VOLUME
18/2010
No. 1
ISSN 1804-1388

COMMUNICATIONS IN MATHEMATICS

Geometric Mechanics and Global Calculus of Variations

Edited by Olga Krupková and Geoff Prince

Available online at
http://cm.osu.cz

UNIVERSITAS OSTRAVIENSIS

Subscriptions

This journal is available for exchange. Information on exchange and/or subscription is available at: *Library of The University of Ostrava, Bráfova 7, 701 03 Ostrava, Czech Republic*

Website: http://cm.osu.cz

The journal is reviewed in Zentralblatt für Mathematik and Mathematical Reviews.

ISSN 1804-1388

ISBN 978-80-7368-958-2

Communications in Mathematics 18 (2010) 1–2

Editorial

From the Editor-in-Chief

Welcome to the first issue of *Communications in Mathematics*!

Let me first introduce the journal. Despite a new name, *Communications in Mathematics* is not a new publication – it has a 17 year tradition, being a continuation of the journal *Acta Mathematica et Informatica Universitatis Ostraviensis*, founded in 1993, and later published under the title *Acta Mathematica Universitatis Ostraviensis* (2003–2009).

The journal is sponsored and published by the University of Ostrava on a non-profit basis, and we are grateful to the Rector of the University, Jiří Močkoř, for his support. With his help the face of the journal has been changed. The aim of these changes has been to transform the original university journal to a more prestigious publication in mathematics and its applications. We have altered the journal's title, and appointed a new Editor-in-Chief and a completely new editorial board. We are proud to have found Editors with established, high reputations in their respective fields.

The scope of the journal has also changed to match its new mission. *Communications in Mathematics* will publish articles in pure and applied mathematics, preferably in the following areas belonging to the research interests of the individual members of the editorial board:

- algebraic structures
- calculus of variations
- combinatorics
- control and optimization
- cryptography
- differential equations
- differential geometry
- fuzzy logic and fuzzy set theory
- global analysis
- mathematical physics
- number theory

The journal will publish original research articles which make a significant contribution to one or more topics, mostly within the above areas, as well as high quality survey papers. We encourage interdisciplinary papers and those developing applications of mathematical methods. The journal will regularly present reviews of recently published books and monographs for the interest of our readers.

This issue, co-edited by Geoff Prince and me, is devoted to geometric mechanics and the global calculus of variations. It contains three original research papers, one survey article and a book review.

From 2011 the journal will usually appear in two issues per year, both in printed form and on-line. Our authors will benefit from constructive peer reviews, a distinguished editorial board ensuring high quality articles, easy electronic submission and online progress tracking, and rapid publication - online within two months of acceptance of the final version. The policy of the journal is to make it as free and accessible as possible: as a matter of course there will be the benefit of full-text access for readers, a free use of colour and multimedia in the online edition, and no page charges for either authors or readers.

Communications in Mathematics will be reviewed in Mathematical Reviews and Zentralblatt MATH, and the editors hope that the high quality of published research will soon produce a high impact factor.

I would like to encourage our readers and potential authors to submit their quality research to *Communications in Mathematics*. I believe that we shall succeed in creating a high level mathematical journal and the open exchange of important results in different branches of mathematics.

Enjoy your reading!

Olga Krupková
Editor-in-Chief

Communications in Mathematics 18 (2010) 3–19

Some geometric aspects of the calculus of variations in several independent variables

David Saunders

Abstract. This paper describes some recent research on parametric problems in the calculus of variations. It explains the relationship between these problems and the type of problem more usual in physics, where there is a given space of independent variables, and it gives an interpretation of the first variation formula in this context in terms of cohomology.

1 Introduction

In this paper we consider some geometrical aspects of those problems in the calculus of variations which are known as 'parametric': see, for example, the classical work [9] for the difference between parametric and non-parametric variational problems. To illustrate this difference in a simple way, consider the following, superficially similar, examples of the two types of problem. For the first problem, suppose we are asked to find the trajectory of a free unit-mass particle in three-dimensional space with coordinates (u^1, u^2, u^3). For the second, suppose we are asked to find the shortest curve between two points in three-dimensional space with differently-labelled coordinates (y^1, y^2, y^3). A solution to the former problem is a map $[0, T] \to \mathbb{R}^3$, $t \mapsto (a^i t + b^i)$, and a Lagrangian for the problem is $\frac{1}{2}\left((\dot{u}^1)^2 + (\dot{u}^2)^2 + (\dot{u}^3)^2\right)$. In contrast, a solution to the latter problem is a straight line segment $[(p^i), (q^i)] \subset \mathbb{R}^3$, and a Lagrangian is $\sqrt{(\dot{y}^1)^2 + (\dot{y}^2)^2 + (\dot{y}^3)^2}$: note that this latter function is 'positively homogeneous'.

More generally, variational problems in physics are commonly defined on fibred manifolds $\pi : E \to M$ (for the free particle, this would be $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$). Extremals are local sections of π, and the Lagrangian is a function (or, more properly, a differential m-form, where $m = \dim M$) defined on a jet bundle $J^1\pi$ (or $J^k\pi$) of jets of local sections of π. But in geometry, variational problems are commonly defined on manifolds E without a given fibration. Extremals are then submanifolds

2010 MSC: 35A15, 58A10, 58A20
Key words: calculus of variations, parametric problems

of E, defined 'parametrically'. To see where the Lagrangian might be defined, we need to consider different types of jet bundle and the relationships between them.

We can illustrate this by examining the relationships between a vector space, an affine space and a projective space. If V is a vector space with $\dim V = n+1$, a basis $(e_0, e_1, \ldots, e_n)$ and corresponding coordinate functions $(\dot{y}^0, \dot{y}^1, \ldots, \dot{y}^n)$, then the set

$$A = \{v \in V : \dot{y}^0(v) = 1\}$$

is an n-dimensional affine space, whereas the set

$$P = (V - \{0\})/(\mathbb{R} - \{0\})$$

is an n-dimensional projective space; there is a natural injection $A \to P$.

Now let $\pi : E \to \mathbb{R}$ be a fibred manifold, with $\dim E = n+1$ and coordinates $(y^0 = t, y^1, \ldots, y^n)$; we can apply the remark above to the fibres of the tangent bundle to E. We write $J^1\pi$ for the manifold of jets of local sections of π, and $J^1(E, 1)$ for the manifold of jets of immersed 1-dimensional submanifolds in E. The bundle $J^1\pi \to E$ is an affine bundle, and there is a canonical injection $J^1\pi \to TE$ whose image is given by $\dot{y}^0 = 1$. On the other hand, the bundle $J^1(E, 1) \to E$ is isomorphic to the projective tangent bundle $PTE \to E$, and we may identify $J^1\pi$ with an open submanifold of $J^1(E, 1)$ by mapping the jet of a local section to the jet of its image. Writing $\mathring{T}E$ for the slit tangent manifold, excluding the zero section, we may see that the bundle $\mathring{T}E \to J^1(E, 1)$ is a principal bundle with structure group $\mathbb{R} - \{0\}$.

As an application of this structure, we mention the study of Finsler geometry (see, for example, [2]), or of its special case, Riemannian geometry. Here, we take a manifold E with local coordinates y^a $(0 \leq a \leq n)$. The Lagrangian (that is, the Finsler function) L is defined on $\mathring{T}E$, and the condition of positive homogeneity is that $\dot{y}^a \partial L / \partial \dot{y}^a = L$. The variational problem is to find extremals γ of the integral

$$\int j^1\gamma^*(L)\, dt$$

subject to suitable boundary conditions. If γ is an extremal then so is $\gamma \circ \phi$ where

$$\phi : \mathbb{R} \to \mathbb{R} \text{ is a diffeomorphism}, \qquad \phi' > 0$$

The problem may also be formulated on the quotient manifold $PTE^+ = \mathring{T}E/\mathbb{R}^+$, which is a double cover of the projective tangent bundle PTE.

Our task in this paper will be to extend these structures to provide a framework for the study of multiple-integral parametric variational problems, of first or higher order. In Section 2 we shall describe a geometrical background which is appropriate for a study of these problems, and in Section 3 we shall introduce a particular class of vector forms which will turn out to useful tools for our investigation. Section 4 contains a brief reminder, for comparison, of an approach to non-parametric problems defined on spaces of jets of fibred manifolds, and finally in Section 5 we show how an analogous approach may be devised for parametric problems with positively homogeneous Lagrangians.

This paper is based upon a talk given first at the University of Ostrava, and then subsequently at the Banach Center of the Polish Academy of Sciences. The author is grateful to Olga Krupková, and to Janusz Grabowski and Paweł Urbański, for the opportunity to present these talks.

2 Geometrical background

In this section we shall describe a geometrical background which may be used for the study of parametric variational problems. Convenient references here are [3], [4]; see also [7].

Finsler geometry, a single-integral problem, is defined on the slit tangent bundle $\mathring{T}E$; first-order multiple integral problems are defined on a sub-bundle of the Whitney sum $\bigoplus^m TE$. The *bundle of regular velocities on* E is

$$\mathring{T}_{(m)}E = \{(\xi_1, \ldots, \xi_m) \in \bigoplus\nolimits^m TE : (\xi_i) \text{ linearly independent}\} \; ;$$

equivalently, we may say that $\mathring{T}_{(m)}E$ is the bundle of 'non-degenerate velocities', 1-jets $j^1_0\sigma$ at $0 \in \mathbb{R}^m$ of non-singular maps $\sigma : \mathbb{R}^m \to E$. If (y^a) are local coordinates on $Y \subset E$, then (y^a, y^a_i) $(1 \leq i \leq m)$ are local coordinates on $Y^1 \subset \mathring{T}_{(m)}E$, where

$$y^a_i(j^1_0\sigma) = \left.\frac{\partial \sigma^a}{\partial t^i}\right|_0 , \qquad y^a_i(\xi_1, \ldots, \xi_m) = \dot{y}^a(\xi_i)$$

and where $Y^1 = \tau_m^{-1}(Y)$ with $\tau_m : \mathring{T}_{(m)}E \to E$ the natural projection.

As with any manifold of jets, we may define contact forms and other related structures on $\mathring{T}_{(m)}E$. We say that a differential form $\omega \in \Omega(\mathring{T}_{(m)}E)$ is a *contact form* if the pull-back $(j^1\sigma)^*\omega$ by the prolongation of any non-singular map $\sigma : \mathbb{R}^m \to E$ always vanishes. In coordinates, contact 1-forms are linear combinations of $(m+1) \times (m+1)$ determinants like

$$\begin{vmatrix} y^{a_1}_1 & y^{a_2}_1 & \cdots & y^{a_{m+1}}_1 \\ y^{a_1}_2 & y^{a_2}_2 & \cdots & y^{a_{m+1}}_2 \\ \vdots & \vdots & & \vdots \\ y^{a_1}_m & y^{a_2}_m & \cdots & y^{a_{m+1}}_m \\ dy^{a_1} & dy^{a_2} & \cdots & dy^{a_{m+1}} \end{vmatrix}$$

and so have a more complicated expression than the contact 1-forms $du^\alpha - u^\alpha_i dx^i$ on a jet bundle.

Next, for each function $f : E \to \mathbb{R}$, define the functions $d_i f : \mathring{T}_{(m)}E \to \mathbb{R}$ by

$$d_i f(j^1_0\sigma) = \frac{\partial(f \circ \sigma)}{\partial t^i} \qquad \text{where } \sigma : \mathbb{R}^m \to E \, ;$$

the operator d_i is a globally-defined vector field along $\tau_m : \mathring{T}_{(m)}E \to E$, called a *total derivative*. It is straightforward to check that a 1-form θ is a contact form exactly when $\langle d_i, \theta \rangle = 0$ for $1 \leq i \leq m$. In coordinates, we see that

$$d_i = y^a_i \frac{\partial}{\partial y^a} \, .$$

Finally, the Whitney sum $\bigoplus^m TE \to E$ is a vector bundle, and so supports a vertical lift operation, arising from the canonical isomorphism between a vector space and its tangent space at any point. Denote the vertical lift to (η_i) by

$$\bigoplus^m T_{\tau_m(\eta_i)} E \to T_{(\eta_i)} \left(\bigoplus^m TE\right) , \qquad (\xi_i) \mapsto (\xi_i)^{\uparrow(\eta_i)} .$$

Then, for each vector $\zeta \in T_{(\eta_i)} \mathring{T}_{(m)} E$, define the vector $S^i\zeta \in T_{(\eta_i)} \mathring{T}_{(m)} E$ by

$$S^i \zeta = (0, \ldots, 0, T\tau_m(\zeta), 0, \ldots, 0)^{\uparrow(\eta_i)} .$$

With this definition S^i is a type $(1, 1)$ tensor field on $\mathring{T}_{(m)} E$, called a *vertical endomorphism*; in coordinates we have

$$S^i = dy^a \otimes \frac{\partial}{\partial y^a_i} .$$

We may also relate the bundle of regular velocities $\mathring{T}_{(m)} E$ with the *Grassmannian bundle* $J^1(E, m)$: the former is a manifold of equivalence classes of non-degenerate maps $\mathbb{R}^m \to E$, whereas the latter is a manifold of equivalence classes of images of such maps, namely of m-dimensional subspaces of TE. We see that two regular velocities $j^1_0 \sigma$, $j^1_0 \hat{\sigma}$ represent the same subspace when

$$j^1_0 \hat{\sigma} = j^1_0 (\sigma \circ \phi)$$

for some diffeomorphism $\phi : \mathbb{R}^m \to \mathbb{R}^m$ with $\phi(0) = 0$. We may also consider the bundle $J^1(E, m)^+$ of oriented Grassmannians, where the diffeomorphism ϕ must preserve the orientation on $\mathbb{R}^m$. The natural projections give principal bundles

$$\begin{aligned} \rho &: \mathring{T}_{(m)} E \to J^1(E, m) && (\text{group } GL(m, \mathbb{R})) \\ \rho^+ &: \mathring{T}_{(m)} E \to J^1(E, m)^+ && (\text{group } GL(m, \mathbb{R})^+) , \end{aligned}$$

where a basis of fundamental vector fields is given by $\{\Delta^i_j = S^i(d_j)\}$. In coordinates, we therefore have

$$\Delta^i_j = y^a_j \frac{\partial}{\partial y^a_i} .$$

Note that any fibration $\pi : E \to M$ defines open submanifolds $J^1\pi \subset J^1(E, M)$ and $J^1\pi \subset J^1(E, M)^+$. If we take $m = 1$ we recover the special cases $J^1(E, 1) = PTE$ and $J^1(E, 1)^+ = PTE^+$.

We can finally, without too much conceptual difficulty although with increased computational complexity, extend these definitions to the case of higher-order regular velocities. We shall take the manifold of k-th order regular velocities $\mathring{T}^k_{(m)} E$ to be the set of all k-jets (at the origin) of non-singular maps $\mathbb{R}^m \to E$, with local coordinates y^a_I on $\mathring{T}^k_{(m)} E$, where $I \in \mathbb{N}^m$ is a symmetric multi-index with $0 \leq |I| \leq k$. The total derivatives d_i and vertical endomorphisms S^i have coordinate representations

$$d_i = \sum_{|I|=0}^{k-1} y^a_{I+1_i} \frac{\partial}{\partial y^a_I} , \qquad S^i = \sum_{|I|=0}^{k-1} \big(I(i) + 1\big) dy^a_I \otimes \frac{\partial}{\partial y^a_{I+1_i}}$$

where different instances of each type of operator commute, so that we may use multi-index notation d_I, S^I where appropriate. We may again construct principal bundles

$$\rho^k : \mathring{T}^k_{(m)}E \to J^k(E,m)\,, \qquad \rho^{k+} : \mathring{T}^k_{(m)}E \to J^k(E,m)^+\,,$$

whose groups are the jet groups L^k_m, L^{k+}_m

$$\begin{aligned} L^k_m &= \{j^k_0\phi : \mathbb{R}^m \xrightarrow{\phi} \mathbb{R}^m \text{ is a diffeomorphism, } \phi(0)=0\} \\ L^{k+}_m &= \{j^k_0\phi \in L^k_m : |\mathcal{J}(\phi)| > 0\} \end{aligned}$$

A basis for the space of fundamental vector fields of the principal bundles is given by

$$\{\Delta^I_j = S^I(d_j) : 1 \leq |I| \leq k\}\,;$$

we put i^I_j for the contraction with Δ^I_j, and d^I_j for the Lie derivative by Δ^I_j.

3 Vector forms

In the study of the parametric calculus of variations we use vectors of operators d_i, tensors S^i, and forms ϑ^i. These fit into a framework of *vector forms* [12], and the use of these forms will provide us with a convenient tool.

We consider forms on $\mathring{T}^k_{(m)}E$ taking values in the vector space $\mathbb{R}^{m*}$ and its exterior powers: put

$$\Omega^{r,s}_k = \left(\Omega^r \mathring{T}^k_{(m)}E\right) \otimes \left(\textstyle\bigwedge^s \mathbb{R}^{m*}\right)\,.$$

Let the standard basis for $\mathbb{R}^{m*}$ be denoted by (dt^i); then a vector form Φ may be written canonically as

$$\Phi = \phi_{i_1\cdots i_s} \otimes dt^{i_1} \wedge \ldots \wedge dt^{i_s} \in \Omega^{r,s}_k$$

where the scalar forms $\phi_{i_1\cdots i_s}$ are skew-symmetric in their indices. Although this looks like a coordinate formula, in fact it is not: the indices $i_1, \ldots, i_s$ refer to a fixed basis of $\mathbb{R}^{m*}$, and so the formula is valid globally on $\mathring{T}^k_{(m)}E$.

We shall consider several operators on vector forms. First, obviously, we may use the de Rham differential $d : \Omega^{r,s}_k \to \Omega^{r+1,s}_k$, defined on decomposable forms by

$$d(\phi \otimes \eta) = d\phi \otimes \eta$$

and extended to arbitrary forms by linearity. But we may also use the total derivatives d_i to define two further operators, a contraction and a Lie derivative, by

$$i_{\mathrm{T}} : \Omega^{r,s}_k \to \Omega^{r-1,s+1}_{k+1}\,, \qquad i_{\mathrm{T}}(\phi \otimes \eta) = (d_i \lrcorner\, \phi) \otimes (dt^i \wedge \eta)$$

and

$$d_{\mathrm{T}} : \Omega^{r,s}_k \to \Omega^{r,s+1}_{k+1}\,, \qquad d_{\mathrm{T}}(\phi \otimes \eta) = d_i\phi \otimes (dt^i \wedge \eta)\,.$$

It is immediate, from the definitions and the properties of contraction and Lie derivative on scalar forms, that

$$dd_{\mathrm{T}} = d_{\mathrm{T}}d\,, \qquad d^2_{\mathrm{T}} = 0\,, \qquad d_{\mathrm{T}} = di_{\mathrm{T}} + i_{\mathrm{T}}d\,.$$

We may therefore construct a bicomplex, where in the first column it is convenient to write $\overline{\Omega}^{0,s}$ to denote the quotient $\Omega^{0,s}/\bigwedge^s \mathbb{R}^{m*}$ of vector-valued functions modulo constant functions. In the diagram we omit explicit mention of the order of the velocity manifolds on which the spaces are defined; if the order of the spaces for a given row is k then the order for the next row will be $k+1$. For small values of k of course only the lower part of the diagram will exist.

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{\Omega}^{0,0} & \xrightarrow{d} & \Omega^{1,0} & \xrightarrow{d} & \Omega^{2,0} & \xrightarrow{d} & \Omega^{3,0} & \longrightarrow & \\
 & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & \\
 & & \cdots & & \cdots & & \cdots & & \cdots & & \\
 & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & \\
0 & \longrightarrow & \overline{\Omega}^{0,m-2} & \xrightarrow{d} & \Omega^{1,m-2} & \xrightarrow{d} & \Omega^{2,m-2} & \xrightarrow{d} & \Omega^{3,m-2} & \longrightarrow & \\
 & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & \\
0 & \longrightarrow & \overline{\Omega}^{0,m-1} & \xrightarrow{d} & \Omega^{1,m-1} & \xrightarrow{d} & \Omega^{2,m-1} & \xrightarrow{d} & \Omega^{3,m-1} & \longrightarrow & \\
 & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & {\scriptstyle d_{\mathrm{T}}}\downarrow & & \\
0 & \longrightarrow & \overline{\Omega}^{0,m} & \xrightarrow{d} & \Omega^{1,m} & \xrightarrow{d} & \Omega^{2,m} & \xrightarrow{d} & \Omega^{3,m} & \longrightarrow &
\end{array}$$

The bicomplex described above might appear to have some relation to the variational bicomplex for differential forms on the jet prolongations of fibred spaces, and the latter, when defined in the usual way on the infinite jet manifold, is locally exact: indeed, its interior columns are globally exact [1], [13], [14] (see [15] for a useful summary). The present bicomplex is, however, defined on (a family of) finite-order velocity manifolds, and the map $d_{\mathrm{T}} : \Omega^{r,s}_k \to \Omega^{r,s+1}_{k+1}$ is *not* exact, even locally. It is, however, globally exact modulo pull-backs (for $r \geq 1$).

There are, perhaps surprisingly, *two* homotopy operators for d_{T} which are similar in formulation but subtly different in effect; the first was described in [12], and the second is a version for velocity manifolds of an operator described in [6]. The operators are $P, \widetilde{P} : \Omega^{r,s}_k \to \Omega^{r,s-1}_{(r+1)k-1}$, defined by

$$P(\Phi) = P^j_{(s)}(\phi_{i_1 \cdots i_s}) \otimes \left\{ \frac{\partial}{\partial t^j} \lrcorner \left(dt^{i_1} \wedge \ldots \wedge dt^{i_s} \right) \right\}$$
$$\widetilde{P}(\Phi) = \widetilde{P}^j_{(s)}(\phi_{i_1 \cdots i_s}) \otimes \left\{ \frac{\partial}{\partial t^j} \lrcorner \left(dt^{i_1} \wedge \ldots \wedge dt^{i_s} \right) \right\}$$

where $P = \widetilde{P}$ when acting on vector 1-forms, or on first-order forms. The scalar operators $P^j_{(s)}$ and $\widetilde{P}^j_{(s)}$ are given by the formulæ

$$P^j_{(s)} = \sum_J \frac{(-1)^{|J|}(m-s)!|J|!}{r(m-s+|J|+1)!J!} d_J S^{J+1_j}\,,$$
$$\widetilde{P}^j_{(s)} = \sum_J \frac{(-1)^{|J|}(m-s)!|J|!}{r^{|J|+1}(m-s+|J|+1)!J!} d_J \widetilde{S}^{J+1_j}$$

where, for a scalar form θ,

$$S^{1_{j_1}+1_{j_2}+\cdots+1_{j_r}}\theta = i_{S^{j_1}S^{j_2}\cdots S^{j_r}}\theta$$
$$\widetilde{S}^{1_{j_1}+1_{j_2}+\cdots+1_{j_r}}\theta = i_{S^{j_1}}i_{S^{j_2}}\cdots i_{S^{j_r}}\theta\,.$$

It is interesting to note that $\widetilde{P}^2 = 0$, but that $P^2 \neq 0$. Proofs that these operators really do act as homotopy operators modulo pullbacks may be found in the references cited (the proof for P is given in [6] for the related operator on jet manifolds, but the proof for velocity manifolds is essentially the same).

4 Variational problems on jet manifolds

For the purposes of comparison, we give a brief summary of the relevant part of variational theory on jet manifolds.

Let $\pi : E \to M$ be a fibred manifold, with $\dim M = m$ and $\dim E = m + n$, where the base manifold M is orientable; we take local coordinates x^i on M and (x^i, u^α) on E. We let $J^k\pi$ denote the manifold of k-th order jets of local sections of π [7], [10]. In this context a Lagrangian of order k is an m-form $\lambda = L\,d^m x$ on $J^k\pi$, horizontal over M. The fixed-boundary variational problem defined by λ is the search for submanifolds $\sigma(C) \subset E$ satisfying

$$\int_C ((j^k\sigma)^* X^k\lambda) = 0$$

for every variation field X on E satisfying $X|_{\sigma(\partial C)} = 0$, where X^k denotes the prolongation of X to $J^k\pi$.

Such a variational problem may be expressed in terms of certain other m-forms called Lepage forms [8]. The m-form θ on $J^l\pi$ (where $l \geq k$) is a *Lepage form* if $i_Y d\theta$ is a contact form whenever the vector field Y is vertical over E. It is a *Lepage equivalent* of λ if it is a Lepage form, and in addition $\pi^*_{l,k}\lambda - \theta$ is a contact form. Every Lagrangian m-form defined on $J^k\pi$ has a Lepage equivalent defined on $J^{2k-1}\pi$, although the question of whether there is a suitable geometric construction depends on the values of m and k.

The simplest cases, as might be expected, are for single-integral problems where $m = 1$. For a first-order Lagrangian $\lambda = L\,dx$ on $J^1\pi$ the 1-form

$$\theta = L\,dx + \frac{\partial L}{\partial \dot{u}^\alpha}(du^\alpha - \dot{u}^\alpha dx)$$

is the unique Lepage equivalent, the *Poincaré-Cartan form*; it is also defined on $J^1\pi$. For a higher-order Lagrangian $\lambda = L\,dx$ on $J^k\pi$ the 1-form

$$\theta = L\,dx + \sum_{p=0}^{k-1}\left(\sum_{q=0}^{k-p-1}(-1)^q\frac{d^q}{dx^q}\frac{\partial L}{\partial u^\alpha_{(p+q+1)}}\right)(du^\alpha_{(p)} - u^\alpha_{(p+1)}dx)$$

is the unique Lepage equivalent, and it is defined on $J^{2k-1}\pi$.

For a multiple integral variational problem where $m \geq 2$, a first-order Lagrangian $\lambda = L\,d^m x$ defined on $J^1\pi$ gives rise to three distinct globally-defined Lepage equivalents

$$\theta_1 = L\,d^m x + \frac{\partial L}{\partial u^\alpha_i}\omega^\alpha \wedge d^{m-1}x_i$$

$$\theta_2 = \frac{1}{L^{m-1}}\bigwedge_{i=1}^{m}\left(L\,dx^i + \frac{\partial L}{\partial u^\alpha_i}\omega^\alpha\right)$$

$$\theta_3 = \sum_{r=0}^{\min\{m,n\}}\frac{1}{(r!)^2}\frac{\partial^r L}{\partial u^{\alpha_1}_{i_1}\cdots\partial u^{\alpha_r}_{i_r}}\omega^{\alpha_1}\wedge\cdots\wedge\omega^{\alpha_r}\wedge d^{m-r}x_{i_1\cdots i_r}$$

where $\omega^\alpha = du^\alpha - u^\alpha_j dx^j$ (of course θ_2 is defined only where the Lagrangian does not vanish). For a second-order Lagrangian, Lepage equivalents similar to θ_1 and θ_2 may again be found; it is not known whether there is a Lepage equivalent similar to θ_3. If $m \geq 3$ then it is known that global Lepage equivalents cannot be constructed in a canonical way without the use of additional data such as a connection. A list of references for these various constructions may be found in [11].

5 Homogeneous problems

We now consider m-dimensional variational problems on E, with fixed boundary conditions. For our purposes it is sufficient to consider submanifolds of the form $\sigma(C)$, where $\sigma : \mathbb{R}^m \to E$ and $C \subset \mathbb{R}^m$ is a compact m-dimensional submanifold: this is because variational problems are local, in the sense that an m-dimensional submanifold of E is extremal with fixed boundary conditions if, and only if, every small piece of it is extremal with fixed boundary conditions.

A vector function $\Lambda = L\,d^m t \in \Omega^{0,m}$ is called a *Lagrangian* for a parametric variational problem. It is called *positively homogeneous* if it is equivariant with respect to the action of the jet group L^{k+}_m, where k is the order of the Lagrangian. If Λ is positively homogeneous then the scalar function L satisfies

$$d^i_j L = \delta^i_j L\,, \qquad d^I_j L = 0 \text{ for } |I| \geq 2\,.$$

The fixed-boundary variational problem defined by Λ is the search for submanifolds $\sigma(C) \subset E$ satisfying

$$\int_C ((j\sigma)^* X^k L) d^m t = 0$$

for every variation field X on E satisfying $X|_{\sigma(\partial C)} = 0$, where X^k denotes the prolongation of X to $\mathring{T}^k_{(m)}E$. We may study this problem by looking for 'equivalents' of Lagrangians.

Definition 1. Let $\Lambda \in \Omega^{0,m}$ be a positively homogeneous Lagrangian. A scalar m-form $\Theta_m \in \Omega^{m,0}$ is called an *integral equivalent* of Λ if

$$\Lambda = \left(\frac{(-1)^{m(m-1)/2}}{m!}\right) i^m_{\mathrm{T}} \Theta_m \, .$$

A vector r-form $\Theta_r \in \Omega^{r,m-r}$ is called an *intermediate equivalent* if

$$\Lambda = \frac{(-1)^{r(r-1)/2}(m-r)!}{m!} \, i^r_{\mathrm{T}} \Theta_r \qquad 0 \le r \le m-1 \, .$$

It is clear that if Θ_{r+1} is an equivalent of Λ then

$$\Theta_r = \frac{(-1)^r}{m-r} \, i_{\mathrm{T}} \Theta_{r+1}$$

is also an equivalent. We use the terminology 'integral equivalent' because if $\sigma : \mathbb{R}^m \to E$ then $(j\sigma)^*\Lambda = (j\sigma)^*\Theta_m$, where by $j\sigma$ we mean the prolongation of σ to a map $\mathbb{R}^m \to \mathring{T}^l_{(m)}E$ for l sufficiently large, so that

$$\int_C (j\sigma)^*\Lambda = \int_C (j\sigma)^*\Theta_m \, ,$$

from which we see that $\Lambda = \Theta_0$ and Θ_m have the same extremals.

We may also define some related forms which are used to obtain the Euler--Lagrange equations for the problem.

Definition 2. Let Θ_m be an integral equivalent of Λ; define the scalar $(m+1)$-form $\mathcal{E}_m \in \Omega^{m+1,0}$ by

$$\mathcal{E}_m = d\Theta_m \, .$$

Now let Θ_r be an intermediate equivalent of Λ for $0 \le r \le m-1$; define the vector form $\mathcal{E}_r \in \Omega^{r+1,m-r}$ by

$$\mathcal{E}_r = d\Theta_r - (-1)^r d_{\mathrm{T}} \Theta_{r+1} \, .$$

By a straightforward calculation we see that, corresponding to the relationships describing a family of intermediate equivalents, we have

$$\mathcal{E}_r = \frac{(-1)^{r+1}}{m-r} \, i_{\mathrm{T}} \mathcal{E}_{r+1} \qquad 0 \le r \le m-1 \, ;$$

the form $\mathcal{E}_0$ is called the *Euler form* of Θ_m. The various forms we have defined inhabit two diagonals of our bicomplex.

