|\|\Delta q\| \geqslant 0.$$

If the process is the segment of a straight line from q to $q + \Delta q$ and the vector Δq points in the direction of $-(q - q_0)$, then

$$\begin{aligned} &U(q_2) - U(q_1) + \rho \times [\text{length of process}] \\ &= -k\|q - q_0\|\|\Delta q\| + \frac{k}{2}\|\Delta q\|^2 + \rho\|\Delta q\| > 0 \end{aligned}$$

In all other cases the value of the expression (3) is higher. It follows that q is a configuration of equilibrium.

The two examples were designed to show that variational formulations have a wider area of applicability if based on Definition B). This definition appears in the Levi-Civita formulations of mechanics. It is not present in modern geometric formulations.

1.2 Precise definitions of local equilibria

Let Q be the *configuration space* of a system. A *virtual displacement trajectory* (a *trajectory* for short) is a submanifold $\mathfrak{c} \subset Q$ homeomorphic to the interval $\mathbb{R}_+ = [0, \infty) \subset \mathbb{R}$. The submanifold $\mathfrak{c}$ the image of an embedding

$$\mathfrak{q}\colon \mathbb{R}_+ \to Q.$$

The point $q = \mathfrak{q}(0)$ is the *initial point* of the trajectory and the trajectory will be denoted by $(q, \mathfrak{c})$.

The embedding $\mathfrak{q}$ is called a *parameterization* of $(q, \mathfrak{c})$. The set of virtual displacement trajectories will be denoted by $\mathcal{P}(Q)$.

There is a *work function*

$$W_{(q,\mathfrak{c})}\colon \mathfrak{c} \to \mathbb{R}$$

defined on each trajectory $(q, \mathfrak{c})$. We introduce the mapping

$$W\colon \mathcal{P}(Q) \to \bigcup_{(q,\mathfrak{c})\in\mathcal{P}(Q)} C^\infty(\mathbb{R}|\mathfrak{c})\colon (q, \mathfrak{c}) \mapsto W_{(q,\mathfrak{c})}.$$

This mapping characterizes the system.

A configuration $q \in Q$ is a *local stable equilibrium configuration* if for each displacement trajectory $(q, \mathfrak{c})$ the work function $W_{(q,\mathfrak{c})}$ has a local minimum at q.

Let $\mathfrak{c}$ be parameterized by an embedding

$$\mathfrak{q}\colon \mathbb{R}_+ \to Q.$$

The work function can be converted to a function

$$\tilde{W}_{\mathfrak{q}}\colon \mathbb{R}_+ \to \mathbb{R}\colon s \mapsto W_{(q,\mathfrak{c})}(\mathfrak{q}(s))$$

of the parameter. The first order necessary condition of equilibrium for a configuration q states that for each trajectory $(q, \mathfrak{c})$ the derivative of the work function $\tilde{W}_{\mathfrak{q}}$ satisfies

$$\mathrm{D}\tilde{W}_{\mathfrak{q}}(0) \geqslant 0. \tag{4}$$

This condition is parameterization independent.

Only the first order differential conditions are used in variational formulations of physical theories. For the purpose of studying the first order differential criteria virtual displacemements are well represented by vectors $\delta q \in \mathsf{T}Q$ tangent to trajectories and the set of work functions is represented by a *work form*

$$\sigma \colon \mathsf{T}Q \to \mathbb{R}$$

derived from the differentials of work functions. The work form is positive homogeneous in the sense that

$$\sigma(k\delta q) = k\sigma(\delta q)$$

if $k \geqslant 0$. The condition (4) assumes the form

$$\sigma(\delta q) \geqslant 0$$

for each vector δq at q.

1.3 Constraints

An unconstrained system is characterized by a work function

$$W_{(q,\mathfrak{c})} \colon \mathfrak{c} \to \mathbb{R} \tag{5}$$

defined on each trajectory $(q, \mathfrak{c})$. The mapping

$$W \colon \mathcal{P}(Q) \to \bigcup_{(q,\mathfrak{c})\in\mathcal{P}(Q)} C^\infty(\mathbb{R}|\mathfrak{c}) \colon (q, \mathfrak{c}) \mapsto W_{(q,\mathfrak{c})}.$$

is also used. *Constraints* are conditions imposed on trajectories by specifying a subset $\mathcal{C}$ of the set of all displacement trajectories. Trajectories in $\mathcal{C}$ are said to be *admissible*. A work function (5) is assigned to admissible trajectories. Let C^0 be the set of initial configurations of all admissible displacement trajectories. Constraints are said to be *holonomic* if $\mathcal{C}$ is the set of all displacement trajectories included in C^0. In other cases constraints are said to be *non holonomic*.

A system is characterized by the pair $(\mathcal{C}, W)$, with

$$W \colon \mathcal{C} \to \bigcup_{(q,\mathfrak{c})\in\mathcal{C}} C^\infty(\mathbb{R}|\mathfrak{c}) \colon (q, \mathfrak{c}) \mapsto W_{(q,\mathfrak{c})}.$$

A configuration $q \in C^0$ is a *local stable equilibrium configuration* of a constrained system if for each displacement trajectory $(q, \mathfrak{c}) \in \mathcal{C}$ the work function $W_{(q,\mathfrak{c})}$ has a local minimum at q.

For the purpose of formulating the first differential order necessary condition of local equilibrium the system is characterized by a *virtual work function*

$$\sigma \colon C^1 \to \mathbb{R}$$

defined on a *constraint set* $C^1 \subset \mathsf{T}Q$. For each

$$q \in C^0 = \tau_Q(C^1)$$

the set

$$C^1_q = C^1 \cap \mathsf{T}_q Q$$

is a cone in the sense that if

$$\delta q \in C^1_q,$$

then

$$\lambda \delta q \in C^1_q$$

for each

$$\lambda \geqslant 0.$$

A vector δq is said to be *tangent* to a set $C^0 \in Q$ if there is a curve

$$\gamma \colon \mathbb{R} \to Q$$

such that $\gamma([0,\infty)) \subset C^0$ and $\delta q = \mathsf{t}\gamma(0)$. The set of vectors tangent to C^0 is the *tangent set* of C^0 denoted by $\mathsf{T}C^0$. Constraints are said to be *holonomic* if $C^1 = \mathsf{T}C^0$. Otherwise constraints are said to be *non holonomic*. The inclusion

$$C^1 \subset \mathsf{T}C^0$$

is usually verified.

The virtual work function is a homogeneous form in the sense that

$$\sigma(\lambda\delta q) = \lambda\sigma(\delta q)$$

if

$$\lambda \geqslant 0.$$

The necessary condition of local equilibrium states that a configuration $q \in C^0$ is an *equilibrium configuration* of the static system

$$(C^1, \sigma)$$

if the inequality

$$\sigma(\delta q) \geqslant 0$$

is satisfied for each virtual displacement

$$\delta q \in C^1_q.$$

2 Control of mechanical system by external forces

2.1 Composed systems

Let two static systems with the same configuration space Q be characterized by

$$({C^1}_1, \sigma_1)$$

and

$$({C^1}_2, \sigma_2)$$

respectively. Then the system constructed by coupling the two systems is characterized by

$$(C^1, \sigma)$$

with

$$C^1 = {C^1}_1 \cap {C^1}_2$$

and

$$\sigma = \sigma_1 | C^1 + \sigma_2 | C^1.$$

Certain regularity is assumed in this construction of the coupled system. Some possible irregularities will be discussed separately. The construction of the coupled system is certainly valid when one of the systems is unconstrained.

2.2 Control

Equilibrium configurations of an isolated system are not of much interest. A static system is usually subjected to *control* by being coupled to an external system. The work function σ together with the constraint set C^1 provides complete information on the response of a static system to control. Equilibrium configurations $q \in C^0 \cap F^0$ of a static system characterized by (C^1, σ) coupled to an external system represented by (F^1, φ) are determined by the *virtual work principle*

$$\sigma(\delta q) + \varphi(\delta q) \geqslant 0 \quad \text{for each virtual displacement } \delta q \in C^1_q \cap F^1_q.$$

2.3 The Legendre-Fenchel transformation, the constitutive set

A static system is said to be *regular* if $C^1 = \mathsf{T}Q$, there is a function

$$U \colon Q \to \mathbb{R},$$

and the virtual work form is derived from the potential U according to

$$\sigma \colon \mathsf{T}Q \to \mathbb{R} \colon \delta q \mapsto \langle \mathrm{d}U, \delta q \rangle.$$

Control by regular external systems is of special interest. Equilibrium configurations $q \in C^0$ of a static system (C^1, σ) controlled by a regular system represented by $(\mathsf{T}Q, \mathrm{d}U)$ are determined by

$$\sigma(\delta q) + \langle \mathrm{d}U, \delta q \rangle \geqslant 0 \quad \text{for each virtual displacement } \delta q \in C^1_q. \tag{6}$$

Note that only the differential $\mathrm{d}U(q)$ of the potential U appears in the virtual work principle (6). Two controlling regular systems $(\mathsf{T}Q, \mathrm{d}U_1)$ and $(\mathsf{T}Q, \mathrm{d}U_2)$ will have the same effect at q if

$$\mathrm{d}U_2(q) = \mathrm{d}U_1(q).$$

This equality establishes an equivalence relation of controlling regular systems at q. A suitable representant of the equivalence class of a system $(\mathsf{T}Q, \mathrm{d}U)$ at q is the covector

$$f = -\mathrm{d}U(q) \in \mathsf{T}^*_q Q. \tag{7}$$

Due to the presence of constraints two different covectors f_1 and f_2 in T^*_qQ will still have the the same effect if

$$\langle f_2, \delta q\rangle = \langle f_1, \delta q\rangle \quad \text{for each virtual displacement } \delta q \in C^1_q.$$

This could lead to a further classification of controlling devices different for different controlled systems. The covector (7) is a completely universal characteristic of a regular controlling system $(\mathsf{T}Q, \mathrm{d}U)$ at q. An *external force* will be the term used for this covector.

