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Partial Fuzzy Metric Space and Some Fixed Point Results

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Abstract. In this paper, we introduce the concept of partial fuzzy metric on a nonempty set X and give the topological structure and some properties of partial fuzzy metric space. Then some fixed point results are provided.

1 Introduction and preliminaries

We recall some basic definitions and results from the theory of fuzzy metric spaces, used in the sequel.

Definition 1. [5] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* if it satisfies the following conditions:

1. $*$ is associative and commutative,
2. $*$ is continuous,
3. $a * 1 = a$ for all $a \in [0,1]$,
4. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norms are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 2. [1] A triple $(X, M, *)$ is called a *fuzzy metric space* (in the sense of George and Veeramani) if X is a nonempty set, $*$ is a continuous t-norm and $M : X^2 \times (0,\infty) \to [0,1]$ is a fuzzy set satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1 \Leftrightarrow x = y$,

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3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$,
5. $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is a continuous mapping

If the fourth condition is replaced by

4′. $M(x, z, \max\{t, s\}) \geq M(x, y, t) * M(y, z, s)$,

then the space $(X, M, *)$ is said to be a *non-Archimedean fuzzy metric space*. It should be noted that any non-Archimedean fuzzy metric space is a fuzzy metric space.

The following properties of M noted in the theorem below are easy consequences of the definition.

Theorem 1. *Let $(X, M, *)$ be a fuzzy metric space.*

1. *$M(x, y, t)$ is nondecreasing with respect to t for each $x, y \in X$,*
2. *If M is non-Archimedean, then $M(x, y, t) \geq M(x, z, t) * M(z, y, t)$ for all $x, y, z \in X$ and $t > 0$.*

Example 1. Let (X, d) be an ordinary metric space and $a*b = ab$ for all $a, b \in [0, 1]$. Then the fuzzy set M on $X^2 \times (0, \infty)$ defined by

$$M(x, y, t) = \exp\left(-\frac{d(x, y)}{t}\right),$$

is a fuzzy metric on X.

Example 2. Let $a * b = ab$ for all $a, b \in [0, 1]$ and M be the fuzzy set on $\mathbb{R}^+ \times \mathbb{R}^+ \times (0, \infty)$ (where $\mathbb{R}^+ = (0, \infty)$) defined by

$$M(x, y, t) = \frac{\min\{x, y\}}{\max\{x, y\}},$$

for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, M, *)$ is a fuzzy metric space.

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with centre $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. This definition of Cauchy sequence is identical with that given by George and Veeramani.

The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent.

The fixed point theory in fuzzy metric spaces started with the paper of Grabiec [2]. Later on, the concept of fuzzy contractive mappings, initiated by Gregori and Sapena in [3], have become of interest for many authors, see, e.g., the papers [3], [7], [8], [9], [10], [11].

In our paper we present the concept of partial fuzzy metric space and some properties of it. Then we give some fundamental fixed point theorem on complete partial fuzzy metric space.

2 Partial fuzzy metric space

In this section we introduce the concept of partial fuzzy metric space and give its properties.

Definition 3. A *partial fuzzy metric* on a nonempty set X is a function

$$P_M : X \times X \times (0, \infty) \to [0, 1]$$

such that for all $x, y, z \in X$ and $t, s > 0$

(PM1) $x = y \Leftrightarrow P_M(x, x, t) = P_M(x, y, t) = P_M(y, y, t)$,

(PM2) $P_M(x, x, t) \geq P_M(x, y, t)$,

(PM3) $P_M(x, y, t) = P_M(y, x, t)$,

(PM4) $P_M(x, y, \max\{t, s\}) * P_M(z, z, \max\{t, s\}) \geq P_M(x, z, t) * P_M(z, y, s)$.

(PM5) $P_M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

A partial fuzzy metric space is a 3-tuple $(X, P_M, *)$ such that X is a nonempty set and P_M is a partial fuzzy metric on X. It is clear that, if $P_M(x, y, t) = 1$, then from (PM1) and (PM2) $x = y$. But if $x = y$, $P_M(x, y, t)$ may not be 1. A basic example of a partial fuzzy metric space is the 3-tuple $(\mathbb{R}^+, P_M, *)$, where

$$P_M(x, y, t) = \frac{t}{t + \max\{x, y\}}$$

for all $t > 0$, $x, y \in \mathbb{R}^+$ and $a * b = ab$.