We shall now impose an additional property on the equivalents of a Lagrangian.

$$\Theta_r = \frac{(-1)^r}{m-r}\, i_{\mathrm{T}}\Theta_{r+1}, \quad \Theta_0 = \Lambda$$

$$\begin{tikzcd}
 & & & \Theta_m \in \Omega^{m,0} \arrow[r, "d"] \arrow[dl] \arrow[d, "d_{\mathrm{T}}"] & \Omega^{m+1,0} \ni \mathcal{E}_m \arrow[dl] \\
 & & \arrow[dl] \arrow[d, "d_{\mathrm{T}}"] & \cdots \arrow[dl] & \\
 & \Theta_2 \in \Omega^{2,m-2} \arrow[r, "d"] \arrow[dl] \arrow[d, "d_{\mathrm{T}}"] & \Omega^{3,m-2} \ni \mathcal{E}_2 \arrow[dl] & & \\
 \Theta_1 \in \Omega^{1,m-1} \arrow[r, "d"] \arrow[d, "d_{\mathrm{T}}"] \arrow[dl] & \Omega^{2,m-1} \ni \mathcal{E}_1 \arrow[dl] & & & \\
 \Lambda \in \overline{\Omega}^{0,m} \arrow[r, "d"] & \Omega^{1,m} \ni \mathcal{E}_0 & & &
\end{tikzcd}$$

$$\mathcal{E}_r = d\Theta_r - (-1)^r d_{\mathrm{T}}\Theta_{r+1}$$

Definition 3. Let Λ be a positively homogeneous Lagrangian, and let Θ_r be an equivalent of Λ $(1 \leq r \leq m)$. We say that Θ_r is *Lepagian* if the corresponding Euler form $\mathcal{E}_0 = \varepsilon_0 \otimes d^m t \in \Omega^{1,m}$ satisfies

$$S\mathcal{E}_0 = (S^i \varepsilon_0) \otimes d^{m-1} t_i = 0\,,$$

so that $\mathcal{E}_0$ is horizontal over E.

So far, we have described conditions which integral (or intermediate) equivalents and their Euler forms must satisfy, but we have not yet indicated whether such forms exist. We shall now remedy that deficiency.

Theorem 1. *The vector 1-form*

$$\Theta_1 = P d\Lambda$$

defined on $\overset{\circ}{T}{}^{2k-1}_{(m)} E$ *is an integral equivalent of* Λ $(m = 1)$ *or an intermediate equivalent* $(m \geq 2)$*, and is Lepagian. It is called the* Hilbert equivalent *of* Λ.

Proof. From the definition of P,

$$P\Phi = P^j \phi \otimes d^{m-1} t_j\,, \quad \text{where} \quad P^j = \sum_J \frac{(-1)^{|J|}}{(|J|+1)J!} d_J S^{J+1_j}\,,$$

so that

$$\begin{aligned}
i_{\mathrm{T}} P d\Lambda &= i_{\mathrm{T}} P(dL \otimes d^m t) \\
&= i_{\mathrm{T}}(P^j dL \otimes d^{m-1} t_j) \\
&= i_k P^j dL \otimes dt^k \wedge d^{m-1} t_j \\
&= i_j P^j dL \otimes d^m t\,.
\end{aligned}$$

Then

$$\begin{aligned} i_j P^j dL &= i_j\Big(\sum_J \frac{(-1)^{|J|}}{(|J|+1)J!} d_J S^{J+1_j} dL\Big) \\ &= \sum_J \frac{(-1)^{|J|}}{(|J|+1)J!} d_J i_j S^{J+1_j} dL \end{aligned}$$

because $[i_k, d_j] = 0$; next

$$i_j P^j dL = \sum_J \frac{(-1)^{|J|}}{(|J|+1)J!} d_J (S^{J+1_j} i_j + S^{J+1_j-1_k} i^k_j) dL$$

because $[i_j, S^k] = i^k_j$ and $[i^J_j, S^k] = i^{J+1_k}_j$, but by homogeneity $i^J_j dL = d^J_j L = 0$ for $(|J| \geq 2)$; consequently

$$i_j P^j dL = i^j_j dL$$

because, when $|K| \geq 1$, S^K vanishes on functions and hence on $i_j dL$ and $i^k_j dL$; and so, finally,

$$i_j P^j dL = mL\,,$$

giving $i_{\mathrm{T}} P d\Lambda = m\Lambda$, so that Θ_1 is indeed an equivalent (integral or intermediate, as appropriate).

To show that Θ_1 is Lepagian, note that

$$\begin{aligned} S d_{\mathrm{T}} \Theta_1 &= S d_{\mathrm{T}} P d\Lambda \\ &= S d_{\mathrm{T}} (P^j dL \otimes d^{m-1} t_j) \\ &= S\big(d_i P^j dL \otimes (dt^i \wedge d^{m-1} t_j)\big) \\ &= S\big(d_j P^j dL \otimes d^m t\big) \\ &= S^i (d_j P^j dL) \otimes d^{m-1} t_i \\ &= S^i\Big(\sum_{|J|\geq 0} \frac{(-1)^{|J|}}{(|J|+1)J!} d_{J+1_j} S^{J+1_j} dL\Big) \otimes d^{m-1} t_i \\ &= S^i\Big(\sum_{|K|\geq 1} \frac{(-1)^{|K|-1}}{K!} d_K S^K dL\Big) \otimes d^{m-1} t_i\,; \end{aligned}$$

but $[S^i, d_k] = \delta^i_k$, giving $[S^i, d_K] = K(i) d_{K-1_i}$, so that

$$\begin{aligned} S d_{\mathrm{T}} \Theta_1 &= \sum_{|K|\geq 1} \frac{(-1)^{|K|-1}}{K!} (d_K S^{K+1_i} + K(i) d_{K-1_i} S^K) dL \otimes d^{m-1} t_i \\ &= S^i dL \otimes d^{m-1} t_i \\ &= S(dL \otimes d^m t) = S d\Lambda \end{aligned}$$

as the two parts of the sum over the multi-index K combine to give a collapsing sum. It it then immediate that $S\mathcal{E}_0 = 0$, as required. □

Theorem 2. *Let Λ be a homogeneous Lagrangian, with Hilbert equivalent Θ_1 and Euler form $\mathcal{E}_0$. If $\widetilde{\Theta}_1$ is any other Lepagian vector 1-form equivalent to Λ, with corresponding Euler form $\widetilde{\mathcal{E}}_0$, then*

$$\widetilde{\mathcal{E}}_0 = \mathcal{E}_0 \qquad \text{and} \qquad \widetilde{\Theta}_1 - \Theta_1 = d_{\mathrm{T}}\Phi$$

for some $\Phi \in \Omega^{r,m-2}$, so that if $m = 1$ then $\widetilde{\Theta}_1 = \Theta_1$.

Proof. It follows straightforwardly from the Lepagian condition $S\widetilde{\mathcal{E}}_0 = 0$ that $P\widetilde{\mathcal{E}}_0 = 0$, so that

$$\begin{aligned} 0 &= P\widetilde{\mathcal{E}}_0 \\ &= P(d\Lambda - d_{\mathrm{T}}\widetilde{\Theta}_1) \\ &= \Theta_1 - Pd_{\mathrm{T}}\widetilde{\Theta}_1 \\ &= \Theta_1 - (1 - d_{\mathrm{T}}P)\widetilde{\Theta}_1 \,, \end{aligned}$$

giving $\widetilde{\Theta}_1 - \Theta_1 = d_{\mathrm{T}}P\widetilde{\Theta}_1$ (or $\widetilde{\Theta}_1 = \Theta_1$ if $m = 1$). Thus

$$\begin{aligned} \widetilde{\mathcal{E}}_0 - \mathcal{E}_0 &= (d\Lambda - d_{\mathrm{T}}\widetilde{\Theta}_1) - (d\Lambda - d_{\mathrm{T}}\Theta_1) \\ &= -d_{\mathrm{T}}^2 P\widetilde{\Theta}_1 \\ &= 0 \,. \end{aligned}$$

□

In coordinates, if $\Lambda = L\, d^m t$ then the Hilbert equivalent and the Euler form are given by

$$\begin{aligned} \Theta_1 &= \sum_I \sum_J \frac{(-1)^{|I|}(I + J + 1_i)!|I|!|J|!}{(|I| + |J| + 1)!I!J!} d_I \left(\frac{\partial L}{\partial y^a_{I+J+1_i}} \right) dy^a_J \otimes d^{m-1}t_i \,, \\ \mathcal{E}_0 &= \sum_I (-1)^{|I|} d_I \left(\frac{\partial L}{\partial y^a_I} \right) dy^a \otimes d^m t \,. \end{aligned}$$

If $m \geq 2$ then there can indeed be Lepagian vector 1-forms which are equivalent to a given Lagrangian but differ from its Hilbert equivalent. To see this, let $\Phi \in \Omega^{0,m-2}$, so that $d_{\mathrm{T}}d\Phi \in \Omega^{1,m-1}$. Then

$$i_{\mathrm{T}}(\Theta_1 + d_{\mathrm{T}}d\Phi) = i_{\mathrm{T}}\Theta_1 - d_{\mathrm{T}}i_{\mathrm{T}}d\Phi = \Lambda - d_{\mathrm{T}}^2\Phi = \Lambda$$

and

$$d\Lambda - d_{\mathrm{T}}(\Theta_1 + d_{\mathrm{T}}d\Phi) = d\Lambda - d_{\mathrm{T}}\Theta_1 - d_{\mathrm{T}}^2 d\Phi = \mathcal{E}_0 \,,$$

although there is no reason why we should have $d_{\mathrm{T}}d\Phi = 0$. For instance, when $m = 2$ we could take $\Phi = y^1 \in \Omega^{0,0}$ and then $d_{\mathrm{T}}d\Phi = dy^1_i \otimes dt^i \neq 0$.

We can now construct a version of the first variation formula for homogeneous variational problems, using the Hilbert equivalent Θ_1 of a Lagrangian Λ. Given a variation field X on E with $X|_{\sigma(\partial C)} = 0$, and its prolongation $\widehat{X}$ on $\mathring{T}^l_{(m)}E$

with l sufficiently large, we may use the standard formula $d_{\widehat{X}} = di_{\widehat{X}} + i_{\widehat{X}}d$, Stokes' Theorem, and the formula $d\Lambda = \mathcal{E}_0 + d_{\mathrm{T}}\Theta_1$ to obtain

$$\begin{aligned}\int_C (j\sigma)^* d_{\widehat{X}}\Lambda &= \int_C (j\sigma)^* di_{\widehat{X}}\Lambda + \int_C (j\sigma)^* i_{\widehat{X}} d\Lambda \\ &= \int_{\partial C} (j\sigma)^* i_{\widehat{X}}\Lambda + \int_C (j\sigma)^* i_{\widehat{X}} d\Lambda \\ &= \int_C (j\sigma)^* i_{\widehat{X}}(\mathcal{E}_0 + d_{\mathrm{T}}\Theta_1)\,.\end{aligned}$$

But

$$\int_C (j\sigma)^* i_{\widehat{X}} d_{\mathrm{T}}\Theta_1 = \int_C (j\sigma)^* d_{\mathrm{T}} i_{\widehat{X}}\Theta_1 = \int_C d\big((j\sigma)^* i_{\widehat{X}}\Theta_1\big) = \int_{\partial C} (j\sigma)^* i_{\widehat{X}}\Theta_1 = 0$$

because prolongations commute with total derivatives, and the pull-back of d_{T} to $\mathbb{R}^m$ is d; thus

$$\int_C (j\sigma)^* d_{\widehat{X}}\Lambda = \int_C (j\sigma)^* i_{\widehat{X}}\mathcal{E}_0 = \int_C (j\sigma)^* i_X\mathcal{E}_0$$

because $\mathcal{E}_0$ is horizontal over E.

Now if $m = 1$ then the Hilbert equivalent is an integral equivalent of Λ. But if $m \geq 2$ then this is no longer true, and we need some further work to find integral equivalents. Let $\Lambda = L\, d^m t$ be a positively homogeneous Lagrangian with $m \geq 2$, and write its Hilbert equivalent Θ_1 as

$$\Theta_1 = \vartheta^i \otimes d^{m-1}t_i\,;$$

the scalar 1-forms ϑ_i are called the *Hilbert forms* of Λ.

Definition 4. If Λ never vanishes, define the *Carathéodory equivalent* $\Theta_m \in \Omega^{m,0}$ by

$$\Theta_m = \frac{1}{L^{m-1}} \bigwedge_{i=1}^{m} \vartheta^i\,.$$

Theorem 3. *The Carathéodory equivalent Θ_m is an integral equivalent of Λ.*

Proof. We must show that $i_{\mathrm{T}}^m\Theta_m = (-1)^{m(m-1)/2}\, m!\Lambda$, so rewrite Θ_m as

$$\Theta_m = \frac{1}{m!L^{m-1}} \sum_{\sigma\in\mathfrak{S}_m} (-1)^\sigma \vartheta^{\sigma(1)} \wedge \cdots \wedge \vartheta^{\sigma(m)}\,,$$

where $\mathfrak{S}_m$ is the permutation group, and use induction. The calculation uses $d_j \lrcorner\, \vartheta^i = \delta^i_j L$, the proof of which is similar to that used to show that $i_{\mathrm{T}}\Theta_1 = m\Lambda$; we also define $\tau_{r,s} \in \mathfrak{S}_m$ by

$$\tau_{r,s}(i) = \begin{cases} m - s & (i = r) \\ i - 1 & (r + 1 \leq i \leq m - s) \\ i & \text{otherwise}\,. \end{cases}$$

Now

$$
\begin{aligned}
& i_{\mathrm{T}}\Bigl(\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\otimes dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\Bigr) \\
&= \sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma d_j \lrcorner \bigl(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\bigr)\otimes dt^j\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)} \\
&= \sum_{r=1}^{m-s}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma(-1)^{r-1}\bigl(\vartheta^{\sigma(1)}\wedge\cdots\wedge(d_j\lrcorner\vartheta^{\sigma(r)})\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\bigr)\otimes \\
&\qquad\qquad \otimes dt^j\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)} \\
&= L\sum_{r=1}^{m-s}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma(-1)^{r-1}\bigl(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(r-1)}\wedge\vartheta^{\sigma(r+1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\bigr)\otimes \\
&\qquad\qquad \otimes dt^{\sigma(r)}\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)} \\
&= L\sum_{r=1}^{m-s}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma(-1)^{r-1}(-1)^{m-r-s}\Bigl\{ \\
&\qquad \bigl(\vartheta^{\sigma\tau_{r,s}(1)}\wedge\cdots\wedge\vartheta^{\sigma\tau_{r,s}(r-1)}\wedge\vartheta^{\sigma\tau_{r,s}(r+1)}\wedge\cdots\wedge\vartheta^{\sigma\tau_{r,s}(m-s)}\bigr)\otimes \\
&\qquad\qquad \otimes dt^{\sigma\tau_{r,s}(r)}\wedge dt^{\sigma\tau_{r,s}(m-s+1)}\wedge\cdots\wedge dt^{\sigma\tau_{r,s}(m)}\Bigr\} \\
&= (-1)^{m-s-1}L\sum_{r=1}^{m-s}\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\bigl(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s-1)}\bigr)\otimes \\
&\qquad\qquad \otimes dt^{\sigma(m-s)}\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)} \\
&= (-1)^{m-s-1}(m-s)L\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\bigl(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s-1)}\bigr)\otimes \\
&\qquad\qquad \otimes dt^{\sigma(m-s)}\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\,,
\end{aligned}
$$

so if

$$
\begin{aligned}
i^s_{\mathrm{T}}\Theta_m = {}& \frac{(-1)^{s(2m-s-1)/2}}{(m-s)!L^{m-s-1}}\Bigl\{ \\
& \sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s)}\otimes dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\Bigr\}
\end{aligned}
$$

then

$$
\begin{aligned}
i^{s+1}_{\mathrm{T}}\Theta_m = {}& \frac{(-1)^{s(2m-s-1)/2}}{(m-s)!L^{m-s-1}}\Bigl\{ \\
& (-1)^{m-s-1}(m-s)L\sum_{\sigma\in\mathfrak{S}_m}(-1)^\sigma\bigl(\vartheta^{\sigma(1)}\wedge\cdots\wedge\vartheta^{\sigma(m-s-1)}\bigr)\otimes \\
& \qquad \otimes dt^{\sigma(m-s)}\wedge dt^{\sigma(m-s+1)}\wedge\cdots\wedge dt^{\sigma(m)}\Bigr\}
\end{aligned}
$$

$$= \frac{(-1)^{(s+1)(2m-s-2)/2}}{(m-s-1)!L^{m-s-2}} \sum_{\sigma\in\mathfrak{S}_m} (-1)^\sigma \big(\vartheta^{\sigma(1)} \wedge \cdots \wedge \vartheta^{\sigma(m-s-1)}\big)\otimes$$
$$\otimes\, dt^{\sigma(m-s)} \wedge dt^{\sigma(m-s+1)} \wedge \cdots \wedge dt^{\sigma(m)}$$

as required. Hence

$$\begin{aligned}
i_{\mathrm{T}}^m \Theta_m &= \frac{(-1)^{m(m-1)/2}}{L^{-1}} \sum_{\sigma\in\mathfrak{S}_m} (-1)^\sigma dt^{\sigma(1)} \wedge \cdots \wedge dt^{\sigma(m)} \\
&= (-1)^{m(m-1)/2}\, m! L\, dt^1 \wedge \cdots \wedge dt^m \\
&= (-1)^{m(m-1)/2}\, m! \Lambda\,.
\end{aligned}$$

□

We see also from the induction formula that

$$i_{\mathrm{T}}^{m-1}\Theta_m = (-1)^{m(m-1)/2}\,(m-1)!\Theta_1$$

where Θ_1 is the Hilbert equivalent; consequently Θ_m is Lepagian. The Carathéodory equivalent of a nonvanishing homogeneous Lagrangian is the ‘parametric’ version of the Lepage equivalent θ_2 for a variational problem on a jet manifold, with the difference that there is no longer a restriction to first or second order Lagrangians.

We can now create a ‘variation formula’ for Θ_m. For a variation field X on E with $X|_{\sigma(\partial C)} = 0$,

$$\begin{aligned}
\int_C (j\sigma)^* d_{\widehat{X}}\Theta_m &= \int_C (j\sigma)^* i_{\widehat{X}} d\Theta_m + \int_C (j\sigma)^* d i_{\widehat{X}}\Theta_m \\
&= \int_C (j\sigma)^* i_{\widehat{X}}\mathcal{E}_m + \int_C d(j\sigma)^* i_{\widehat{X}}\Theta_m \\
&= \int_C (j\sigma)^* i_{\widehat{X}}\mathcal{E}_m + \int_{\partial C} (j\sigma)^* i_{\widehat{X}}\Theta_m \\
&= \int_C (j\sigma)^* i_{\widehat{X}}\mathcal{E}_m \\
&= \int_C (j\sigma)^* i_{\mathrm{T}}^m i_{\widehat{X}}\mathcal{E}_m \\
&= (-1)^m \int_C (j\sigma)^* i_{\widehat{X}} i_{\mathrm{T}}^m \mathcal{E}_m \\
&= (-1)^{m(m-1)/2} m! \int_C (j\sigma)^* i_{\widehat{X}}\mathcal{E}_0 \\
&= (-1)^{m(m-1)/2} m! \int_C (j\sigma)^* i_X\mathcal{E}_0
\end{aligned}$$

which is independent of the ‘prolonged’ part of $\widehat{X}$.

Finally, we consider the possibility of other integral equivalents of a Lagrangian Λ. We can, of course, obtain such an equivalent by adding any contact form to Θ_m; but it is of greater interest to see if we can obtain equivalents which have particular desirable properties. The analogy of Lepage equivalents for variational problems on jet manifolds suggests that there might be other possibilities in the parametric case, and there is indeed a version of θ_3 for first-order homogeneous Lagrangians. Let Λ be such a Lagrangian, and put

$$\widehat{\Theta}_{r+1} = (-1)^r P d\widehat{\Theta}_r \qquad (1 \leq r < m)\,,$$

so that each $\widehat{\Theta}_r$ is a first-order vector form. Using commutator relations as before, we obtain

$$\widehat{\Theta}_r = \frac{(-1)^r}{m-r}\, i_{\mathrm{T}}\widehat{\Theta}_{r+1}$$

so that $\widehat{\Theta}_m$ is a Lepagian integral equivalent of Λ, the *fundamental equivalent* of Λ. Thus $d\widehat{\Theta}_m = \widehat{\mathcal{E}}_m = 0$ if, and only if, $\widehat{\mathcal{E}}_0 = \mathcal{E}_0 = 0$, the same property satisfied by θ_3 in the jet manifold case [5].

6 Conclusions

Parametric variational problems are often studied on Grassmannian bundles. There is, however, some interest in considering the versions of the problem defined on velocity manifolds, subject to the homogeneity condition. The bicomplex of vector forms performs a similar rôle to the variational bicomplex in the jet bundle theory, but the intermediate and integral equivalents corresponding to the Lepage equivalents may be defined globally for forms of arbitrary order. It seems reasonable to expect that further study of the subject in this context would produce useful results concerning related concepts such as regularity, symmetry and the Helmholtz equations for the inverse problem of the calculus of variations.

Acknowledgements

The author wishes to acknowledge the support of grant no. 201/09/0981 for Global Analysis and the Geometry of Fibred Spaces from the Czech Science Foundation.

References

[1] I.M. Anderson: *The variational bicomplex*. book preprint, technical report of the Utah State University, 1989. Available at `http://www.math.usu.edu/~fg_mp/`

[2] D. Bao, S.-S. Chern, Z. Shen: *An Introduction to Riemann-Finsler Geometry*. Springer (2000).

[3] M. Crampin, D.J. Saunders: The Hilbert-Carathéodory form for parametric multiple integral problems in the calculus of variations. *Acta Appl. Math.* 76 (1) (2003) 37–55.

[4] M. Crampin, D.J. Saunders: The Hilbert-Carathéodory and Poincaré-Cartan forms for higher-order multiple-integral variational problems. *Houston J. Math.* 30 (3) (2004) 657–689.

[5] M. Crampin, D.J. Saunders: On null Lagrangians. *Diff. Geom. Appl.* 22 (2) (2005) 131–146.

[6] M. Crampin, D.J. Saunders: Homotopy Operators for the Variational Bicomplex, Representations of the Euler-Lagrange Complex, and the Helmholtz-Sonin Conditions. *Lobachevskii J. Math.* 30 (2) (2009) 107–123.

[7] I. Kolář, P.W. Michor, J. Slovák: *Natural Operations in Differential Geometry*. Springer (1993).

[8] D. Krupka: Lepagean forms and higher order variational theories. In: S. Benenti, M. Francaviglia, A. Lichnerowicz (eds.): *Proceedings of the IUTAM-ISIMM Symposium on Modern Developments in Analytical Mechanics*. Tecnoprint (1983) 197–238.

[9] H. Rund: *The Hamilton-Jacobi Equation in the Calculus of Variations*. Krieger (1973).

[10] D.J. Saunders: *The geometry of jet bundles*. Cambridge University Press (1989).

[11] D.J. Saunders: Jet manifolds and natural bundles. In: D. Krupka, D.J. Saunders (eds.): *Handbook of Global Analysis*. Elsevier (2008) 1035–1068.

[12] D.J. Saunders: Homogeneous variational complexes and bicomplexes. *J. Geom. Phys.* 59 (2009) 727–739.

[13] W.M. Tulczyjew: The Euler-Lagrange resolution. In: *Lecture Notes in Mathematics 836*. Springer (1980) 22–48.

[14] A.M. Vinogradov: The $\mathcal{C}$-spectral sequence, Lagrangian formalism and conservation laws. *J. Math. Anal. Appl.* 100 (1984) 1–129.

[15] R. Vitolo: Variational sequences. In: D. Krupka, D.J. Saunders (eds.): *Handbook of Global Analysis*. Elsevier (2008) 1115–1163.

Author's address:
Department of Mathematics, Faculty of Science, The University of Ostrava, 30. dubna 22, 701 03 Ostrava, Czech Republic

E-mail address: david@symplectic.demon.co.uk

Received: 30 November, 2010
Accepted for publication: 15 December, 2010

Communications in Mathematics 18 (2010) 21–35

Generalized Birkhoffian realization of nonholonomic systems

Yong-Xin Guo, Chang Liu and Shi-Xing Liu

Abstract. Based on the Cauchy-Kowalevski theorem for a system of partial differential equations to be integrable, a kind of generalized Birkhoffian systems (GBSs) with local, analytic properties are put forward, whose manifold admits a presymplectic structure described by a closed 2-form which is equivalent to the self-adjointness of the GBSs. Their relations with Birkhoffian systems, generalized Hamiltonian systems are investigated in detail. Analytic, algebraic and geometric properties of GBSs are formulated, together with their integration methods induced from the Birkhoffian systems. As an important example, nonholonomic systems are reduced into GBSs, which gives a new approach to some open problems of nonholonomic mechanics.

1 Introduction

As it is well known, making use of the calculus of variations, any analytic, regular, holonomic, conservative mechanical systems can be formulated by Lagrange's equations or Hamilton's equations, which are basis of establishing, simplifying and integrating the equations of motion. Thus it is important to find out the solutions of inverse problems of the calculus of variations for different dynamical systems so as to make the most of the Lagrange's equations and Hamilton's equations. However, the Lagrangian or Hamiltonian formulation for a dynamical system, limited by the conditions of self-adjointness, such as the Helmholtz's conditions [10], [13], [15], [18], is not directly universal if the physical variables remain without use of Darboux transformations. Based on the Cauchy-Kowalevski theorem of the integrability conditions for partial differential equations and the converse of the Poincaré lemma, it can be proved that there exists a direct universality of Birkhoff's equations for local Newtonian systems by means of reduction of Newton's equations to a first-order form, which means all local, analytic, regular, finite-dimensional,

2010 MSC: 70F17, 70F25, 70G45

Key words: Birkhoff's equations, generalized Birkhoffian Systems, Cauchy-Kowalevski theorem, Nonholonomic constraints, inverse problem, conditions of self-adjointness

unconstrained or holonomic, conservative or non-conservative, and self-adjoint or non-self-adjoint systems always admit, in a star-shaped neighborhood of a regular point of their variables, a representation in terms of first-order Birkhoff's equations in the coordinate and time variables of the experimenter [11], [14]. The systems whose equations of motion are represented by the first-order Birkhoff's equations on a symplectic or a contact manifold spanned by the physical variables are called Birkhoffian systems, which are self-adjoint. The self-adjointness of local, analytic, regular, holonomic mechanical systems means the existence of symplectic or contact structure of the manifold. The Lie algebraic structure only exists for the autonomous Birkhoffian systems.

The inverse problem of the calculus of variations for nonholonomic systems is very complicated [7], [12]. Only some special nonholonomic systems, such as some Chaplygin's systems, can admit a homogenous Lagrangian or Hamiltonian formulation. Since the Chaplygin's systems can be reduced into a kind of holonomic nonconservative systems, it is suitable to formulate such nonholonomic systems in Birkhoffian mechanics [9], [11]. For a general nonholonomic system, i.e. a n-dimensional mechanical system constrained by l nonlinear nonholonomic constraints which is a coupled dynamical system, whose equations of motion are $n+l$ fixed first- and second-order ordinary differential equations, their inverse problem of the calculus of variations can be geometrically analyzed in a singular Lagrangian [6] or represented in Birkhoffian framework on an $2n$-dimensional phase space [4], [11]. For the latter case, the nonholonomic systems are reduced into the conditional holonomic systems on a $2n$-dimensional phase space, whose initial conditions are not arbitrary but confined by the nonholonomic constraints. Because the conditional holonomic systems are of symmetry determined by the constraints, it is necessary to reduce the Birkhoff's equations on the $2n$-dimensional phase space to those on its constraint submanifold of minimal dimension $2n-l$. Such a symmetry reduction strongly relies on the dimension $2n-l$ of the constraint submanifold or the number l of the constraints acted upon the system. Therefore, in order to directly universally analyze the inverse problem of the calculus of variations for general nonholonomic systems, we need to generalize the Birkhoffian mechanics. This problem can arise in other coupled dynamical systems, such as control theory for systems, supermechanics, etc.

In section 2, we will review Birkhoffian formulation of Newtonian Systems, emphasizing its analytic, algebraic and geometric properties. Its relation with generalized Hamiltonian mechanics is pointed out. In section 3, generalized Birkhoff's equations for all analytic, regular first-order dynamical systems are constructed based on the Cauchy-Kowalevski theorem for existence theory of partial differential equations. The integration methods induced from Birkhoffian mechanics are listed in section 4. In section 5, general nonholonomic systems are reduced into generalized Birkhoffian systems(GBSs), whose equations of motion are represented by the generalized Birkhoff's equations.

2 Review of Birkhoffian formulation of Newtonian systems

Consider a *holonomic* dynamical system on a contact manifold $R \times TQ$ with local coordinates $\{q^i, \dot{q}^i\}$ $(i = 1, 2, \ldots, n)$ where Q is a n-dimensional configuration manifold. Let a *regular* Lagrangian be denoted by $L(t, q, \dot{q})$. Suppose the system is subject to non-conservative forces $f_i(t, q, \dot{q})$ which are *analytic*. The equations of motion for the system can be represented by non-homogeneous Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = f_i \tag{1}$$

The regularity condition $L_{ij} = \det(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}) \neq 0$ guarantees that these n second-order differential equations for q^i on the contact manifold $R \times TQ$ can be reduced into $2n$ first-order non-homogeneous Hamilton's equations on the contact manifold $R \times T^*Q$ with local coordinates $\{t, a^\mu\}$ $(\mu = 1, 2, \ldots, 2n)$:

$$\omega_{\mu\nu}\dot{a}^\nu - \frac{\partial H(t,a)}{\partial a^\mu} = F_\mu(t,a) \tag{2}$$

where $\{a^\mu\} = \{q^i, p_i\}$, $p_i = \frac{\partial L}{\partial \dot{q}^i}$, $H = \frac{\partial L}{\partial \dot{q}^i}\dot{q}^i - L$ is the Hamiltonian for the system and simple symplectic matrix is

$$\omega = (\omega_{\mu\nu})_{2n\times 2n} = \begin{pmatrix} 0_{n\times n} & -1_{n\times n} \\ +1_{n\times n} & 0_{n\times n} \end{pmatrix}_{2n\times 2n} \tag{3}$$

In general, equations (2) are non-self-adjoint. By means of the Cauchy-Kowalevski theorem, it can be proved that there exist integrating factors $\{h^\mu_\lambda\}$ for the equations (2) to become self-adjoint equations

$$\Omega_{\mu\nu}\dot{a}^\nu - \left[\frac{\partial B\,(t,a)}{\partial a^\mu} + \frac{\partial R_\mu\,(t,a)}{\partial t}\right] = 0 \tag{4}$$

in a star-shaped region of a regular point (t, a), where B is a Birkhoffian usually taken as the energy function of the system, R_μ are a set of Birkhoffian functions usually related with the function F_μ and $\Omega_{\mu\nu}$ is the covariant Birkhoff's tensor defined by

$$\Omega_{\mu\nu}\,(t,a) = \frac{\partial R_\nu\,(t,a)}{\partial a^\mu} - \frac{\partial R_\mu\,(t,a)}{\partial a^\nu} \tag{5}$$

is symplectic. The regularity condition $\det(\Omega_{\mu\nu}) \neq 0$ means that the $2n$ equations (4) are independent and can be transformed into the contravariant form

$$\dot{a}^\mu = \Omega^{\mu\nu}\left[\frac{\partial B\,(t,a)}{\partial a^\nu} + \frac{\partial R_\nu\,(t,a)}{\partial t}\right] \tag{6}$$

where $\Omega^{\mu\nu} = \Omega^{-1}_{\mu\nu}$.