An alternative representation of a static system (C^1, σ) is provided by the *constitutive set*

$$S = \{f \in \mathsf{T}^*Q; q = \pi_Q(f) \in C^0, \forall_{\delta q \in C^1_q} \sigma(\delta q) - \langle f, \delta q\rangle \geqslant 0\} \tag{8}$$

The passage from the objects (C^1, σ) characterizing a system to the constitutive set S is the *Legendre-Fenchel transformation* known in convex analysis. The constitutive set provides a complete characterization of a *convex* system. For a convex system the objects C^1 and σ can be reconstructed from the constitutive set.

3 Examples of static systems

The geometric structure used in formulations of statics with external forces is the diagram

$$\begin{array}{c} (\mathsf{T}^*Q, \langle\,,\rangle) \\ {\scriptstyle \pi_Q}\big\downarrow \\ Q \end{array} \tag{9}$$

It is the cotangent fibration of the configuration space Q with the canonical pairing

$$\langle\,,\rangle\colon \mathsf{T}^*Q \underset{(\pi_Q, \tau_Q)}{\times} \mathsf{T}Q \to \mathbb{R}.$$

If Q is an affine space modelled on a vector space V, then the cotangent bundle is identified with $Q \times V^*$ and the mapping π_Q is the canonical projection

$$\pi_Q\colon Q \times V^* \to Q\colon (q, f) \mapsto q.$$

The component f of an element (q, f) of the phase space T^*Q is the external force applied to the material point at configuratiom q. The tangent bundle $\mathsf{T}Q$ is identified with the product $Q \times V$ and the tangent projection is represented by the canonical projection

$$\tau_Q\colon Q \times V \to Q\colon (q, \delta q) \mapsto q.$$

The fibre product of the cotangent bundle wth the tangent bundle is the space of elements $(q, f), (q, \delta q)$ in $(Q \times V^*) \times (Q \times V)$. The pairing $\langle\,,\rangle$ is defined by

$$\langle (q, f), (q, \delta q)\rangle = \langle f, \delta q\rangle.$$

The diagram (9) takes the form

$$\begin{array}{c}(Q \times V^*, \langle , \rangle) \\ \pi_Q \Big\downarrow \\ Q\end{array}$$

The response of a static system to control by esternal forces is described by the constitutive set (8).

Example 3. A material point with configuration q in an affine space Q is tied to a fixed point $q_0 \in Q$ with a spring of spring constant k. The model space is a Euclidean vector space V with a metric tensor

$$g \colon V \to V^*.$$

The system is regular. The internal energy of the system is the function

$$U \colon Q \to \mathbb{R} \colon q \mapsto \frac{k}{2}\|q - q_0\|^2.$$

This function generates the constitutive set

$$S = \{(q, f) \in Q \times V^*; f = kg(q - q_0)\}.$$

Example 4. A material point with configuration q in a Euclidean affine space Q is tied to a fixed point with configuration q_0 with a rigid rod of length a. The configuration q is constrained to the sphere

$$C^0 = \{q \in Q; \|q - q_0\| = a\}.$$

This is a system with a holonomic bilateral constraint. The set

$$C^1 = \{(q, \delta q) \in Q \times V; \|q - q_0\| = a, \langle g(q - q_0), \delta q\rangle = 0\}$$

of admissible virtual displacements is the tangent set $\mathsf{T}C^0$ of the holonomic constraint C^0. With the virtual work form $\sigma = 0$ the constitutive set is the set

$$S = \{(q, f) \in Q \times V^*; \|q - q_0\| = a, f = a^{-2}\langle f, q - q_0\rangle g(q - q_0)\}.$$

Example 5. The rigid rod of the Example 4 is replaced by a flexible string of length a. The configuration q is constrained to the closed ball

$$C^0 = \{q \in Q; \|q - q_0\| \leqslant a\}.$$

This is a system with a holonomic unilateral constraint. The set

$$C^1 = \{(q, \delta q) \in Q \times V; \|q - q_0\| \leqslant a, \langle g(q - q_0), \delta q\rangle \leqslant 0 \text{ if } \|q - q_0\| = a\}$$

of admissible virtual displacements is the tangent set $\mathsf{T}C^0$ of the configuration constraint C^0. With the virtual work form $\sigma = 0$ the constitutive set is the set

$$\begin{aligned} S = \{(q, f) \in Q \times V^*; \|q - q_0\| \leqslant a, f = 0 \text{ if } \|q - q_0\| \leq a, \\ f = \|f\| a^{-1} g(q - q_0) \text{ if } \|q - q_0\| = a\}.\end{aligned}$$

Example 6. Let Q be a Riemannian manifold with a metric tensor

$$g\colon \mathsf{T}Q \to \mathsf{T}^*Q.$$

A material point with configuration $q \in Q$ is subject to homogeneous, isotropic friction. The virtual work form is the mapping

$$\sigma\colon \mathsf{T}Q \to \mathbb{R}\colon \delta q \mapsto \rho\sqrt{\langle g(\delta q), \delta q\rangle}$$

with $\rho \geqslant 0$. The principle of virtual work is the inequality

$$\rho\sqrt{\langle g(\delta q), \delta q\rangle} \geqslant 0$$

satisfied for each virtual displacement $\delta q \in \mathsf{T}Q$. This inequality is obviously satisfied at each $q \in Q$ for each virtual displacement $\delta q \in \mathsf{T}_qQ$. Hence each configuration is an equilibrium configuration of the system. A covector $f \in \mathsf{T}^*Q$ is in the constitutive set if the inequality

$$\rho\|\delta q\| - \langle f, \delta q\rangle \geqslant 0$$

is satisfied for each virtual displacement δq such that $\tau_Q(\delta q) = \pi_Q(f)$. Let f be in the constitutive set. By using $\delta q = g^{-1}(f)$ in the preceding inequality we arrive at

$$\|f\|^2 \leqslant \rho\|f\|.$$

Hence,

$$\|f\| \leqslant \rho.$$

The inequality

$$\langle f, \delta q\rangle \leqslant \|f\|\|\delta q\|$$

is the result of the Schwarz inequality applied to the pair of vectors $g^{-1}(f)$ and δq such that $\tau_Q(\delta q) = \pi_Q(f)$. If $\|f\| \leqslant \rho$, then

$$\langle f, \delta q\rangle \leqslant \rho\|\delta q\|.$$

Hence, f is in the constitutive set. We conclude that the constitutive set of the system is the set

$$S = \{f \in \mathsf{T}^*Q; \|f\| \leqslant \rho\}.$$

Example 7. This is the affine version of Example 6. Let the configuration space Q be an affine space modelled on a Euclidean vector space V. The material point is not constrained and is subject to isotropic static friction. The virtual work is the function

$$\sigma\colon Q \times V \to \mathbb{R}\colon (q, \delta q) \mapsto \rho(q)\|\delta q\| = \rho(q)\sqrt{\langle g(\delta q), \delta q\rangle}.$$

The set

$$S = \{(q, f) \in Q \times V^*; \forall_{\delta q \in V}\, \rho(q)\|\delta q\| \geqslant \langle f, \delta q\rangle\} \tag{10}$$

is the constitutive set. Let $(q, f) \in S$. By setting $\delta q = g^{-1}(f)$ in the inequality

$$\rho(q)\|\delta q\| \geqslant \langle f, \delta q\rangle$$

we obtain the inequality

$$\rho(q)\|f\| \geqslant \|f\|^2.$$

Hence,

$$S \subset \{(q,f) \in Q \times F; \|f\| \leqslant \rho(q)\}.$$

Let (q,f) satisfy the inequality

$$\|f\| \leqslant \rho(q).$$

The relation

$$\langle f, \delta q\rangle \leqslant |\langle f, \delta q\rangle| \leqslant \|f\|\|\delta q\| \leqslant \rho(q)\|\delta q\|$$

is derived from the Schwarz inequality

$$|\langle f, \delta q\rangle| \leqslant \|f\|\|\delta q\|.$$

We have shown that

$$S = \{(q,f) \in Q \times F; \|f\| \leqslant \rho(q)\}.$$

Example 8. The material point with configuration $q \in Q$ in Example 3 is subject to friction. The virtual work form is the mapping

$$\sigma\colon Q \times V \to \mathbb{R}\colon (q, \delta q) \mapsto k\langle g(q - q_0), \delta q\rangle + \rho(q)\|\delta q\|.$$

The constitutive set is the set

$$S = \{(q,f) \in Q \times V^*; \forall_{\delta q \in V} k\langle g(q - q_0), \delta q\rangle + \rho(q)\|\delta q\| \geqslant \langle f, \delta q\rangle\}.$$

This set is the set constitutive set (10) of Example 7 with f replaced by

$$f - kg(q - q_0).$$

The expression

$$S = \{(q,f) \in Q \times V^*; \|f - kg(q - q_0)\| \leqslant \rho(q)\}$$

for the constitutive set is the result.