From (PM4) for all $x, y, z \in X$ and $t > 0$, we have:

$$P_M(x, y, t) * P_M(z, z, t) \geq P_M(x, z, t) * P_M(z, y, t).$$

Let $(X, M, *)$ and $(X, P_M, *)$ be a fuzzy metric space and partial fuzzy metric space, respectively. Then mappings $P_{M_i} : X \times X \times (0, \infty) \to [0, 1]$ $(i \in \{1, 2\})$ defined by

$$P_{M_1}(x, y, t) = M(x, y, t) * P_M(x, y, t)$$

and

$$P_{M_2}(x, y, t) = M(x, y, t) * a$$

are partial fuzzy metrics on X, where $0 < a < 1$.

Theorem 2. *The partial fuzzy metric $P_M(x, y, t)$ is nondecreasing with respect to t for each $x, y \in X$ and $t > 0$, if the continuous t-norm $*$ satisfies the following condition for all $a, b, c \in [0, 1]$*

$$a * b \geq a * c \Rightarrow b \geq c.$$

Proof. From (PM4) for all $x, y, z \in X$ and $t, s > 0$, we have:

$$P_M(x, y, \max\{t, s\}) * P_M(z, z, \max\{t, s\}) \geq P_M(x, z, s) * P_M(z, y, t).$$

Let $t > s$, then taking $z = y$ in above inequality we have

$$P_M(x, y, t) * P_M(y, y, t) \geq P_M(x, y, s) * P_M(y, y, t),$$

hence by assume we get $P_M(x, y, t) \geq P_M(x, y, s)$. □

It is easy to see that every fuzzy metric is a partial fuzzy metric, but the converse may not be true. In the following examples, the partial fuzzy metrics fails to satisfy properties of fuzzy metric.

Example 3. Let (X, p) is a partial metric space in the sense of Matthews [6] and $P_M : X \times X \times (0, \infty) \to [0, 1]$ be a mapping defined as

$$P_M(x, y, t) = \frac{t}{t + p(x, y)},$$

or

$$P_M(x, y, t) = \exp\left(-\frac{p(x, y)}{t}\right).$$

If $a * b = ab$ for all $a, b \in [0, 1]$, then clearly P_M is a partial fuzzy metric, but it may not be a fuzzy metric.

Lemma 1. *Let $(X, P_M, *)$ be a partial fuzzy metric space with $a * b = ab$ for all $a, b \in [0, 1]$. If we define $p : X^2 \to [0, \infty)$ by*

$$p(x, y) = \sup_{\alpha \in (0,1)} \int_{\alpha}^{1} \log_a(P_M(x, y, t))\, \mathrm{d}t,$$

then p is a partial metric on X for fixed $0 < a < 1$.

Proof. It is clear from the definition that $p(x, y)$ is well defined for each $x, y \in X$ and $p(x, y) \geq 0$ for all $x, y \in X$.

1. For all $t > 0$

$$p(x, x) = p(x, y) = p(y, y) \Leftrightarrow P_M(x, x, t) = P_M(x, y, t) = P_M(y, y, t) \Leftrightarrow x = y.$$

2.
$$\begin{aligned} p(x, x) &= \sup_{\alpha \in (0,1)} \int_{\alpha}^{1} \log_a(P_M(x, x, t))\, \mathrm{d}t \\ &\leq \sup_{\alpha \in (0,1)} \int_{\alpha}^{1} \log_a(P_M(x, y, t))\, \mathrm{d}t \\ &= p(x, y). \end{aligned}$$

3.
$$\begin{aligned} p(x,y) &= \sup_{\alpha\in(0,1)} \int_{\alpha}^{1} \log_a(P_M(x,y,t))\,\mathrm{d}t \\ &= \sup_{\alpha\in(0,1)} \int_{\alpha}^{1} \log_a(P_M(y,x,t))\,\mathrm{d}t \\ &= p(y,x). \end{aligned}$$