The Birkhoff's equations (4) are analytic in the sense that they are derivable from the most general possible linear first-order action functional, the Pfaffian action

$$\mathcal{A}(\tilde{E}) = \int_{t_1}^{t_2} dt\,[R_\nu\,(t,a)\,\dot{a}^\nu - B\,(t,a)]\,(\tilde{E}) \tag{7}$$

where $\tilde{E}$ is a possible pass in the contact manifold. The discussion of gauge freedom and some methods for integrating the Birkhoff's equations can be found in [11], [14].

The conditions of self-adjointness of Birkhoff's Equations (4)

$$\Omega_{\mu\nu} + \Omega_{\nu\mu} = 0 \tag{8a}$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial a^\tau} + \frac{\partial \Omega_{\nu\tau}}{\partial a^\mu} + \frac{\partial \Omega_{\tau\mu}}{\partial a^\nu} = 0 \tag{8b}$$

$$\frac{\partial \Omega_{\mu\nu}}{\partial t} = \frac{\partial}{\partial a^\nu}\left[\frac{\partial B\left(t,a\right)}{\partial a^\mu} + \frac{\partial R_\mu\left(t,a\right)}{\partial t}\right] - \frac{\partial}{\partial a^\mu}\left[\frac{\partial B\left(t,a\right)}{\partial a^\nu} + \frac{\partial R_\nu\left(t,a\right)}{\partial t}\right] \tag{8c}$$

are equivalent to the the integrability conditions for the 2-form on the contact manifold $R \times T^*Q$

$$\hat{\Omega} = \frac{1}{2}\hat{\Omega}_{\mu\nu}\left(\hat{a}\right) d\hat{a}^\mu \wedge d\hat{a}^\nu; \qquad \mu = 0, 1, 2, \ldots, 2n \tag{9}$$

to be closed, i.e.,

$$d\hat{\Omega} = 0 \tag{10}$$

where $\hat{\Omega}_{\mu\nu} = \Omega_{\mu\nu}$, $\hat{\Omega}_{0\nu} = -\hat{\Omega}_{\nu 0} = \frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}$ $(\mu, \nu = 1, 2, \ldots, 2n)$. The above nonautonomous Birkhoff's equations (4) can be globally represented by a *general, local, Birkhoffian vector field* $\tilde{X}$ on $R \times T^*Q$ verifying the properties

$$i_{\tilde{X}}\hat{\Omega} = 0, \quad dt(\tilde{X}) = 1 \tag{11}$$

Locally the vector field admits

$$\tilde{X} = \frac{\partial}{\partial t} + \Omega^{\mu\nu}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right)\frac{\partial}{\partial a^\mu} \tag{12}$$

Evidently the universality of the Birkhoff's equations is not only direct but also global.

For the autonomous Birkhoffian systems where[1] $\frac{\partial R_\mu}{\partial t} = 0$, the Birkhoff's equations are equivalent to the generalized Hamilton's equations [8] on an even-dimensional Poisson manifold

$$\dot{a}^\mu = \Omega^{\mu\nu}\frac{\partial B\left(t,a\right)}{\partial a^\nu} \tag{13}$$

In these cases the Poisson brackets can be defined by time evolution

$$\dot{A}\left(a\right) = \frac{\partial A}{\partial a^\mu}\dot{a}^\mu = \frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}\frac{\partial B}{\partial a^\nu} \stackrel{\text{def}}{=} [A, B] \tag{14}$$

verifying the Lie algebra axioms

$$[A, B] + [B, A] = 0 \tag{15a}$$

$$\left[[A, B], C\right] + \left[[B, C], A\right] + \left[[C, A], B\right] = 0 \tag{15b}$$

It should be pointed out that the Lie algebraic structure does not exist for a non-autonomous Birkhoffian system for the dependence of $R(t, a)$ on time t if we take Birkhoffian B to be total energy of the system.

[1] As will be seen in section 3, the so-called semi-autonomous Birkhoffian systems mentioned in [11], [14] are really autonomous ones. So the Birkhoffian systems can be classified into autonomous ones and non-autonomous ones.

3 Generalized Birkhoff's equations

In order to generalize the theory of Birkhoffian systems, it is necessary to analyze the fundamental conditions for the second-order dynamical systems to be capable of reduction to first-order self-adjoint Birkhoffian formulation, especially their role in the course of reduction to first-order systems.

(1) **Locality.** By locality we mean that the systems considered can be formulated by ordinary or partial differential equations, in which the interactions are independent of integro-differential type. This condition is necessary for the Lie algebra and symplectic geometry to be suitable to analyze the theories of some dynamical systems.

(2) **Analyticity.** Analyticity of a function means that it admits a convergent, multiple, power series expansion in a neighborhood of a point. The analyticity can be remained for a second-order dynamical system to be reduced into a first-order system. Evidently analyticity is based on locality. Analyticity does not depend on the self-adjointness of a dynamical system.

(3) **Regularity.** A system is called regular when it is of full rank or maximal rank, i.e., its functional Jacobi determinant is everywhere non-null in the region of a point, with the possible exception of finite number of isolated points. Regularity means maximal independency and invertibility. Thus a regular system can be recovered from its non-degenerately transformed form. Regularity is not an invariant character with respect to symmetry reduction, e.g., the canonical and Eulerian representations for the rotation of a rigid body with respect to a fixed point.

(4) **Holonomicity.** By a holonomic system we mean that constraints the system is subject to are integrable in the sense of Frobenius theorem. A holonomically constrained system can be reduced into a constraint-free system of lower dimensions. Therefore, holonomicity ensures the phase space for the first-order systems reduced from the second-order regular dynamical systems are even dimensional.

The universality of analytic/Lie/symplectic formulation in the most general possible form, i.e., Birkhoffian realization, does not depend on whether or not the original systems are conservative or self-adjoint. It should be pointed out that this symbiosis among analytic, Lie and symplectic techniques is comparatively flimsy because any one of the four legs under the symbiosis may be possibly broken. For example, the nonlocal type of interactions often occurs in several branches of physics, whose dynamical equations are integro-differential equations. Moreover, nonholonomic systems largely exist in the fields of physics, mechanics and engineering. Therefore it should be encouraged to generalize the Birkhoffian formulation to a new symbiosis among analytic, algebraic, geometric form in order to keep up with the process of mathematical and physical advances.

In this section, we consider a kind of GBSs from which the nonholonomic systems may be recovered if the second-order dynamical systems are reconstructed. Consider a dynamical system described by first-order differential equations

$$\dot{a}^I = \Xi^I(t, a^J), \qquad I, J = 1, 2, \ldots, m \tag{16}$$

on an m-dimensional manifold M with local coordinates $\{t, a\}$ where $\Xi^I(t, a)$ are analytic on the regular points. It can be proved that the equations (16) admit an analytic and presymplectic structure whether they are self-adjoint or not. Two methods can be utilized to realize this goal. The first one is to find out an integrating factor matrix with the help of Cauchy-Kowalevski theorem for the partial differential equations to be integrable, so as to obtain a self-adjoint genotopic transformed covariant form of the equations (16) or a presymplectic form on the manifold M. Then the converse of Poincaré lemma is used to get the final result. The second method is a direct use of the following Cauchy-Kowalevski theorem.

Theorem 1. *Consider an initial problem consisting of $n+1$ first-order partial differential equations of the Cauchy-Kowalevski form*

$$\frac{\partial R_\alpha(t,a)}{\partial t} = \sum_{\beta=0}^{n}\sum_{I=1}^{m} \Xi_\alpha^{I\beta}(t,a)\frac{\partial R_\beta(t,a)}{\partial a^I} + \sum_{\beta=0}^{n}\Pi_\alpha^\beta(t,a)R_\beta(t,a) + \Theta(t,a) \tag{17}$$

in $n+1$ unknown functions R_α ($\alpha = 0,1,2,\ldots,n$) and in $m+1$ independent variables $\{t,a^I\}$ ($I = 1,2,\ldots,m$), and the $n+1$ initial conditions

$$R_\alpha(0,a^1,a^2,\ldots,a^m) = \mathcal{R}_\alpha(a^1,a^2,\ldots,a^m) \tag{18}$$

If the functions $\Xi_\alpha^{I\beta}(t,a)$, $\Pi_\alpha^\beta(t,a)$, $\Theta(t,a)$ and $\mathcal{R}_\alpha(a)$ are real analytic functions at the regular point $A(a)$, then a unique analytic solution $R_0, R_1, \ldots, R_n$ of the initial problem (17) *and* (18) *exists in a neighborhood of the point $A(a)$.*

Considering the need of analytic and presymplectic structure, we set $n = m$, $\Pi_\alpha^\beta = 0$, $\Theta = 0$, $R_0 = -B$, $a^0 = t$, $\Xi_J^{I\beta} = 2\delta_J^{[I}\Xi^{\beta]} = \delta_J^I\Xi^\beta - \delta_J^\beta\Xi^I$ (where $\Xi^0 = 1$, so $\Xi_J^{I0} = \delta_J^I$). Then equations (17) become

$$\frac{\partial R_I(t,a)}{\partial t} = \left[\frac{\partial R_J(t,a)}{\partial a^I} - \frac{\partial R_I(t,a)}{\partial a^J}\right]\Xi^J(t,a) - \frac{\partial B(t,a)}{\partial a^I} \tag{19a}$$

$$\frac{\partial B(t,a)}{\partial t} = \Xi_0^{I0}\frac{\partial B(t,a)}{\partial a^I} - \Xi_0^{IJ}\frac{\partial R_J(t,a)}{\partial a^I} \tag{19b}$$

The solution for $\{R_I, -B\}$ is uniquely determined by known functions $\Xi^J, \Xi_0^{I0}, \Xi_0^{IJ}$ due to the Cauchy-Kowalevski theorem. However, for the definite functions Ξ^I, different choices of functions Ξ_0^{I0}, Ξ_0^{IJ} can produce different solutions $\{R_I, -B\}$, which is not of physical meaning in general. Evidently there exist infinite solutions for the equations (19a). If the quantity B in the equations (19a) is given, the equations (19a) are complete and the theorem ensures unique existence of the functions R_I. In this case the equation (19b) for the functions Ξ_0^{I0}, Ξ_0^{IJ}, which is in fact algebraic, is not complete.

This analysis may be useful to easily find out the solution $\{R_I, -B\}$ based on a suitable choice of functions Ξ_0^{I0}, Ξ_0^{IJ}. For example, we can suppose that $\Xi_0^{I0} = 0, \Xi_0^{IJ} = \delta^{IJ}$. Then the equation (19b) becomes

$$\frac{\partial B(t,a)}{\partial t} + \frac{\partial R_I(t,a)}{\partial a^I} = 0 \tag{20}$$

Now we will observe the analytic, algebraic and geometric characteristic of the following generalized Birkhoff's equations

$$\left[\frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}\right]\dot{a}^J - \left(\frac{\partial B}{\partial a^I} + \frac{\partial R_I}{\partial t}\right) = 0, \quad I,J = 1,2,\ldots,m \tag{21}$$

from which it is easy to infer that

$$\frac{dB}{dt} = \frac{\partial B}{\partial t} - \frac{\partial R_I}{\partial t}\dot{a}^I \tag{22}$$

The equations (21) are analytic because they are derivable from the Pfaffian action

$$\mathcal{A}(\tilde{E}) = \int_{t_1}^{t_2} dt \left[R_\nu (t,a)\,\dot{a}^\nu - B\,(t,a)\right](\tilde{E}) \tag{23}$$

with the end points condition,where $\tilde{E}$ is a possible pass in the contact manifold M.

We can define 1-form $\Theta(t,a) = R_I(t,a)da^I - B(t,a)dt$ on the manifold M subject to the condition that its exterior derivative

$$\Omega = d(R_I\, da^I - B\, dt) = \frac{1}{2}\left(\frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}\right) da^I \wedge da^J + \left(\frac{\partial B}{\partial a^I} + \frac{\partial R_I}{\partial t}\right) dt \wedge da^I \tag{24}$$

is of maximal rank. Making use of the notation $\Omega_{IJ} = \frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}$, $\Gamma_I = \frac{\partial B}{\partial a^I} + \frac{\partial R_I}{\partial t}$, it is easy to verify the equivalence relation between the closure of the 2-form Ω and self-adjointness of equations (21), i.e.,

$$d\Omega = 0 \Longleftrightarrow \begin{cases} \Omega_{IJ} + \Omega_{JI} = 0 \\ \dfrac{\partial \Omega_{IJ}}{\partial a^K} + \dfrac{\partial \Omega_{JK}}{\partial a^I} + \dfrac{\partial \Omega_{KI}}{\partial a^J} = 0 \\ \dfrac{\partial \Omega_{IJ}}{\partial t} = \dfrac{\partial \Gamma_I}{\partial a^I} - \dfrac{\partial \Gamma_J}{\partial a^I} \end{cases} \tag{25}$$

It inferred that the self-adjointness of the systems is independent of the non-degeneracy of 2-form Ω.

Definition 1. A presymplectic structure on a manifold M can be defined by a closed 2-form Ω, which may be degenerate in the sense that for all vector fields $V \in \Gamma(M)$, $\exists X \neq 0$, $X \in \Gamma(M)$, such that $\Omega(X,V) = 0$. The pair (M,Ω) is called a presymplectic manifold.

Because a 2-form on the manifold M with odd dimension is degenerate,[2] such a closed 2-form can not define a one-to-one and onto map between the tangent space $T_{\{t,a\}}M$ and the cotangent space $T^*_{\{t,a\}}M$ unless the dimension of the manifold M is even, i.e., $m = 2n$ and $\det(\Omega_{IJ}) \neq 0$, which is the case of a Birkhoffian system. Therefore, if we denote the set of smooth real-valued functions on M by $\mathcal{F}(M)$, there does not, in general, exist the unique Hamiltonian vector field X_f on M such that $i_{X_f}\Omega = df$, $f \in \mathcal{F}(M)$. Although the dynamical vector field

[2]Denote the transpose of the matrix (Ω_{IJ}) by $(\Omega_{IJ})^T = (\Omega_{JI})$, then $\det(\Omega_{IJ})^T = \det(\Omega_{IJ})$. Since the matrix (Ω_{IJ}) is antisymmetric, i.e., $\Omega_{IJ} = -\Omega_{JI}$, we have $\det(-\Omega_{IJ}) = \det(\Omega_{IJ})$. It can be referred that $\det(-\Omega_{IJ}) = (-1)^m \det(\Omega_{IJ})$ from the relation $\det(\Omega_{IJ}) = \varepsilon^{IJ\ldots K}\Omega_{1I}\Omega_{2J}\cdots\Omega_{mK}$ where $\varepsilon^{IJ\ldots K}$ is the totally antisymmetric *Levi-Civita* symbol. Thus $\det(\Omega_{Ij}) = 0$ if m is odd.

$X = \frac{\partial}{\partial t} + \Xi^I(t,a)\frac{\partial}{\partial a^I}$ cannot locally represented by the Birkhoffian as equation (12), we fortunately still have a global formulation of the equations (21), i.e.,

$$i_X\Omega = 0, \quad i_X\, dt = 1. \tag{26}$$

which also include the time variation of the generalized Birkhoffian function B, i.e, the equation (22). It is interesting that the global form (26) is not only suitable to formulate Hamiltonian or Birkhoffian systems with contact (symplectic) structure but also enables to represent GBSs and other systems as nonholonomic systems [2], [3] of non-symplectic structure.

If the closed 2-form Ω is regular, the system is reduced to the Birkhoffian system. If the GBS is autonomous or semi-autonomous, the equation

$$\Omega_{IJ}\dot{a}^J - \frac{\partial B}{\partial a^I} = 0, \quad I, J = 1, 2, \ldots, m \tag{27}$$

can be globally formulated by

$$i_X\Omega = -dB \tag{28}$$

where $\Omega = \frac{1}{2}\Omega_{IJ}da^I \wedge da^J$. Combining the equation (22) with equation (28), yields that $\frac{\partial B}{\partial t} = 0$, i.e., the so-called semiautonomous Birkhoffian systems are really autonomous ones. Furthermore, if the closed 2-form is regular, the Lie algebra can be constructed by the Poisson bracket

$$\{f, g\} = \Omega^{-1}(df, dg) = \Omega(X_f, X_g) \tag{29}$$

in accordance with the time evolution law $\dot{a}^I = \Omega^{IJ}\frac{\partial B}{\partial a^J} = 0$. It should pointed out that Lie algebra structure does not generally exist for a GBS unless $\frac{\partial R_I}{\partial t} = 0$, $\det(\Omega_{IJ}) \neq 0$.

4 Integration of generalized Birkhoff's equations

As Birkhoffian mechanics, the most important tasks to study the generalized Birkhoffian mechanics mainly focus on both constructing the Birkhoffian, Birkhoffian functions and integrating the generalized Birkhoff's equations. The first procedure should be checking whether the methods utilized in Birkhoffian mechanics can be generalized to GBSs or not. Fortunately, all the existing methods to construct Birkhoffian functions can be used in GBSs because that the methods only rely upon the locality, analyticity of the integrand, independent of the regularity of the matrix Ω_{IJ} mentioned above.

A given first-order system verifying the conditions of locality, analyticity and regularity always admits infinite varieties of equivalent generalized Birkhoffian representations characterized by the gauge transformations

$$R_I(t,a) \to R'_I(t,a) = R_I(t,a) + \frac{\partial G(t,a)}{\partial a^I} \tag{30a}$$

$$B(t,a) \to B'(t,a) = B(t,a) - \frac{\partial G(t,a)}{\partial t} \tag{30b}$$

All the Birkhoffian representations are equivalent in the sense that Birkhoff's equations are the same for all possible functions (30), i.e.,

$$\left(\frac{\partial R'_\nu}{\partial a^\mu} - \frac{\partial R'_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B'}{\partial a^\mu} + \frac{\partial R'_\mu}{\partial t}\right) = \left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu - \left(\frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t}\right) \quad (31)$$

The practical meaning is that we can choose a gauge function $G(t,a)$ to make the Birkhoffian be the physical energy of the system.

Therefore, we can outline the three methods to construct Birkhoffian functions as follows.

Method 1. Let B be the total energy of the system and then solve the Cauchy--Kowalevski equations (19a) in the functions R_I.

Method 2. Via the method of the genotopic transformations starting from the equation (16), construct a self-adjoint covariant form

$$[\Omega_{IJ}(t,a)\,\dot{a}^J + \Gamma_I(t,a)]_{SA} = 0, \qquad I,J = 1,2,\ldots,m \quad (32)$$

The functions R_I are then given by

$$R_I(t,a) = \left[\int_0^1 d\tau\,\tau\Omega_{IJ}(t,\tau a)\right] a^J \quad (33)$$

and the Birkhoffian is provided by the rule

$$B(t,a) = -\left[\int_0^1 d\tau\left(\Gamma_I + \frac{\partial R_I}{\partial t}\right)(t,\tau a)\right] a^I \quad (34)$$

This method is recommended when no physical condition is imposed on the meaning of the Birkhoffian and on the prescriptions for the construction of the first-order form. It is often preferable in practice, because of the greater freedom in the Birkhoffian functions.

Method 3. Suppose that the m obtained first integrals $\mathcal{I}^J$ of the first-order system (16) are independent in the sense that $\det(\partial\mathcal{I}^J/\partial a^I) \neq 0$. Then

$$R_I(t,a) = G_J\frac{\partial\mathcal{I}^J}{\partial a^I}, \quad B(t,a) = -G_J\frac{\partial\mathcal{I}^J}{\partial t} \quad (35)$$

where G_J are functions of the integrals $\mathcal{I}^I$, which are not constrained by the regularity condition $\det(\partial G_I/\partial\mathcal{I}^J - \partial G_J/\partial\mathcal{I}^I) \neq 0$.

5 Application of generalized Birkhoffian formulation to nonholonomic systems

As an important example of GBS, we consider a mechanical system on the contact manifold $R \times TQ$ with local coordinates $\{t, q^i, \dot{q}^i\}$ $(i = 1,2,\ldots,n)$. Denote the Lagrangian of the system by $\mathcal{L}(t,q,\dot{q})$ and suppose the system is subject to the nonholonomic constraints

$$\dot{q}^\alpha = \varphi^\alpha(t, q^i, \dot{q}^\mu), \quad \alpha = 1,2,\ldots,l;\ \mu = 1,2,\ldots,k = n-l \quad (36)$$

The dynamics of the nonholonomic system is uniquely determined by the four factors: (1) Lagrange-d'Alembert principle, (2) ideal constraints, (3) Chetaev's condition for the virtual displacement, and (4) the regularity of the Hessian matrix $\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j}\right)$. Of course, the locality and analyticity are understood. The equations of motion for the system form a set of mixed second- and first-order ordinary differential equations [6], [16], [17], [19]

$$\ddot{q}^\mu = f^\mu(t, q^\nu, \dot{q}^\nu, q^\alpha), \quad \nu = 1, 2, \ldots, k = n - l \tag{37a}$$

$$\dot{q}^\alpha = \varphi^\alpha(t, q^\nu, \dot{q}^\nu, q^\beta), \quad \beta = 1, 2, \ldots, l \tag{37b}$$

The Lagrangian and Hamiltonian inverse problem for such a coupled system is not universal [1], [7], [12]. The nonholonomicity of the system makes the Birkhoffian realization for the system to be not universal. However, the universality of self-adjointness for the nonholonomic system can be realized in the generalized Birkhoffian framework based on the conditions of locality, analyticity and regularity of the system.

Introduce l regular coordinates $\{x^\mu\}$

$$x^\mu = \xi^\mu(t, q^\nu, \dot{q}^\nu, q^\alpha), \quad \det(\partial \xi^\mu / \partial \dot{q}^\nu) \neq 0 \tag{38}$$

whose inverse transformation is

$$\dot{q}^\mu = \zeta^\mu(t, q^\nu, x^\nu, q^\alpha) \tag{39}$$

Substituting equation (39) into the equation (37) we get the following first-order system

$$\dot{q}^\mu = \zeta^\mu(t, q^\nu, x^\nu, q^\alpha) \tag{40a}$$

$$\dot{x}^\mu = \psi^\mu(t, q^\nu, x^\nu, q^\alpha) \tag{40b}$$

$$\dot{q}^\alpha = \varphi^\alpha(t, q^\nu, x^\nu, q^\beta) \tag{40c}$$

Sometimes we directly choose x^μ to be generalized velocity $\dot{q}^\mu$ or generalized momentum p_μ. Denote the $m = 2k + l$ local coordinates $\{q^\nu, x^\nu, q^\alpha\}$ on constraint manifold M by $\{a^I\}$ $(I = 1, 2, \ldots, m = 2k + l)$. Then the equations (37) can be reformulated by

$$\dot{a}^I = \Xi^I(t, a^J), \quad I, J = 1, 2, \ldots, m = 2k + l \tag{41}$$

The locality, analyticity and regularity of the functions $\Xi^I(t, a)$ make the equations (41) admit a generalized Birkhoffian formulation

$$\left[\frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}\right] \dot{a}^J - \left(\frac{\partial B}{\partial a^I} + \frac{\partial R_I}{\partial t}\right) = 0, \quad I, J = 1, 2, \ldots, m = 2k + l \tag{42}$$

where the total energy of the system can be taken as the Birkhoffian B and the functions R_I are related with the nonholonomic constraint forces. It should be remarked that the regularity of Hessian matrix for the original nonholonomic mechanical system does not assure the regularity of the matrix (Ω_{IJ}) which is determined by the integrability of the constraints or the nonholomicity of odd number of constraints. If l is even the system is a Birkhoffian system. For the case of odd l, no symplectic and Lie algebra structure exist on the constraint manifold M. However, the self-adjointness for the equation (41) is independent of parity of the number m of the nonholonomic constraints.

If the second-order equations are decoupled with the constraints, e.g., the Chaplygin's system

$$\ddot{q}^{\mu} = f^{\mu}(t, q^{\nu}, \dot{q}^{\nu}), \quad \nu = 1, 2, \ldots, k = n - l \tag{43a}$$

$$\dot{q}^{\alpha} = \varphi^{\alpha}(t, q^{\nu}, \dot{q}^{\nu}), \quad \alpha = 1, 2, \ldots, l \tag{43b}$$

the Birkhoffian formulation can be realized on a $2k$-dimensional subspace [9], [11].

Example 1. [9] Consider the motion of a simplified sleigh with unit mass and unit moment of inertia in $\mathbb{R}^2 \times T^1$ with coordinates (x, y, φ), subjected to the nonholonomic constraint $\dot{y} = \dot{x} \tan \varphi$. The Lagrangian is $L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2\right)$ and the Lagrangian embedded in the constraint is $\mathcal{L} = \frac{1}{2}\left(\dot{x}^2 \sec^2 \varphi + \dot{\varphi}^2\right)$. Obviously the system is a Chaplygin's system and thus reduced to a holonomic nonconservative subsystem for x and φ on a submanifold $h^s_\tau \subset T\left(\mathbb{R}^2 \times T^1\right)$, decouped with the constraint. The Chaplygin's equations of motion are

$$\ddot{x} + \dot{x}\dot{\varphi} \tan \varphi = 0, \quad \ddot{\varphi} = 0, \quad \dot{y} - \dot{x} \tan \varphi = 0$$

Utilizing the Legendre transformation $\dot{x} = p_x \cos^2 \varphi$, $\dot{\varphi} = p_\varphi$, the Hamiltonian embedded in the constraint is $\mathcal{H} = \frac{1}{2}\left(p_x^2 \cos^2 \varphi + p_\varphi^2\right)$. The equations of motion are given by the matrix form

$$\begin{pmatrix} 0 & p_x \tan \varphi & -1 & 0 \\ -p_x \tan \varphi & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\varphi} \\ \dot{p}_x \\ \dot{p}_\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} p_x^2 \sin 2\varphi \\ p_x \cos^2 \varphi \\ p_\varphi \end{pmatrix}$$

with four independent first integrals

$$I^1 = p_\varphi, \quad I^2 = \varphi - p_\varphi t, \quad I^3 = p_x \cos \varphi, \quad I^4 = \frac{1}{2}\left[\omega^2 x^2 + p_x^2 \cos^4 \varphi\right]$$

where $\omega = \dot{\varphi}$ is constant.