Example 9. Let M be an affine plane modelled on a Euclidean vector space V. The configuration space of a skate is the set $Q = M \times D$, where D is the projective space of directions in the affine space M. We use the Euclidean metric in M to identify the space D with the unit circle

$$D = \{\vartheta \in V; \langle g(\vartheta), \vartheta\rangle = 1\}.$$

Virtual displacements are elements of the space $M \times V \times \mathsf{T}D$, where

$$\mathsf{T}D = \{(\vartheta, \delta\vartheta) \in D \times V; \langle g(\vartheta), \delta\vartheta\rangle = 0\}.$$

The skate is a system with non holonomic constraints. The set C^0 is the entire space Q. The constraint consists in restricting virtual displacements in M to those parallel to the direction specified by an element of D. Thus

$$C^1 = \{(x, \delta x, \vartheta, \delta\vartheta) \in M \times V \times \mathsf{T}D; \exists_{\lambda \in \mathbb{R}} \delta x = \lambda\vartheta\}.$$

The constitutive set is a subset of the space $Q \times V^* \times \mathsf{T}^*D$. The space T^*D is specified as the set of pairs (ϑ, τ), where ϑ is in D and τ is in the quotient space $V^*/\mathsf{T}^\circ_\vartheta D$, where the space $\mathsf{T}^\circ_\vartheta D$ is the polar of the space $\mathsf{T}_\vartheta D \subset V$. The quotient space $V^*/\mathsf{T}^\circ_\vartheta D$ is dual to $\mathsf{T}_\vartheta D$. The set

$$\begin{aligned} S &= \{(x, f, \vartheta, \tau) \in Q \times V^* \times \mathsf{T}^*D; \langle f, \delta x\rangle + \langle \tau, \delta\vartheta\rangle = 0 \\ &\quad \text{for each } (x, \delta x, \vartheta, \delta\vartheta) \in C^1\} \\ &= \{(x, f, \vartheta, \tau) \in \mathsf{T}^*Q; \langle f, \vartheta\rangle = 0, \tau = 0\} \end{aligned}$$

is the constitutive set of the system with the virtual work form $\sigma = 0$. Let the skate be subject to friction represented by a non negative function $\rho\colon Q \to \mathbb{R}$. The virtual work is the function

$$\sigma\colon C^1 \to \mathbb{R}\colon (x, \delta x, \vartheta, \delta\vartheta) \mapsto \rho(x, \vartheta)\|\delta x\| = \rho(x, \vartheta)\sqrt{\langle g(\delta x), \delta x\rangle}.$$

The set

$$S = \{(x, f, \vartheta, \tau) \in \mathsf{T}^*Q; \forall_{(x,\delta x,\vartheta,\delta\vartheta) \in C^1} \rho(x, \vartheta)\|\delta x\| \geqslant \langle f, \delta x\rangle + \langle \tau, \delta\vartheta\rangle\}$$

is the constitutive set. The equality $\tau = 0$ is obtained by setting $\delta x = 0$ in the inequality

$$\rho(x, \vartheta)\|\delta x\| \geqslant \langle f, \delta x\rangle + \langle \tau, \delta\vartheta\rangle$$

with arbitrary $\delta\vartheta$. By setting $\delta x = \lambda\vartheta$ we arrive at the inequality

$$\rho(x, \vartheta)|\lambda| \geqslant \lambda\langle f, \vartheta\rangle$$

for each $\lambda \in \mathbb{R}$. The inequality must be satisfied for $\lambda = \langle f, \vartheta\rangle$. Hence

$$\rho(x, \vartheta)|\langle f, \vartheta\rangle| \geqslant \langle f, \vartheta\rangle^2$$

and $|\langle f, \vartheta\rangle| \leqslant \rho(x, \vartheta)$. If $|\langle f, \vartheta\rangle| \leqslant \rho(x, \vartheta)$, then

$$\rho(x, \vartheta)|\lambda| \geqslant |\lambda||\langle f, \vartheta\rangle| \geqslant \langle f, \lambda\vartheta\rangle$$

for each $\lambda \in \mathbb{R}$. It follows that the virtual work principle is satisfied. In conclusion we obtain the expression

$$S = \{(x, f, \vartheta, \tau) \in \mathsf{T}^*Q; |\langle f, \vartheta\rangle| \leqslant \rho(x, \vartheta), \tau = 0\}$$

for the constitutive set of the system.

Example 10. Let Q be the affine physical space. The example gives a formal description of experiments performed by Coulomb in his study of static friction. Let a material point be constrained to the set

$$C^0 = \{q \in Q; \langle g(k), q - q_0 \rangle \geqslant 0\},$$

where q_0 is a point in Q and $k \in V$ is a unit vector. The boundary

$$\partial C^0 = \{q \in Q; \langle g(k), q - q_0 \rangle = 0\}$$

is a plane passing through q_0 and orthogonal to k. In its displacements along the boundary the point encounters friction proportional to the component of the external force pressing the point against the boundary. The system is characterized by the virtual work function $\sigma = 0$ defined on the non holonomic constraint

$$\begin{aligned} C^1 = \{(q, \delta q) \in Q \times V; &\langle g(k), q - q_0 \rangle \geqslant 0, \\ &\langle g(k), \delta q \rangle \geqslant \nu \sqrt{\|\delta q\|^2 - \langle g(k), \delta q)\rangle^2} \text{ if } \langle g(k), q - q_0 \rangle = 0\}, \end{aligned}$$

where $\nu > 0$ is the coefficient of friction. The inequality

$$\langle g(k), \delta q \rangle \geqslant \nu \sqrt{\|\delta q\|^2 - \langle g(k), \delta q)\rangle^2}$$

defines a cone in the tangent space $\mathsf{T}_q Q$. The axis of the cone is the vector k and the angle 2ϑ such that $\nu = \cot \vartheta$ is the aperture. The principle of virtual work states that (q, f) is in the constitutive set S if and only if the inequality

$$\langle f, \delta q \rangle \leqslant 0$$

is satisfied for each $(q, \delta q) \in C^1$. If the material point is not on the boundary, then $\langle g(k), q - q_0 \rangle > 0$. The virtual displacements are not constrained and a pair $(q, f) \in Q \times V^*$ is in the constitutive set S if and only if $f = 0$. If the material point is on the boundary, then $\langle g(k), q - q_0 \rangle = 0$. We show that in this case a pair (q, f) is in the constitutive set if and only if the inequality

$$\sqrt{\|f\|^2 - \langle f, k \rangle^2 \rangle} + \nu \langle f, k \rangle \leqslant 0$$

is satisfied. If $f = -\|f\| g(k)$, then (q, f) is in the constitutive set and $\|f\|^2 - \langle f, k \rangle^2 = 0$. Let (q, f) be in the constitutive set and let $\|f\|^2 - \langle f, k \rangle^2 \neq 0$. The virtual displacement $(q, \delta q)$ with

$$\delta q = g^{-1}(f) - \langle f, k \rangle k + \nu \sqrt{\|f\|^2 - \langle f, k \rangle^2} k$$

is in C^1 since

$$\langle g(k), \delta q \rangle = \nu \sqrt{\|f\|^2 - \langle f, k \rangle^2}.$$

From the principle of virtual work and

$$\langle f, \delta q \rangle = \|f\|^2 - \langle f, k \rangle^2 + \nu \sqrt{\|f\|^2 - \langle f, k \rangle^2} \langle f, k \rangle$$

it follows that

$$\|f\|^2 - \langle f,k\rangle^2 + \nu\sqrt{\|f\|^2 t - \langle f,k\rangle^2}\langle f,k\rangle \leqslant 0$$

and

$$\sqrt{\|f\|^2 - \langle f,k\rangle^2)} + \nu\langle f,k\rangle \leqslant 0$$

since $\|f\|^2 - \langle f,k\rangle^2 > 0$.

The Schwarz inequality

$$|\langle g(u),v\rangle - \langle g(k),u\rangle\langle g(k),v\rangle| \leqslant \sqrt{\|u\|^2 - \langle g(k),u\rangle^2}\sqrt{\|v\|^2 - \langle g(k),v\rangle^2}$$

for the bilinear symmetric form

$$(u,v) \mapsto (u - \langle g(k),u\rangle k | v - \langle g(k),v\rangle k) = \langle g(u),v\rangle - \langle g(k),u\rangle\langle g(k),v\rangle$$

applied to the pair $(g^{-1}(f), \delta q)$ leads to the inequality

$$\langle f,\delta q\rangle - \langle f,k\rangle\langle g(k),\delta q\rangle \leqslant \sqrt{\|f\|^2 - \langle f,k\rangle^2}\sqrt{\|\delta q\|^2 - \langle g(k),\delta q\rangle^2}.$$

If

$$\sqrt{\|f\|^2 - \langle f,k\rangle^2)} + \nu\langle f,k\rangle \leqslant 0$$

and

$$\langle g(k),\delta q\rangle \geqslant \nu\sqrt{\|\delta q\|^2 - \langle g(k),\delta q\rangle^2},$$

then

$$\sqrt{\|f\|^2 - \langle f,k\rangle^2}\sqrt{\|\delta q\|^2 - \langle g(k),\delta q\rangle^2} \leqslant -\langle f,k\rangle\langle g(k),\delta q\rangle.$$

It follows that $\langle f,\delta q\rangle \leqslant 0$. Hence, (q,f) is in the constitutive set S. We have shown that the set

$$\begin{aligned} S = \{(q,f) \in Q \times V^*; \langle g(k), q - q_0\rangle \geqslant 0, f = 0 \text{ if } \langle g(k), q - q_0\rangle > 0 \\ \text{and } \sqrt{\|f\|^2 - \langle f,k\rangle^2} + \nu\langle f,k\rangle \leqslant 0 \text{ if } \langle g(k), q - q_0\rangle = 0\} \end{aligned}$$

is the constitutive set of the system. The inequality

$$\sqrt{\|f\|^2 - \langle f,k\rangle^2} + \nu\langle f,k\rangle \leqslant 0$$

means that the vector $g^{-1}(f)$ is inside a cone in the tangent space $\mathsf{T}_q Q$. The vector $-k$ is the axis of the cone and the angle 2ϑ such that $\nu = \cot\vartheta$ is the aperture.