4. Since
$$P_M(x,y,t)P_M(z,z,t) \geq P_M(x,z,t)P_M(z,y,t),$$
and $\log_a$ is decreasing, it follows that
$$\log_a(P_M(x,y,t)) + \log_a(P_M(z,z,t)) \leq \log_a(P_M(x,z,t)) + \log_a(P_M(z,y,t)),$$
hence
$$\begin{aligned} p(x,y)+p(z,z) &= \sup_{\alpha\in(0,1)} \int_{\alpha}^{1} \log_a(P_M(x,y,t))\,\mathrm{d}t + \sup_{\alpha\in(0,1)} \int_{\alpha}^{1} \log_a(P_M(z,z,t))\,\mathrm{d}t \\ &\leq \sup_{\alpha\in(0,1)} \int_{\alpha}^{1} \log_a(P_M(x,z,t))\,\mathrm{d}t + \sup_{\alpha\in(0,1)} \int_{\alpha}^{1} \log_a(P_M(z,y,t))\,\mathrm{d}t \\ &= p(x,z)+p(z,y). \end{aligned}$$
This proves that p is a partial metric on X. □

Definition 4. Let $(X, P_M, *)$ be a partial fuzzy metric space.

1. A sequence $\{x_n\}$ in a partial fuzzy metric space $(X, P_M, *)$ *converges* to x if and only if $P_M(x,x,t) = \lim_{n\to\infty} P_M(x_n,x,t)$ for every $t > 0$.
2. A sequence $\{x_n\}$ in a partial fuzzy metric space $(X, P_M, *)$ is called a *Cauchy sequence* if $\lim_{n,m\to\infty} P_M(x_n,x_m,t)$ exists.
3. A partial fuzzy metric space $(X, P_M, *)$ is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$.

Suppose that $\{x_n\}$ is a sequence in partial fuzzy metric space $(X, P_M, *)$, then we define $L(x_n) = \{x \in X : x_n \to x\}$. In the following example shows that every convergent sequence $\{x_n\}$ in a partial fuzzy metric space $(X, P_M, *)$ fails to satisfy Cauchy sequence. In particular, it shows that the limit of a convergent sequence is not unique.

Example 4. Let $X = [0,\infty)$ and $P_M(x,y,t) = \frac{t}{t+\max\{x,y\}}$, then it is clear that $(X, P_M, *)$ is a partial fuzzy metric space where $a * b = ab$ for all $a, b \in [0,1]$. Let $\{x_n\} = \{1,2,1,2,\dots\}$. Then clearly it is convergent sequence and for every $x \geq 2$ we have
$$\lim_{n\to\infty} P_M(x_n,x,t) = P_M(x,x,t),$$
therefore
$$L(x_n) = \{x \in X : x_n \to x\} = [2,\infty).$$
but $\lim_{n,m\to\infty} P_M(x_n,x_m,t)$ is not exist, that is, $\{x_n\}$ is not Cauchy sequence.

The following Lemma shows that under certain conditions the limit of a convergent sequence is unique.

Lemma 2. *Let $\{x_n\}$ be a convergent sequence in partial fuzzy metric space $(X, P_M, *)$ such that $a * b \geq a * c \Rightarrow b \geq c$ for all $a, b, c \in [0, 1]$, $x_n \to x$ and $x_n \to y$. If*

$$\lim_{n\to\infty} P_M(x_n, x_n, t) = P_M(x, x, t) = P_M(y, y, t),$$

then $x = y$.

Proof. As

$$P_M(x, y, t) * P_M(x_n, x_n, t) \geq P_M(x, x_n, t) * P_M(y, x_n, t),$$

taking limit as $n \to \infty$, we have

$$P_M(x, y, t) * P_M(x, x, t) \geq P_M(x, x, t) * P_M(y, y, t).$$

By given assumptions and from (PM2), we have

$$P_M(y, y, t) \geq P_M(x, y, t) \geq P_M(y, y, t),$$

which shows that $P_M(x, y, t) = P_M(y, y, t) = P_M(x, x, t)$, therefore $x = y$. □