Taking the conventional notations $a^J = \{x, \varphi, p_x, p_\varphi\}$ $(J = 1, 2, 3, 4)$, we find out a set of Birkhoffian functions by means of Hojman's method [5], [14]

$$\begin{aligned}
R_1 &= a^1 a^3 \left(a^4\right)^2 \cos a^2 \\
R_2 &= \frac{1}{2} a^4 + \frac{a^3 \cos a^2}{2a^4} - \left(a^3\right)^2 \cos^3 a^2 \sin 2a^2 \\
R_3 &= \left(a^3\right)^2 \cos^5 a^2 \\
R_4 &= -a^2 + \frac{1}{2} a^4 t - \frac{t a^3 \cos a^2}{2a^4} + \left(a^1\right)^2 a^3 a^4 \cos a^2 \\
B &= \frac{1}{2}\left[\left(a^4\right)^2 + a^3 \cos a^2\right]
\end{aligned}$$

Thus the symplectic tensor $\Omega_{IJ} = \frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}$ is given by the matrix elements

$$\begin{aligned}
&\Omega_{11} = \Omega_{22} = \Omega_{33} = \Omega_{44} = 0 \\
&\Omega_{12} = -\Omega_{21} = a^1 a^3 \left(a^4\right)^2 \sin a^2 \\
&\Omega_{13} = -\Omega_{31} = -a^1 \left(a^4\right)^2 \cos a^2 \\
&\Omega_{14} = -\Omega_{41} = 0 \\
&\Omega_{23} = -\Omega_{32} = \left(a^3\right)^2 \cos^4 a^2 \sin a^2 - \frac{\cos a^2}{2a^4} \\
&\Omega_{24} = -\Omega_{42} = -\frac{3}{2} - \left(a^1\right)^2 a^3 a^4 \sin a^2 + \frac{a^3 \cos a^2 + t a^3 a^4 \sin a^2}{2(a^4)^2} \\
&\Omega_{34} = -\Omega_{43} = \left(a^1\right)^2 a^4 \cos a^2 - \frac{t \cos a^2}{2a^4}
\end{aligned}$$

which satisfies the conditions of self-adjointness. It can be verified that the equations of motion can be represented by the nonautonomous Birkhoff's equations

$$\Omega_{IJ} \dot{a}^J - \frac{\partial B}{\partial a^I} - \frac{\partial R_I}{\partial t} = 0$$

Example 2. Consider a nonholonomic system whose configuration is denoted by $\{q^1, q^2\}$. The Lagrangian of the system is $L = \frac{1}{2}\left((\dot{q}^1)^2 + (\dot{q}^2)^2\right)$. Suppose the system is constrained by a nonholonomic constraint

$$\dot{q}^1 + t\dot{q}^2 - q^2 + t = 0.$$

Then the differential equations of motion for the system are

$$\begin{gathered}
(1+t^2)\ddot{q}^2 + 2t\dot{q}^2 + 2\dot{q}^1 - 2q^2 + 3t = 0 \\
\dot{q}^1 + t\dot{q}^2 - q^2 + t = 0
\end{gathered}$$

Let $a^1 = q^2$, $a^2 = \dot{q}^2$, $a^3 = q^1$, then the equations can be transformed into the first-order differential equations

$$\begin{aligned}
&\dot{a}^1 = a^2, \\
&\dot{a}^2 = \frac{-t}{1+t^2}, \\
&\dot{a}^3 = a^1 - ta^2 - t
\end{aligned}$$

with three independent first integrals

$$\begin{aligned}
&I^1 = a^3 - t\left(a^1 - ta^2 - t\right) - \frac{1}{2}\ln\left(1+t^2\right), \\
&I^2 = a^1 - ta^2 - t + \arctan t, \\
&I^3 = a^2 + \frac{1}{2}\ln\left(1+t^2\right)
\end{aligned}$$

By using the Hojman's method, we can get the Birkhoffian functions

$$\begin{aligned}
R_1 &= G_1\frac{\partial I^1}{\partial a^1} + G_2\frac{\partial I^2}{\partial a^1} + G_3\frac{\partial I^3}{\partial a^1} = -G_1 t + G_2\\
R_2 &= G_1\frac{\partial I^1}{\partial a^2} + G_2\frac{\partial I^2}{\partial a^2} + G_3\frac{\partial I^3}{\partial a^2} = G_1 t^2 - G_2 t + G_3\\
R_3 &= G_1\frac{\partial I^1}{\partial a^3} + G_2\frac{\partial I^2}{\partial a^3} + G_3\frac{\partial I^3}{\partial a^3} = G_1\\
B &= -\left[G_1\frac{\partial I^1}{\partial t} + G_2\frac{\partial I^2}{\partial t} + G_3\frac{\partial I^3}{\partial t}\right]\\
&= -G_1\left(2ta^2 + 2t - a^1 - \frac{t}{1+t^2}\right) - G_2\left(\frac{1}{1+t^2} - a^1 - 1\right) - G_3\frac{t}{1+t^2}
\end{aligned}$$

Set $G_1 = I^2, G_2 = 0, G_3 = I^3$. Then

$$\begin{aligned}
R_1 &= -t\left(a^1 - ta^2 - t + \arctan t\right)\\
R_2 &= t^2\left(a^1 - ta^2 - t + \arctan t\right) + a^2 + \frac{1}{2}\ln\left(1+t^2\right)\\
R_3 &= a^1 - ta^2 - t + \arctan t\\
B &= \left(2ta^2 + 2t - a^1 - \frac{t}{1+t^2}\right)\left(-a^1 + ta^2 + t - \arctan t\right)\\
&\quad - \frac{t}{1+t^2}\left[a^2 + \frac{1}{2}\ln\left(1+t^2\right)\right]
\end{aligned}$$

Thus the presymplectic tensor $\Omega_{IJ} = \frac{\partial R_J}{\partial a^I} - \frac{\partial R_I}{\partial a^J}$ is given by the matrix

$$(\Omega_{IJ})_{3\times 3} = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & -t\\ -1 & t & 0 \end{pmatrix}$$

It can be verified that the equations of motion can be represented by the generalized Birkhoff equations

$$\Omega_{IJ}\dot{a}^J - \frac{\partial B}{\partial a^I} - \frac{\partial R_I}{\partial t} = 0$$

Concluding remarks

As shown above the inverse problem of the calculus of variations for a dynamical system is characterized essentially by the self-adjointness conditions of the equations of motion in first-order form, which is equivalent to a closed 2-form on the manifold. Any local, analytic, regular, finite-dimensional, nonholonmic, self-adjoint or non-self-adjoint dynamical systems in first-order form always admit a generalized Birkhoffian formulation in a contractible region of regular point of variables. The sequence from self-adjointness to symplecticity and to Lie algebra of the formulation for the dynamics is a sequence for the conditions to become more and more strict. The symbiosis of self-adjoint/symplectic/Lie algebraic/physical formulation is only suitable to the local, analytic, regular, holonomic, autonomous dynamical systems.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (10932002, 10872084, 10472040), the Outstanding Young Talents Training Fund of Liaoning Province of China (3040005), the Research Program of Higher Education of Liaoning Province, China (2008S098), the Program of Supporting Elitists of Higher Education of Liaoning Province, China (2008RC20) and the Program of Constructing Liaoning Provincial Key Laboratory, China (2008403009).

References

[1] A.M. Bloch, O.E. Fernandez, T. Mestdag: Hamiltonization of nonholonomic systems and the inverse problem of the calculus of variations. *Rep. Math. Phys.* 63 (2009) 225–249.

[2] A.M. Bloch, J. Baillieul, P. Crouch, J. Marsden: *Nonholonomic Mechanics and Control*. Springer, London (2003).

[3] J.M. Cortes: *Geometric, Control and Numerical Aspects of Nonholonomic Systems*. Springer, Berlin (2002).

[4] Y.X. Guo, S.K. Luo, M. Shang, F.X. Mei: Birkhoffian formulation of nonholonomic constrained systems. *Rep. Math. Phys.* 47 (2001) 313–322.

[5] S. Hojman: Construction of genotopic transformations for first order systems of differential equations. *Hadronic J.* 5 (1981) 174–184.

[6] L.A. Ibort, J.M. Solano: On the inverse problem of the calculus of variations for a class of coupled dynamical systems. *Inverse Problems* 7 (1991) 713–725.

[7] O. Krupková, J. Musilová: Non-holonomic variational systems. *Rep. Math. Phys.* 55 (2005) 211–220.

[8] J.B. Li, X.H. Zhao, Z.R. Liu: *Theory of Generalized Hamiltonian Systems and Its Applications*. Science Press of China Beijing (2007).

[9] C. Liu, S.X. Liu, Y.X. Guo: Inverse problem for Chaplygin's nonholonomic. *Sci. Chin. G* 53 (2010). (to appear)

[10] E. Massa, E. Pagani: Jet bundle geometry, dynamical connections, and the inverse problem of Lagrangian mechanics. *Ann. Inst. Henri Poincaré: Phys. Theor.* 61 (1994) 17–62.

[11] F.X. Mei, R.C. Shi, Y.F. Zhang, H.B. Wu: *Dynamics of Birkhoffian systems*. Press of Beijing Institute of Technology Beijing (1996). (in Chinese)

[12] P. Morando, S. Vignolo: A geometric approach to constrained mechanical systems, symmetries and inverse problems. *J. Phys. A: Math. Gen.* 31 (1998) 8233–8245.

[13] R.M. Santilli: *Foundations of Theoretical Mechanics I*. Springer-Verlag, New York (1978).

[14] R.M. Santilli: *Foundations of Theoretical Mechanics II*. Springer-Verlag, New York (1983).

[15] W. Sarlet: The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics. *J. Phys. A: Math. Gen.* 15 (1982) 1503–1517.

[16] W. Sarlet, F. Cantrijn, D.J. Saunders: A differential geometric setting for mixed first- and second-order ordinary differential equations. *J. Phys. A: Math. Gen.* 30 (1997) 4031–4052.

[17] W. Sarlet, F. Cantrijn, D.J. Saunders: A geometrical framework for the study of non-holonomic Lagrangian systems. *J. Phys. A: Math. Gen.* 28 (1995) 3253–3268.

[18] W. Sarlet, G. Thompson, G.E. Prince: The inverse problem in the calculus of variations: the use of geometrical calculus in Douglas's analysis. *Trans. Amer. Math. Soc.* 354 (2002) 2897–2919.

[19] D.J. Saunders, W. Sarlet, F. Cantrijn: A geometrical framework for the study of non-holonomic Lagrangian systems II. *J. Phys. A: Math. Gen.* 29 (1996) 4265–4274.

Authors' addresses:

Yong-Xin Guo, Shi-Xing Liu: College of Physics, Liaoning University, Shenyang 110036, China

Chang Liu: School of Aerospace Engineering, Beijing Institute of Technology, Beijing 100081, China

E-mail address: yxguo@lnu.edu.cn

Received: 5 December, 2010

Accepted for publication: 19 December, 2010

Communications in Mathematics 18 (2010) 37–50

Homogeneous systems of higher-order ordinary differential equations

Mike Crampin

Abstract. The concept of homogeneity, which picks out sprays from the general run of systems of second-order ordinary differential equations in the geometrical theory of such equations, is generalized so as to apply to equations of higher order. Certain properties of the geometric concomitants of a spray are shown to continue to hold for higher-order systems. Third-order equations play a special role, because a strong form of homogeneity may apply to them. The key example of a single third-order equation which is strongly homogeneous in this sense states that the Schwarzian derivative of the dependent variable vanishes. This equation is of importance in the theory of the association between third-order equations and pseudo-Riemannian manifolds due to Newman and his co-workers.

1 Introduction

In the geometrical theory of systems of second-order ordinary differential equations an important role is played by a class of equations which are homogeneous in a certain sense. In the theory envisaged here a system of second-order equations in m dependent variables is represented by a vector field Γ of a special type on the tangent bundle $T(M)$ of an m-dimensional smooth manifold M. A tangent bundle comes equipped with a canonical vertical vector field Δ, the Liouville field, which is the infinitesimal generator of dilations of the fibres. Then the equations are homogeneous if the corresponding differential equation field Γ satisfies $[\Delta, \Gamma] = \Gamma$ (so one might say it is homogeneous of degree 1). Such differential equation fields, which may in fact be defined not on the whole of $T(M)$ but only on the slit tangent bundle ($T(M)$ with its zero section deleted), are often called sprays. Examples include the geodesic field of a Finsler space, and a fortiori that of a Riemannian space, or indeed of any affine connection. It should be mentioned that

2010 MSC: 34A26, 34C14, 53A55, 53B15, 83C80

Key words: Differential equation field, dynamical covariant derivative, horizontal distribution, Jacobi endomorphism, Liouville field, multiconnection, n-velocity, spray, Wuenschmann invariant.

this paper deals only with equations which do not explicitly involve the independent variable; in a dynamical context such equations could be called autonomous, and the examples mentioned above are clearly of this type. For a textbook account of the geometry of sprays see [13].

The aim of the present paper is to propose a definition — or as it turns out, definitions — of homogeneity for systems of higher-order ordinary differential equations, where 'higher-order' means of course 'of order higher than the second'; and to establish under these definitions properties that parallel those enjoyed by sprays in the second-order case.

To describe the properties of sprays that will be generalized I first need to say somewhat more about the geometrical theory of systems of second-order differential equations. A standard reference here is [3]. The main aim of the theory may be described as the formulation of differential-geometric concomitants of the equation field Γ, or in other words associated geometric objects which do not depend on the choice of coordinates. (Despite what has just been said, I must admit to the rather frequent use of coordinates below — but only in order to simplify or speed up the exposition.) For my purposes here there are three important concomitants of a second-order differential field Γ. These involve in their specifications so-called vector fields along the tangent bundle projection $\tau : T(M) \to M$, or in other words sections of the pull-back bundle $\tau^*T(M) \to T(M)$. I make a short detour to mention a couple of useful features of this construction. There is a canonical section of $\tau^*T(M)$, given in local coordinates by $\dot{y}^i\partial/\partial y^i$, where (y^i), $i = 1, 2, \dots, m$ are coordinates on M and $(\dot{y}^i)$ the corresponding fibre cordinates on $T(M)$; this section is sometimes called the total derivative; I shall denote it by T. Given any section Y of $\tau^*T(M)$ there is a canonical associated vertical vector field on $T(M)$, its vertical lift, denoted by Y^V. The three geometric objects associated with a second-order differential equation field are

1. its *dynamical covariant derivative*: this is an $\mathbb{R}$-linear operator
$$\nabla : \operatorname{sect}\tau^*T(M) \to \operatorname{sect}\tau^*T(M)$$
satisfying the covariant-derivative-like property $\nabla(fY) = f\nabla Y + \Gamma(f)Y$ for $f \in C^\infty(T(M))$;

2. its *horizontal distribution*: this is a distribution on $T(M)$, complementary to the vertical distribution, which can be thought of as defining a nonlinear connection, and leads to a horizontal lift operation taking $Y \in \operatorname{sect}\tau^*T(M)$ to Y^H, a horizontal vector field on $T(M)$;

3. its *Jacobi endomorphism*: this is a $C^\infty(T(M))$-linear map
$$\Phi : \operatorname{sect}\tau^*T(M) \to \operatorname{sect}\tau^*T(M)\,,$$
so called because Jacobi's equation, which is the equation satisfied by variation fields along an integral curve of Γ, can be expressed as $\nabla^2 Y + \Phi(Y) = 0$.

The three are related as follows:
$$[\Gamma, Y^V] = -Y^H + (\nabla Y)^V, \qquad [\Gamma, Y^H] = (\nabla Y)^H + \Phi(Y)^V.$$

When Γ is a spray these objects have the following properties (see for example [5], [13]):

1. the horizontal distribution is invariant under Δ: if $Y \in \mathfrak{X}(M)$ then $[\Delta, Y^H] = 0$;
2. the Jacobi endomorphism is homogeneous of degree 2: $\mathcal{L}_\Delta \Phi = 2\Phi$ (where the Lie derivative is used in a sense to be explained fully later);
3. Γ is horizontal with respect to the horizontal distribution it defines, and in fact $\Gamma = T^H$;
4. $\Phi(T) = 0$.

In Section 3 below I describe how one may define analogues of the dynamical covariant derivative, the horizontal distribution and the Jacobi endomorphism for a differential equation field of any order. There are in fact several inequivalent ways of extending these concepts to equations of higher order, which are discussed in [1], [2], [4], [6], [8], [12]. The one adopted here is based on an approach first given for fourth-order equations in [1], and extended to equations of arbitrary order in [8]. It is to be preferred, in my opinion, because it gets most quickly to the generalized Jacobi equation.

In Section 4 I give a definition of homogeneity for a differential equation field of any order, and show that properties 1 and 2 above hold, mutatis mutandis, for homogeneous fields with this definition. There is a stronger sense of homogeneity, but it applies only to equations of the third order. I explain this in Section 5, and show that the remaining two properties then hold.

Finally in Section 5 I obtain the most general strongly homogeneous third-order differential equation in a single dependent variable. This turns out to be essentially the vanishing of the Schwarzian derivative of the dependent variable. Single third-order differential equations are of interest because of their association in certain cases with pseudo-Riemannian spaces, an association rediscovered as a result of an approach to problems in general relativity initiated by E. T. Newman and his co-workers in [9]. The equations which feature in these studies are picked out by the vanishing of a certain invariant called the Wuenschmann invariant: this has a simple expression in terms of the Jacobi endomorphisms, as has been shown in [8], and also more recently in [4]. It turns out that each member of the class of strongly homogeneous third-order differential equations in a single dependent variable mentioned above has vanishing Jacobi endomorphisms, and ipso facto vanishing Wuenschmann invariant. The properties of pseudo-Riemannian spaces associated with these equations have been studied in [10], [11].

The paper proper begins with an account of the space which plays the role of the tangent bundle for equations of order n, namely the bundle of n-velocities.

2 The bundle of n-velocities

Let M be a smooth manifold of dimension m, with local coordinates (y^i), $i = 1, 2, \ldots, m$. Let $T^n(M)$ be the bundle of n-velocities on M: a point of $T^n(M)$ is an equivalence class of curves σ in M, defined on an open interval containing 0, under the equivalence relation $\sigma \equiv \rho$ if in one, and hence any, coordinate representation

$$\frac{d^r\sigma^i}{dx^r}(0) = \frac{d^r\rho^i}{dx^r}(0), \quad r = 0, 1, \dots, n.$$

Denote by $(y^i_r) = (y^i_0, y^i_1, \dots, y^i_n)$ the natural coordinates on $T^n(M)$, so that (y^i_r) are the coordinates of the equivalence class of the curve

$$y^i(x) = y^i_0 + xy^i_1 + \cdots + \frac{1}{r!}x^r y^i_r + \cdots + \frac{1}{n!}x^n y^i_n.$$

Evidently $T^n(M)$ is fibred over $T^r(M)$ for $r = 0, 1, \dots, n-1$, where $T^0(M) = M$. The corresponding projections are denoted by $\tau_r : T^n(M) \to T^r(M)$; we have $\tau_r(y^i_0, y^i_1, \dots, y^i_n) = (y^i_0, y^i_1, \dots, y^i_r)$. A vector $v \in T(T^n(M))$ such that $\tau_{0*}v = 0$ is said to be vertical; one such that $\tau_{r*}v = 0$ to be vertical over $T^r(M)$; on occasion, one vertical over $T^{n-1}(M)$ to be very vertical. At any point of $T^n(M)$, the space of vectors vertical over $T^r(M)$ is spanned by the coordinate vectors $\partial/\partial y^i_s$ for $s > r$. For $n \geq r \geq 1$, denote the vector sub-bundle of $T(T^n(M))$ consisting of vectors vertical over $T^{r-1}(M)$ by V_r, so that in particular the vector sub-bundle consisting of vertical vectors (vectors vertical over M) is V_1. Then

$$V_n \subset V_{n-1} \subset \cdots \subset V_1 \subset T(T^n(M))$$

is a filtration of $T(T^n(M))$. One can identify V_n, and V_{r-1}/V_r, with $\tau_0^*(T(M))$, the pull-back by $\tau_0 : T^n(M) \to M$ of the tangent bundle $T(M) \to M$. I shall denote by $\mathcal{V}_r$ the module of vector fields on $T^n(M)$ which are vertical over $T^{r-1}(M)$, or in other words, sections of $V_r \to T^n(M)$. Then

$$\mathcal{V}_n \subset \mathcal{V}_{n-1} \subset \cdots \subset \mathcal{V}_1 \subset \mathfrak{X}(T^n(M)).$$

The type $(1,1)$ tensor field S on $T^n(M)$ given by

$$S = \sum_{r=1}^{n} r \frac{\partial}{\partial y^i_r} \otimes dy^i_{r-1}$$

is called the vertical endomorphism. Evidently $S(\mathfrak{X}(T^n(M))) = \mathcal{V}_1$, and for $1 \leq r \leq n-1$, $S(\mathcal{V}_r) = \mathcal{V}_{r+1}$, while $S(\mathcal{V}_n) = \{0\}$.

The additive group $\mathbb{R}$ acts on $T^n(M)$ as follows. For any curve σ in M, and for $t \in \mathbb{R}$, we may define a curve σ_t by $\sigma_t(x) = \sigma(e^t x)$. This map of curves defines a map of n-velocities, given in coordinates by $(y^i_r) \mapsto (e^{rt}y^i_r)$. The corresponding (vertical) vector field on $T^n(M)$ is

$$\Delta = y^i_1\frac{\partial}{\partial y^i_1} + 2y^i_2\frac{\partial}{\partial y^i_2} + \cdots + ny^i_n\frac{\partial}{\partial y^i_n}.$$

It is the fundamental vector field of the action corresponding to the vector field $t\partial/\partial t$ on $\mathbb{R}$.

Vector fields Δ^r on $T^n(M)$, $r = 1, 2, \ldots, n$, are defined as follows:

$$\Delta^{r+1} = S(\Delta^r), \quad \Delta^1 = \Delta.$$

Then $S(\Delta^n) = 0$. It may be shown that these vector fields satisfy

$$[\Delta^r, \Delta^s] = \begin{cases} (r-s)\Delta^{r+s-1} & \text{for } r+s-1 \leq n \\ 0 & \text{otherwise.} \end{cases}$$

In particular, they form an n-dimensional Lie algebra, say $\mathfrak{D}$. The generators of this algebra may be related to vector fields on $\mathbb{R}$ in a way that extends the relation between Δ and $t\partial/\partial t$ described above. Let $\mathfrak{p}$ be the Lie algebra of vector fields on $\mathbb{R}$ whose coefficients are (formal) power series in t, and let $\mathfrak{p}^n$ be the subalgebra of those vector fields which vanish to order n at 0, that is, whose coefficient begins with t^{n+1}. Then $\mathfrak{p}^0$ is the subalgebra of vector fields which vanish at 0; for $n > 0$, $\mathfrak{p}^n$ is an ideal in $\mathfrak{p}^0$; and $\mathfrak{D}$ is anti-isomorphic to $\mathfrak{p}^0/\mathfrak{p}^n$. In fact the map $\mathfrak{p}^0 \to \mathfrak{X}(T^n(M))$ by $t^r\partial/\partial t \mapsto S^{r-1}(\Delta)$ for $r \geq 1$ is an anti-homomorphism, with kernel $\mathfrak{p}^n$. Now the group D of diffeomorphisms of $\mathbb{R}$ which leave 0 fixed acts on $T^n(M)$ by reparametrization; the action leaves the fibres invariant and induces the identity on M. We may think of $\mathfrak{p}^0$ as playing the role of the Lie algebra of D, and the vector fields Δ^r as the fundamental vector fields of the action.

The space $\mathfrak{q}$ of all quadratic vector fields on $\mathbb{R}$ is a Lie algebra, and no space of all polynomial vector fields of some higher degree has that property. Then $\mathfrak{p}^0/\mathfrak{p}^2$ is isomorphic to $\mathfrak{q}^0$, the subalgebra of $\mathfrak{q}$ consisting of quadratic vector fields that vanish at 0. Correspondingly, $\{\Delta^1, \Delta^2\}$ span a 2-dimensional Lie algebra of vector fields on $T^n(M)$, which is anti-isomorphic to $\mathfrak{q}^0$.

3 Systems of $(n+1)$st-order differential equations

A differential equation field of order $n+1$ is a vector field Γ on $T^n(M)$ of the form

$$\Gamma = y_1^i \frac{\partial}{\partial y_0^i} + y_2^i \frac{\partial}{\partial y_1^i} + \cdots + y_n^i \frac{\partial}{\partial y_{n-1}^i} + f^i \frac{\partial}{\partial y_n^i}.$$

The integral curves of such a vector field, projected onto M, are the solutions of the system of $(n+1)$st-order ordinary differential equations

$$y_{n+1}^i = f^i(y^j, y_1^j, \ldots, y_n^j), \qquad y_r^i = \frac{d^r y^i}{dx^r}.$$

The vector field Γ is a geometrical expression for the system of differential equations. One geometrical approach to the study of systems of ordinary differential equations is to work with the corresponding vector field: this is the approach adopted here.

Notice that Γ is a differential equation field if and only if $S(\Gamma) = \Delta$.

If Γ is a differential equation field of order $n+1$ and $\hat{\Gamma}$ is another vector field on $T^n(M)$) then $\hat{\Gamma}$ is also a differential equation field of order $n+1$ if and only if differs from Γ by a very vertical vector field.

Associated with a differential equation field Γ there is a linear differential operator ∇ with properties reminiscent of those of a covariant derivative; accordingly, it is called the dynamical covariant derivative. It acts on $\operatorname{sect}\tau_0^*(T(M))$, and satisfies $\nabla(fY) = f\nabla Y + \Gamma(f)Y$ for $f \in C^\infty(T^n(M))$ and $Y \in \operatorname{sect}\tau_0^*(T(M))$. In terms of coordinate fields ∇ is given by

$$\nabla\left(\frac{\partial}{\partial y^i}\right) = -\frac{1}{n+1}\frac{\partial f^j}{\partial y_n^i}\frac{\partial}{\partial y^j} = \Gamma_i^j\frac{\partial}{\partial y^j}.$$

One can easily check from this formula that ∇ is well-defined, in the sense that Γ_i^j transforms appropriately under a coordinate transformation $\bar{y}^i = \bar{y}^i(y^j)$ on M. Then (taking account of the Leibniz-like rule)

$$\nabla Y = \nabla\left(Y^i\frac{\partial}{\partial y^i}\right) = (\Gamma(Y^i) + Y^j\Gamma_j^i)\frac{\partial}{\partial y^i} = (\nabla Y^i)\frac{\partial}{\partial y^i}.$$

The dynamical covariant derivative may of course be made to act on other geometrical objects. In particular, it acts on $T(M)$-valued 1-forms on $T^n(M)$, that is to say, sections of $T^*(T^n(M)) \otimes \tau_0^*(T(M)) \to T^n(M)$, as follows: for $\theta \in \operatorname{sect} T^*(T^n(M))$ and $Y \in \operatorname{sect}\tau_0^*(T(M))$ set

$$\nabla(\theta \otimes Y) = \mathcal{L}_\Gamma\theta \otimes Y + \theta \otimes \nabla Y.$$

For convenience of calculation one may proceed as follows. Any section of

$$T^*(T^n(M)) \otimes \tau_0^*(T(M))$$

can be expressed as $\theta^i \otimes \partial/\partial y^i$ where (θ^i) is an m-tuple of 1-forms on $T^n(M)$, and the index i may be thought of as tensorial so far as coordinate transformations on M are concerned. Set

$$\nabla\left(\theta^i \otimes \frac{\partial}{\partial y^i}\right) = (\nabla\theta^i) \otimes \frac{\partial}{\partial y^i}.$$

This defines a new m-tuple of 1-forms on $T^n(M)$, $(\nabla\theta^i)$, and again the index is tensorial; explicitly,

$$\nabla\theta^i = \mathcal{L}_\Gamma\theta^i + \Gamma_j^i\theta^j.$$

It follows that for any vector field Z on $T^n(M)$

$$\theta^i([\Gamma, Z]) = \nabla(\theta^i(Z)) - (\nabla\theta^i)(Z).$$

This operation will now be applied repetitively, starting with $\theta^i = dy_0^i = \tau_{0*}(dy^i)$, in other words with the section of $T^*(T^n(M)) \otimes \tau_0^*(T(M))$ which, as a map, takes a vector v on $T^n(M)$ to $\tau_{0*}v$ considered as an element of $\tau_0^*(T(M))$ located at the same point of $T^n(M)$ as v. That is to say, we define 1-forms Θ_r^i, $0 \le r \le n$, by

$$\Theta_{r+1}^i = \nabla\Theta_r^i, \qquad \Theta_0^i = dy_0^i,$$

or in terms of $T(M)$-valued 1-forms, $\Theta_{r+1} = \nabla\Theta_r$. Now for $0 \le r < n$, $\mathcal{L}_\Gamma dy_r^i = dy_{r+1}^i$, whence for $0 \le r \le n$ we may write

$$\Theta_r^i = dy_r^i + \sum_{s=0}^{r-1}(C_r^s)_j^i dy_s^j$$

for some coefficients $(C^s_r)^i_j$ which are (local) functions on $T^n(M)$. These satisfy the recurrence relations

$$\begin{aligned}
(C^r_{r+1})^i_j &= (C^{r-1}_r)^i_j + \Gamma^i_j \\
(C^s_{r+1})^i_j &= \Gamma(C^s_r)^i_j + \Gamma^i_k(C^s_r)^k_j + (C^{s-1}_r)^i_j \quad 1 \le s \le r-1 \\
(C^0_{r+1})^i_j &= \Gamma(C^0_r)^i_j + \Gamma^i_k(C^0_r)^k_j.
\end{aligned}$$

In particular,

$$(C^r_{r+1})^i_j = (r+1)\Gamma^i_j.$$

From the formula $\Theta^i_r = dy^i_r + \sum_{s<r}(C^s_r)^i_j dy^j_s$ we see that $\{\Theta^i_r\}$, $0 \le r \le n$, is a local basis of 1-forms on $T^n(M)$. Let $\{Y^r_i\}$ be the dual basis. Then

$$Y^n_i = \frac{\partial}{\partial y^i_n}, \qquad Y^r_i = \frac{\partial}{\partial y^i_r} + \sum_{s=r+1}^n (D^r_s)^j_i \frac{\partial}{\partial y^i_s} \quad 0 \le r \le n-1,$$

for coefficients $(D^r_s)^j_i$ which may easily be expressed in terms of the $(C^s_r)^i_j$.

For $r = 0, 1, \dots, n-1$ let $\mathcal{H}_r$ be the distribution on $T^n(M)$ spanned locally by the vector fields Y^r_i. We obtain in this way n m-dimensional distributions, each of which can be identified with $\operatorname{sect}\tau_0^*(T(M))$, such that $\mathcal{H}_r \cap \mathcal{H}_s = \{0\}$ for $r \ne s$. For $r \ge 1$, $\mathcal{H}_r$ is a complement to $\mathcal{V}_{r+1}$ in $\mathcal{V}_r$, while $\mathcal{H}_0$ is a complement to $\mathcal{V}_1$ in $\mathfrak{X}(T^n(M))$. If we set $\bar{\mathcal{H}}_r = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_r$ for $r < n$ then $\bar{\mathcal{H}}_r \oplus \mathcal{V}_{r+1} = \mathfrak{X}(T^n(M))$, and

$$\bar{\mathcal{H}}_0 \subset \bar{\mathcal{H}}_1 \subset \cdots \subset \bar{\mathcal{H}}_{n-1} \subset \mathfrak{X}(T^n(M))$$

is a filtration of $\mathfrak{X}(T^n(M))$ complementary to the filtration by vertical distributions. It is called a horizontal filtration, and the whole construction a multiconnection, a term coined in [12]. A vector field on $T^n(M)$ which lies in $\mathcal{H}_0$ is said to be horizontal with respect to the multiconnection, and $\mathcal{H}_0$ itself is the horizontal distribution.

For any section Y of $\tau_0^*(T(M))$ denote by Y^r the corresponding element of $\mathcal{H}_r$, $0 \le r \le n-1$, and by Y^n the corresponding element of $\mathcal{V}_n$ (which it is often convenient to think of as $\mathcal{H}_n$, however counter-intuitive this may be). Then

$$Y^r = Y^i Y^r_i \quad \text{where} \quad Y = Y^i \frac{\partial}{\partial y^i}.$$

Any vector field Z on $T^n(M)$ may be expresssed as a sum of its components in the $\mathcal{H}_r$, and each component identified with an element of $\operatorname{sect}\tau_0^*(T(M))$ which will be denoted by Z_r. Then $Z = \sum_{r=0}^n (Z_r)^r$, and $Z_r = \Theta_r(Z)$.