4 Partial control of static systems

We have considered control of static systems through interaction with systems with the same configuration space. This is not always the case. One can in general associate three distinct configuration spaces with a static system: the *internal configuration space* $\bar{Q}$, the *control configuration space* Q, and the *observed configuration space* $\tilde{Q}$. There are differential relations connecting the three spaces.

We will consider the cases when a static system with a configuration space $\bar{Q}$ is controlled by external devices in a configuration space Q and the relation between the two spaces is a differential fibration $\eta\colon \bar{Q} \to Q$. The configuration space $\bar{Q}$ of the controlled system is the internal configuration space and the configuration space Q of the controlling devices is the control configuration space. We will refer to such situations as cases of *partial control*. The observed configuration space $\tilde{Q}$ will coincide either with Q or with $\bar{Q}$.

4.1 Families of functions

An *internal energy function*

$$\bar{U} \colon \bar{Q} \to \mathbb{R}$$

is interpreted as a *family* of functions defined on fibres of the fibration η. The symbol $(\bar{U}, \eta)$ is used to denote this family.

A generating family $(\bar{U}, \eta)$ generates the constitutive set

$$S = \{f \in \mathsf{T}^*Q; \exists_{\bar{q} \in \bar{Q}} \eta(\bar{q}) = \pi_Q(f) \forall_{\delta\bar{q} \in \mathsf{T}_{\bar{q}}\bar{Q}} \langle \mathrm{d}\bar{U}, \delta\bar{q} \rangle = \langle f, \mathsf{T}\eta(\delta\bar{q}) \rangle\} \tag{11}$$

of a partially controlled system.

We denote by $\mathsf{V}\bar{Q}$ the subbundle

$$\{\delta\bar{q} \in \mathsf{T}\bar{Q}; \mathsf{T}\eta(\delta\bar{q}) = 0\}$$

of vertical vectors. The set

$$Cr(\bar{U}, \eta) = \{\bar{q} \in \bar{Q}; \langle \mathrm{d}\bar{U}, \delta\bar{q} \rangle = 0 \text{ for each } \delta\bar{q} \in \mathsf{V}_{\bar{q}}\bar{Q}\}$$

is called the *critical set* of the family. If $\bar{q}$ satisfies the conditions stated in the definition of S, then the equality $\langle \mathrm{d}\bar{U}(\bar{q}), \delta\bar{q} \rangle = 0$ is obtained with $\delta q = 0$ and any vertical vector $\delta\bar{q} \in \bar{Q}_{\bar{q}}$. It follows that $\bar{q} \in Cr(\bar{U}, \eta)$.

There is a mapping

$$\kappa(\bar{U}, \eta) \colon Cr(\bar{U}, \eta) \to \mathsf{T}^*Q$$

characterized by

$$\langle \kappa(\bar{U}, \eta)(\bar{q}), \delta q \rangle = \langle \mathrm{d}\bar{U}, \delta\bar{q} \rangle$$

for each $\delta q \in \mathsf{T}_{\eta(\bar{q})}Q$ and each $\delta\bar{q} \in \mathsf{T}_{\bar{q}}\bar{Q}$ such that $\mathsf{T}\eta(\delta\bar{q}) = \delta q$. The constitutive set is the image of $\kappa(\bar{U}, \eta)$. Note that if

$$\kappa(\bar{U}, \eta)(\bar{q}) = f,$$

then

$$\pi_Q(f) = \eta(\bar{q}).$$

The constitutive set (11) describes the relation between the controlling force and the controlled configuration. It is used when the controlled configuration is the observed configuration. If the internal configuration is observed, then the constitutive set

$$\tilde{S} = \{(\bar{q}, f) \in \bar{Q} \times \mathsf{T}^*Q; \bar{q} \in Cr(\bar{U}, \eta), f = \kappa(\bar{U}, \eta)(\bar{q})\}$$

should be used.

4.2 Reduction of generating families

Let $(\bar{U}, \eta)$ be a family generating the set (11). We have the following obvious proposition.

Proposition 1. *Let $\bar{q} \in Cr(\bar{U}, \eta)$. The single point set*

$$S_{\bar{q}} = \{f \in \mathsf{T}^*Q; \pi_Q(f) = \eta(\bar{q}) \forall_{\delta\bar{q} \in \mathsf{T}_{\bar{q}}\bar{Q}} \mathrm{d}\bar{U}(\delta\bar{q}) = \langle f, \mathsf{T}\eta(\delta\bar{q})\rangle\}.$$

is represented in the form

$$S_{\bar{q}} = \{f \in \mathsf{T}^*Q; \pi_Q(f) = \eta(\bar{q}) \forall_{\delta q \in \mathsf{T}_{\eta(\bar{q})}Q} \sigma_{\bar{q}}(\delta q) = \langle f, \delta q\rangle\},$$

where

$$\sigma_{\bar{q}} \colon \mathsf{T}_{\eta(\bar{q})}Q \to \mathbb{R} \colon \delta q \mapsto \mathrm{d}\bar{U}(\delta\bar{q}), \delta\bar{q} \in \mathsf{T}_{\bar{q}}\bar{Q}, \mathsf{T}\eta(\delta\bar{q}) = \delta q. \tag{12}$$

It follows from the above proposition that if $Cr(\bar{U}, \eta)$ is the image of a section $\zeta \colon Q \to \bar{Q}$ of the fibration η then the family $(\bar{U}, \eta)$ generating the set S in (11) can be replaced by the function

$$\sigma \colon \mathsf{T}Q \to \mathbb{R} \colon (\delta q) \mapsto \sigma_{\zeta(\tau_Q(\delta q))}(\delta q),$$

where $\sigma_{\zeta(\tau_Q(\delta q))}$ is the function $\sigma_{\bar{q}}$ defined in the the formula (12) with $\bar{q} = \zeta(\tau_Q(\delta q))$. It is obvious that $\sigma = \mathrm{d}(\bar{U} \circ \zeta)$. Thus the set S is generated by the function $U = \bar{U} \circ \zeta$.

4.3 Examples

Example 11. Three material points with configurations q_0, q, and q' in the affine space Q are interconnected with springs with spring constants k_1, k_2, and k_3. The point q_0 is fixed and not controlled. The two points q and q' are not constrained. The configuration q' is not controlled. The internal configuration space is the affine space $\bar{Q} = Q \times Q$ of internal configurations $\bar{q} = (q, q')$ modelled on $V \times V$. The control configuration space is the space Q of controlled configurations q and V is the model space. The canonical projection

$$\eta \colon \bar{Q} \to Q \colon \bar{q} = (q, q') \mapsto q$$

is the relation between the two spaces. The internal energy is the function

$$\bar{U} \colon \bar{Q} \to \mathbb{R} \colon \bar{q} = (q, q') \mapsto \frac{k_1}{2}\|q - q_0\|^2 + \frac{k_2}{2}\|q' - q_0\|^2 + \frac{k_3}{2}\|q' - q\|^2.$$

The internal energy defines a family $(\bar{U}, \eta)$ of functions on fibres of the projection η. The critical set

$$Cr(\bar{U}, \eta) = \{\bar{q} = (q, q') \in \bar{Q}; (k_2 + k_3)(q' - q_0) - k_3(q - q_0) = 0\}$$

of the family is the image of the section

$$\zeta \colon Q \to \bar{Q} \colon q = q \mapsto (q, q_0 + k_3(k_2 + k_3)^{-1}(q - q_0))$$

of the projection η. The constitutive set is the set

$$S = \left\{(q, f) \in Q \times V^*; f = \frac{k_1k_2 + k_1k_3 + k_2k_3}{k_2 + k_3} g(q - q_0)\right\}.$$

Note that the presence of the material point with configuration q' can be ignored. This is due to the fact that the critical set is the image of a section of the projection η. The constitutive set is generated by the reduced internal energy function

$$U = \bar{U} \circ \zeta \colon Q \to \mathbb{R} \colon q \mapsto \frac{1}{2} \frac{k_1 k_2 + k_1 k_3 + k_2 k_3}{k_2 + k_3} \|q - q_0\|^2.$$

This is the internal energy function

$$U \colon Q \to \mathbb{R} \colon q \mapsto \frac{k}{2} \|q - q_0\|^2.$$

of Example 3 with

$$k = \frac{k_1 k_2 + k_1 k_3 + k_2 k_3}{k_2 + k_3}.$$

Example 12. The present example gives a simplified discrete model of the buckling of a rod. One end of the rod is a point in an affine space Q with configuration q constrained to the half-line

$$L = \{q \in Q; q - q_0 = \langle g(u), q - q_0 \rangle u, \langle g(u), q - q_0 \rangle > 0\}$$

starting at a point q_0 in the direction of a unit vector u. The other end is a point with configuration q' constrained to the plane

$$P = \{q' \in Q; \langle g(u), q' - q_0 \rangle = 0\}$$

through q_0 perpendicular to u. The rod can be compressed or extended in length but not bent. Its relaxed length is a and the elastic constant is k. The buckling of the rod is simulated by displacements of its end point in the plane P tied elastically to the point q_0 with a spring of spring constant k'. The configuration space $\bar{Q}$ is the product $Q \times Q$ with holonomic constraints represented by

$$C^0 = \{(q, q') \in \bar{Q}; q \in L, q' \in P\}.$$

The set

$$\begin{aligned} C^1 = \{&(q, q', \delta q, \delta q') \in \mathsf{T}\bar{Q}; q \in L, q' \in P, \\ &\delta q = \langle g(u), \delta q \rangle u, \langle g(u), \delta q' \rangle = 0\} \end{aligned}$$

of admissible virtual displacements is the tangent set of C^0. The internal energy of the system is the function

$$\bar{U} \colon C^0 \to \mathbb{R} \colon (q, q') \mapsto \frac{k}{2}(\|q - q'\| - a)^2 + \frac{k'}{2}\|q' - q_0\|^2.$$

