COMMUNICATIONS IN MATHEMATICS

VOLUME 23/2015
No. 2

ISSN 1804-1388 (Print)
ISSN 2336-1298 (Online)

Available online at
http://cm.osu.cz

UNIVERSITAS OSTRAVIENSIS

Communications in Mathematics 23 (2015) 119–129

On the equivalence of control systems on Lie groups

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Abstract. We consider state space equivalence and feedback equivalence in the context of (full-rank) left-invariant control systems on Lie groups. We prove that two systems are state space equivalent (resp. detached feedback equivalent) if and only if there exists a Lie group isomorphism relating their parametrization maps (resp. traces). Local analogues of these results, in terms of Lie algebra isomorphisms, are also found. Three illustrative examples are provided.

1 Introduction

Geometric control theory began in the late 1960s with the study of (nonlinear) control systems by using concepts and methods from differential geometry (cf. [14], [21]). In the spirit of Klein's *Erlanger Programm*, a way of understanding the structure of a class of (geometric) objects is to define equivalence relations (or group actions) and then to study their invariants. In order to understand the local geometry of general control systems one needs to introduce natural equivalence relations in the class of such systems or in various distinguished subclasses. We will consider (smooth) control systems of the form

$$\dot{x} = \Xi(x, u), \qquad x \in \mathsf{M},\ u \in U \tag{1}$$

where the state space M and the space of control parameters (shortly the input space) U are smooth manifolds, and the map $\Xi : \mathsf{M} \times U \to T\mathsf{M}$ is smooth. (Ξ defines a family of smooth vector fields on M, smoothly parametrized by the controls.) The class $\mathcal{U}$ of admissible controls is contained in the space of all U-valued measurable maps defined on intervals of the real line $\mathbb{R}$ (see, e.g., [2], [14], [21]). We shall denote a control system (1) by (M, Ξ) (cf. [3]). Let $\mathcal{X} = (\Xi_u = \Xi(\cdot, u))_{u \in U}$ be the associated family of vector fields (on M). The control system $\Sigma = (\mathsf{M}, \Xi)$

2010 MSC: 93B27, 22E60
Key words: left-invariant control system, state space equivalence, detached feedback equivalence

satisfies the Lie algebra rank condition (LARC) at $x_0 \in \mathsf{M}$ provided the Lie algebra (of vector fields on M) generated by $\mathcal{X}$ spans the whole tangent space $T_{x_0}\mathsf{M}$.

The most natural equivalence relation for such control systems is equivalence up to coordinate changes in the state space. This is called *state space equivalence*. Two control systems (M, Ξ) and (M', Ξ') are called state space equivalent (shortly S-equivalent) if there exists a diffeomorphism $\phi : \mathsf{M} \to \mathsf{M}'$ which transforms Σ to Σ'; this amounts to saying that the diffeomorphism ϕ conjugates the families $\mathcal{X}$ and $\mathcal{X}'$ (see [11]). S-equivalence is well understood. It establishes a one-to-one correspondence between the trajectories of the equivalent systems. However, this equivalence relation is very strong. We recall the following result due to Krener [16] and Sussmann [20] (see also [2], [21]).

Proposition 1. *Let Σ and Σ' be two analytic control systems having the same input space $U = U'$ and satisfying the LARC at x_0 and x_0', respectively. Then they are (locally) S-equivalent at x_0 and x_0', respectively, if and only if there exists a linear isomorphism $\psi : T_{x_0}\mathsf{M} \to T_{x_0'}\mathsf{M}'$ such that the equality*

$$\psi\big[\cdots[\Xi_{u_1}, \Xi_{u_2}], \ldots, \Xi_{u_k}\big](x_0) = \big[\cdots[\Xi'_{u_1}, \Xi'_{u_2}], \ldots, \Xi'_{u_k}\big](x_0')$$

holds for any $k \geq 1$ and any $u_1, \ldots, u_k \in U$. Furthermore, if in addition M and M' are simply connected and the vector fields Ξ_u and Ξ'_u are complete, then local state space equivalence implies global state space equivalence.

Therefore, there are so many S-equivalence classes that any general classification appears to be very difficult if not impossible. However, there is a chance for some reasonable classification in low dimensions.

Another fundamental equivalence relation for control systems is that of *feedback equivalence*. We say that two control systems (M, Ξ) and (M', Ξ') are feedback equivalent (shortly F-equivalent) if there exists a diffeomorphism $\Phi : \mathsf{M} \times U \to \mathsf{M}' \times U'$ of the form

$$\Phi(x, u) = \big(\phi(x), \varphi(x, u)\big)$$

which transforms the first system to the second. Note that the map ϕ plays the role of a change of coordinates (in the state space), while the *feedback transformation* φ changes coordinates in the input space in a way which is state dependent. Two feedback equivalent control systems have the same set of trajectories (up to a diffeomorphism in the state space) which are parametrized differently by admissible controls. F-equivalence has been extensively studied in the last few decades (see [18] and the references therein). There are a few basic methods used in the study of F-equivalence. These methods are based either on (studying invariant properties of) associated distributions or on Cartan's method of equivalence [9] or inspired by the Hamiltonian formalism [12]; also, another fruitful approach is closely related to Poincaré's technique for linearization of dynamical systems. Feedback transformations play a crucial role in control theory, particularly in the important problem of *feedback linearization* [13]. The study of F-equivalence of general control systems can be reduced to the case of control affine systems (cf. [11]). For a thorough study of the equivalence and classification of (general) control affine systems, see [8].