Lemma 3. *Let $\{x_n\}$ and $\{y_n\}$ be two sequences in partial fuzzy metric space $(X, P_M, *)$ such that $a * b \geq a * c \Rightarrow b \geq c$ for all $a, b, c \in [0, 1]$,*

$$\lim_{n\to\infty} P_M(x_n, x, t) = \lim_{n\to\infty} P_M(x_n, x_n, t) = P_M(x, x, t),$$

and

$$\lim_{n\to\infty} P_M(y_n, y, t) = \lim_{n\to\infty} P_M(y_n, y_n, t) = P_M(y, y, t),$$

then $\lim_{n\to\infty} P_M(x_n, y_n, t) = P_M(x, y, t)$. In particular, for every $z \in X$

$$\lim_{n\to\infty} P_M(x_n, z, t) = \lim_{n\to\infty} P_M(x, z, t).$$

Proof. As

$$P_M(x_n, y_n, t) * P_M(x, x, t) \geq P_M(x_n, x, t) * P_M(x, y_n, t),$$

therefore

$$\begin{aligned} P_M(x_n, y_n, t) * P_M(x, x, t) * P_M(y, y, t) &\geq P_M(x_n, x, t) * P_M(x, y_n, t) * P_M(y, y, t) \\ &\geq P_M(x_n, x, t) * P_M(x, y, t) * P_M(y, y_n, t). \end{aligned}$$

Thus

$$\begin{aligned} &\limsup_{n\to\infty} P_M(x_n, y_n, t) * P_M(x, x, t) * P_M(y, y, t) \\ &\qquad \geq \limsup_{n\to\infty} P_M(x_n, x, t) * P_M(x, y, t) * \limsup_{n\to\infty} P_M(y, y_n, t) \\ &\qquad = P_M(x, x, t) * P_M(x, y, t) * P_M(y, y, t), \end{aligned}$$

hence

$$\limsup_{n\to\infty} P_M(x_n, y_n, t) \geq P_M(x, y, t).$$

Also, as

$$P_M(x, y, t) * P_M(x_n, x_n, t) \geq P_M(x, x_n, t) * P_M(x_n, y, t),$$

therefore

$$\begin{aligned} P_M(x, y, t) * P_M(x_n, x_n, t) * P_M(y_n, y_n, t) \\ &\geq P_M(x, x_n, t) * P_M(x_n, y, t) * P_M(y_n, y_n, t) \\ &\geq P_M(x, x_n, t) * P_M(x_n, y_n, t) * P_M(y_n, y, t) \end{aligned}$$

Thus

$$\begin{aligned} &P_M(x, y, t) * P_M(x, x, t) * P_M(y, y, t) \\ &\quad = P_M(x, y, t) * \limsup_{n\to\infty} P_M(x_n, x_n, t) * \limsup_{n\to\infty} P_M(y_n, y_n, t) \\ &\quad \geq \limsup_{n\to\infty} P_M(x, x_n, t) * \limsup_{n\to\infty} P_M(x_n, y_n, t) * \limsup_{n\to\infty} P_M(y_n, y, t) \\ &\quad = P_M(x, x, t) * \limsup_{n\to\infty} P_M(x_n, y_n, t) * P_M(y, y, t). \end{aligned}$$

Therefore

$$P_M(x, y, t) \geq \limsup_{n\to\infty} P_M(x_n, y_n, t).$$

That is,

$$\limsup_{n\to\infty} P_M(x_n, y_n, t) = P_M(x, y, t).$$

Similarly, we have

$$\limsup_{n\to\infty} P_M(x_n, y_n, t) = P_M(x, y, t).$$

Hence the result follows. □

Definition 5. Let $(X, P_M, *)$ be a partial fuzzy metric space. P_M is said to be upper semicontinuous on X if for every $x \in X$,

$$P_M(p, x, t) \geq \limsup_{n\to\infty} P_M(x_n, x, t),$$

whenever $\{x_n\}$ is a sequence in X which converges to a point $p \in X$.

3 Fixed point results

Let $(X, P_M, *)$ be a partial fuzzy metric space and $\emptyset \neq S \subseteq X$. Define

$$\delta_{P_M}(S, t) = \inf\{P_M(x, y, t) : x, y \in S\}$$

for all $t > 0$. For an $A_n = \{x_n, x_{n+1}, \dots\}$ in partial fuzzy metric space $(X, P_M, *)$, let $r_n(t) = \delta_{P_M}(A_n, t)$. Then $r_n(t)$ is finite for all $n \in \mathbb{N}$, $\{r_n(t)\}$ is nonincreasing, $r_n(t) \to r(t)$ for some $0 \leq r(t) \leq 1$ and also $r_n(t) \leq P_M(x_l, x_k, t)$ for all $l, k \geq n$.