A Jacobi field for Γ is a vector field Z on $T^n(M)$ such that $\mathcal{L}_\Gamma Z = 0$. Thus along any integral curve γ of Γ, Z is the infinitesimal generator of variations of γ to nearby integral curves of Γ. We have

$$\Theta_r([\Gamma, Z]) = \nabla(\Theta_r(Z)) - (\nabla\Theta_r)(Z),$$

so that Z is a Jacobi field if and only if $Z_{r+1} = \nabla Z_r$ for $r = 0, 1, \dots, n-1$ and $\nabla Z_n = (\nabla\Theta_n)(Z)$. Now the 1-forms $\nabla\Theta^i_n$ must be linearly dependent on the Θ^i_r with $0 \le r \le n$: say

$$\nabla\Theta^i_n + (\Phi_n)^i_j\Theta^j_n + (\Phi_{n-1})^i_j\Theta^j_{n-1} + \cdots + (\Phi_1)^i_j\Theta^j_1 + (\Phi_0)^i_j\Theta^j_0 = 0.$$

For each r, $(\Phi_r)^i_j$ are the components of an endomorphism Φ_r of $\operatorname{sect}\tau_0^*(T(M))$. In fact $\Phi_n = 0$: we have

$$\Theta^i_n = dy^i_n + n\Gamma^i_j dy^j_{n-1} + \sum_{s=0}^{n-2}(C^s_r)^i_j dy^j_s,$$

whence

$$\nabla\Theta^i_n \sim \mathcal{L}_\Gamma dy^i_n + (n+1)\Gamma^i_j dy^j_n \sim df^i - \frac{\partial f^i}{\partial y^j_n} dy^j_n \sim 0,$$

where $\sim$ indicates equality modulo terms in dy^i_r with $r < n$; the result follows. So we may write

$$\nabla\Theta_n + \Phi_{n-1}\circ\Theta_{n-1} + \cdots + \Phi_1\circ\Theta_1 + \Phi_0\circ\Theta_0 = 0.$$

Thus a Jacobi field Z is determined by a single element of $\operatorname{sect}\tau_0^*(T(M))$, say Y, with $Z_0 = Y$, $Z_1 = \nabla Y$, ..., $Z_n = \nabla^n Y$; $Z = \sum_0^n (\nabla^r Y)^r$; and Y must satisfy

$$\nabla^{n+1}Y + \Phi_{n-1}(\nabla^{n-1}Y) + \cdots + \Phi_1(\nabla Y) + \Phi_0(Y) = 0.$$

This equation, or more precisely its restriction to any integral curve of Γ, is called Jacobi's equation for Γ, and the Φ_r are the Jacobi endomorphisms relative to the multiconnection.

Evidently Γ itself is a Jacobi field, with

$$\Gamma_0 = y^i_1 \frac{\partial}{\partial y^i},$$

which is a canonical element of $\operatorname{sect}\tau_0^*(T(M))$, and will be denoted by T. The Jacobi endomorphisms therefore satisfy

$$\nabla^{n+1}T + \Phi_{n-1}(\nabla^{n-1}T) + \cdots + \Phi_1(\nabla T) + \Phi_0(T) = 0.$$

Using the formula for $\Theta_r([\Gamma, Z])$ and the defining formula for the Jacobi endomorphisms one finds that for $Y \in \operatorname{sect}\tau_0^*(T(M))$

$$\begin{aligned}
[\Gamma, Y^0] &= (\nabla Y)^0 + \Phi_0(Y)^n \\
[\Gamma, Y^r] &= -Y^{r-1} + (\nabla Y)^r + \Phi_r(Y)^n, \qquad 1 \le r \le n-1 \\
[\Gamma, Y^n] &= -Y^{n-1} + (\nabla Y)^n.
\end{aligned}$$

4 Homogeneous differential equation fields

A differential equation field Γ will be said to be *homogeneous* if $[\Delta, \Gamma] = \Gamma$ ($\Delta = \Delta^1$). I show that, as for second-order fields, if Γ is homogeneous then its horizontal distribution is invariant under Δ, and in an appropriate sense the multiconnection defined above, and the Jacobi endomorphisms, are homogeneous.

For this purpose it is helpful to have a notion of Lie derivative with respect to a vector field on $T^n(M)$ acting on $\operatorname{sect}\tau_0^*(T(M))$. A general theory of Lie derivatives of vector fields along fibre bundle projections $\pi : E \to M$ by vector fields on E

projectable to M was given in [7]. Here we need only the case of the Lie derivative with respect to a π-vertical vector field, which is somewhat simpler. A section Y of $\pi^*(T(M)) \to E$ can be regarded as a linear operator $C^\infty(M) \to C^\infty(E)$ which obeys a Leibniz rule: for $\varphi_1, \varphi_2 \in C^\infty(M)$, $Y(\varphi_1\varphi_2) = Y(\varphi_1)\varphi_2 + \varphi_1 Y(\varphi_2)$. Then for any π-vertical vector field Z on E, $Z \circ Y$ is a linear operator $C^\infty(M) \to C^\infty(E)$, and it obeys the Leibniz rule because $Z(\varphi_1) = Z(\varphi_2) = 0$. This operator is defined to be $\mathcal{L}_Z Y$. Clearly $\mathcal{L}_Z Y$ depends $\mathbb{R}$-linearly on Y, and for $f \in C^\infty(E)$, $\mathcal{L}_Z(fY) = f\mathcal{L}_Z Y + Z(f)Y$. Moreover, if $Y \in \mathfrak{X}(M)$, $\mathcal{L}_Z Y = 0$. In fact the coordinate representation of $\mathcal{L}_Z$ is very simple:

$$\mathcal{L}_Z\left(Y^i \frac{\partial}{\partial y^i}\right) = Z(Y^i)\frac{\partial}{\partial y^i}:$$

in a sense, the foregoing discussion merely serves to justify the claim that when Z is vertical, differentiating the coefficients of Y along Z is a tensorial operation.

This Lie derivative operation can be extended to related geometric objects in the usual way. In particular, if Φ is an endomorphism of $\pi^*(T(M))$ then $(\mathcal{L}_Z\Phi)(Y) = \mathcal{L}_Z(\Phi(Y)) - \Phi(\mathcal{L}_Z Y)$; and again, one calculates $\mathcal{L}_Z\Phi$ by merely differentiating the components of Φ along Z.

In the case of interest $E = T^n(M)$ and $Z = \Delta$. A simple calculation shows that $[\Delta, Y^n] = (\mathcal{L}_\Delta Y - nY)^n$, and in particular if $Y \in \mathfrak{X}(M)$ then $[\Delta, Y^n] = -nY^n$. Here, as before, Y^n is the element of $\mathcal{V}_n$ corresponding to $Y \in \operatorname{sect}\tau_0^*(T(M))$; and we conclude (as indeed is otherwise obvious) that $[\Delta, \mathcal{V}_n] \subset \mathcal{V}_n$.

Theorem 1. *If Γ is homogeneous then $[\Delta, \mathcal{H}_r] \subset \mathcal{H}_r$ for $r = 0, 1, \ldots, n-1$, and for any $Y \in \mathfrak{X}(M)$, $[\Delta, Y^r] = -rY^r$, where Y^r is the element of $\mathcal{H}_r$ corresponding to $Y \in \operatorname{sect}\tau_0^*(T(M))$. Furthermore, the Jacobi endomorphisms satisfy $\mathcal{L}_\Delta\Phi_r = (n+1-r)\Phi_r$.*

Proof. First of all, note that when it is expressed in terms of coordinates the homogeneity condition amounts to $\Delta(f^i) = (n+1)f^i$. It is an easy consequence that $\Delta(\Gamma^i_j) = \Gamma^i_j$, and hence that for any $Y \in \operatorname{sect}\tau_0^*(T(M))$, $[\mathcal{L}_\Delta, \nabla]Y = \nabla Y$. It further follows that for any $T(M)$-valued 1-form Θ, $[\mathcal{L}_\Delta, \nabla]\Theta = \nabla\Theta$. The $T(M)$-valued 1-forms Θ_r that define the multiconnection satisfy $\Theta_{r+1} = \nabla\Theta_r$ for $r = 0, 1, \ldots, n-1$, and therefore

$$\mathcal{L}_\Delta\Theta_{r+1} = \nabla(\mathcal{L}_\Delta\Theta_r) + \nabla\Theta_r = \nabla(\mathcal{L}_\Delta\Theta_r) + \Theta_{r+1}.$$

Now $\mathcal{L}_\Delta\Theta_0 = 0$, whence $\mathcal{L}_\Delta\Theta_r = r\Theta_r$. Thus for any vector field Z,

$$\Theta_r([\Delta, Z]) = \mathcal{L}_\Delta(\Theta_r(Z)) - r\Theta_r(Z).$$

In particular, if $Y \in \operatorname{sect}\tau_0^*(T(M))$, for any $s = 0, 1, \ldots, n$

$$\Theta_s([\Delta, Y^r]) = \mathcal{L}_\Delta(\Theta_s(Y^r)) - s\Theta_s(Y^r),$$

from which it follows that $[\Delta, Y^r] = (\mathcal{L}_\Delta Y)^r - rY^r \in \mathcal{H}_r$, and in particular that if $Y \in \mathfrak{X}(M)$ then $[\Delta, Y^r] = -rY^r$.

The Jacobi endomorphisms are determined by the formula

$$\nabla\Theta_n + \Phi_{n-1} \circ \Theta_{n-1} + \cdots + \Phi_1 \circ \Theta_1 + \Phi_0 \circ \Theta_0 = 0;$$

that is to say, for any $Y \in \operatorname{sect} \tau_0^*(T(M))$ and for $0 \leq r \leq n-1$

$$\Phi_r(Y) = -(\nabla\Theta_n)(Y^r).$$

Thus

$$\begin{aligned}
(\mathcal{L}_\Delta \Phi_r)(Y) &= -\mathcal{L}_\Delta((\nabla\Theta_n)(Y^r)) - \Phi_r(\mathcal{L}_\Delta Y) \\
&= -\mathcal{L}_\Delta(\nabla\Theta_n)(Y^r) - (\nabla\Theta_n)([\Delta, Y^r]) - \Phi_r(\mathcal{L}_\Delta Y) \\
&= -\nabla(\mathcal{L}_\Delta\Theta_n)(Y^r) - (\nabla\Theta_n)(Y^r) + r(\nabla\Theta_n)(Y^r) \\
&\qquad - (\nabla\Theta_n)((\mathcal{L}_\Delta Y)^r) - \Phi_r(\mathcal{L}_\Delta Y) \\
&= -n(\nabla\Theta_n)(Y^r) - (\nabla\Theta_n)(Y^r) + r(\nabla\Theta_n)(Y^r) \\
&\qquad - (\nabla\Theta_n)((\mathcal{L}_\Delta Y)^r) - \Phi_r(\mathcal{L}_\Delta Y) \\
&= (n+1-r)\Phi_r(Y)
\end{aligned}$$

as claimed. □

5 Strongly homogeneous third-order differential equation fields

In order for a differential equation field Γ to be worthy of the description homogeneous it must certainly satisfy $[\Delta, \Gamma] = \Gamma = [\Delta^1, \Gamma]$. But for $n > 1$ we have the whole algebra $\mathfrak{D}$ at our disposal, and one might imagine that one could impose some conditions on the brackets $[\Delta^r, \Gamma]$ for all $r = 1, 2, \ldots, n$. A little experimentation using coordinates suggests that such conditions would have to take the form

$$[\Delta^1, \Gamma] = \Gamma, \qquad [\Delta^r, \Gamma] = r\Delta^{r-1}, \quad r = 2, 3, \ldots, n.$$

Unfortunately, in general these conditions are inconsistent, for if $n + 2 \geq r + s$ and $r \neq s$ we would have

$$[\Delta^r, [\Delta^s, \Gamma]] - [\Delta^s, [\Delta^r, \Gamma]] = (r - s)(r + s - 1)\Delta^{r+s-2} \neq 0,$$

while if also $r + s > n + 1$, $[\Delta^r, \Delta^s] = 0$, and Jacobi's identity would be violated. For $n > 2$ the values $r = n$, $s = 2$ satisfy both of the given inequalities (or indeed any r, s with $1 < r, s \leq n$, $r \neq s$ and $r + s = n + 2$). However, the conditions

$$[\Delta^1, \Gamma] = \Gamma, \qquad [\Delta^2, \Gamma] = 2\Delta^1$$

are consistent, and can be imposed whenever $n \geq 2$. The case of greatest interest is that with $n = 2$, in other words, the case of third-order differential equations.

Accordingly, a third-order differential equation field

$$\Gamma = y_1^i \frac{\partial}{\partial y_0^i} + y_2^i \frac{\partial}{\partial y_1^i} + f^i \frac{\partial}{\partial y_2^i}$$

is defined to be *strongly homogeneous* if it satisfies the conditions

$$[\Delta^1, \Gamma] = \Gamma, \qquad [\Delta^2, \Gamma] = 2\Delta^1,$$

where

$$\Delta^1 = y_1^i \frac{\partial}{\partial y_1^i} + 2y_2^i \frac{\partial}{\partial y_2^i}, \qquad \Delta^2 = 2y_1^i \frac{\partial}{\partial y_2^i}.$$

That is, Γ is strongly homogeneous if the assignment $\partial/\partial t \mapsto \Gamma$ extends the anti-isomorphism of Lie algebras $\mathfrak{q}^0 \to \mathfrak{D}$ to an anti-isomorphism of Lie algebras $\mathfrak{q} \to \mathfrak{D}^+$, where $\mathfrak{D}^+ = \langle\Gamma\rangle \oplus \mathfrak{D}$ with the brackets above. (The remarks in the opening paragraph of this section show that in general one cannot extend the anti-homomorphism $\mathfrak{p}^0 \to \mathfrak{X}(T^n(M)) : t^r\partial/\partial t \mapsto \Delta^r$ to an anti-homomorphism $\mathfrak{p} \to \mathfrak{X}(T^n(M))$ by a similar move.)

When expressed in terms of the coefficients f^i the conditions become

$$\Delta^1(f^i) = 3f^i, \qquad \Delta^2(f^i) = 6y_2^i.$$

Theorem 2. *Let Γ be a strongly homogeneous third-order differential equation field. Then Γ is horizontal with respect to the multiconnection defined by its dynamical covariant derivative; and the corresponding Jacobi endomorphisms satisfy $\Phi_r(T) = 0$, $r = 0, 1$.*

Proof. We have to show that $\Gamma \in \mathcal{H}_0$. Using the notation from the previous section, this is equivalent to $\Gamma_1 = \Gamma_2 = 0$. Now $\Gamma_0 = T$, $\Gamma_1 = \nabla T$, $\Gamma_2 = \nabla^2 T$. We have

$$\nabla T = \left(\Gamma(y_1^i) + y_1^j \Gamma_j^i\right) \frac{\partial}{\partial y^i},$$

and

$$\Gamma(y_1^i) + y_1^j \Gamma_j^i = y_2^i - \tfrac{1}{3} y_1^j \frac{\partial f^i}{\partial y_2^j}.$$

But $\Delta^2(f^i) = 6y_2^i$, which is to say that

$$y_1^j \frac{\partial f^i}{\partial y_2^j} = 3y_2^i,$$

and so $\nabla T = 0$, $\Gamma_1 = \Gamma_2 = 0$, and $\Gamma = T^0$.

From the formula $\Theta_r([\Gamma, Z]) = \nabla(\Theta_r(Z)) - (\nabla\Theta_r)(Z)$ with $Z = \Delta^1$, together with the fact that $\Delta_0^1 = 0$, using the homogeneity conditions in bracket form one finds that

$$\Delta_1^1 = \Gamma_0 = T, \quad \Delta_2^1 = \nabla\Gamma_0 + \Gamma_1 = 2\nabla T = 0,$$

so $\Delta^1 \in \mathcal{H}_1$, in fact $\Delta^1 = T^1$. The Jacobi endomorphisms are defined by the formula

$$\nabla\Theta_2 + \Phi_1 \circ \Theta_1 + \Phi_0 \circ \Theta_0 = 0.$$

It is easy to see that $\nabla\Theta_2$ vanishes on Γ and Δ^1, from which it follows that $\Phi_0(T) = \Phi_1(T) = 0$. $\square$

For completeness' sake it's worth pointing out that $\Delta^2_0 = \Delta^2_1 = 0$. Moreover, $\Delta^2_2 = T$, so $\Delta^2 = T^2 \in \mathcal{H}_2$, from which one can check independently that $\nabla\Theta_2(\Delta^2) = 0$.

As an instructive example of a strongly homogeneous third-order differential equation field I discuss the case of a single dependent variable, that is (in more natural notation) the equation

$$y''' = f(y, y', y'')$$

considered as defining a differential equation field on $T^2(\mathbb{R})$.

Theorem 3. *There is no strongly homogeneous third-order differential equation field defined on the whole of $T^2(\mathbb{R})$. If we restrict attention to the submanifold of $T^2(\mathbb{R})$ where $y' > 0$ then the most general strongly homogeneous third-order differential equation is*

$$y''' = \tfrac{3}{2}\frac{(y'')^2}{y'} + \kappa(y)(y')^3$$

where κ is an arbitrary smooth function of a single variable.

Proof. Denote by ϕ^1 and ϕ^2 the flows generated by Δ^1 and Δ^2 on $T^2(\mathbb{R})$. Then

$$\phi^1_t(y, y', y'') = (y, e^t y', e^{2t} y''), \qquad \phi^2_t(y, y', y'') = (y, y', y'' + 2ty').$$

Any point with $y' = 0$ is of course invariant under ϕ^2_t. From the condition $\Delta^1(f) = 3f$ it follows that $f(\phi^1_t(y, y', y'')) = e^{3t} f(y, y', y'')$. Along any integral curve of Δ^2, on the other hand, we have

$$\frac{df}{dt} = 6(y'' + 2ty'),$$

whence

$$f(\phi^2_t(y, y', y'')) = f(y, y', y'') + 6(ty'' + t^2 y').$$

This condition cannot be satisfied with $y' = 0$. For $y' > 0$, however, one can find values of s and t such that $(y, y', y'') = \phi^2_s(\phi^1_t(y, 1, 0))$, namely $t = \log y'$, $s = \frac{1}{2}y''/y'$. Then

$$\begin{aligned} f(y, y', y'') &= f(\phi^2_s(\phi^1_t(y, 1, 0))) = f(\phi^2_s(y, y', 0)) \\ &= (y')^3 f(y, 1, 0) + 6\left(\tfrac{1}{2}\frac{y''}{y'}\right)^2 y' \\ &= \tfrac{3}{2}\frac{(y'')^2}{y'} + \kappa(y)(y')^3 \end{aligned}$$

with $\kappa(y) = f(y, 1, 0)$. □

Since the equation is satisfied by $y(-x)$ if it is satisfied by $y(x)$, a similar result holds for $y' < 0$.

This result should not really come as a surprise, at least so far as the equation with $\kappa = 0$ is concerned. This equation says that the Schwarzian derivative of y is zero. Thus, or directly, the general solution is

$$y = \frac{ax+b}{cx+d}$$

(a factor may be taken to get the right number of arbitrary constants). Note that the solutions are invariant under reparametrizations

$$x \mapsto \frac{Ax+B}{Cx+D}, \quad AD - BC \neq 0,$$

and this is just the pseudo-group of local diffeomorphisms of $\mathbb{R}$ generated by the quadratic vector fields — the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}$ by fractional-linear or Möbius transformations.

The general analysis shows that since the differential equation field, even with κ nonzero, is strongly homogeneous it is horizontal. Moreover, the corresponding Jacobi endomorphisms Φ_0 and Φ_1 satisfy $\Phi_r(T) = 0$, which means in this case that they are both zero. It follows immediately that the Wuenschmann invariant, which in the present notation is $\nabla\Phi_1 - 2\Phi_0$ (see [4], [8]), vanishes.

Finally, it should be pointed out that the property of having solutions invariant under reparametrizations by fractional-linear transformations holds for all strongly homogeneous third-order systems. That is to say, if $\xi \mapsto (y^i(\xi))$ is a solution of such a system, and $x \mapsto \xi(x)$ is a fractional-linear transformation, then $x \mapsto (y^i(\xi(x)))$ is also a solution (where it is defined). For

$$\begin{aligned}
\frac{dy^i}{dx} &= \xi' \frac{dy^i}{d\xi} \\
\frac{d^2y^i}{dx^2} &= (\xi')^2 \frac{d^2y^i}{d\xi^2} + \xi'' \frac{dy^i}{d\xi} \\
\frac{d^3y^i}{dx^3} &= (\xi')^3 \frac{d^3y^i}{d\xi^3} + 3\xi'\xi'' \frac{d^2y^i}{d\xi^2} + \xi''' \frac{dy^i}{d\xi}.
\end{aligned}$$

On the other hand, the use of ϕ_t^1 and ϕ_t^2 as in the theorem leads to the result that for a strongly homogeneous system the functions f^i satisfy

$$f^i(y_0^j, ky_1^j, k^2y_2^j + ly_1^j) = k^3 f^i(y_0^j, y_1^j, y_2^j) + 3kly_2^i + \tfrac{3}{2}\frac{l^2}{k} y_1^i$$

for any $k, l \in \mathbb{R}$ with $k > 0$. It follows (taking $k = \xi'$ and $l = \xi''$) that

$$\begin{aligned}
\frac{d^3y^i}{dx^3} - f^i\left(y^i, \frac{dy^i}{dx}, \frac{d^2y^i}{dx^2}\right) &= (\xi')^3 \left(\frac{d^3y^i}{d\xi^3} - f^i\left(y^i, \frac{dy^i}{d\xi}, \frac{d^2y^i}{d\xi^2}\right)\right) \\
&\quad + \left(\xi''' - \tfrac{3}{2}\frac{(\xi'')^2}{\xi'}\right)\frac{dy^i}{d\xi},
\end{aligned}$$

and the final term vanishes if $\xi(x)$ is a fractional-linear function of x.

This is the analogue for strongly homogeneous third-order systems of the fact that the solutions of the second-order equations defined by a spray are invariant under affine reparametrizations.

Acknowledgement

I am a Guest Professor at Ghent University: I am grateful to the Department of Mathematics for its hospitality. I thank David Saunders for helpful comments.

References

[1] I. Anderson, G. Thompson: The inverse problem of the calculus of variations for ordinary differential equations. *Mem. Amer. Math. Soc.* 98 (1992) No. 473.

[2] L.C. de Andrés, M. de León, P.R. Rodrigues: Canonical connections associated with regular Lagrangians of higher order. *Geom. Dedicata* 39 (1991) 17–28.

[3] P. Antonelli, I. Bucataru: KCC theory of a system of second order differential equations. In: Antonelli (ed.): *Handbook of Finsler Geometry Vol. 1.* Kluwer (2003) 83–174.

[4] I. Bucataru, O. Constantinescu, M.F. Dahl: *A geometric setting for systems of ordinary differential equations.* preprint: arXiv:1011.5799 [math.DG].

[5] M. Crampin: Connections of Berwald type. *Publ. Math. Debrecen* 57 (2000) 455–473.

[6] M. Crampin, W. Sarlet, F. Cantrijn: Higher-order differential equations and higher-order Lagrangian mechanics. *Math. Proc. Cam. Phil. Soc.* 99 (1986) 565–587.

[7] M. Crampin, D.J. Saunders: Affine and projective transformations of Berwald connections. *Diff. Geom. Appl.* 25 (2007) 235–250.

[8] M. Crampin, D.J. Saunders: On the geometry of higher-order ordinary differential equations and the Wuenschmann invariant. In: Clemente-Gallardo and Martínez (eds.): *Groups, Geometry and Physics.* Monografía 29, Real Academia de Ciencias de Zaragoza (2007) 79–92.

[9] S. Fritelli, C. Kozameh, E.T. Newman: Differential geometry from differential equations. *Comm. Math. Phys.* 223 (2001) 383–408.

[10] M. Godliński, P. Nurowski: Third order ODEs and four-dimensional split signature Einstein metrics. *J. Geom. Phys.* 56 (2006) 344–357.

[11] M. Godliński, P. Nurowski: *Geometry of third-order ODEs.* preprint: arXiv:0902.4129v1 [math.DG].

[12] D.J. Saunders: On the inverse problem for even-order ordinary differential equations in the higher-order calculus of variations. *Diff. Geom. Appl.* 16 (2002) 149–166.

[13] Z. Shen: *Differential Geometry of Spray and Finsler Spaces.* Kluwer (2001).

Author's address:
Department of Mathematics, Ghent University, Krijgslaan 281, B–9000 Gent, Belgium

Address for correspondence: 65 Mount Pleasant, Aspley Guise, Beds MK17 8JX, UK

E-mail address: Crampin@btinternet.com

Received: 12 December, 2010

Accepted for publication: 23 December, 2010

Communications in Mathematics 18 (2010) 51–77

Geometric mechanics on nonholonomic submanifolds

Olga Krupková

Abstract. In this survey article, nonholonomic mechanics is presented as a part of geometric mechanics. We follow a geometric setting where the constraint manifold is a submanifold in a jet bundle, and a nonholonomic system is modelled as an exterior differential system on the constraint manifold. The approach admits to apply coordinate independent methods, and is not limited to Lagrangian systems under linear constraints. The new methods apply to general (possibly nonconservative) mechanical systems subject to general (possibly nonlinear) nonholonomic constraints, and admit a straightforward generalization to higher order mechanics and field theory. In particular, we are concerned with the following topics: the geometry of nonholonomic constraints, equations of motion of nonholonomic systems on constraint manifolds and their geometric meaning, a nonholonomic variational principle, symmetries, a nonholonomic Noether theorem, regularity, and nonholonomic Hamilton equations.

1 Introduction

Nonholonomic mechanics is concerned with study of systems the motion of which is subject to constraints on time, positions and velocities. The interest to investigate mechanical systems with holonomic and nonholonomic constraints goes back to the 19th century, when D'Alembert's principle of virtual work and Gauss' principle of least action in presence of constraints were considered. It was discovered that holonomically constrained dynamics can be understood as motions subject to reactive forces of a gradient form, given by the constraints. As conjectured by Chetaev in early 30's of the last century, nonholonomic equations of motion could have a similar form, but now the reactive forces should take the form of derivatives with respect to the velocities [9]. Since that time, Chetaev's equations have been tested in many situations and on many examples in mechanics and engineering,

2010 MSC: 70G45, 70G75, 37J60, 70F25, 70H30
Key words: Jet bundle, holonomic constraint, nonholonomic constraint, the constraint distribution, generalized D'Alembert principle, Chetaev equations, reduced equations, the nonholonomic first variation formula, symmetry, nonholonomic Noether theorem, regularity, nonholonomic Hamilton equations.

and it turned out that (contrary to the so-called vakonomic equations proposed as alternative equations of motion), they really do describe motions of nonholonomic mechanical systems (see e.g. [6], [12]).

Within the classical analysis approach, only Lagrangian systems subject to linear integrable (semi-holonomic) constraints have been well-understood. In case of non-integrable, or even non-liner constraints, a satisfactory, complete theory, similar to the analytical dynamics of unconstrained systems, has been missing. On the other hand, during the last 20 years, in connection with the developments of geometric mechanics and global calculus of variations, methods of differential geometry and global analysis have turned out be well suited and helpful for understanding nonholonomic systems. There have been proposed several geometric models, appropriate in different situations, applicable to Lagrangian systems in tangent bundles or in jet bundles. It should be stressed, however, that almost all the work on noholonomic systems is concerned with the case of constraints linear (affine) in the velocities. The bibliography is very extensive and it is not possible to list here all important contributions; we refer at least to [2], [7], [10], [11], [13], [14], [15], [20], [29], [30], [33], [43], [44], [46], [47], [49] and references therein.

In this article we present the nonholonomic mechanics as a part of geometric mechanics. However, we should emphasize that we follow the setting where a nonholonomic system is modeled as an exterior differential system *on a constraint manifold* (subbundle of a jet bundle) [29], [30], [33], [35], [39], [51] (i.e., motion equations appear in the "reduced form", without Lagrange multipliers). This approach consistently reflects the geometric character of nonholonomic constraints. It naturally admits to apply coordinate independent methods and transfer standard concepts and techniques of differential geometry and the calculus of variations on manifolds to the situation when differential constraints are present. Moreover, this approach is *not limited* to *Lagrangian* systems under *linear constraints*. In fact, both Lagrangian and non-conservative systems are treated in a unique way, and similarly, a unique geometric model of differential constraints (whatever they are: linear integrable or non-integrable, or nonlinear) is presented. Within this setting, a generalization to higher-order systems and constraints, and extension of nonholonomic mechanics to field theory is straightforward [31], [32], [34], [35], [36], [41], [42]. Remarkably, the new way of treating and understanding nonholonomic systems brings new methods for investigating concrete examples of nonholonomic systems, either with linear constraints (see [19]), or with nonlinear constraints (see [50] for problems of mechanics and engineering and [38] for applications in the special relativity theory).

The aim of the present article is to survey, in a consistent way, some of the recent results on first order mechanical systems. After a brief introduction to the standard geometric theory of first order mechanical systems in jet bundles (to be found e.g. in [29] or [40]) we turn to include nonholonomic constraints into the picture. We are concerned with the geometry of nonholonomic constraints, equations of motion of nonholonomic systems on constraint manifolds and their geometric meaning, including also the case of "implicit equations", a nonholonomic variational principle, symmetries of nonholonomic systems and a nonholonomic Noether theorem, and finally we discuss regularity of nonholonomic equations, and nonholonomic

Hamilton equations. We note that there are also other interesting topics studied within nonholonomic mechanics, not included in this article, as e.g. the inverse variational problem in the nonholonomic setting [3], [39], nonholonomic reduction in presence of symmetries [4], [5], [8], integrability of nonholonomic systems and Hamilton-Jacobi theory [1], nonholonomic mechanics on Lie algebroids [11], [17], [44], etc.

2 Mechanical systems in jet bundles

Compared to the classical approach, geometric methods bring a new quality into the study of mechanical systems. The geometric language leads to an elegant and transparent formulation of results. It is important that concepts and formulas can be introduced in an *intrinsic* (coordinate independent) form: this is not only convenient for computations, but clarifies the geometric content and enables to distinguish between local and global results.

In this section we introduce structures for mechanics on fibred manifolds. We shall deal with both Lagrangian and nonconservative, generally time-dependent systems, the dynamics of which is described by systems of second order ordinary differential equations. In our approach, geometric concepts related with differential equations on manifolds play a central role. For more detailed exposition we refer to [21], [23], [48], and especially to the book [28] devoted to higher-order mechanics.

2.1 Basic structures

Throughout the paper we consider smooth manifolds and mappings. In coordinate formulas summation over repeated indices applies.

A smooth mapping $Y \to X$ between differentiable manifolds is called *submersion* if its rank is equal to $\dim X$ at each point $y \in Y$. A *surjective* submersion $\pi : Y \to X$ is called a *fibred manifold*. The manifold X is called *base*, Y *total space*, and the map π itself *projection*. The submanifold $\pi^{-1}(x)$ of Y, where $x \in X$, is called *fibre* over x. In case that all the fibres are diffeomorphic to each other, we speak about a *bundle* over X.

We shall consider fibred manifolds where $\dim X = 1$. This means that if X is connected, it is diffeomorphic either to $\mathbb{R}$ or S^1. We denote $\dim Y = m + 1$, hence m denotes the dimension of the fibres. From the definition of submersion it follows that to every point $y \in Y$ there exists a chart (V, ψ) on Y and (U, φ) on X such that V is a neighbourhood of y, $U = \pi(V)$, and the coordinate functions are of the form $\varphi = (t)$, $\psi = (t, q^\sigma)$, $1 \leq \sigma \leq m$. Charts of this kind are called *fibred charts*. Mostly we shall assume that $X = \mathbb{R}$: in this case we choose t on $\mathbb{R}$ to be a global coordinate.