The configuration q' is not controlled. The internal energy defines a family $(\bar{U}, \eta)$ of functions on fibres of

$$\eta \colon C^0 \to L \colon (q, q') \mapsto q.$$

The critical set is the union of sets

$$Cr_1(\bar{U},\eta) = \{(q,q') \in \bar{Q}; q \in L, q' = q_0\}$$

and

$$Cr_2(\bar{U},\eta) = \{(q,q') \in \bar{Q}; q \in L, q' \in P, (k+k')\|q'-q\| = ka\}.$$

The critical set $Cr_1(\bar{U},\eta)$ is the image of the section

$$\zeta_1 \colon L \to C^0 \colon q \mapsto (q, q_0).$$

The reduced internal energy

$$U_1 = \bar{U} \circ \zeta_1 \colon L \to \mathbb{R} \colon q \mapsto \frac{k}{2}(\|q-q_0\| - a)^2$$

generates the constitutive set

$$S_1 = \{(q,f) \in Q \times V^*; q \in L, \langle f, u\rangle = k(\|q-q_0\| - a)\}$$

The critical set $Cr_2(\bar{U},\eta)$ is not the image of a section of η. A reducion of the internal energy is still possible since the internal energy written in the form

$$\bar{U} \colon C^0 \to \mathbb{R} \colon (q,q') \mapsto \frac{k}{2}(\|q-q'\| - a)^2 + \frac{k'}{2}(\|q-q'\|^2 - \|q-q_0\|^2)$$

is a function only of the distance $\|q-q'\|$, and on the critical set $Cr_2(\bar{U},\eta)$ this distance is determined by

$$\|q-q'\| = \frac{ka}{k+k'}.$$

The result of the reduction is the function

$$U_2 \colon L \to \mathbb{R} \colon q \mapsto -k'\|q-q_0\|^2 + \text{Constant}.$$

It generates the constitutive set

$$S_2 = \{(q,f) \in Q \times V^*; q \in L, (k+k')\|q-q_0\| < ka, \langle f,u\rangle = -k'\|q-q_0\|\}.$$

The constitutive set $S = S_1 \cup S_2$ is not a submanifold of $Q \times V^*$.

Example 13. A material point with configuration q' in the affine space Q is connected to a fixed point q_0 with a rigid rod of length a. A second material point with configuration q is tied elastically to q' with a spring of spring constant k. The configuration q' is not controlled. The internal configuration space $\bar{Q}$ is the product $Q \times Q$ with holonomic constraints represented by

$$C^0 = \{(q,q') \in \bar{Q}; \|q'-q_0\| = a\}.$$

The set

$$C^1 = \{(q,q',\delta q,\delta q') \in Q \times Q \times V \times V; \|q'-q_0\| = a, \langle g(q'-q_0), \delta q'\rangle = 0\}$$

is the tangent set of C^0. The control configuration space is the space Q and the canonical projection

$$\eta\colon \bar{Q} \to Q\colon (q, q') \mapsto q$$

is the relation between the two spaces. The internal energy is the function

$$\bar{U}\colon C^0 \to \mathbb{R}\colon (q, q') \mapsto \frac{k}{2}\|q - q'\|^2$$

and

$$\begin{aligned} Cr(\bar{U}, \eta) &= \{(q, q') \in \bar{Q}; \|q' - q_0\| = a, \\ &\quad q' - q = \langle g(q' - q_0), q' - q\rangle a^{-2}(q' - q_0)\}. \\ &= \{(q, q') \in \bar{Q}; \|q' - q_0\| = a, \\ &\quad q' - q_0 = \pm a(q - q_0)\|q - q_0\|^{-1} \text{ if} q \neq q_0\}. \end{aligned}$$

is the critical set. The set

$$\begin{aligned} S = \{&(q, f) \in Q \times V^*; \|f\| = ka \text{ if } q = q_0, \\ &f = k(1 \pm a\|q - q_0\|^{-1})g(q - q_0) \text{ if } q \neq q_0\} \end{aligned}$$

is the constitutive set of the family $(\bar{U}, \eta)$. Note that the critical set is not the image of a section of η. For each control configuration q we have two different internal equilibrium configurations (q, q') if $q \neq q_0$ and an infinity of internal equilibrium configurations if $q = q_0$. The external force necessary to maintain the control configuration q depends on the internal configuration. Thus even if the internal configuration is not directly observed its presence can not be ignored. The constitutive set is the image of the injective mapping

$$\kappa_{(\bar{U},\eta)}\colon Cr(\bar{U}, \eta) \to Q \times V^*\colon (q, q') \mapsto (q, kg(q - q')).$$

If the internal configuration is observed, then the set

$$\tilde{S} = \{(q, q', f) \in Q \times Q \times V^*; (q, q') \in Cr(\bar{U}, \eta), f = kg(q - q')\}$$

can be used to describe the relation between the controlling force and the observed internal cofiguration.

4.4 Families of forms

A *generating family* of forms consists of a differential fibration

$$\eta\colon \bar{Q} \to Q$$

and a form

$$\bar{\sigma}\colon \mathsf{T}\bar{Q} \to \mathbb{R}.$$

The form $\bar{\upsilon}\sigma$ defines a family $(\bar{\upsilon}\sigma, \eta)$ of forms $\bar{\upsilon}\sigma_q$ on fibres of the fibration η. Each form $\bar{\upsilon}\sigma_q$ is the restriction of the form $\bar{\upsilon}\sigma$ to the set

$$\{\delta\bar{q} \in \mathsf{T}\bar{Q}; \eta(\tau_{\bar{Q}}(\delta\bar{q})) = q\}.$$

We denote by $\mathsf{V}\bar{Q}$ the subbundle

$$\{\delta\bar{q} \in \mathsf{T}\bar{Q}; \mathsf{T}\eta(\delta\bar{q}) = 0\}$$

of vertical vectors. The set

$$Cr(\bar{\sigma}, \eta) = \{\bar{q} \in \bar{Q}; \bar{\sigma}(\delta\bar{q}) \geqslant 0 \text{ for each } \delta\bar{q} \in \mathsf{V}_{\bar{q}}\bar{Q}\}$$

is called the *critical set* of the family.

A generating family $(\bar{\sigma}, \eta)$ generates the set

$$\begin{aligned} S = \{f \in \mathsf{T}^*Q; q = \pi_Q(f) \in Q, \exists_{\bar{q}\in\bar{Q}_q} \text{ if } \delta q \in \mathsf{T}_q Q, \\ \delta\bar{q} \in \mathsf{T}_{\bar{q}}\bar{Q}, \text{ and } \mathsf{T}\eta(\delta\bar{q}) = \delta q, \text{ then } \bar{\sigma}(\delta\bar{q}) \geqslant \langle f, \delta q\rangle\}. \end{aligned}$$

If $\bar{q}$ satisfies the conditions stated in the definition of S, then the inequality $\bar{\sigma}(\delta\bar{q}) \geqslant 0$ is obtained with $\delta q = 0$ and any vertical vector $\delta\bar{q} \in \mathsf{V}_{\bar{q}}\bar{Q}$. It follows that $\bar{q} \in Cr(\bar{\sigma}, \eta)$. Consequently,

$$S = \bigcup_{\bar{q}\in Cr(\bar{\sigma},\eta)} S_{\bar{q}} \ ,$$

where

$$\begin{aligned} S_{\bar{q}} = \{f \in \mathsf{T}^*Q; q = \pi_Q(f) = \eta(\bar{q}), \text{ if } \delta q \in \mathsf{T}_q Q, \delta\bar{q} \in \mathsf{T}_{\bar{q}}\bar{Q} \\ \text{and } \mathsf{T}\eta(\delta\bar{q}) = \delta q, \text{ then } \bar{\sigma}(\delta\bar{q}) \geqslant \langle f, \delta q\rangle\}. \end{aligned}$$

It can be shown that if $\bar{q} \in Cr(\bar{\sigma}, \eta)$, then the set $S_{\bar{q}}$ is not empty. The relation

$$\kappa(\bar{\sigma}, \eta)\colon Cr(\bar{\sigma}, \eta) \to \mathsf{T}^*Q$$

defined by

$$\operatorname{graph} \kappa(\bar{\sigma}, \eta) = \{(\bar{q}, f) \in Cr(\bar{\sigma}, \eta) \times \mathsf{T}^*Q; f \in S_{\bar{q}}\}$$

generalizes the mapping $\kappa(\bar{U}, \eta)$ introduced in Section 4.1. The constitutive set is the image of the relation. We refer to the set $S_{\bar{q}}$ as the *contribution* to the constitutive set S from the critical point $\bar{q}$.