Let $\mathcal{F}$ be the set of all continuous functions $F : [0,1]^3 \times [0,1] \to [-1,1]$ such that F is nondecreasing on $[0,1]^3$ satisfying the following condition:

- $F((u,u,u),v) \leq 0$ implies that $v \geq \gamma(u)$ where $\gamma : [0,1] \to [0,1]$ is a nondecreasing continuous function with $\gamma(s) > s$ for $s \in [0,1)$.

Example 5. Let $\gamma(s) = s^h$ for $0 < h < 1$, then the functions F defined by

$$F\big((t_1,t_2,t_3),t_4\big) = \gamma\big(\min\{t_1,t_2,t_3\}\big) - t_4$$

and

$$F\big((t_1,t_2,t_3),t_4\big) = \gamma\left(\sum_{i=1}^{3} a_i t_i\right) - t_4,$$

where $a_i \geq 0$, $\sum\limits_{i=1}^{3} a_i = 1$, belong to $\mathcal{F}$.

Now we give our main theorem.

Theorem 3. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space, P_M is upper semicontinuous function on X and T be a self map of X satisfying*

$$F\big(P_M(x,y,t), P_M(Tx,x,t), P_M(Tx,y,t), P_M(Tx,Ty,t)\big) \leq 0 \tag{1}$$

for all $x, y \in X$, where $F \in \mathcal{F}$. Then T has a unique fixed point p in X and T is continuous at p.

Proof. Let $x_0 \in X$ and $Tx_n = x_{n+1}$. Let $r_n(t) = \delta_{P_M}(A_n,t)$, where $A_n = \{x_n, x_{n+1}, \dots\}$. Then we know $\lim\limits_{n\to\infty} r_n(t) = r(t)$ for some $0 \leq r(t) \leq 1$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then T has a fixed point. Assume that $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ be fixed. Taking $x = x_{n-1}$, $y = x_{n+m-1}$ in (1) where $n \geq k$ and $m \in \mathbb{N}$, we have

$$\begin{aligned} F&\begin{pmatrix} P_M(x_{n-1}, x_{n+m-1}, t), P_M(Tx_{n-1}, x_{n-1}, t), \\ P_M(Tx_{n-1}, x_{n+m-1}, t), P_M(Tx_{n-1}, Tx_{n+m-1}, t) \end{pmatrix} \\ &= F\begin{pmatrix} P_M(x_{n-1}, x_{n+m-1}, t), P_M(x_n, x_{n-1}, t), \\ P_M(x_n, x_{n+m-1}, t), P_M(x_n, x_{n+m}, t) \end{pmatrix} \leq 0 \end{aligned}$$

Thus we have

$$F\big(r_{n-1}(t), r_{n-1}(t), r_n(t), P_M(x_n, x_{n+m}, t)\big) \leq 0,$$

since F is nondecreasing on $[0,1]^3$. Also, since $r_n(t)$ is nonincreasing, we have

$$F\big(r_{k-1}(t), r_{k-1}(t), r_{k-1}(t), P_M(x_n, x_{n+m}, t)\big) \leq 0,$$

which implies that

$$P_M(x_n, x_{n+m}, t) \geq \gamma(r_{k-1}(t)).$$

Thus for all $n \geq k$, we have

$$\inf_{n\geq k}\big\{P_M(x_n, x_{n+m}, t)\big\} = r_k(t) \geq \gamma(r_{k-1}(t)).$$