When dealing with dynamics of mechanical systems, we are concerned with a special kind of mappings between the base and the total space, called *sections*. By a section of the fibred manifold $\pi : Y \to X$ one means a (smooth) mapping $\gamma : X \to Y$, defined possibly on an open subset W of X, such that $\pi \circ \gamma = \mathrm{id}_W$. Also, it is necessary to work with quantities dependent on first or higher derivatives of the corresponding sections. A precise mathematical setting is based on the concept of a *jet manifold*. We say that sections γ_1 and γ_2 defined on an open set $W \subset X$ have *contact of order one* at a point $x \in W$ if $\gamma_1(x) = \gamma_2(x)$, and if there is a fibred chart around $\gamma_1(x) = \gamma_2(x)$ such that the derivatives of the components $\gamma_1^\sigma = q^\sigma \gamma_1 \varphi^{-1}$ and $\gamma_2^\sigma = q^\sigma \gamma_2 \varphi^{-1}$ of the sections γ_1 and γ_2 at the point $\varphi(x)$ coincide. The latter condition does not depend on the choice of fibred coordinates. In this way there arises an equivalence relation: the equivalence class can be easily visualized as a family of sections passing through the same point $y \in Y$ and possessing the same tangent vector. The equivalence class containing a section γ is called the 1-*jet* of γ at x and is denoted by $J^1_x\gamma$. Collecting all the equivalence classes for all the points $x \in X$ one obtains a set naturally endowed with a structure of a smooth manifold of dimension $2m+1$, denoted by J^1Y, and called *the first jet prolongation* of the fibred manifold $\pi : Y \to X$. Moreover, the manifold J^1Y is fibred over X (the fibred projection is denoted by π_1) as well as over Y (with the projection denoted by $\pi_{1,0}$). Consequently, one has on J^1Y coordinates, associated with fibred coordinates on Y. They are denoted by $(t, q^\sigma, \dot{q}^\sigma)$. The construction can be easily generalized to obtain *higher-order* jets: For every $x \in X$ one considers equivalence classes of sections passing through the same point over x and having at x the same derivatives up to the order r. In this way one gets a manifold J^rY, called the *manifold of* r*-jets of local sections of* π, or briefly the r*-jet prolongation of* π. Similarly as in the first-order case, one has on J^rY coordinates naturally associated with fibred coordinates on Y denoted by $(t, q^\sigma, q_1^\sigma, q_2^\sigma, \dots, q_r^\sigma)$. Instead of q_1^σ and q_2^σ one often writes $\dot{q}^\sigma$ and $\ddot{q}^\sigma$. From the definition of $J^r_x\gamma$ (which is a point in J^rY) one can see that the values of the coordinate functions at $J^r_x\gamma$ can be regarded as the coefficients of the r-th order Taylor polynomial of the mapping γ around x. The manifold J^rY is fibred over X, Y, and all J^sY, $s = 1, \dots, r-1$. The corresponding projections are denoted by $\pi_r : J^rY \to X$, $\pi_{r,0} : J^rY \to Y$, $\pi_{r,s} : J^rY \to J^sY$, where $s < r$. For simplicity of notations, we also write $J^0Y = Y$. In this paper we mostly use the first and second jet prolongations, J^1Y and J^2Y.

If γ is a section of $\pi : Y \to X$ then the mapping $x \to J^r_x\gamma$ is a section of the fibred manifold $\pi_r : J^rY \to X$; it is called the r*-jet prolongation* of γ and denoted by $J^r\gamma$. It is important to note that a section of π_r *need not* be of the form of an r-jet prolongation of a section of π. A section δ of π_r such that $\delta = J^r\gamma$ is called *holonomic*. For example, in fibred coordinates, a section of J^1Y is a mapping $\delta(t) = (t, f^\sigma(t), g^\sigma(t))$ while a holonomic section takes the form $J^1\gamma(t) = (t, f^\sigma(t), df^\sigma/dt)$.

Remark 1. Classical mechanics is often modeled on fibred manifolds of the form $\pi : \mathbb{R} \times M \to \mathbb{R}$, where M is a manifold of dimension m (called the *configuration space*). In this case $J^1Y = \mathbb{R} \times TM$, $J^2Y = \mathbb{R} \times T^2M$ and sections of π are *graphs of curves* $c : \mathbb{R} \to M$.

In fibred manifolds, there are distinguished vector fields and differential forms, adapted to the fibred and prolongation structure.

A vector field ξ on Y is called π*-projectable* if there exists a vector field ξ_0 on X such that $T\pi.\xi = \xi_0 \circ \pi$, and π*-vertical* if it projects onto a zero vector field on X, i.e., $T\pi.\xi = 0$. In fibred coordinates, projectable vector fields have their $\partial/\partial t$ component dependent on t only, and vertical vector fields have this component equal to zero.

Similarly one defines a $\pi_{r,s}$*-projectable* or a $\pi_{r,s}$*-vertical* vector field on J^rY, where $r > s$.

Local flows of projectable vector fields transfer sections into sections. Consequently, π-projectable vector fields on Y can be naturally prolonged to vector fields on J^rY. The procedure is as follows: Let ξ be a π-projectable vector field, ξ_0 its projection, and denote $\{\phi_u\}$ and $\{\phi_{0u}\}$ the corresponding local one-parameter groups. For every u, the mapping ϕ_u is an isomorphism of the fibred manifold π meaning that $\pi \circ \phi_u = \phi_{0u} \circ \pi$. Then for every section γ, the composition $\gamma_u = \phi_u \circ \gamma \circ \phi_{0u}^{-1}$ is again a section and we can define the *r-jet prolongation* of ϕ_u by $J^r\phi_u(J^r_x\gamma) = J^r_{\phi_{0u}(x)}(\phi_u\gamma\phi_{0u}^{-1})$. Then $J^r\phi$ is a local flow corresponding to a vector field on J^rY, denoted by $J^r\xi$ and called the *r-jet-prolongation* of ξ. The vector field $J^r\xi$ is both π_r-projectable and $\pi_{r,s}$-projectable for $0 \le s < r$, and its π_r-projection, (resp. $\pi_{r,s}$-projection) is ξ_0 (resp. ξ, resp. $J^s\xi$, $1 \le s \le r-1$). In fibred coordinates, where

$$\xi = \xi^0(t)\,\frac{\partial}{\partial t} + \xi^\sigma(t, q^\nu)\,\frac{\partial}{\partial q^\sigma}\,, \tag{1}$$

one has for $1 \le k \le r$

$$J^r\xi = \xi^0\frac{\partial}{\partial t} + \xi^\sigma\frac{\partial}{\partial q^\sigma} + \sum_{k=1}^{r}\xi^\sigma_k\,\frac{\partial}{\partial q^\sigma_k}\,, \quad \text{where} \quad \xi^\sigma_k = \frac{d\xi^\sigma_{k-1}}{dt} - q^\sigma_k\frac{d\xi^0}{dt}. \tag{2}$$

Above

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \dot q^\sigma\frac{\partial}{\partial q^\sigma} + \ddot q^\sigma\frac{\partial}{\partial \dot q^\sigma} + \dddot q^\sigma\frac{\partial}{\partial \ddot q^\sigma} + \dots \tag{3}$$

denotes the total derivative operator.

A differential k-form η on J^rY is called π_r*-horizontal* (resp. $\pi_{r,s}$*-horizontal*) if it vanishes whenever at least one of its arguments is a π_r-vertical (resp. $\pi_{r,s}$-vertical) vector field. A k-form η on J^rY is called *contact* if for every section γ of π

$$J^r\gamma^*\eta = 0\,. \tag{4}$$

Putting

$$\omega^\sigma = dq^\sigma - \dot q^\sigma dt,\ \ \dot\omega^\sigma = d\dot q^\sigma - \ddot q^\sigma dt,\ \ \dots,\ \ \omega^\sigma_{r-1} = dq^\sigma_{r-1} - q^\sigma_r\,dt \tag{5}$$

$1 \le \sigma \le m$, we obtain a family of local contact 1-forms on J^rY. Remarkably, the *contact ideal* on J^rY is locally generated by these one-forms and their exterior derivatives. We also note that one-forms (5) can be completed to a *basis of linear forms*

$$(dt, \omega^\sigma, \dots, \omega^\sigma_{r-1}, dq^\sigma_r) \tag{6}$$

well adapted to the structure of J^rY. Working in coordinates, it is much more convenient to use this basis instead of the canonical basis $(dt, dq^\sigma, \dots, dq^\sigma_r)$.

We have an important property of differential forms in jet bundles: Every k-form η on J^rY, if lifted to $J^{r+1}Y$, admits a unique and invariant decomposition into two parts such that in the adapted basis the first and the second part contains wedge products of exactly $k-1$ and k basic contact forms (5), respectively. We write

$$\pi^*_{r+1,r}\eta = h\eta + p_1\eta, \quad \pi^*_{r+1,r}\eta = p_{k-1}\eta + p_k\eta \tag{7}$$

if $k = 1$ and $k \geq 2$, respectively. $h\eta$ is a horizontal form on $J^{r+1}Y$, called the *horizontal part* of η, $p_i\eta$ is then called the *i-contact part* of η (we also speak about a *i-contact form*). Note that for a function f we get $hdf = \frac{df}{dt}dt$.

2.2 Fibred mechanics

In what follows, let us consider a fibred manifold $\pi : Y \to \mathbb{R}$ with $\dim Y = m+1$, and fibred coordinates denoted by (t, q^σ), where t is a global coordinate on $\mathbb{R}$.

A *dynamical form* of order r is defined to be a 2-*form* E on J^rY which is 1-contact, and horizontal with respect to the projection onto Y. In fibred coordinates $E = E_\sigma \omega^\sigma \wedge dt$, where $E_1, \dots, E_m$ are functions on an open subset of J^rY. Dynamical forms are appropriate objects to represent systems of *ordinary differential equations* on manifolds. In this paper we shall be interested in (at most) *second-order* ODE's. Then E is defined on J^2Y and its components E_σ depend upon $t, q^\nu, \dot{q}^\nu, \ddot{q}^\nu$ $(1 \leq \sigma, \nu \leq m)$. Equation $E = 0$ determines a submanifold of J^2Y of codimension m. A section γ of π is called a *path* of E if it satisfies $E \circ J^2\gamma = 0$. In fibred coordinates this is a system of m (possibly *implicit*) second order ODE's

$$E_\sigma\Big(t, \gamma^\nu(t), \frac{d\gamma^\nu}{dt}, \frac{d^2\gamma^\nu}{dt^2}\Big) = 0, \quad 1 \leq \sigma \leq m \tag{8}$$

for components of γ.

In what follows it will be sufficient to restrict to the case of so-called J^1Y-pertinent dynamical forms that are distinguished by a significant property: the corresponding *dynamics proceeds in the manifold J^1Y* (sometimes called the *evolution space*).

Given a dynamical form E, we say that a 2-form α defined on an open subset $U \subset J^2Y$ is an *extension* of E on U if $E|_U = p_1\alpha$. E is called *pertinent with respect to* J^1Y if around every point in J^2Y it has a local extension α that is projectable onto an open subset of J^1Y. A second order dynamical form E is pertinent with respect to J^1Y if an only if

$$E_\sigma = A_\sigma(t, q^\rho, \dot{q}^\rho) + B_{\sigma\nu}(t, q^\rho, \dot{q}^\rho)\ddot{q}^\nu. \tag{9}$$

Then every local projectable extension of E takes the form

$$\alpha = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot{q}^\nu + F, \tag{10}$$

where F is a 2-contact 2-form on an open subset of J^1Y. The class $[\alpha]$ of 2-forms (10) is then called the (first-order) *Lepage class* of the dynamical form E. The corresponding ODE's are *affine in the second derivatives (accelerations)*,

$$A_\sigma\Big(t, \gamma^\rho(t), \frac{d\gamma^\rho}{dt}\Big) + B_{\sigma\nu}\Big(t, \gamma^\rho(t), \frac{d\gamma^\rho}{dt}\Big)\frac{d^2\gamma^\nu}{dt^2} = 0. \tag{11}$$

Equations (11) can be represented in a form of a *Pfaffian system*, or vector distribution Δ on J^1Y, called *dynamical distribution* of E [27], [32], as follows:

$$\begin{aligned}\Delta &= \mathrm{span}\{i_\xi\alpha \,|\, \text{where } \xi \text{ runs over all vertical vector fields on } J^1Y\} \\ &= \mathrm{span}\{A_\sigma dt + F_{\sigma\nu}\omega^\nu + B_{\sigma\nu}d\dot q^\nu,\ B_{\sigma\nu}\omega^\sigma\},\end{aligned} \tag{12}$$

where $F_{\sigma\nu}$ are the components of F.

Remarkably, Δ need not have a constant rank, and need not be completely integrable. We say that a dynamical form E is *regular* if around every point in J^1Y it has a dynamical distribution which is everywhere of *rank one* [26], [27]. It can be shown that *every regular dynamical form E has a unique global rank one dynamical distribution.* It is locally annihilated by $2m$ one-forms ω^σ and $A_\sigma dt + B_{\sigma\nu}d\dot q^\nu$, or, equivalently, *spanned by one vector field*

$$\zeta = \frac{\partial}{\partial t} + \dot q^\sigma \frac{\partial}{\partial q^\sigma} - B^{\sigma\nu}A_\nu \frac{\partial}{\partial \dot q^\sigma}\,. \tag{13}$$

This geometrical model is used to study non-conservative time-dependent mechanical systems, to classify ODE's according to their dynamical properties, to study structure of solutions of both regular ODE's and ODE's "in implicit form" (non-representable by a vector field), to generalize Hamilton's equations to non-variational and non-regular equations, to study transformations of ODE's, and symmetries and first integrals, to develop exact integration methods based on symmetries and transformations (eg. generalized Liouville and Jacobi theorem of the calculus of variations), to study relations between variational and non-variational equations (the inverse problem of the calculus of variations, the problem of existence of variational multipliers), and much more (see eg. [24], [28], [32], [37], [40] and references therein).

Let us now turn to *variational equations.*

By a *Lagrangian* of order r, $r \geq 1$, we mean a *horizontal form* λ on J^rY. In fibred coordinates a Lagrangian reads $\lambda = L\,dt$, where L is a function on an open subset of J^rY. To every Lagrangian there exists a *unique* 1-form θ_λ on $J^{2r-1}Y$ such that $h\theta_\lambda = \lambda$ and $p_1 d\theta_\lambda$ is a dynamical form [21]. The 1-form θ_λ is called the *Lepage equivalent* or the *Cartan form* of λ, and the related dynamical form $E_\lambda = p_1 d\theta_\lambda$ is then called the *Euler-Lagrange form* of λ. We shall be mostly interested in first order Lagrangians. In this case $\lambda = L\,dt$ where L depends upon t, q^σ, and $\dot q^\sigma$, $1 \leq \sigma \leq m$, and

$$\theta_\lambda = L\,dt + \frac{\partial L}{\partial \dot q^\sigma}\omega^\sigma, \quad E_\lambda = \Big(\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^\sigma}\Big)\omega^\sigma \wedge dt\,. \tag{14}$$

The components of E_λ are familiar as *Euler-Lagrange expressions*, and equations for paths of an Euler-Lagrange form are called *Euler-Lagrange equations.* We note that the same Euler-Lagrange form can arise from different Lagrangians, possibly even of different orders. Such Lagrangians are called *equivalent*; it is known that Lagrangians λ_1 of order r and λ_2 of order $k \geq r$ are equivalent iff around every point there is a function f of order $k-1$ such that $\lambda_2 = \lambda_1 + hdf$ (where, precisely, on the place of λ_1 one has to consider its lift by the projection $\pi_{k,r}$).

Let us recall some important properties of Euler-Lagrange dynamical forms [26], [28]. First, every second order Euler-Lagrange form is J^1Y-pertinent, since the Euler-Lagrange expressions are affine in the second derivatives. This means that E_λ on J^2Y is represented by a Lepage class $[\alpha]$ defined on J^1Y. Moreover, the Lepage class has a distinguished representative, independent upon the choice of a particular Lagrangian for $E = E_\lambda$, as follows:

Theorem 2. *Given an Euler-Lagrange form E on J^2Y, the associated Lepage class contains a unique global and closed representative α_E, defined on J^1Y. The 2-form α_E can be expressed by means of the Euler-Lagrange expressions as follows:*

$$\begin{aligned}\alpha_E &= E_\sigma\omega^\sigma \wedge dt + \frac{1}{4}\Big(\frac{\partial E_\sigma}{\partial \dot q^\nu} - \frac{\partial E_\nu}{\partial \dot q^\sigma}\Big)\omega^\sigma \wedge \omega^\nu + \frac{\partial E_\sigma}{\partial \ddot q^\nu}\omega^\sigma \wedge \dot\omega^\nu \\ &= A_\sigma\omega^\sigma \wedge dt + \frac{1}{4}\Big(\frac{\partial A_\sigma}{\partial \dot q^\nu} - \frac{\partial A_\nu}{\partial \dot q^\sigma}\Big)\omega^\sigma \wedge \omega^\nu + B_{\sigma\nu}\omega^\sigma \wedge d\dot q^\nu,\end{aligned} \tag{15}$$

where

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{d'}{dt}\frac{\partial L}{\partial \dot q^\sigma}, \quad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot q^\sigma \partial \dot q^\nu}. \tag{16}$$

Moreover, for every (possibly local) Lagrangian λ of order $r \geq 1$ for E, the Cartan form θ_λ satisfies the following property: $d\theta_\lambda$ is projectable onto an open set in J^1Y, and on this set,

$$d\theta_\lambda = \alpha_E. \tag{17}$$

Above,

$$\frac{d'}{dt} = \frac{d}{dt} - \ddot q^\nu \frac{\partial}{\partial \dot q^\nu} = \frac{\partial}{\partial t} + \frac{\partial}{\partial q^\nu}\dot q^\nu \tag{18}$$

denotes so-called "cut total derivative" applied to functions on J^1Y.

Remarkably, also the converse holds true [26], [28], giving us a *one-to-one* relationship between *variational equations* and a class of *closed* 2-*forms*:

Theorem 3. *Let α be a 2-form on J^1Y such that $E = p_1\alpha$ is a dynamical form. If α is closed then E is locally variational, meaning that around every point in J^1Y there exists a Lagrangian λ such that over the domain of λ, $E = E_\lambda$.*

We note that the existence of a *global* Lagrangian for a locally variational dynamical form is related with topological properties of the manifold Y [23].

Euler-Lagrange equations can be obtained from the *variational principle*. Let us briefly recall the procedure. Denote by $\mathcal{S}_{[a,b]}(\pi)$ the set of sections of π with domains around an interval $[a, b] \subset \mathbb{R}$. Given a Lagrangian λ on J^1Y, consider the function

$$\mathcal{S}_{[a,b]}(\pi) \ni \gamma \to \int_a^b J^1\gamma^*\lambda = \int_a^b J^1\gamma^*\theta_\lambda \in \mathbb{R} \tag{19}$$

called the *action function* of λ over $[a, b]$. To get a correct concept of *variation* (one-parametric deformation) of a section γ, one has to restrict to consider π-*projectable* vector fields on Y: if ξ is a projectable vector field on Y with projection ξ_0, and $\{\phi_u\}$, resp. $\{\phi_{0u}\}$ are the corresponding local one-parameter groups, we get a one-parameter family $\{\gamma_u\}$ of *sections* where $\gamma_u = \phi_u \gamma \phi_{0u}^{-1}$ is defined in a neighbourhood of $\phi_{0u}([a,b]) \subset \mathbb{R}$, called *variation of the section* γ *induced by* ξ. The arising function

$$\mathcal{S}_{[a,b]}(\pi) \ni \gamma \to \left(\frac{d}{du} \int_{\phi_{0u}([a,b])} J^1\gamma_u^* \, \lambda \right)_{u=0} = \int_a^b J^1\gamma^* \, \mathcal{L}_{J^1\xi}\lambda \in \mathbb{R} \tag{20}$$

is called the *first variation of the action function* of the Lagrangian λ over the interval $[a,b]$, *induced by* ξ. *The First Variation Formula* is a splitting of the above integral into a sum of two terms such that the first one does not depend upon "derivations of variations" (the Euler-Lagrange term) and the second one is a boundary term. With the Cartan form θ_λ the decomposition is available *directly* (without the integration by parts procedure), and in an *invariant* way [21]:

$$\begin{aligned}
\int_a^b J^1\gamma^* \, \mathcal{L}_{J^1\xi}\lambda &= \int_a^b J^1\gamma^* \, \mathcal{L}_{J^1\xi}\theta_\lambda \\
&= \int_a^b J^1\gamma^* \, i_{J^1\xi} d\theta_\lambda + \int_a^b d(i_{J^1\xi}\theta_\lambda \circ J^1\gamma) \\
&= \int_a^b J^2\gamma^* \, i_{J^2\xi} E_\lambda \ + \ \text{the above boundary term.}
\end{aligned} \tag{21}$$

A section γ of π is called an *extremal of* λ *on* $[a,b]$ if the first variation of the action of λ on the interval $[a,b]$ vanishes for every vertical vector field ξ on Y with the support in $\pi^{-1}([a,b])$ (such a vector field is often called a *fixed-endpoints variation*). γ is called *extremal of* λ if it is an extremal on every interval $[a,b] \subset \mathbb{R}$.

With help of the First Variation Formula one obtains necessary and sufficient conditions for extremals as follows [21]:

Theorem 4. *Let λ be a Lagrangian on J^1Y. A section γ of π is an extremal of λ if and only if γ satisfies one of the following equivalent conditions:*

(1) *$E_\lambda \circ J^2\gamma = 0$, i.e. γ is a path of the Euler-Lagrange form of λ.*

(2) *For every vertical vector field ξ on Y, $J^1\gamma^* \, i_{J^1\xi} d\theta_\lambda = 0$.*

(3) *In every fibred chart γ satisfies the system of m second-order ordinary differential equations*
$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma} = 0, \quad 1 \le \sigma \le m\,. \tag{22}$$

Notice the meaning of condition (2) of the above theorem: it is a geometric interpretation of the Euler-Lagrange equations in terms of a dynamical distribution of the dynamical form $E = E_\lambda$. Namely, accounting Theorem 2 we can see that every Euler-Lagrange dynamical form possesses a distinguished *global* dynamical distribution related with the Lepage 2-form α_E,

$$\Delta_E = \mathrm{annih}\{ i_\xi \alpha_E \,|\, \xi \text{ runs over all vertical vector fields on } J^1Y \} \tag{23}$$

called the *Euler-Lagrange distribution* [25], [27]. By condition (2) of the above theorem, extremals (solutions of the Euler-Lagrange equations) are *holonomic* integral sections of the distribution Δ_E.

Equations for (all) integral sections of the Euler-Lagrange distribution Δ_E, i.e. equations

$$\delta^* i_\xi \alpha_E = 0 \quad \text{for every } \pi_1\text{-vertical vector field } \xi \text{ on } J^1Y \tag{24}$$

for sections δ of the fibred manifold $\pi_1 : J^1Y \to \mathbb{R}$ are then called *Hamilton equations* [16], [27]. In case that $\operatorname{rank}\Delta_E = 1$, i.e., E is *regular* (as a dynamical form), $\operatorname{rank}\alpha_E$ is maximal (equal $2m$), and the Euler-Lagrange distribution is spanned by one vector field ζ. It is, up to a multiplier f, a unique solution of the equation $i_\zeta \alpha_E = 0$, and is called *Euler-Lagrange field* [16], or *Hamiltonian vector field*. The condition for regularity can be expressed by means of Lagrangians as follows:

(i) If Δ_E is defined on J^1Y (this means that the Euler-Lagrange equations are nontrivially second-order equations) the regularity condition takes the form

$$\det\Big(\frac{\partial^2 L}{\partial \dot q^\sigma \partial \dot q^\nu}\Big) \neq 0\,. \tag{25}$$

(ii) If α_E is projectable onto Y, i.e. Δ_E is defined on Y, then the regularity condition takes the form [26]

$$\det\Big(\frac{\partial^2 L}{\partial q^\sigma \partial \dot q^\nu} - \frac{\partial^2 L}{\partial \dot q^\sigma \partial q^\nu}\Big) \neq 0\,. \tag{26}$$

This is the case when the Euler-Lagrange equations are first-order equations, i.e. the corresponding Lagrangians are *affine* functions in the velocities.

For regular Lagrangians, i.e. satisfying either (25) or (26), the Cauchy problem has a unique solution, i.e., *through every point in the dynamical space* (J^1Y, respectively Y) *there passes a unique maximal solution of the Euler-Lagrange equations.*

The Cartan form θ_λ takes the coordinate form (14). Expressing the same form in the canonical basis $(dt, dq^\sigma, d\dot q^\sigma)$ one obtains

$$\theta_\lambda = -H\,dt + p_\sigma dq^\sigma, \quad \text{where} \quad p_\sigma = \frac{\partial L}{\partial \dot q^\sigma}, \quad H = -L + p_\sigma \dot q^\sigma. \tag{27}$$

If L is not affine in velocities (meaning that Δ_E is defined on J^1Y) then the "momenta" p_σ are (local) functions on J^1Y. If, moreover, L is regular, we get on J^1Y local coordinates (t, q^σ, p_σ), called *Legendre coordinates.* In these coordinates, Hamilton equations (24) take the "canonical form"

$$\frac{dp_\sigma}{dt} = -\frac{\partial H}{\partial q^\sigma}, \quad \frac{dq^\sigma}{dt} = \frac{\partial H}{\partial p_\sigma}\,. \tag{28}$$

Up to now we have been interested in the meaning of the first term in the decomposition of the first variation (21). The second term, however, is important as well, since it is connected with *conservation laws.*

We say that a π-projectable vector field ξ on Y is a *point symmetry* of a Lagrangian λ if

$$\mathcal{L}_{J^1\xi}\lambda = 0\,, \tag{29}$$

and a *generalized point symmetry of* λ if it is a point symmetry of its Euler-Lagrange form, i.e.,

$$\mathcal{L}_{J^2\xi}E_\lambda = 0\,. \tag{30}$$

Within the terminology of the classical calculus of variations, point symmetries of a Lagrangian correspond to infinitesimal transformations that leave invariant the action integral; similarly point symmetries of the Euler-Lagrange form correspond to transformations leaving the action integral invariant "up to a divergence". Equation (29) and (30) is called *Noether equation* and *Noether–Bessel-Hagen equation*, respectively. It is known that every point symmetry of λ is a point symmetry of E_λ [22].

Substituting into the First Variation Formula (21) the symmetry condition and taking account of the extremal condition (2) in Theorem 4 we immediately obtain the following famous result [45], [22]:

Theorem 5. (Noether Theorem)

(1) *Assume that a π-projectable vector field ξ on Y is a point symmetry of a Lagrangian λ. Then, along every extremal of λ, the function $F = i_{J^1\xi}\theta_\lambda$ is constant.*

(2) *Assume that a π-projectable vector field ξ on Y is a generalized point symmetry of a Lagrangian λ. Then (locally) $\mathcal{L}_{J^1\xi}\lambda = hdf$ for a function f, and along every extremal of λ, the function $F = i_{J^1\xi}\theta_\lambda - f$ is constant.*

Within fibred mechanics one can easily consider also Lagrangian systems subject to external forces that need not be variational (so-called *nonconservative systems*) [32]. More precisely, by a *mechanical system* on a fibred manifold π we shall mean a pair (λ, Φ) where λ is a Lagrangian on J^1Y and Φ is a *first-order dynamical form*, called a *force*. It is generally assumed that λ is not affine in velocities (provides Euler-Lagrange equations that are nontrivially of order two). The corresponding dynamical form is then $E = E_\lambda - \pi^*_{2,1}\Phi$, and equations for paths of E take the form

$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^\sigma} = \Phi_\sigma, \quad 1 \le \sigma \le m\,. \tag{31}$$

The corresponding Lepage class is represented by the Lepage 2-form

$$\begin{aligned}\alpha &= A_\sigma\omega^\sigma \wedge dt + \frac{1}{4}\Big(\frac{\partial A_\sigma}{\partial \dot q^\nu} - \frac{\partial A_\nu}{\partial \dot q^\sigma}\Big)\omega^\sigma \wedge \omega^\nu + B_{\sigma\nu}\omega^\sigma \wedge d\dot q^\nu \\ &= d\theta_\lambda - \Phi - \frac{1}{4}\Big(\frac{\partial \Phi_\sigma}{\partial \dot q^\nu} - \frac{\partial \Phi_\nu}{\partial \dot q^\sigma}\Big)\omega^\sigma \wedge \omega^\nu,\end{aligned} \tag{32}$$

where A_σ and $B_{\sigma\nu}$ are defined as above by $E_\sigma = A_\sigma + B_{\sigma\nu}\ddot q^\nu$ and take the form

$$A_\sigma = \frac{\partial L}{\partial q^\sigma} - \frac{\partial^2 L}{\partial t\,\partial \dot q^\sigma} - \frac{\partial^2 L}{\partial q^\nu \partial \dot q^\sigma}\dot q^\nu - \Phi_\sigma\,, \quad B_{\sigma\nu} = -\frac{\partial^2 L}{\partial \dot q^\sigma \partial \dot q^\nu}\,. \tag{33}$$

The motion is described by the dynamical distribution $\Delta = \text{annih}\{i_\xi \alpha\}$ where ξ runs over all vertical vector fields on J^1Y.

Using theorems 1 and 2 we can see that the force Φ is conservative (potential) if and only if it is variational (as a first order dynamical form), i.e. if and only if the 2-form α is closed, hence

$$\Phi + \frac{1}{4}\Big(\frac{\partial \Phi_\sigma}{\partial \dot{q}^\nu} - \frac{\partial \Phi_\nu}{\partial \dot{q}^\sigma}\Big)\omega^\sigma \wedge \omega^\nu \tag{34}$$

is closed. It can be easily verified that Φ satisfies *Helmholtz conditions* [18]. Recall that in this case the Helmholtz conditions take the form

$$\begin{aligned} \frac{\partial \Phi_\sigma}{\partial \dot{q}^\nu} + \frac{\partial \Phi_\nu}{\partial \dot{q}^\sigma} &= 0, \\ \frac{\partial \Phi_\sigma}{\partial q^\nu} - \frac{\partial \Phi_\nu}{\partial q^\sigma} + \frac{d}{dt}\frac{\partial \Phi_\nu}{\partial \dot{q}^\sigma} &= 0. \end{aligned} \tag{35}$$

Since we assume the force Φ be of the first order, the latter condition gives

$$\frac{\partial^2 \Phi_\sigma}{\partial \dot{q}^\nu \partial \dot{q}^\rho} = 0, \tag{36}$$

i.e. that the force is affine in velocities,

$$\Phi_\sigma = a_{\sigma\rho}\dot{q}^\rho + b_\sigma, \tag{37}$$

and the first condition (35) then immediately means that the matrix of the coefficients $(a_{\sigma\rho})$ is skew-symmetric. Substituting now (37) to the second condition (35) we obtain the Helmholtz conditions for a force Φ in the familiar form

$$\begin{aligned} a_{\sigma\nu} &= -a_{\nu\sigma}, \\ \frac{\partial a_{\sigma\rho}}{\partial q^\nu} + \frac{\partial a_{\nu\sigma}}{\partial q^\rho} + \frac{\partial a_{\rho\nu}}{\partial q^\sigma} &= 0, \\ \frac{\partial b_\sigma}{\partial q^\nu} - \frac{\partial b_\nu}{\partial q^\sigma} + \frac{\partial a_{\nu\sigma}}{\partial t} &= 0. \end{aligned} \tag{38}$$

We note that conditions (38) mean that Φ is a *Lorentz-type force*. From the geometric point of view, (38) indeed are exactly the closedness conditions for the 2-form (34).