4.5 Examples

Example 14. Let the point with configuration q' of Example 11 be subject to friction. The virtual work form is the family $(\bar{s}, \eta)$ with

$$\begin{aligned} \bar{\sigma}\colon Q \times Q \times V \times V \to \mathbb{R}\colon (q, q', \delta q, \delta q') \to k_1\langle g(q - q_0), \delta q\rangle \\ + k_2\langle g(q' - q_0), \delta q'\rangle + k_3\langle g(q' - q), \delta q' - \delta q\rangle + \rho\|\delta q'\| \end{aligned}$$

and

$$\eta\colon \bar{Q} \to Q\colon (q, q') \mapsto q.$$

With a suitable choice of $\delta q'$ the expression

$$\langle -(k_2 + k_3)g(q' - q_0) + k_3 g(q - q_0), \delta q'\rangle$$

in the definition

$$\begin{aligned} Cr(\bar{\sigma},\eta) &= \{(q,q')\in\bar{Q}; \forall_{\delta q'\in V}\langle k_2 g(q'-q_0)+k_3 g(q'-q),\delta q'\rangle+\rho\|\delta q'\|\geqslant 0\} \\ &= \{(q,q')\in\bar{Q}; \forall_{\delta q'\in V}\langle -(k_2+k_3)g(q'-q_0)+k_3 g(q-q_0),\delta q'\rangle\leqslant\rho\|\delta q'\|\} \end{aligned}$$

of the critical set can reach its maximum

$$\|(k_2+k_3)(q'-q_0)-k_3(q-q_0)\|\|\delta q'\|.$$

Hence,

$$Cr(\bar{\sigma},\eta)=\{(q,q')\in\bar{Q}; \|(k_2+k_3)(q'-q_0)-k_3(q-q_0)\|\leqslant\rho\}$$

The critical set is not the image of a section of η. The pair $(q,f)\in Q\times V^*$ is in the constitutive set if the inequality

$$\begin{aligned} k_1\langle g(q-q_0),\delta q\rangle &+ k_2\langle g(q'-q_0),\delta q'\rangle \\ &+ k_3\langle g(q'-q),\delta q'-\delta q\rangle+\rho\|\delta q'\|-\langle f,\delta q\rangle\geqslant 0 \end{aligned}$$

is satisfied for some $q'\in Q$ and all $(\delta q,\delta q')\in V\times V$. If the inequality is satisfied, then (q,q') is in the critical set and $\delta q'$ can be set to 0. The resulting inequality

$$(k_1+k_3)\langle g(q-q_0),\delta q\rangle-k_3\langle g(q'-q_0),\delta q\rangle-\langle f,\delta q\rangle\geqslant 0$$

has the solution

$$f=(k_1+k_3)g(q-q_0)-k_3 g(q'-q_0).$$

Combining this result with the definition of the critical set we obtain the final expression

$$S=\left\{(q,f)\in Q\times V^*; \left\|f-\frac{k_1k_2+k_1k_3+k_2k_3}{k_2+k_3}g(q-q_0)\right\|\leqslant\frac{k_3}{k_2+k_3}\rho\right\}$$

for the constitutive set. The presence of the internal configuration q' can not be ignored. If it is known, then the force f is obtained from (14). The internal configuration q' can be observed. The set

$$\begin{aligned} \tilde{S}=\{(q,q',f)\in\bar{Q}\times V; &\|(k_2+k_3)(q'-q_0)-k_3(q-q_0)\|\leqslant\rho, \\ & f=(k_1+k_3)g(q-q_0)-k_3 g(q'-q_0)\} \end{aligned}$$

includes the information about the internal configuration.

Example 15. The material point with configuration q' in Example 13 is subject to friction. The family $(\bar{U},\eta)$ of functions is replaced by a family of forms $(\bar{\sigma},\eta)$ with

$$\bar{\sigma}\colon C^1\to\mathbb{R}\colon (q,q',\delta q,\delta q')\mapsto k\langle g(q-q'),\delta q-\delta q'\rangle+\rho\|\delta q'\|.$$

The set

$$\begin{aligned} Cr(\bar{\sigma},\eta)=\{(q,q')\in\bar{Q}; &\|q'-q_0\|=a, \forall_{\delta q'\in V} \\ &\text{if } \langle g(q'-q_0),\delta q'\rangle=0, \text{ then } k\langle g(q-q'),\delta q'\rangle\leqslant\rho\|\delta q'\|\} \end{aligned}$$

is the critical set of the family. The maximum value of the expression

$$k\langle g(q-q'),\delta q'\rangle$$

is

$$k\|(q-q')-a^{-2}\langle g(q-q'),q'-q_0\rangle(q'-q_0)\|\|\delta q'\|.$$

Hence,

$$Cr(\bar{\sigma},\eta)=\{(q,q')\in\bar{Q};k\|(q-q')-a^{-2}\langle g(q-q'),q'-q_0\rangle(q'-q_0)\|\leqslant\rho\}. \quad (13)$$

The critical set is not a section of the projection η. The expression

$$(q-q')-a^{-2}\langle g(q-q'),q'-q_0\rangle(q'-q_0)$$

is the component of $q-q'$ orthogonal to $q'-q_0$. If $q\neq q_0$, then $q'\in C^0$ must be such that the length this of component does not exceed ρ/k. If $q=q_0$, then all configurations $q'\in C^0$ are in the critical set. The pair $(q,f)\in Q\times V^*$ is in the constitutive set if the inequality

$$k\langle g(q-q'),\delta q-\delta q'\rangle+\rho\|\delta q'\|-\langle f,\delta q\rangle\geqslant 0$$

is satisfied for some $q'\in C^0$ and all $(\delta q,\delta q')\in V\times V$ such that $\langle g(q'-q_0),\delta q'\rangle=0$. If the inequality is satisfied, then (q,q') is in the critical set and terms with $\delta q'$ can be discarded. The resulting inequality

$$k\langle g(q-q'),\delta q\rangle-\langle f,\delta q\rangle\geqslant 0$$

leads to

$$f=kg(q-q'). \quad (14)$$

The set

$$\tilde{S}=\{(q,q',f)\in\bar{Q}\times V^*;k\|(q-q')-a^{-2}\langle g(q-q'),q'-q_0\rangle(q'-q_0)\|\leqslant\rho \\ f=kg(q-q')\}$$

contains the information about the force in terms of the internal configuration q'. The description of the constitutive set obtained from (13) and (14) is too complicated to be useful.

5 Clean composition

Let C_1 and C_2 be subsets of Q. If the intersection $C_1\cap C_2$ is not empty, we say that it is *clean* if

$$\mathsf{T}C_1\cap\mathsf{T}C_2=\mathsf{T}(C_1\cap C_2).$$

Example 16. We consider the composition of two holonomic systems. The constraints and the constitutive set for the first system are represented by the sets

$$C^0{}_1=\{q\in Q;\|q-q_1\|=a\},$$

$$C^1{}_1 = \{(q, \delta q) \in Q \times V; \|q - q_1\| = a, \langle g(q - q_1), \delta q\rangle = 0\},$$

and

$$S_1 = \{(q, f) \in Q \times V^*; \|q - q_1\| = a, f = a^{-2}\langle f, q - q_1\rangle g(q - q_1)\}.$$

For the second system we have

$$C^0{}_2 = \{q \in Q; \|q - q_2\| = a\},$$

$$C^1{}_2 = \{(q, \delta q) \in Q \times V; \|q - q_2\| = a, \langle g(q - q_2), \delta q\rangle = 0\},$$

and

$$S_2 = \{(q, f) \in Q \times V^*; \|q - q_2\| = a, f = a^{-2}\langle f, q - q_2\rangle g(q - q_2)\}.$$

If the distance $\|q_2 - q_1\|$ between the centres of the spheres $C^0{}_1$ and $C^0{}_2$ is less than $2a$, then the composed system is a system with holonomic constraints. The intersection of the constraints is clean since

$$\begin{aligned} C^1 = C^1{}_1 \cap C^1{}_2 = \{(q, \delta q) \in Q \times V; \|q - q_1\| = a, \|q - q_2\| = a, \\ \langle g(q - q_1), \delta q\rangle = 0, \langle g(q - q_2), \delta q\rangle = 0\} \end{aligned}$$

is the tangent set $\mathsf{T}C^0$ of the intersection

$$C^0 = C^0{}_1 \cap C^0{}_2 = \{q \in Q; \|q - q_1\| = a, \|q - q_2\| = a\}.$$

The constitutive set

$$\begin{aligned} S = \{(q, f) \in Q \times V^*; \|q - q_1\| = a, \|q - q_2\| = a, \langle f, \delta q\rangle = 0 \text{ for each} \\ \delta q \in V \text{ such that } \langle g(q - q_1), \delta q\rangle = 0 \text{ and } \langle g(q - q_2), \delta q\rangle = 0\}. \end{aligned}$$

is obtained from the principle of virtual work. At each $q \in C^0$ the set

$$S_q = \{f \in V^*; (q, f) \in S\}$$

is the sum

$$\{f \in V^*; (q, f) \in S_1\} + \{f \in V^*; (q, f) \in S_2\}.$$

If $\|q_2 - q_1\| = 2a$, then the set

$$C^0 = C^0{}_1 \cap C^0{}_2 = \{q \in Q; \|q - q_1\| = a, \|q - q_2\| = a\}$$

has only one element $q = q_1 + \frac{1}{2}(q_2 - q_1)$. The intersection $C^1{}_1 \cap C^1{}_2$ is the set

$$\left\{(q, \delta q) \in Q \times V; q = q_1 + \frac{1}{2}(q_2 - q_1), \langle g(q_2 - q_1), \delta q\rangle = 0\right\}.$$

The intersection of constraints is not clean since this intersection is not the tangent set of C^0. With

$$C^1 = \mathsf{T}C^0 = \left\{(q, f) \in Q \times V^*; q = q_1 + \frac{1}{2}(q_2 - q_1), \delta q = 0\right\}$$

the principle of virtual work produces the constitutive set

$$S = \left\{ (q, f) \in Q \times V^*; q = q_1 + \frac{1}{2}(q_2 - q_1) \right\}.$$

This is not the correct constitutive set for the composed system. The reason of this failure is that the approximative assumption of perfect rigidity of the separate constraints is no longer realistic in the case of a composition with

$$\mathsf{T}{C^0}_1 \cap \mathsf{T}{C^0}_2 \neq \mathsf{T}({C^0}_1 \cap {C^0}_2).$$

To obtain a complete description of the composed system the precise elastic properties of the constraints must be known. A partial characterization of the system is provided by the constitutive set

$$S_1 \cap S_2 = \left\{ (q, f) \in Q \times V^*; q = q_1 + \frac{1}{2}(q_2 - q_1), f = a^{-2}\langle f, q - q_1 \rangle g(q - q_1) \right\}$$

generated by the non holonomic constraint $C^1 = {C^1}_1 \cap {C^1}_2$. Note that this constraint is not integrable since the inclusion $C^1 \subset \mathsf{T}C^0$ does not hold.