Letting $k \to \infty$, we get $r(t) \geq \gamma(r(t))$. If $r(t) \neq 1$, then $r(t) \geq \gamma(r(t)) > r(t)$, which is a contradiction. Thus $r(t) = 1$ and hence $\lim_{n\to\infty} \gamma_n(t) = 1$. Thus given $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $r_n(t) > 1 - \varepsilon$. Then we have for $n \geq n_0$ and $m \in \mathbb{N}$, $P_M(x_n, x_{n+m}, t) > 1 - \varepsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists a $p \in X$ such that

$$\lim_{n\to\infty} P_M(x_n, p, t) = P_M(p, p, t).$$

Taking $x = x_n, y = p$ in (1), we have

$$\begin{aligned}&F\big(P_M(x_n, p, t), P_M(Tx_n, p, t), P_M(Tx_n, x_n, t), P_M(Tx_n, Tp, t)\big)\\ &\qquad = F\big(P_M(x_n, p, t), P_M(x_{n+1}, p, t), P_M(x_{n+1}, x_n, t), P_M(x_{n+1}, Tp, t)\big) \leq 0.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\limsup_{n\to\infty} F\big(P_M(x_n, p, t), P_M(x_{n+1}, p, t), P_M(x_{n+1}, x_n, t), P_M(x_{n+1}, Tp, t)\big)\\ &\qquad = F\big(P_M(p, p, t), P_M(p, p, t), 1, \limsup_{n\to\infty} P_M(x_{n+1}, Tp, t)\big) \leq 0.\end{aligned}$$

Since

$$\begin{aligned}&F\big(P_M(p, p, t), P_M(p, p, t), P_M(p, p, t), \limsup_{n\to\infty} P_M(x_{n+1}, Tp, t)\big)\\ &\qquad \leq F\big(P_M(p, p, t), P_M(p, p, t), 1, \limsup_{n\to\infty} P_M(x_{n+1}, Tp, t)\big) \leq 0,\end{aligned}$$

which implies

$$P_M(p, Tp, t) \geq \limsup_{n\to\infty} P_M(x_{n+1}, Tp, t) \geq \gamma(P_M(p, p, t)).$$

On the other hand, we have

$$P_M(p, p, t) \geq P_M(p, Tp, t) \geq \gamma(P_M(p, p, t)).$$

Hence $P_M(p, p, t) = 1$. Also, since

$$P_M(p, Tp, t) \geq \gamma(P_M(p, p, t)) = \gamma(1) = 1,$$

this implies that $P_M(p, Tp, t) = 1$, therefore, we get $Tp = p$.

For the uniqueness, let p and w be fixed points of T. Taking $x = p, y = w$ in (1), we have

$$\begin{aligned}&F\big(P_M(p, w, t), P_M(Tp, p, t), P_M(Tp, w, t), P_M(Tp, Tw, t)\big)\\ &\qquad = F\big(P_M(p, w, t), P_M(p, p, t), P_M(p, w, t), P_M(p, w, t)\big) \leq 0.\end{aligned}$$

Since F is nondecreasing on $[0,1]^3$, we have

$$F\big(P_M(p,w,t), P_M(p,w,t), P_M(p,w,t), P_M(p,w,t)\big) \le 0,$$

which implies

$$P_M(p,w,t) \ge \gamma(P_M(p,w,t)) > P_M(p,w,t)$$

which is a contradiction. Thus we have $P_M(p,w,t) = 1$, therefore, $p = w$. Now, we show that T is continuous at p. Let $\{y_n\}$ be a sequence in X and $\lim_{n\to\infty} y_n = p$. Taking $x = p, y = y_n$ in (1), we have

$$\begin{aligned} &F\big(P_M(p,y_n,t), P_M(Tp,p,t), P_M(Tp,y_n,t), P_M(Tp,Ty_n,t)\big) \\ &\qquad = F\big(P_M(p,y_n,t), P_M(p,p,t), P_M(p,y_n,t), P_M(p,Ty_n,t)\big) \le 0, \end{aligned}$$

hence

$$\begin{aligned} &F\big(P_M(p,p,t), P_M(p,p,t), P_M(p,p,t), \limsup_{n\to\infty} P_M(p,Ty_n,t)\big) \\ &\qquad = F\begin{pmatrix} \limsup_{n\to\infty} P_M(p,y_n,t), \limsup_{n\to\infty} P_M(p,p,t), \\ \limsup_{n\to\infty} P_M(p,y_n,t), \limsup_{n\to\infty} P_M(p,Ty_n,t) \end{pmatrix} \le 0, \end{aligned}$$