3 Holonomic constraints

Holonomic constraints on the motion appear in numerous applications in physics and engineering. Due to their importance they have been considered within analytical mechanics since the very beginning going back to Lagrange and Hamilton.

In classical mechanics holonomic constraints are constraints on positions of particles; they may be time-independent (not explicitly depending on time), or time dependent. In local coordinates $(q^1, \dots, q^m)$ in $\mathbb{R}^m$, holonomic constraints are given by a system of (algebraic) equations

$$u^a(t, q^\sigma) = 0, \quad a = 1, 2, \dots, k < m, \tag{39}$$

satisfying the rank condition

$$\operatorname{rank}\Big(\frac{\partial u^a}{\partial q^\sigma}\Big) = k. \tag{40}$$

The latter condition means that constraint conditions (39) can be expressed in a form

$$q^{m-k+a} = w^a(t, q^1, \dots, q^{m-k}). \tag{41}$$

There are two ways for considering constrained motions:

- *External – with Lagrange multipliers:*

  The influence of the constraint on the motion is modeled via an external attractive force, called *constraint force*, proportional to $\operatorname{grad} u$. *Equations of motion* of a Lagrangian system L subject to constraints (39) then take the form

  $$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^\sigma} = -\mu_a \frac{\partial u^a}{\partial q^\sigma}, \quad 1 \le \sigma \le m, \tag{42}$$

  where μ_a, $1 \le a \le k$, are Lagrange multipliers. Solutions to the problem are both curves in $\mathbb{R}^m$ satisfying simultaneously the constraint conditions and the above motion equations, and Lagrange multipliers as functions of time.

  Remarkably, equations of motion (42) can be obtained as standard Euler--Lagrange equations from the Lagrangian

  $$\hat{L} = L + \mu_a u^a. \tag{43}$$

- *Internal (geometric) – without Lagrange multipliers:*

  The point is that the constraints given by equations (39) with the accompanying rank condition have the *geometric meaning* of a *submanifold* of codimension k in $\mathbb{R}\times\mathbb{R}^m$. In terms of the fibred manifolds terminology, the constraint conditions define a fibred submanifold $\pi : \bar{Y} \to \mathbb{R}$ of the fibred manifold $pr_1 : \mathbb{R}\times\mathbb{R}^m \to \mathbb{R}$. If we denote by ι the canonical embedding of the constraint submanifold $\bar{Y}$ into $\mathbb{R}\times\mathbb{R}^m$, and by (t, q^s), $1 \le s \le m-k = \dim \bar{Y} - 1$, adapted coordinates on $\bar{Y}$, then *equations of motion* of a Lagrangian system $L(t, q^\sigma, \dot q^\sigma)$ subject to constraints (39) take the form of "standard" Euler--Lagrange equations

  $$\frac{\partial \bar{L}}{\partial q^s} - \frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot q^s} = 0, \quad 1 \le s \le m-k, \tag{44}$$

  for the Lagrangian $\bar{L} = L \circ J^1\iota$ on the manifold $J^1\bar{Y}$. It is essential that the latter equations and the equations with Lagrange multipliers above are, as equations for sections passing in the constraint submanifold (i.e. satisfying the constraint conditions), *equivalent.*

  In analytical mechanics the manifold $\bar{Y}$ is called "space of events", (q^s) are "generalized coordinates", $m-k$ is the "number of degrees of freedom", $J^1\bar{Y}$ is called "evolution space" or "phase space", and $\bar{L}$ is the "constrained Lagrangian".

Due to the geometric nature of holonomic constraints, holonomic systems are very well understood within the fibred mechanics setting presented in the previous section. Indeed, they completely fit with the general scheme of the theory - and this concerns both Lagrangian and nonconservative systems. In a full generality, if $\pi : Y \to X$ is a fibred manifold, a *holonomic constraint* in Y is a fibred submanifold $\bar{\pi} : \bar{Y} \to X$ of π. Given a dynamical form E on J^2Y, on $J^2\bar{Y}$ there arises dynamical form $\bar{E} = J^2\iota^* E$. In particular, given a Lagrangian λ on J^1Y we obtain a Lagrangian $\bar{\lambda} = J^1\iota^*\lambda$ on $J^1\bar{Y}$, and the Euler-Lagrange equations of $\bar{\lambda}$ come from the restricted Euler-Lagrange form $E_{\bar{\lambda}} = J^2\iota^* E_\lambda$.

Note that if $\pi : \mathbb{R} \times M \to \mathbb{R}$, and $\bar{Y}$ is of the form $\mathbb{R} \times N$ where N is a submanifold in M we can speak about a *time-independent holonomic constraint*, otherwise $\bar{\pi} : \bar{Y} \to \mathbb{R}$ is a *time-dependent holonomic constraint* in $\mathbb{R} \times M$.

4 Nonholonomic systems on constraint manifolds

In what follows we shall consider constraints on the motion that depend on time, positions and velocities, called *nonholonomic constraints.* In this case equations defining a constraint are *first order differential equations.* In terms of jet bundles constraints with this property are *submanifolds of the first jet manifold.*

As above, let us consider a fibred manifold $\pi : Y \to \mathbb{R}$, where $\dim Y = m + 1$. Precisely speaking, by a *nonholonomic constraint* in J^1Y we shall mean a submanifold $Q \subset J^1Y$, fibred over Y. When appropriate, we shall use notation $\iota : Q \to J^1Y$ for the canonical embedding. A constraint of codimension k ($1 \leq k < m$) in J^1Y is locally defined by *a system of k first order ordinary differential equations*

$$f^a(t, q^\sigma, \dot{q}^\sigma) = 0, \quad 1 \leq a \leq k\,, \tag{45}$$

where the functions f^a satisfy the rank condition

$$\operatorname{rank}\Big(\frac{\partial f^a}{\partial \dot{q}^\sigma}\Big) = k\,. \tag{46}$$

Due to (46), equations of the constraint take a normal form

$$\dot{q}^{m-k+a} = g^a(t, q^\sigma, \dot{q}^1, \ldots, \dot{q}^{m-k}), \quad 1 \leq a \leq k\,. \tag{47}$$

Similarly as in the case of holonomic constraints we can approach the nonholonomic dynamics in two ways:

- *External – with Lagrange multipliers:*

  The influence of the constraint on the motion should be modeled via a *constraint force.* The problem now, however, is that it is not clear how such a force should look like. In [9] Chetaev proposed the following formula for the constraint force:

  $$\mathcal{F}_\sigma = -\mu_a \frac{\partial f^a}{\partial \dot{q}^\sigma}, \tag{48}$$

  where μ_a, $1 \leq a \leq k$, are Lagrange multipliers. *Equations of motion* of a Lagrangian system L subject to nonholonomic constraints (45) then read

  $$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma} = -\mu_a \frac{\partial f^a}{\partial \dot{q}^\sigma}, \quad 1 \leq \sigma \leq m, \tag{49}$$

and are called *Chetaev equations.* In this case, the integration problem means to solve a system of $m+k$ mixed first and second order ordinary differential equations (45) and (49) for m components $q^\sigma(t)$ of the nonholonomic curves and k Lagrange multipliers $\mu_a(t)$.

It should be stressed, that in this case, rather surprisingly, Chetaev equations do not arise as Euler-Lagrange equations from a Lagrangian analogous to (43), i.e.

$$\hat{L} = L + \mu_a f^a, \tag{50}$$

since equations for extremals of (50), called *vakonomic equations*, take a different form

$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma} = -\mu_a\Big(\frac{\partial f^a}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial f^a}{\partial \dot{q}^\sigma}\Big) - \frac{d\mu_a}{dt}\frac{\partial f^a}{\partial \dot{q}^\sigma}, \quad 1 \le \sigma \le m. \tag{51}$$

Solutions of Chetaev and vakonomic equations are different unless the constraints are *semiholonomic* (linear, integrable), satisfying the integrability conditions

$$f^a = \frac{du^a}{dt}. \tag{52}$$

Investigations of different examples indicated that nonholonomic dynamics obey Chetaev equations. On the other hand, vakonomic equations seem to be valuable in control theory.

- *Internal (geometric) – without Lagrange multipliers:*

  The second approach explores the geometric meaning of nonholonomic constraints as submanifolds in jet bundles. In what follows, we shall present namely this model and the arising geometric structures, first considered in our paper [29]. Remarkably, within this setting *nonholonomic systems and their dynamics are described by geometric structures on a corresponding constraint submanifold*, which has the physical meaning of a *constrained phase space.* The dynamics are governed by so-called *reduced equations* which represent a system of $m-k$ second order ordinary differential equations for sections of the constraint submanifold (as expected no Lagrange multipliers enter in these equations). For the study of the constrained systems concepts and techniques of fibred mechanics can be directly used or quite easily generalized. In this way, nonholonomic mechanics is a direct extension of fibred mechanics and admits a straightforward generalization to higher order and field theory.

  Moreover, within the geometric model there arises a *new possibility* to understand and study constrained systems. Indeed, one can distinguish *two different situations* [39]:

  - the constrained system arises from an unconstrained system defined in a neighbourhood of the constraint

  - an internally defined constrained system on the constraint manifold is given, without reference to the ambient space J^1Y; in this case *a corresponding unconstrained system need not exist.*

4.1 Constraint submanifolds in jet bundles

Consider a constraint submanifold $\iota : Q \to J^1Y$ of codimension $k < m$. This means that we have fibred manifolds $\bar\pi_{1,0} : Q \to Y$ where $\bar\pi_{1,0}$ is the restriction of the projection $\pi_{1,0} : J^1Y \to Y$ to Q, and $\bar\pi_1 : Q \to X$, where $\bar\pi_1 = \pi_1|_Q$.

We define the *first prolongation* $\hat Q$ of the constraint Q to be a submanifold in J^2Y, consisting of all points $J^2_x\gamma$ such that $J^1_x\gamma \in Q$, $x \in X$. Locally $\hat Q$ is defined by the equations of the constraint and their derivatives:

$$f^a = 0, \quad \frac{df^a}{dt} = 0, \quad 1 \le a \le k\,, \tag{53}$$

respectively, in normal form,

$$\dot q^{m-k+a} = g^a, \quad \ddot q^{m-k+a} = \frac{dg^a}{dt}\,. \tag{54}$$

We also use notation $\hat\iota : \hat Q \to J^2Y$ for the corresponding canonical embedding. The manifold $\hat Q$ is fibred over Q, Y and X, the fibred projections are simply restrictions of the corresponding canonical projections of the underlying fibred manifolds. We write $\bar\pi_2 : \hat Q \to X$, $\bar\pi_{2,1} : \hat Q \to Q$ and $\bar\pi_{2,0} : \hat Q \to Y$.

Usually we shall use on Q adapted coordinates $(t, q^\sigma, \dot q^s)$, where $1 \le s \le m-k$, and on $\hat Q$ associated coordinates $(t, q^\sigma, \dot q^s, \ddot q^s)$, $1 \le \sigma \le m$, $1 \le s \le m-k$.

The contact ideal on Q respectively $\hat Q$, is locally generated by one-forms

$$\bar\omega^s = dq^s - \dot q^s dt, \quad \bar\omega^{m-k+a} = dq^{m-k+a} - g^a dt, \tag{55}$$

respectively,

$$\bar\omega^s = dq^s - \dot q^s dt, \quad \bar\omega^{m-k+a} = dq^{m-k+a} - g^a dt, \quad \hat\omega^s = d\dot q^s - \ddot q^s dt\,, \tag{56}$$

and their exterior derivatives.

Due to the existence of the contact structure on constraint manifolds, it is possible to prolong projectable vector fields from the total space Y to the constraint and to its prolongations. The procedure was described in [35] and is as follows:

Let ξ be a projectable vector field on Y. A vector field ζ on Q (resp. $\hat Q$) is called the *first* (resp. *second*) *constrained prolongation* of ξ, and is denoted by $J^1_{\mathrm{c}}\xi$ (resp. $J^2_{\mathrm{c}}\xi$), if ζ is a symmetry of the contact ideal on Q (resp. $\hat Q$) and projects onto ξ. It should be stressed that not every projectable vector field on Y admits a constrained prolongation; conditions and formulas can be found in [35].

Similarly as in the unconstrained case, for every q-form η on Q one has a *unique decomposition* into a sum of a $\bar\pi_2$-horizontal form and i-contact forms, $i = 1, 2, \dots q$, on $\hat Q$ [35]; we write

$$\bar\pi^*_{2,1}\eta = \bar h\eta + \bar p_1\eta + \cdots + \bar p_q\eta\,. \tag{57}$$

Applying this decomposition to (locally) exact one-forms on Q we get an invariant *splitting of the exterior derivative* d to the horizontal and contact part, $\bar\pi^*_{2,1}d = \bar hd + \bar p_1 d$. The operator $\bar hd$ has the component

$$\frac{d_{\mathrm{c}}}{dt} = \frac{\partial}{\partial t} + \dot q^s \frac{\partial}{\partial q^s} + g^a \frac{\partial}{\partial q^{m-k+a}} + \ddot q^s \frac{\partial}{\partial \dot q^s}\,, \tag{58}$$

and is called the *constraint total derivative*.

For convenience of notations we also put

$$\frac{d'_{\rm c}}{dt} = \frac{\partial}{\partial t} + \dot q^s \frac{\partial}{\partial q^s} + g^a \frac{\partial}{\partial q^{m-k+a}} \,. \tag{59}$$

4.2 The canonical distribution

The most important object in the constraint geometry is the *canonical distribution* (also called *Chetaev bundle*) [29] (see also [43]). Remarkably, it is an internal object – a bundle naturally arising over every nonholonomic constraint. The canonical distribution gives a geometric meaning to *virtual displacements* in the space of positions and velocities, and to the concept of *reactive (Chetaev) forces*; for more details and introduction of a *nonholonomic D'Alembert principle* we refer to [29] and [34].

The *canonical distribution* for a nonholonomic constraint $Q \subset J^1Y$ is a corank k distribution $\mathcal{C}$ *on the manifold* Q, where $k = \operatorname{codim} Q$, locally annihilated by the system of k linearly independent 1-forms

$$\varphi^a = \left(\frac{\partial f^a}{\partial \dot q^\sigma} \circ \iota\right) \bar\omega^\sigma = \bar\omega^{m-k+a} - \sum_{s=1}^{m-k} \frac{\partial g^a}{\partial \dot q^s} \bar\omega^s, \quad 1 \le a \le k, \tag{60}$$

or, equivalently, locally spanned by the system of $2(m-k)+1$ independent vector fields

$$\begin{aligned} \frac{\partial_c}{\partial t} &\equiv \frac{\partial}{\partial t} + \sum_{a=1}^{k} \Big(g^a - \sum_{l=1}^{m-k} \frac{\partial g^a}{\partial \dot q^l} \dot q^l \Big) \frac{\partial}{\partial q^{m-k+a}} \\ \frac{\partial_c}{\partial q^s} &\equiv \frac{\partial}{\partial q^s} + \sum_{a=1}^{k} \frac{\partial g^a}{\partial \dot q^s} \frac{\partial}{\partial q^{m-k+a}} \\ &\frac{\partial}{\partial \dot q^s} \end{aligned} \tag{61}$$

where $1 \le s \le m-k$.

The annihilator of $\mathcal{C}$ is denoted by $\mathcal{C}^0$.

The ideal in the exterior algebra on Q locally generated by the 1-forms φ^a, $1 \le a \le k$, is called the *constraint ideal*, and denoted by $\mathcal{I}(\mathcal{C}^0)$. Differential forms belonging to the constraint ideal are called *constraint forms*.

Vector fields belonging to the canonical distribution are called *Chetaev vector fields*. Note that every Chetaev vector field takes a form

$$\begin{aligned} Z &= Z^0 \frac{\partial_c}{\partial t} + Z^s \frac{\partial_c}{\partial q^s} + \tilde Z^s \frac{\partial}{\partial \dot q^s} \\ &= Z^0 \frac{\partial}{\partial t} + Z^s \frac{\partial}{\partial q^s} + \sum_{a=1}^{k} \Big(Z^0 \Big(g^a - \sum_{l=1}^{m-k} \frac{\partial g^a}{\partial \dot q^l} \dot q^l \Big) + Z^s \frac{\partial g^a}{\partial \dot q^s} \Big) \frac{\partial}{\partial q^{m-k+a}} + \tilde Z^s \frac{\partial}{\partial \dot q^s}. \end{aligned} \tag{62}$$

We stress that the family of Chetaev vector fields *need not contain*

- vector fields *projectable* onto Y,
- *prolongations* of vector fields defined on Y, even if the canonical distribution is projectable.

Remarkably, the following theorem holds [29]:

Theorem 6. *The constraint Q is given by equations affine in the first derivatives if and only if the canonical distribution $\mathcal{C}$ on Q is $\bar\pi_{1,0}$-projectable (i.e. the projection $\mathcal{D}$ of $\mathcal{C}$ is* a distribution *on Y).*

The distribution $\mathcal{D}$ on Y is then locally spanned by vector fields

$$\frac{\partial}{\partial t} + \sum_{a=1}^{k} A^a \frac{\partial}{\partial q^{m-k+a}}\,, \quad \frac{\partial}{\partial q^s} + \sum_{a=1}^{k} B^a_s \frac{\partial}{\partial q^{m-k+a}}\,, \; 1 \le s \le m-k\,, \tag{63}$$

or, annihilated by 1-forms $A^a dt + B^a_s dq^s - dq^{m-k+a}$, $1 \le a \le k$, where $g^a = A^a + B^a_s \dot q^s$.

The canonical distribution need not be completely integrable. We call a nonholonomic constraint Q *semiholonomic* if its canonical distribution $\mathcal{C}$ is completely integrable. Properties of semiholonomic constraints can be summarized as follows [29], [33], [35]:

Theorem 7. *The following conditions are equivalent:*

(1) *Q is semiholonomic.*

(2) *The canonical distribution $\mathcal{C}$ on Q is projectable onto Y, and its projection is completely integrable.*

(3) *The constraint ideal is closed.*

(4) *Functions g^a defining locally the constraint satisfy*

$$\frac{\partial_c g^a}{\partial q^s} - \frac{d_c}{dt}\frac{\partial g^a}{\partial \dot q^s} = 0\,, \qquad 1 \le s \le m-k\,. \tag{64}$$

Theorem 8. *The canonical distribution $\mathcal{C}$ of a semiholonomic constraint is spanned by vector fields $J^1_c\xi$, where ξ belongs to the projection $\mathcal{D}$ of $\mathcal{C}$, and $\bar\pi_{1,0}$-vertical vector fields.*

We have seen that *constraints linear or affine in velocities can be alternatively modeled by means of a distribution $\mathcal{D}$ on Y*, defined by (63) (that is completely integrable in case of semiholonomic constraints). The geometric description of nonholonomic constraints by a distribution on Y (on a "configuration space", or "space of events") is quite popular and frequently used. The reader should, however, keep in mind that using such a model means *restriction* to constraints *affine in velocities.*

The canonical distribution is naturally lifted to the distribution $\hat{\mathcal{C}}$ on $\hat Q$, defined with help of its annihilator by $\hat{\mathcal{C}}^0 = \bar\pi^*_{2,1}\mathcal{C}^0$.

4.3 Dynamics of nonholonomic systems: Reduced equations

Consider a nonholonomic constraint $\iota : Q \to J^1Y$ endowed with the canonical distribution $\mathcal{C}$ as above. Let E be a J^1Y-pertinent dynamical form on J^2Y and $[\alpha]$ its Lepage class. Recall that $[\alpha]$ consists of local 2-forms on J^1Y, and contains a closed (global) 2-form if and only if the dynamical form E comes as the Euler--Lagrange form from (possibly local) Lagrangians. We keep notations used above, i.e. $E = E_\sigma \omega^\sigma \wedge dt$, $E_\sigma = A_\sigma + B_{\sigma\nu}\ddot{q}^\nu$, and

$$\alpha = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot{q}^\nu + F, \tag{65}$$

where F is a 2-contact 2-form on an open subset of J^1Y.

According to [29], a *constrained mechanical system* associated with $[\alpha]$ is defined to be the class

$$[\bar{\alpha}] = \iota^*\alpha \mod \mathcal{I}(\mathcal{C}^0). \tag{66}$$

This means that $[\bar{\alpha}]$ is defined on the constraint Q and consists of all possibly local 2-forms on Q such that

$$\bar{\alpha} = \bar{A}_l \omega^l \wedge dt + \bar{B}_{ls}\omega^l \wedge d\dot{q}^s + F + \varphi, \tag{67}$$

where F is a 2-contact and φ is a constraint 2-form on Q, and

$$\begin{aligned} \bar{A}_l &= \Big(A_l + A_{m-k+j}\frac{\partial g^j}{\partial \dot{q}^l} + \Big(B_{l,m-k+i} + B_{m-k+j,m-k+i}\frac{\partial g^j}{\partial \dot{q}^l}\Big)\frac{d' g^i}{dt}\Big) \circ \iota, \\ \bar{B}_{ls} &= \Big(B_{ls} + B_{l,m-k+i}\frac{\partial g^i}{\partial \dot{q}^s} + B_{m-k+i,s}\frac{\partial g^i}{\partial \dot{q}^l} + B_{m-k+j,m-k+i}\frac{\partial g^j}{\partial \dot{q}^l}\frac{\partial g^i}{\partial \dot{q}^s}\Big) \circ \iota. \end{aligned} \tag{68}$$

Note that if the matrix B is symmetric then so is $\bar{B}$, however, regularity of B does not imply regularity of $\bar{B}$. The latter has important consequences on dynamical properties of nonholonomically constrained systems making them much different from the holonomic ones. We shall discuss it in more detail below when dealing with the associated exterior differential systems.

If in particular α is related with a mechanical system (λ, Φ), we have

$$\bar{A}_s = \frac{\partial_c \bar{L}}{\partial q^s} - \frac{d'_c}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^s} - \Big(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota\Big)\Big(\frac{\partial_c g^a}{\partial q^s} - \frac{d'_c}{dt}\frac{\partial g^a}{\partial \dot{q}^s}\Big) - \bar{\Phi}_s - \bar{\Phi}_{m-k+a}\frac{\partial g^a}{\partial \dot{q}^s} \tag{69}$$

$$\bar{B}_{sr} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^r \partial \dot{q}^s} + \Big(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota\Big)\frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s} \tag{70}$$

with the notation $\bar{L} = L \circ \iota$, $\bar{\Phi}_\sigma = \Phi_\sigma \circ \iota$.

In place of a single dynamical form $E = p_1\alpha$ we have for the constrained system rather the class $[\bar{E}]$, on $\hat{Q}$, with

$$\bar{E} = \bar{p}_1\bar{\alpha} = \hat{\iota}^*E + \varphi^a \wedge \nu_a \tag{71}$$

where φ^a are the canonical constraint 1-forms defined above and ν_a are horizontal forms.

Since $\mathcal{C} \to Q$ is a subbundle of the tangent bundle $TQ \to Q$, the class $[\bar{E}]$ gives rise to a *dynamical form along the canonical distribution*, called *constrained dynamical form*, $\bar{E}^{\mathrm{c}} = (\hat{\iota}^* E)|_{\hat{\mathcal{C}}} \in \Lambda^2(\hat{\mathcal{C}})$ (see [29] for the definition and more details on forms along a distribution); we note that $\bar{E}^{\mathrm{c}}$ is the same for all $\bar{E} \in [\bar{E}]$.

Computations in adapted fibred coordinates yield the following formula:

$$\bar{E}^{\mathrm{c}} = (\bar{A}_s + \bar{B}_{sr}\ddot{q}^r)\bar{\omega}^s \wedge dt\,. \tag{72}$$

We shall be interested in *constrained sections* of π, that is in sections $\gamma : I \to Y$ such that $J^1\gamma(I) \subset Q$. Constrained sections satisfy the system of k first order ODE's of the constraint. In particular, every such a section is an integral section of the canonical distribution $\mathcal{C}$.

We have the following theorem (cf. [29]):

Theorem 9. *Equations of motion of a mechanical system α constrained to Q are equations for constrained sections of π, taking one of the following two equivalent intrinsic forms:*

$$\bar{E}^{\mathrm{c}} \circ J^2\gamma = 0\,, \tag{73}$$

$$J^1\gamma^* i_Z\bar{\alpha} = 0 \quad \textit{for every } \bar{\pi}_1\textit{-vertical Chetaev vector field } Z \textit{ on } Q \tag{74}$$

(where $\bar{\alpha}$ is (any) representative of the class $[\bar{\alpha}]$).

In coordinates,

$$\bar{A}_s + \bar{B}_{sr}\ddot{q}^r = 0, \tag{75}$$

or, if α is given by means of a Lagrangian λ and a force Φ,

$$\frac{\partial_{\mathrm{c}}\bar{L}}{\partial q^s} - \frac{d_{\mathrm{c}}}{dt}\frac{\partial\bar{L}}{\partial\dot{q}^s} - \Big(\frac{\partial L}{\partial\dot{q}^{m-k+a}} \circ \iota\Big)\Big(\frac{\partial_{\mathrm{c}}g^a}{\partial q^s} - \frac{d_{\mathrm{c}}}{dt}\frac{\partial g^a}{\partial\dot{q}^s}\Big) = \bar{\Phi}_s + \bar{\Phi}_{m-k+a}\frac{\partial g^a}{\partial\dot{q}^s}\,, \tag{76}$$

where $1 \le s \le m-k$.

It should be stressed that the above motion equations for nonholonomic systems are *differential equations on the constraint manifold Q*. They are called *reduced nonholonomic equations* ("without Lagrange multipliers") [29].

Further it should be emphasized that the motion equations for nonholonomic systems are generally *equations in implicit form.* However, due to their interpretation as equations for an exterior differential system on the constraint manifold Q, apparent from (74), they are investigated with the same methods as motion equations in the unconstrained/holonomic case (see e.g. [29], [32], [40]).

4.4 The nonholonomic variational principle

Let us turn to the special case, when the force $\Phi(t, q^\nu, \dot{q}^\nu)$ is *conservative* (potential). Note that equations (31) are then variational being Euler-Lagrange equations of a Lagrangian $L' = L - V$, where V is a potential for Φ. Hence, without loss of generality, and for simplicity of notations, let us consider to have a Lagrangian system on J^1Y, given by a Lagrangian λ. The nonholonomic equations of motion then obviously take one of the equivalent forms:

$$\bar{E}^{\mathrm{c}}_\lambda \circ J^2\gamma = 0\,, \tag{77}$$

$$J^1\gamma^* i_Z \iota^* d\theta_\lambda = 0 \quad \text{for every } \bar{\pi}_1\text{-vertical Chetaev vector field } Z \text{ on } Q, \tag{78}$$

$$\frac{\partial_c \bar{L}}{\partial q^s} - \frac{d_c}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^s} - \Big(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota\Big)\Big(\frac{\partial_c g^a}{\partial q^s} - \frac{d_c}{dt}\frac{\partial g^a}{\partial \dot{q}^s}\Big) = 0\,, \quad 1 \le s \le m-k\,. \tag{79}$$

Reduced equations for constrained Lagrangian systems (79) were first considered in [46], and are equivalent with Chetaev equations.

In [35] a *variational principle for systems subject to nonholonomic constraints* was found, providing the above reduced equations as equations for "constrained extremals". A generalization of the standard variational principle is in no case trivial or straightforward, and needs a careful review of basic variational concepts. Main points are as follows:

- The variational principle is formulated for the fibred manifold $\bar{\pi}_1 : Q \to \mathbb{R}$, endowed with the canonical distribution $\mathcal{C}$.
- "Admissible paths" are sections of the fibred manifold $\bar{\pi}_1 : Q \to \mathbb{R}$. (Note that they need not be holonomic, however, every admissible section δ has a counterpart in Y: it is a section γ of $\pi : Y \to \mathbb{R}$, given by $\gamma = \bar{\pi}_{1,0} \circ \delta$).
- "Admissible variations" are $\bar{\pi}_1$-projectable vector fields belonging to the canonical distribution (Chetaev vector fields). (Note that the requirement of projectability onto the base is essential, since variations of this kind provide a one-parametric family of maps that all are *sections* of the constraint manifold. Also note that the family of admissible sections $\delta_u = \phi_u\, \delta\, \phi_{0u}^{-1}$, arising by deformation of a holonomic section $\delta = J^1\gamma$, may contain non-holonomic sections (which is a violation of the "classical" principle of virtual displacements); moreover, the projection of the family $\{\delta_u\}$, i.e. the family of sections of π of the form $\gamma_u = \bar{\pi}_{1,0}\, \phi_u\, J^1\gamma\, \phi_{0u}^{-1}$ is not induced by a vector field on Y unless the canonical distribution is projectable (meaning that the constraints are affine in velocities)–however, even in this case, $J^1\gamma_u = (J^1\gamma)_u$ need not be true).
- The integrand of the action function (taking the place of a "constrained Lagrangian") is the 1-form $\iota^*\theta_\lambda$.

Definition 10. [35] Denote by $\mathcal{S}_{[a,b]}(\bar{\pi}_1)$ the set of sections of $\bar{\pi}_1$, defined around an interval $[a,b] \subset \mathbb{R}$, $a < b$. Given a Lagrangian λ on J^1Y, the function

$$\mathcal{S}_{[a,b]}(\bar{\pi}_1) \ni \delta \to \int_a^b \delta^* \iota^* \theta_\lambda \in \mathbb{R}\,, \tag{80}$$

is called *constrained* (*to* Q) *action function* of the Lagrangian λ over $[a,b]$.