6 A geometric setting for catastrophe theory

6.1 The framework

The traditonal approach to statics consists in studying equilibrium configurations of isolated systems. Catastrophe theory introduces elements of control to this approach. Families of isolated static systems are considered instead of separate single systems. Variations of equilibria within the family are studied. Applicability of this theory is somewhat limited since only unconstrained potential systems are considered.

We adapt the framework established in Section 4.1. to the catastrophe theory point of view. The base Q of the differential fibration

$$\eta \colon \bar{Q} \to Q$$

is the *control space*. The control configurations are not controlled by external forces. They are directly set by an external control mechanism. Fibres of the fibration are *behaviour spaces*. An internal energy function

$$\bar{U} \colon \bar{Q} \to \mathbb{R}$$

is interpreted as a family $Cr(\bar{U}, \eta)$ of potentials on the behaviour spaces parameterized by control configurations. The potential

$$U_q \colon \bar{Q}_q \to \mathbb{R}$$

corresponding to a control configuration $q \in Q$ is the restriction of $\bar{U}$ to the fibre $\bar{Q}_q = \eta^{-1}(q)$. The critical set

$$Cr(\bar{U}, \eta) = \{\bar{q} \in \bar{Q}; \langle \mathrm{d}\bar{U}, \delta\bar{q} \rangle = 0 \text{ for each } \delta\bar{q} \in \mathsf{V}_{\bar{q}}\bar{Q}\}$$

with

$$\mathsf{V}\bar{Q} = \{\delta\bar{q} \in \mathsf{T}\bar{Q}; \mathsf{T}\eta(\delta\bar{q}) = 0\}$$

is the *catastrophe manifold*. Each element $\bar{q}$ of the catastrophe manifold is an equilibrium configuration for the potential $U_{\eta(\bar{q})}$. A *catastrophe* is a singularity of the *catastrophe map*

$$\chi\colon Cr(\bar{U}, \eta) \to Q$$

obtained as the restriction of the projection η to $Cr(\bar{U}, \eta)$. A singularity occurs at a point $\bar{q} \in Cr(\bar{U}, \eta)$ at which the rank of the tangent mapping

$$\mathsf{T}\chi\colon \mathsf{T}Cr(\bar{U}, \eta) \to \mathsf{T}Q$$

changes. The change of multiplicity of critical points projecting onto the same configuration q is also an indication of a singularity.

The framework requires a obvious extension to families of holonomically constrained potentials in order to accomodate examples we want to present.

6.2 Examples

Example 17. In Example 13 we used the internal energy

$$\bar{U}\colon C^0 \to \mathbb{R}\colon (q, q') \mapsto \frac{k}{2}\|q - q'\|^2$$

defined on the holonomic constraint

$$C^0 = \{(q, q') \in \bar{Q}; \|q' - q_0\| = a\}.$$

The critical set

$$\begin{aligned} Cr(\bar{U}, \eta) = \{(q, q') \in \bar{Q}; \|q' - q_0\| = a,& \\ q' - q = \langle g(q' - q_0), q' - q\rangle a^{-2}(q' - q_0)\}.& \end{aligned}$$

was obtained. This set is now interpreted as the catastrophe manifold. Let D be the unit sphere

$$\{\vartheta \in V; \langle g(\vartheta), \vartheta\rangle = 1\}.$$

The critical set is the image of the injective mapping

$$\gamma\colon \mathbb{R} \times D \to \bar{Q}\colon (r, \vartheta) \mapsto (q_0 + (a + r)\vartheta, q_0 + a\vartheta).$$

The set

$$R \times \mathbb{R} \times \{(\vartheta, \delta\vartheta) \in V \times V; \vartheta \in D, \langle g(\vartheta), \delta\vartheta\rangle = 0\}$$

is the tangent set $\mathsf{T}(\mathbb{R} \times D)$. The tangent mapping

$$\begin{aligned} \mathsf{T}\gamma\colon & \mathsf{T}(\mathbb{R} \times D) \to Q \times Q \times V \times V \\ & \colon (r, \delta r, \vartheta, \delta\vartheta) \mapsto (q_0 + (a + r)\vartheta, q_0 + a\vartheta, \delta r\vartheta + (a + r)\delta\vartheta, a\delta\vartheta) \end{aligned}$$

is injective. It follows that γ is an injective immersion. The mapping

$$\chi\colon \mathbb{R}\times D \to Q\colon (r,\vartheta)\mapsto q_0+(a+r)\vartheta$$

represents the catastrophe map. It is obtained as the composition $\eta\circ\gamma$. The rank of the tangent mapping

$$\mathsf{T}\chi\colon \mathsf{T}(\mathbb{R}\times D)\to Q\times V\colon (r,\delta r,\vartheta,\delta\vartheta)\mapsto (q_0+(a+r)\vartheta, \delta r\vartheta+(a+r)\delta\vartheta)$$

is 3 if $a+r\neq 0$ and 1 if $a+r=0$. This indicates a singularity at $q=q_0$. Specialists will refuse to recognize this singularity as a catastrophe since, as we will see in the next example, it is not stable.

Example 18. We consider a modified version of Example 13. Let

$$k\colon V\to V$$

be a linear mapping positve and symmetric in the sense that

$$\langle g(k(\delta q_1)),\delta q_2\rangle = \langle g(k(\delta q_2)),\delta q_1\rangle$$

for each pair of vectors δq_1 and δq_2 and

$$\langle g(k(\delta q)),\delta q\rangle > 0$$

unless $\delta q=0$. We use the internal energy

$$\bar{U}\colon C^0\to\mathbb{R}\colon (q,q')\mapsto \frac{1}{2}\langle g(k(q-q')),q-q'\rangle$$

defined on the holonomic constraint

$$C^0=\{(q,q')\in\bar{Q};\|q'-q_0\|=a\}.$$

The critical set

$$\begin{aligned}Cr(\bar{U},\eta)=\{(q,q')\in\bar{Q};\|q'-q_0\|=a,\\ q'-q=a^{-2}\langle g(k(q'-q)),q'-q_0\rangle k^{-1}(q'-q_0)\}.\end{aligned}$$

is obtained. If $(q,q')\in Cr(\bar{U},\eta)$ and $q=q_0$, then

$$\|q'-q_0\|=a \tag{15}$$

and

$$q'-q_0=a^{-2}\langle g(k(q'-q_0)),q'-q_0\rangle k^{-1}(q'-q_0). \tag{16}$$

A configuration q' in the set

$$\{q'\in Q;\|q'-q_0\|=a\}$$

satisfies the equality (16) if $q'-q_0$ is an eigenvector of k. The number of such eigenvectors depends on the number of eigenvalues of k. If k has three distinct eigenvalues, then the number is 6. For q sufficiently far from q_0 there are two configurations

$$q' = q_0 \pm a\|k(q - q_0)\|^{-1}k(q - q_0)$$

satisfying (15) and the equation

$$k(q - q') = a^{-2}\langle g(k(q - q')), q' - q_0\rangle(q' - q_0)$$

approximated by

$$k(q - q_0) = a^{-2}\langle g(k(q - q_0)), q' - q_0\rangle(q' - q_0).$$

It is clear that the system described in Example 13 and Example 17 is not topologically stable.

Author's address:
Valle San Benedetto 2, 62036 Monte Cavallo, Italy
Associated with Division of Mathematical Methods in Physics, University of Warsaw, Hoża 74, 00-682 Warszawa

E-mail: `tulczy@libero.it`

Received: 15 December, 2011
Accepted for publication: 21 December, 2011

Communications in Mathematics 19 (2011) 207–210

Book Review

Jaroslav Dittrich

Mathematical results in quantum physics. Edited by P. Exner. World Scientific, Singapore (2011), 274 pages, ISBN 978-981-4350-35-8

In 1980s a group of Czech mathematical physicists worked at Dubna. In 1987, a distinguished member of the group Pavel Exner together with his collaborator Petr Šeba organized there a conference on mathematical physics oriented especially to the Schrödinger operators with contact interactions. After its success the conference has been repeated and its scope has enlarged. It has moved to other places and also other groups take part in the organization. The name established as "Mathematical Results in Quantum Physics" and the acronym as QMath. On the 6–10 September 2010, QMath11 took place at Hradec Králové in the Czech Republic. 130 participant affiliated in 22 countries registered at the conference.

The Proceedings contain contributions based on the most Plenary Talks and the Invited Talks at the topical sessions. Abstracts of the other invited as well as of the contributed talks are included. A DVD with presentations of most of the talks delivered to the conference is attached for the convenience of the reader. It contains also some photographs illustrating the atmosphere. The published plenary talks are the following.

N. Datta: Relative entropies and entanglement monotones, giving the two new definitions of relative entropies and showing their use in the quantum information theory. The description of quantum states by density matrix operator on a finite dimensional Hilbert space is used.