which implies

$$\limsup_{n\to\infty} P_M(p,Ty_n,t)) \ge \gamma(P_M(p,p,t)) = \gamma(1) = 1.$$

Thus,

$$\limsup_{n\to\infty} P_M(p,Ty_n,t) = 1.$$

Similarly, taking limit inf, we have

$$\limsup_{n\to\infty} P_M(p,Ty_n,t) = 1.$$

Therefore, $\limsup_{n\to\infty} P_M(Ty_n,p,t) = 1$, this implies that

$$\limsup_{n\to\infty} P_M(Ty_n,Tp,t) = 1 = P_M(p,p,t) = P_M(Tp,Tp,t).$$

Thus $\lim_{n\to\infty} Ty_n = p = Tp$. Hence T is continuous at p. □

Corollary 1. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space, $m \in \mathbb{N}$ and T be a self map of X satisfying for all $x, y \in X$,*

$$F\big(P_M(x,y,t), P_M(T^m x,x,t), P_M(T^m x,y,t), P_M(T^m x,T^m y,t)\big) \le 0$$

where $F \in \mathcal{F}$. Then T has a unique fixed point p in X and T^m is continuous at p.

Proof. From Theorem 3, T^m has a unique fixed point p in X and T^m is continuous at p. Since $Tp = TT^m p = T^m Tp$, Tp is also a fixed point of T^m, By the uniqueness it follows $Tp = p$. □

In Theorem 3, if we take $F((t_1,t_2,t_3),t_4) = \gamma(\min\{t_1,t_2,t_3\}) - t_4$ then we have the next result.

Corollary 2. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space and T be a self map of X satisfying for all $x, y \in X$,*

$$P_M(Tx,Ty,t) \geq \gamma\big(\min\big\{P_M(x,y,t), P_M(Tx,x,t), P_M(Tx,y,t)\big\}\big).$$

Then T has a unique fixed point p in X and T is continuous at p.

Example 6. Let $X = \mathbb{R}^+$. Define $P_M : X^2 \times [0,\infty) \to [0,1]$ by

$$P_M(x,y,t) = \exp\left(-\frac{\max\{x,y\}}{t}\right)$$

for all $x, y \in X$ and $t > 0$. Then $(X, P_M, *)$ is a complete partial fuzzy metric space where $a * b = ab$. Define map $T : X \to X$ by $Tx = \frac{x}{2}$ for $x \in X$ and let $\gamma : [0,1] \to [0,1]$ defined by $\gamma(s) = s^{\frac{1}{2}}$. It is easy to see that

$$\begin{aligned} P_M(Tx,Ty,t) &= \exp\left(-\frac{\max\{\frac{x}{2},\frac{y}{2}\}}{t}\right) \\ &= \sqrt{\exp\left(-\frac{\max\{x,y\}}{t}\right)} \\ &= \sqrt{P_M(x,y,t)} \\ &\geq \sqrt{\min\big\{P_M(x,y,t), P_M(Tx,x,t), P_M(Tx,y,t)\big\}}. \end{aligned}$$

Thus T satisfy all the hypotheses of Corollary 2 and hence T has a unique fixed point.

Corollary 3. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space, $m \in \mathbb{N}$ and T be a self map of X satisfying for all $x, y \in X$,*

$$P_M(T^m x, T^m y, t) \geq \gamma\big(\min\big\{P_M(x,y,t), P_M(T^m x,x,t), P_M(T^m x,y,t)\big\}\big).$$

Then T has a unique fixed point p in X and and T^m is continuous at p.

Corollary 4. *Let $(X, P_M, *)$ be a complete bounded partial fuzzy metric space and T be a self map of X satisfying for all $x, y \in X$,*

$$P_M(Tx,Ty,t) \geq \sqrt{a_1 P_M(x,y,t) + a_2 P_M(Tx,x,t) + a_3 P_M(Tx,y,t)},$$

such that for every $a_i \geq 0$, $\sum\limits_{i=1}^{3} a_i = 1$. Then T has a unique fixed point p in X and T is continuous at p.

Corollary 5. *Let $(X, M, *)$ be a complete bounded fuzzy metric space and T be a self map of X satisfying for all $x, y \in X$ the*

$$F\big(M(x,y,t), M(Tx,x,t), M(Tx,y,t), M(Tx,Ty,t)\big) \leq 0$$

where $F \in \mathcal{F}$. Then T has a unique fixed point p in X and T is continuous at p.

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