Let $Z \in \mathcal{C}$ be a $\bar{\pi}_1$-projectable vector field, and denote by ϕ and ϕ_0 the flows of Z and its projection Z_0, respectively. The one-parameter family $\{\delta_u\}$ of sections of $\bar{\pi}_1$, where $\delta_u = \phi_u\, \delta\, \phi_{0u}^{-1}$, is called *constrained variation of* δ induced by Z. The function

$$\mathcal{S}_{[a,b]}(\bar{\pi}_1) \ni \delta \to \left(\frac{d}{du}\int_{\phi_{0u}([a,b])} \delta_u^*\, \iota^*\theta_\lambda\right)_{u=0} = \int_a^b \delta^*\, \mathcal{L}_Z \iota^*\theta_\lambda \in \mathbb{R} \tag{81}$$

is then called the *first constrained variation* of the action function of λ over $[a,b]$, *induced by* Z.

To study *constrained sections of the fibred manifold* π, we have to *restrict the domain of definition* $\mathcal{S}_{[a,b]}(\bar{\pi}_1)$ of the function (81) to the subset $\mathcal{S}^h_{[a,b]}(\bar{\pi}_1)$ of *holonomic* sections of the projection $\bar{\pi}_1$, i.e. $\delta = J^1\gamma$ where $\gamma \in \mathcal{S}_{[a,b]}(\pi)$. Then the first constrained variation (81) can be regarded as a function

$$\mathcal{S}_{[a,b],Q}(\pi) \ni \gamma \to \int_a^b J^1\gamma^* \, \mathcal{L}_Z \iota^* \theta_\lambda \in \mathbb{R} \tag{82}$$

defined on a *subset of sections of the projection* $\pi : Y \to \mathbb{R}$.

We stress that (due to the properties of admissible variations mentioned above) the restricted first constrained variation *cannot* be obtained via a "variation procedure" from an action defined directly on the set $\mathcal{S}_{[a,b],Q}(\pi)$.

Applying to (82) Cartan's formula for the decomposition of Lie derivative we obtain the *nonholonomic first variation formula*

$$\int_a^b J^1\gamma^* \, \mathcal{L}_Z \iota^* \theta_\lambda = \int_a^b J^1\gamma^* \, i_Z \iota^* d\theta_\lambda + \int_a^b J^1\gamma^* \, d i_Z \iota^* \theta_\lambda \, , \tag{83}$$

where Z is a $\bar{\pi}_1$-projectable Chetaev vector field.

Formula (83) gives the splitting of the first constrained variation to a "constrained Euler-Lagrange term" and a boundary term. One should notice that on the left-hand side of the nonholonomic first variation formula one cannot put the Lie derivative of the "constrained Lagrangian" $\bar{\lambda} = \iota^*\lambda$ instead of $\mathcal{L}_Z \iota^* \theta_\lambda$, since the difference $\mathcal{L}_Z \iota^* \theta_\lambda - \mathcal{L}_Z \bar{\lambda}$ need not be a contact form.

A section γ of $\pi : Y \to \mathbb{R}$ is called a *constrained extremal* of λ on $[a,b]$ if $\operatorname{Im} J^1\gamma \subset Q$, and if the first constraint variation of the action on the interval $[a,b]$ vanishes for every "fixed endpoints" variation Z over $[a,b]$. γ is called a *constrained extremal of* λ if it is its constrained extremal on every interval $[a,b] \subset \operatorname{Dom}\gamma$. With help of the nonholonomic first variation formula one proves that γ *is a constrained extremal of* λ *if and only if it satisfies equations of the constraint, and one of the (equivalent) equations* (77)–(79) [35]. Therefore we call any of these equations *nonholonomic Euler-Lagrange equations.*

Notice that for semiholonomic constraints equations (79) simplify to

$$\frac{\partial_c \bar{L}}{\partial q^s} - \frac{d_c}{dt}\frac{\partial \bar{L}}{\partial \dot{q}^s} = 0 \, , \quad 1 \le s \le m-k \, , \tag{84}$$

completely determined by the "constrained Lagrangian" $\bar{\lambda} = \iota^*\lambda$.

Similarly as in the unconstrained/holonomic case, the second term on the right-hand side of the nonholonomic first variation formula (83) is connected with conservation laws. Let us recall a generalization of Noether theorem to nonholonomic systems, due to [36]:

A Chetaev vector field $Z \in \mathcal{C}$ is called a *constrained symmetry* of a Lagrangian λ if $\mathcal{L}_Z \iota^* \theta_\lambda$ is a constraint form.

Directly from (83) we obtain:

Theorem 11. **(Nonholonomic Noether theorem)**

Let λ be a Lagrangian on J^1Y, and Z be a constrained symmetry of λ. Then along every constrained extremal of λ, the function $F = i_Z \iota^* \theta_\lambda = i_Z \theta_{\bar{\lambda}}$ is constant.

4.5 Regularity and Hamilton equations of nonholonomic systems

Consider a nonholonomic mechanical system (λ, Φ, Q). Equations (74) represent an important form of the nonholonomic motion equations, since they provide a representation in form of an *exterior differential system* (particularly, a *distribution*) *on the constraint* Q. More precisely, solutions of equations (74) are *holonomic* integral sections of the distribution $\Delta^{\mathrm{c}}_{\bar{\alpha}}$, locally annihilated by the system of 1-forms on Q,

$$\varphi^a, \quad i_Z \bar{\alpha}, \tag{85}$$

where $1 \leq a \leq k$, and Z runs over all vertical vector fields in $\mathcal{C}$, called *constrained dynamical distribution* (note that $\Delta^{\mathrm{c}}_{\bar{\alpha}}$ is a subdistribution of the canonical distribution $\mathcal{C}$). We shall call equations for (all) integral sections of the distribution $\Delta^{\mathrm{c}}_{\bar{\alpha}}$ *nonholonomic Hamilton equations* (cf. [32], [42] for Lagrangian systems). Note that in this context, the constraint manifold Q has the meaning of a genuine *evolution space* for the constrained system.

The constrained dynamical distribution need not have a constant rank, and even if the rank is constant it need not be equal to one. We say that the nonholonomic system $[\bar{\alpha}]$ is *regular* if $\operatorname{rank} \Delta^{\mathrm{c}}_{\bar{\alpha}} = 1$ [29]. From (76) we conclude that for a constrained mechanical system (λ, Φ, Q) the *regularity condition* reads

$$\det \left(\frac{\partial^2 \bar{L}}{\partial \dot{q}^r \partial \dot{q}^s} - \left(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota \right) \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s} \right) \neq 0 \,, \tag{86}$$

i.e. the matrix $(\bar{B}_{sr})$ (70) is regular, If the constrained system is regular then the distribution $\Delta^{\mathrm{c}}_{\bar{\alpha}}$ is locally spanned by one *vector field* (constrained semispray)

$$\zeta = \frac{\partial}{\partial t} + \sum_{l=1}^{m-k} \dot{q}^l \frac{\partial}{\partial q^l} + \sum_{a=1}^{k} g^a \frac{\partial}{\partial q^{m-k+a}} - \sum_{l,s=1}^{m-k} \bar{B}^{ls} \bar{A}_s \frac{\partial}{\partial \dot{q}^l} \,, \tag{87}$$

where $(\bar{B}^{ls})$ is the inverse matrix to $(\bar{B}_{ls})$ and $\bar{A}_s$ are given by (69), and the nonholonomic Hamilton equations are equivalent with the nonholonomic motion equations in Theorem 9.

Let us turn again to the case when the original mechanical system is Lagrangian. Then the nonholonomic Hamilton equations take the form

$$\begin{aligned} \delta^* i_Z \iota^* d\theta_\lambda = 0 \quad & \text{for every } \bar{\pi}_1\text{-vertical Chetaev vector field } Z \text{ on } Q, \\ \delta^* \varphi^a = 0 \,, \ \ & 1 \leq a \leq k \,. \end{aligned} \tag{88}$$

If the constrained system (λ, Q) is regular then the nonholonomic Hamilton equations are equivalent with the nonholonomic Euler-Lagrange equations ((77) or (78) or (79)). In this case we can introduce a *nonholonomic Legendre transformation* [51]:

Theorem 12. *Let $x \in Q$ be a point. Suppose that in a neighbourhood of x,*

$$\frac{\partial \bar{B}_{ls}}{\partial \dot{q}^r} = \frac{\partial \bar{B}_{lr}}{\partial \dot{q}^s}, \quad 1 \leq l, r, s \leq m-k\,. \tag{89}$$

Then there exists a neighbourhood $U \subset Q$ of x, and, on U, functions P_l, $1 \leq l \leq m-k$, and a 1-form η, such that

$$\iota^* d\theta_\lambda = \eta \wedge dt + dP_l \wedge dq^l + F\,, \tag{90}$$

where F is a 2-contact form on Q. If, moreover, the constrained system (λ, Q) is regular, then $(t, q^\sigma, \dot{q}^l) \to (t, q^\sigma, P_l)$ is a coordinate transformation on U.

The integrability condition for the $\bar{B}_{sl}$'s (89) ensures that one can express functions P_l explicitly. To this purpose we consider a mapping $\chi : [0,1] \times W \to W$ defined by $(u, t, q^\sigma, \dot{q}^l) \to (t, q^\sigma, u\dot{q}^l)$, where $W \subset Q$ is an appropriate open set. Then Poincaré Lemma gives us a solution

$$P_l = -\dot{q}^s \int_0^1 (\bar{B}_{ls} \circ \chi)\, du = \frac{\partial \bar{L}}{\partial \dot{q}^l} - \dot{q}^s \int_0^1 \left(\left(\frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota \right) \frac{\partial^2 g^a}{\partial \dot{q}^l \partial \dot{q}^s} \right) \circ \chi\, du\,. \tag{91}$$

We call the above functions P_l, $1 \leq l \leq k$, *nonholonomic momenta*, and the corresponding coordinate transformation *nonholonomic Legendre transformation* of λ. The 1-form η in (90) is called a *nonholonomic energy* 1*-form.*

The 1-form η is *determined up to a constraint* 1*-form*, and *need not be closed.* In constraint Legendre coordinates we can write

$$\eta = \eta_0\, dt + \eta_l\, dq^l + \eta^l\, dP_l \mod \mathcal{I}(\mathcal{C}^0)\,. \tag{92}$$

In nonholonomic Legendre coordinates the nonholonomic Hamilton equations take the following canonical form

$$\frac{d}{dt}(P_l \circ \delta) = \eta_l, \quad \frac{d}{dt}(q^l \circ \delta) = -\eta^l\,, \quad \frac{d}{dt}(q^{m-k+a} \circ \delta) = g^a\,, \tag{93}$$

where $1 \leq l \leq m-k$, $1 \leq a \leq k$.

For non-holonomic constraints *affine in velocities* the situation essentially simplifies: Indeed, then (89) is fulfilled identically and the nonholonomic momenta are defined by

$$P_l = \frac{\partial \bar{L}}{\partial \dot{q}^l}, \quad 1 \leq l \leq m-k\,. \tag{94}$$

The regularity condition takes the form

$$\det \left(\frac{\partial^2 \bar{L}}{\partial \dot{q}^l \partial \dot{q}^s} \right) \neq 0. \tag{95}$$

Moreover, if the constraint Q is semiholonomic then the family of energy 1-forms (92) contains a closed 1-form equal to $-d\bar{H}$, where

$$\bar{H} = -\bar{L} + P_l \dot{q}^l. \tag{96}$$

Acknowledgments

Research supported by grant GA 201/09/0981 of the Czech Science Foundation.

References

[1] P. Balseiro, J.C. Marrero, D. Martín de Diego, E. Padrón: A unified framework for mechanics: Hamilton-Jacobi equation and applications. *Nonlinearity* 23 (2010) 1887–1918.

[2] A.M. Bloch: *Nonholonomic Mechanics and Control.* Springer Verlag, New York (2003).

[3] A.M. Bloch, O.E. Fernandez, T. Mestdag: Hamiltonization of nonholonomic systems and the inverse problem of the calculus of variations. *Rep. Math. Phys.* 63 (2009) 225–249.

[4] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden, R.M. Murray: Nonholonomic systems with symmetry. *Arch. Ration. Mech. Anal.* 136 (1996) 21–99.

[5] F. Cantrijn, M. de León, J.C. Marrero, D. Martín de Diego: Reduction of constrained systems with symmetry. *J. Math. Phys.* 40 (1999) 795–820.

[6] F. Cardin, M. Favretti: On nonholonomic and vakonomic dynamics of mechanical systems with nonintegrable constraints. *J. Geom. Phys.* 18 (1996) 295–325.

[7] J.F. Cariñena, M.F. Rañada: Lagrangian systems with constraints: a geometric approach to the method of Lagrange multipliers. *J. Phys. A: Math. Gen.* 26 (1993) 1335–1351.

[8] H. Cendra, S. Ferraro, S. Grillo: Lagrangian reduction of generalized nonholonomic systems. *J. Geom. Phys.* 58 (2008) 1271–1290.

[9] N.G. Chetaev: On the Gauss principle. *Izv. Kazan. Fiz.-Mat. Obsc.* 6 (1932–33) 323–326. (in Russian)

[10] J. Cortés: *Geometric, Control and Numerical Aspects of Nonholonomic Systems.* Lecture Notes in Mathematics, Vol. 1793, Springer Verlag, New York (2002).

[11] J. Cortés, M. de León, J.C. Marrero, E. Martínez: Nonholonomic Lagrangian systems on Lie algebroids. *Discrete Contin. Dyn. Syst. A* 24 (2009) 213–271.

[12] J. Cortés, M. de León, D. Martín de Diego, S. Martínez: Geometric description of vakonomic and nonholonomic dynamics. Comparison of solutions. *SIAM J. Control Optim.* 41 (2003) 1389–1412.

[13] M. de León, D.M. de Diego: On the geometry of non-holonomic Lagrangian systems. *J. Math. Phys.* 37 (1996) 3389–3414.

[14] M. de León, J.C. Marrero, D.M. de Diego: Non-holonomic Lagrangian systems in jet manifolds. *J. Phys. A: Math. Gen.* 30 (1997) 1167–1190.

[15] G. Giachetta: Jet methods in nonholonomic mechanics. *J. Math. Phys.* 33 (1992) 1652–1665.

[16] H. Goldschmidt, S. Sternberg: The Hamilton-Cartan formalism in the calculus of variations. *Ann. Inst. Fourier* 23 (1973) 203–267.

[17] K. Grabowska, J. Grabowski: Variational calculus with constraints on general algebroids. *J. Phys. A: Math. Theor.* 41 (2008) No. 175204.

[18] H. Helmholtz: Ueber die physikalische Bedeutung des Prinzips der kleinsten Wirkung. *J. für die reine u. angewandte Math.* 100 (1887) 137–166.

[19] J. Janová, J. Musilová: Non-holonomic mechanics: A geometrical treatment of general coupled rolling motion. *Int. J. Non-Linear Mechanics* 44 (2009) 98–105.

[20] W.S. Koon, J.E. Marsden: The Hamiltonian and Lagrangian approaches to the dynamics of nonholonomic systems. *Reports on Math. Phys.* 40 (1997) 21–62.

[21] D. Krupka: Some geometric aspects of variational problems in fibered manifolds. *Folia Fac. Sci. Nat. UJEP Brunensis* 14 (1973) 1–65. Electronic transcription: arXiv:math-ph/0110005

[22] D. Krupka: A geometric theory of ordinary first order variational problems in fibered manifolds, II. Invariance. *J. Math. Anal. Appl.* 49 (1975) 469–476.

[23] D. Krupka: Global variational theory in fibred spaces. In: D. Krupka, D. Saunders (eds.): *Handbook of Global Analysis.* Elsevier (2008) 773–836.

[24] D. Krupka, O. Krupková, G. Prince, W. Sarlet: Contact symmetries of the Helmholtz form. *Diff. Geom. Appl.* 25 (2007) 518–542.

[25] D. Krupka, J. Musilová: Hamilton extremals in higher order mechanics. *Arch. Math. (Brno)* 20 (1984) 21–30.

[26] O. Krupková: Lepagean 2-forms in higher order Hamiltonian mechanics, I. Regularity. *Arch. Math. (Brno)* 22 (1986) 97–120.

[27] O. Krupková: Lepagean 2-forms in higher order Hamiltonian mechanics, II. Inverse Problem. *Arch. Math. (Brno)* 23 (1987) 155–170.

[28] O. Krupková: *The Geometry of Ordinary Variational Equations.* Lecture Notes in Mathematics, Vol. 1678. Springer Verlag, Berlin (1997).

[29] O. Krupková: Mechanical systems with nonholonomic constraints. *J. Math. Phys.* 38 (1997) 5098–5126.

[30] O. Krupková: On the geometry of non-holonomic mechanical systems. In: O. Kowalski, I. Kolář, D. Krupka, J. Slovák (eds.): *Differential Geometry and Applications.* Proc. Conf. Brno 1998. Masaryk Univ., Brno (1999) 533–546.

[31] O. Krupková: Higher-order mechanical systems with constraints. *J. Math. Phys.* 41 (2000) 5304–5324.

[32] O. Krupková: Differential systems in higher-order mechanics. In: D. Krupka (ed.): *Proceedings of the Seminar on Differential Geometry.* Mathematical Publications, Vol. 2. Silesian Univ., Opava (2000) 87–130.

[33] O. Krupková: Recent results in the geometry of constrained systems. *Reports on Math. Phys.* 49 (2002) 269–278.

[34] O. Krupková: Partial differential equations with differential constraints. *J. Differential Equations* 220 (2006) 354–395.

[35] O. Krupková: The nonholonomic variational principle. *J. Phys. A: Math. Theor.* 42 (2009) No. 185201.

[36] O. Krupková: Noether Theorem, 90 years on. In: *Geometry and Physics, Proc. XVII International Fall Workshop on Geometry and Physics, Castro Urdiales, Spain, 2008.* AIP Conf. Proceedings, American Institute of Physics, New York (2009) 159–170.

[37] O. Krupková: Variational Equations on Manifolds. In: A.R. Baswell (ed.): *Advances in Mathematics Research.* Vol. 9. Nova Science Publishers, New York (2009) 201–274.

[38] O. Krupková, J. Musilová: The relativistic particle as a mechanical systems with non-holonomic constraints. *J. Phys. A: Math. Gen.* 34 (2001) 3859–3875.

[39] O. Krupková, J. Musilová: Non-holonomic variational systems. *Reports on Math. Phys.* 55 (2005) 211–220.

[40] O. Krupková, G.E. Prince: Second Order Ordinary Differential Equations in Jet Bundles and the Inverse Problem of the Calculus of Variations. In: D. Krupka, D. Saunders (eds.): *Handbook of Global Analysis*. Elsevier (2008) 841–908.

[41] O. Krupková, P. Volný: Euler-Lagrange and Hamilton equations for non-holonomic systems in field theory. *J. Phys. A: Math. Gen.* 38 (2005) No. 8715.

[42] O. Krupková, P. Volný: Differential equations with constraits in jet bundles: Lagrangian and Hamiltonian systems. *Lobachevskii J. Math.* 23 (2006) 95–150.

[43] E. Massa, E. Pagani: A new look at classical mechanics of constrained systems. *Ann. Inst. Henri Poincaré* 66 (1997) 1–36.

[44] T. Mestdag, B. Langerock: A Lie algebroid framework for nonholonomic systems. *J. Phys. A: Math. Gen.* 38 (2005) 1097–1111.

[45] E. Noether: Invariante Variationsprobleme. *Nachr. kgl. Ges. Wiss. Göttingen, Math. Phys. Kl.* (1918) 235–257.

[46] W. Sarlet: A direct geometrical construction of the dynamics of non-holonomic Lagrangian systems. *Extracta Mathematicae* 11 (1996) 202–212.

[47] W. Sarlet, F. Cantrijn, D.J. Saunders: A geometrical framework for the study of non-holonomic Lagrangian systems. *J. Phys. A: Math. Gen.* 28 (1995) 3253–3268.

[48] D.J. Saunders: *The Geometry of Jet Bundles*. London Math. Soc. Lecture Notes Series, Vol. 142. Cambridge Univ. Press, Cambridge (1989).

[49] D.J. Saunders, W. Sarlet, F. Cantrijn: A geometrical framework for the study of non-holonomic Lagrangian systems II. *J. Phys. A.: Math. Gen.* 29 (1996) 4265–4274.

[50] M. Swaczyna: Several examples of nonholonomic mechanical systems. *Communications in Mathematics*, to appear.

[51] P. Volný, O. Krupková: Hamilton equations for non-holonomic mechanical systems. In: O. Kowalski, D. Krupka, J. Slovák (eds.): *Differential Geometry and Its Applications*. Proc. Conf., Opava, 2001. Mathematical Publications, Vol. 3. Silesian Univ., Opava, Czech Republic (2001) 369–380.

Author's address:

Department of Mathematics, Faculty of Science, The University of Ostrava, 30. dubna 22, 701 03 Ostrava, Czech Republic

Department of Mathematics and Statistics, La Trobe University, Melbourne, Victoria 3086, Australia

E-mail address: olga.krupkova@osu.cz, O.Krupkova@latrobe.edu.au

Received: 3 December, 2010

Accepted for publication: 10 December, 2010

Communications in Mathematics 18 (2010) 79–82

Book review

Geoff Prince

Classical Mechanics: Hamiltonian and Lagrangian Formalism by Alexei Deriglazov. Springer (2010), 308 pages, ISBN 978-3-642-14036-5, e-ISBN 978-3-642-14037-2.

Many modern books on classical mechanics are coloured by other areas of mathematical or theoretical physics. Quantum mechanics, quantum field theory, continuum mechanics and special and general relativity all exert their influence. On the positive side this means that the fundamental ideas of force, inertia and inertial frames of reference, often neglected by mathematicians, are all thoroughly explored. On the negative side it can lead to undue dependence on the historical development of the parts of the subject which the author doesn't favour. For example, in V.I. Arnold's famous *Mathematical Methods of Classical Mechanics* differential forms are not introduced until after Lagrangian dynamics is treated. Although the title of the work under review indicates a study of both the post Newtonian formalisms in mechanics the author makes it clear in the introduction that he prefers the Hamiltonian framework, not least because of its role in quantum theory. As a result Lagrangian dynamics is unfavourably compared to Hamiltonian mechanics and much of its modern formulation is untouched. In writing this review I will try and indicate some of the current trends in the Lagrangian theory.

So classical mechanics is one of those areas having multiple ownership. This can be productive because the subject has inputs from many areas which should stimulate cross fertilisation. On the other hand it has inhibited mathematicians from developing the subject as their own. We have all had the experience of learning classical mechanics as a stand alone subject with many idiosyncratic methods, not bearing any resemblance to subjects like linear algebra which have an axiomatic basis and a body of theorems applicable to a wide range of situations. Our undergraduate experience of the subject constrains us from seeing it as an area in which the beautiful theory of ordinary differential equations due to Lie, Cartan and others applies. And of course the multiple ownership of the subject will forever prevent us from teaching it as such. Nonetheless, we should at least attempt to see the differential equations aspect of classical mechanics in this light.

This book has been developed from lectures aimed at graduate students in theoretical physics, but implicitly at those with an interest in quantum theory. This may be the reason for the very limited use of differential geometric ideas, especially those of exterior calculus and the theory of connections. These two components of the calculus of manifolds are central to many of the 20^{th} century developments in Lagrangian mechanics. The reader can get a sense of the current situation (albeit that of the reviewer and this journal's editor-in-chief) from the chapter *Second Order Ordinary Differential Equations in Jet Bundles and the Inverse Problem of the Calculus of Variations* in *Handbook of Global Analysis*, edited by D. Krupka and D. Saunders, Elsevier (2007). I will describe two of these developments: Noether's theorem and progress in the inverse problem in the calculus of variations. References can be found in the aforementioned article.

The central object in the modern Lagrangian theory in one independent variable, t, and n dependent ones, x^a, is the *Cartan two-form*. For a given non-degenerate Lagrangian this form is

$$d\theta_L = d\left(Ldt + \frac{\partial L}{\partial u^a}(dx^a - u^a dt)\right) = \frac{\partial^2 L}{\partial u^a u^b}(du^a - f^a dt) \wedge (dx^b - u^b dt).$$

Here (t, x^a, u^a) are local co-ordinates on the evolution space, $E := \mathbb{R} \times TM$, M being the configuration manifold, and the Euler-Lagrange equations in normal form are

$$\ddot{x}^a = f^a(t, x^b, u^b).$$

Of course this two-form has a geometric definition which can be found in the literature, but its most important intrinsic property is that it has a one-dimensional kernel spanned by the semi spray, known as the Euler-Lagrange field,

$$\Gamma = \frac{\partial}{\partial t} + u^a \frac{\partial}{\partial x^a} + f^a \frac{\partial}{\partial u^a}.$$

Noether's theorem in this setting relates a non-trivial symmetry, $X \in \mathfrak{X}(E)$, of $d\theta_L$ to a non-trivial first integral, F, of Γ (so that $\Gamma(F) = 0$):

$$\mathcal{L}_X d\theta_L = 0 \iff X \lrcorner\, d\theta_L = dF.$$

This relation between X and F fixes X up to a multiple of Γ giving a converse to Noether's theorem which is not available if we restrict ourselves to so-called point symmetries. This remarkably simple approach to the famous theorem should be contrasted to the lengthy account given in Deriglazov's book in which the converse to the theorem is discovered in the Hamiltonian context and pulled back to the Lagrangian picture by the Legendre transformation without reference to the point symmetry issue.

The inverse problem in the calculus of variations is the problem of identifying all, if any, Lagrangians whose Euler-Lagrange field is a given semi spray Γ. The Fields medallist Jesse Douglas solved this problem for $n = 2$ in 1941. While special cases have been solved for arbitrary n the solution for $n = 3$, for example, has not yet been produced. Douglas himself, undoubtedly a modest man, said "*the problem ...... is one of the most important hitherto unsolved problems of the calculus of variations*". Apart from its intrinsic value this problem has given birth to deep results on second order differential equations. The question is of interest to physicists because if a problem admits more than one Lagrangian it may admit more than one quantisation, not all of which are equivalent.

The theorem which geometrises the *Helmholtz conditions* (due to Douglas) is

Theorem 1. *Given a semi spray, Γ, necessary and sufficient conditions for the existence of a regular Lagrangian, whose Euler-Lagrange field is Γ, are that there exists $\Omega \in \bigwedge^2(E)$:*

1. *Ω has maximal rank*
2. *$\Omega(V_1, V_2) = 0, \ \forall \ V_1, V_2 \in V(E)$, the vertical sub-bundle on E*
3. *$\Gamma \lrcorner \, \Omega = 0$*
4. *$d\Omega = 0$*

The Lagrangians are recovered from the observation that Ω is a Cartan two-form $d\theta_L$. The inverse problem is not touched upon in the book under review.

This is a comprehensive book in its own way. The chapter headings are: 1. Sketch of Lagrangian Formalism, 2. Hamiltonian Formalism, 3. Canonical Transformations of Two-Dimensional Phase Space, 4. Properties of Canonical Transformations, 5. Integral Invariants, 6. Potential Motion in a Geometric Setting, 7. Transformations, Symmetries and Noether Theorem, 8. Hamiltonian Formalism for Singular Theories. However, and as indicated earlier, the mathematical setting is not modern and the influences lie in theoretical physics outside classical mechanics. For example, of the 50 references only 10 are post the year 2000 and of those 7 are works of the author and the other 3 lie outside classical mechanics. Nonetheless, it provides interesting reading and the detailed level of discussion reflects the extensive nature of the graduate lecture course on which the book is based. For example, the discussion of Dirac's theory of constraints in chapter 8 is quite deep and provides a natural end point of the author's development of the Hamiltonian and Lagrangian frameworks. Special relativity and quantum mechanics are both represented through the examples and the quasi-Riemannian geometric formulation is developed in chapter 6 in the context of the Principle of Maupertius. Rather surprisingly this formulation of the Newtonian equations of motion as quasi-geodesic equations is developed without reference to general relativity or to Cartan's formulation of the Newtonian equations as the auto-parallel equations of a non-metric affine connection.

There are informative and serious exercises scattered throughout the text although not as many as one would find in an undergraduate text on the subject. However, I believe that the approach here is too idiosyncratic for the book to be widely accepted as a basis for an advanced course on classical mechanics. The situation is not improved by the personal mathematical style adopted, albeit consistently, by the author for dealing with the co-ordinate transformations which abound in classical mechanics along with the use of a non-standard mathematical vocabulary.

Author's address:
Department of Mathematics and Statistics, La Trobe University, Melbourne, Victoria 3086, Australia

E-mail address: G.Prince@latrobe.edu.au

Aims and Scope

Communications in Mathematics publishes original research papers and survey papers in pure and applied mathematics, preferably in the following areas: *algebraic structures*, *calculus of variations*, *combinatorics*, *control and optimization*, *cryptography*, *differential equations*, *differential geometry*, *fuzzy logic and fuzzy set theory*, *global analysis*, *history of mathematics*, *mathematical physics*, *number theory*.

To be acceptable for publication papers must make original and significant contributions to one or more of the topics within these areas. High quality review papers of interest to a wide range of scientists in mathematics and its applications are welcome.

Instructions for Authors

Articles submitted for publication must be written in English, and produced in TeX, preferably using Publisher's document style based upon LaTeX.

The first page should contain the title, authors' names and addresses, full contact details of the corresponding author, the abstract, keywords, and 2010 MS classification numbers.

Footnotes should be avoided if possible, but if they are included they should be brief, and numbered consecutively.

References must be as complete as possible and put in alphabetical order.

Submission to the journal proceeds totally on-line through the web page `http://cm.osu.cz`, and the author is guided stepwise through the procedure. To complete the submission successfully, the author is required to choose one of the Editors to handle the paper. He/she should choose an Editor whose areas fit as closely as possible with the topic of the paper. All correspondence, including notification of the Editor's decision, requests for revision, and proofs takes place by e-mail.

Communications in Mathematics, Volume 18, 2010, is the continuation of the journal *Acta Mathematica et Informatica Universitatis Ostraviensis*, ISSN 1211-4774 (1993–2003), and of the journal *Acta Mathematica Universitatis Ostraviensis*, ISSN 1214-8148 (2004–2009).

Published by The University of Ostrava
Typeset by TeX

ISSN 1804-1388
ISBN 978-80-7368-958-2

Geometric Mechanics and Global Calculus of Variations

Edited by Olga Krupková and Geoff Prince

Contents

Editorial

From the Editor-in-Chief 1

Research papers

David Saunders: Some geometric aspects of the calculus of variations in several independent variables 3

Yong-Xin Guo, Chang Liu and Shi-Xing Liu: Generalized Birkhoffian realization of nonholonomic systems 21

Mike Crampin: Homogeneous systems of higher-order ordinary differential equations 37

Survey paper

Olga Krupková: Geometric mechanics on nonholonomic submanifolds . 51

Book review

Geoff Prince: Classical Mechanics: Hamiltonian and Lagrangian Formalism by Alexei Deriglazov 79