R.L Frank, E.H. Lieb, R. Seiringer, L.E. Thomas: Binding, stability, and non-binding of multi-polaron systems. Polaron is an object consisting of an electron moving in a crystal and interacting with the crystal lattice excitations modeled by a quantized boson field. Electron is assumed localized in a region large with respect to crystal lattice spacing so it can be assumed as moving in continuum.

The authors consider a model Hamiltonian for the system of N polarons

$$\begin{aligned} H_U^{(N)} =& \sum_{j=1}^{N} \mathbf{p}_j^2 + \int a^\dagger(\mathbf{k}) a(\mathbf{k}) d\mathbf{k} \\ &+ \frac{\sqrt{\alpha}}{\sqrt{2}\pi} \sum_{j=1}^{N} \int \frac{1}{|k|} [a(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{x}_j) + a^\dagger(\mathbf{k}) \exp(-i\mathbf{k}\cdot\mathbf{x}_j)] d\mathbf{k} \\ &+ U \sum_{1\le i<j\le N} |\mathbf{x}_i - \mathbf{x}_j|^{-1} . \end{aligned}$$

Here $\mathbf{x}_j$ are coordinates of the electrons, $p_j = -i\nabla_j$ their momenta, a and $a^\dagger$ bosonic field annihilation and creation operators. Constants α and U are parameters of the model, some other constants are simplified by the choice of units. For the three-dimensional case, the Hamiltonian acts in the Hilbert space $L^2(\mathbb{R}^{3N}) \otimes \mathcal{F}$ where $\mathcal{F}$ is the Fock space of the boson field. The infimum $E_U^{(N)}(\alpha)$ of the $H_U^{(N)}$ spectrum is not an eigenvalue. The existence of binding, i.e. the relation between $E_U^{(N)}(\alpha)$ and $NE_U^{(1)}(\alpha)$, is studied for some ranges of parameters. The authors give up to date review of the known results and their own recent results. Thermodynamic stability, $N^{-1}E_U^{(N)}(\alpha) \ge -constant$ independent of N, holds for $U > 2\alpha > 0$.

A. Giuliani: Interacting electrons on the honeycomb lattice. As a model of graphene, two-dimensional one-atom thick graphite layer, a Hubbard model on a honeycomb lattice is considered. It describes hopping of electrons between lattice vertices. Interaction with the three-dimensional electromagnetic field is further introduced. Mainly the properties of ground state are discussed.

M. Lewin: Renormalization of Dirac's polarized vacuum. A mean field theory for the electrons in a atom or molecule is developed. The Hamiltonian of the Dirac equation with the self-consistent Coulomb field generated by the atomic nuclei, finite number of real electrons and virtual electrons of the Dirac sea is studied. Its spectral projection to the energies below the Fermi level is looked for. Its existence can be proved under an ultraviolet cut-off only. Renormalization of the charge is discussed. The existence of the asymptotic expansion in the renormalized coupling constant for the renormalized nuclear charge density is shown.

O. Post: Convergence result for thick graphs. The problem of approximation of the Laplacian spectral properties on a domain X_ε containing a graph X_0 by that on the graph itself (with an appropriate boundary conditions at the vertices) is discussed. The domain is assumed to be in a sense close to the graph, shrinking to the graph if a small parameter ε approaches zero. Its geometry is explained, especially the shape of neighborhoods of vertices and edges. The two Laplace-like non-negative operators H_ε and H_0 are defined in different Hilbert spaces $\mathcal{H}_\varepsilon = L^2(X_\varepsilon)$ and $\mathcal{H}_0 = L^2(X_0)$. A linear bounded operator $J : \mathcal{H}_0 \to \mathcal{H}_\varepsilon$ is needed for their comparison. Typically, the range of J contains transversally constant functions. It is said that $H_\varepsilon \to H_0$ in the generalized norm resolvent sense of order $\mathcal{O}(\varepsilon^{1/2})$ if and only if there exists J such that

$$J^*J = \mathrm{id}_0 , \quad \|(\mathrm{id}_\varepsilon - JJ^*)R_\varepsilon\| = \mathcal{O}(\varepsilon^{1/2}) , \quad \|JR_0 - R_\varepsilon J\| = \mathcal{O}(\varepsilon^{1/2})$$

where $R_\varepsilon = (H_\varepsilon + 1)^{-1}$ denotes the resolvent. Some conditions sufficient for the validity of the last relation are given and the consequences for the spectra are discussed. In the Dirichlet case, the first eigenvalue of the transverse Laplacian diverging as $\varepsilon \to 0$ must be subtracted from the operator of course.

B. Schlein: Spectral properties of Wigner matrices. Hermitian Wigner matrices are finite $N \times N$ Hermitian matrices, the entries of which are random variables, up to the hermiticity independent, with the same distribution law for the diagonal entries and the real and the imaginary parts of the nondiagonal entries. Wigner introduced these matrices as a model of heavy nuclei, they are useful as models of other complex or chaotic systems as well. A review of know spectral properties is given and the new author's result on the statistics of the spectrum is formulated.

R. Sims: Lieb-Robinson bounds and quasi-locality for the dynamics of many-body quantum systems. Roughly speaking, the velocity of disturbances propagation in a lattice of quantum systems is studied.

M. Aizenman, S. Warzel: Disorder-induced delocalization on tree graphs. Random Schrödinger operator on a regular tree graph is shown to have absolutely continuous spectrum in a suitable regime.

T. Weidl: Semiclassical spectral bounds and beyond. The Schrödinger like operator

$$H(V) = (-\Delta)^l - V(x)\,, \quad l > 0\,, \quad V(x) \geq 0\,, \quad x \in \mathbb{R}^d$$

in $L^2(\mathbb{R}^d)$ is considered and the sum

$$S_{d,\gamma}(V) = \sum_j \lambda_j^\gamma = \mathrm{Tr}(H(V))_-^\gamma\,, \quad \gamma \geq 0$$

is defined where $-\lambda_j$ are negative eigenvalues of $H(V)$. The ranges of validity of Lieb-Thirring estimates

$$S_{d,\gamma}(V) \leq R(d,\gamma,l) S_{d,\gamma}^{cl}(V)$$

are discussed where classical phase space average

$$S_{d,\gamma}^{cl}(V) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (|\xi|^{2l} - V(x))_- \,\frac{dx\,d\xi}{(2\pi)^d}\,.$$

Possible values of constants $R(d,\gamma,l)$ are studied. Similar estimate from below and several generalizations of the quantity $S_{d,\gamma}(V)$, also in a subdomain of $\mathbb{R}^d$, are investigated.

Among seven topical sessions the one devoted to the honour of Ari Laptev, the president of European Mathematical Society in 2007-2010, on the occasion of his sixtieth birthday had a special significance. Contributions on the talks of *R.D. Benguria* on spectral problems in spaces of constant curvature, *R.L. Frank and L. Geisinger* on the two-term spectral asymptotic for the Dirichlet Laplacian, *B. Helffer* on the Ginzburg-Landau functional, *V. Dinu, A. Jensen and G. Nenciu* on the resonance decay law, and *M. Loss and G. Stolz* on the localization for the random displacement model are included in the proceedings.

More evolved contributions from the other topical sessions are those of *J. Dolbeault and M.J. Esteban*, and *J. Lampart, S. Teufel and J. Wachsmuth* on the spectral theory, *W.D. Roeck*, *D. Hasler and I. Herbst*, and *G.A. Hagedorn and A. Joye* on the many-body quantum systems, *L.F. Santos and M. Rigol*, and *S. Nonnenmacher* on the quantum chaos, *M. Fiałkowski, A. Bitner and R. Hołyst*, *B. Steffen, A. Seyfried and M. Boltes*, and *D. Vašata, P. Exner and P. Šeba* on the physics of social systems.

As most of other proceedings, the book can be recommended to the reader who is looking for a brief, still up to date, information on the above mentioned topics with the references to detailed proofs.

Author's address:
Nuclear Physics Institute ASCR, CZ-250 68 Řež, Czech Republic

E-mail: dittrich@ujf.cas.cz

Received: 22 November, 2011

Aims and Scope

Communications in Mathematics publishes original research papers and survey papers in pure and applied mathematics, preferably in the following areas: *algebraic structures*, *calculus of variations*, *combinatorics*, *control and optimization*, *cryptography*, *differential equations*, *differential geometry*, *fuzzy logic and fuzzy set theory*, *global analysis*, *history of mathematics*, *mathematical physics*, *number theory*.

To be acceptable for publication papers must make original and significant contributions to one or more of the topics within these areas. High quality review papers of interest to a wide range of scientists in mathematics and its applications are welcome.

Instructions for Authors

Articles submitted for publication must be written in English, and produced in TEX, preferably using Publisher's document style based upon LATEX.

Articles should contain the title, authors' names and addresses, full contact details of the corresponding author, the abstract, keywords, and 2010 MS classification numbers.

Footnotes should be avoided if possible, but if they are included they should be brief, and numbered consecutively.

References must be as complete as possible and put in alphabetical order.

Submission to the journal proceeds totally on-line through the web page `http://cm.osu.cz`, and the author is guided stepwise through the procedure. To complete the submission successfully, the author is required to choose one of the Editors to handle the paper. He/she should choose an Editor whose areas fit as closely as possible with the topic of the paper. All correspondence, including notification of the Editor's decision, requests for revision, and proofs takes place by e-mail.

Communications in Mathematics, Volume 19, 2011, is the continuation of the journal *Acta Mathematica et Informatica Universitatis Ostraviensis*, ISSN 1211-4774 (1993–2003), and of the journal *Acta Mathematica Universitatis Ostraviensis*, ISSN 1214-8148 (2004–2009).

Published by The University of Ostrava
Typeset by TEX

ISSN 1804-1388

Geometrical aspects of variational calculus on manifolds

Guest Editor: László Kozma

Contents