In the context of left-invariant control systems, state space equivalence and feedback equivalence have not yet been considered in a general and systematic manner; we do so in this paper. Characterizations of state space equivalence and (detached) feedback equivalence are obtained: globally, in terms of Lie group isomorphisms (Theorems 1 and 3, respectively) and locally, in terms of Lie algebra isomorphisms (Theorems 2 and 4, respectively). A few examples exhibiting the use of (local) equivalences are provided.

2 Left-invariant control systems

Invariant control systems on Lie groups were first considered in 1972 by Brockett [7] and by Jurdjevic and Sussmann [15]. A *left-invariant control system* $\Sigma = (\mathsf{G}, \Xi)$ is a control system evolving on some (real, finite-dimensional) Lie group G, whose dynamics $\Xi : \mathsf{G} \times U \to T\mathsf{G}$ are invariant under left translations, i.e., the push-forward $(L_g)_*\Xi_u$ equals Ξ_u for all $g \in \mathsf{G}$ and $u \in U$. (The tangent bundle $T\mathsf{G}$ is identified with $\mathsf{G} \times \mathfrak{g}$, where $\mathfrak{g} = T_{\mathbf{1}}\mathsf{G}$ denotes the associated Lie algebra.) Such a control system is described as follows (cf. [14], [2], [19], [17])

$$\dot{g} = \Xi(g, u), \qquad g \in \mathsf{G},\ u \in U$$

where $\Xi(g, u) = g\Xi(\mathbf{1}, u) \in T_g\mathsf{G}$. (The notation $g\Xi(\mathbf{1}, u)$ stands for the image of the element $\Xi(\mathbf{1}, u) \in \mathfrak{g}$ under the tangent map of the left translation $dL_g = T_{\mathbf{1}}L_g : \mathfrak{g} \to T_g\mathsf{G}$.) The input space U is a smooth manifold and admissible controls are piecewise continuous U-valued maps, defined on compact intervals $[0, T]$. The family $\mathcal{X} = (\Xi_u = \Xi(\cdot, u))_{u \in U}$ consists of left-invariant vector fields on G. We further assume that the parametrization map $\Xi(\mathbf{1}, \cdot) : U \to \mathfrak{g}$ is an embedding. This means that the image set $\Gamma = \operatorname{im} \Xi(\mathbf{1}, \cdot)$, called the *trace* of Σ, is a submanifold of $\mathfrak{g}$. By identifying (the left-invariant vector field) $\Xi(\cdot, u)$ with $\Xi(\mathbf{1}, u) \in \mathfrak{g}$, we have that $\Gamma = \{\Xi_u : u \in U\}$. We say that Σ has *full rank* if its trace generates the Lie algebra $\mathfrak{g}$ (i.e., $\mathsf{Lie}(\Gamma) = \mathfrak{g}$). We note that Σ satisfies the LARC at identity (and hence everywhere) if and only if Σ has full rank.

A *trajectory* for an admissible control $u(\cdot) : [0, T] \to U$ is an absolutely continuous curve $g(\cdot) : [0, T] \to \mathsf{G}$ such that $\dot{g}(t) = g(t)\Xi(\mathbf{1}, u(t))$ for almost every $t \in [0, T]$. We say that a system Σ is *controllable* if for any $g_0, g_1 \in \mathsf{G}$, there exists a trajectory $g(\cdot) : [0, T] \to \mathsf{G}$ such that $g(0) = g_0$ and $g(T) = g_1$. Necessary conditions for controllability are that the group G is connected and the system has full rank. Henceforth, we shall assume that all the systems under consideration have full rank and that all Lie groups under consideration are connected.

Left-invariant control *affine* systems are those systems for which the parametrization map $\Xi(\mathbf{1}, \cdot) : \mathbb{R}^\ell \to \mathfrak{g}$ is affine. When the state space G is fixed, we specify such a system Σ by its parametrization map and simply write

$$\Sigma : \quad A + u_1B_1 + \cdots + u_\ell B_\ell.$$

Σ is said to be *homogeneous* if $A = \Xi(\mathbf{1}, 0) \in \operatorname{span}(B_1, \ldots, B_\ell)$, i.e., Γ is a linear subspace of $\mathfrak{g}$; otherwise Σ is *inhomogeneous*.

3 State space equivalence

Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be left-invariant control systems with the same input space U. Then Σ and Σ' are called *locally state space equivalent* (shortly $\mathrm{S_{loc}}$-equivalent) at points $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if there exist open neighbourhoods N and N' of a and a', respectively, and a diffeomorphism $\phi : N \to N'$ (mapping a to a') such that $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), u)$ for $g \in N$ and $u \in U$. Systems Σ and Σ' are called *state space equivalent* (shortly S-equivalent) if this happens globally (i.e., $N = \mathsf{G}$ and $N' = \mathsf{G}'$).

Firstly, we characterize (global) S-equivalence.

Lemma 1. *Let $\phi : \mathsf{G} \to \mathsf{G}'$ be a diffeomorphism. The push-forward $\phi_* X$ of any left-invariant vector field X on G is left invariant if and only if ϕ is the composition of a Lie group isomorphism $\bar{\phi} : \mathsf{G} \to \mathsf{G}'$ and a left translation $L_{\phi(\mathbf{1})}$ on G', i.e., $\phi = L_{\phi(\mathbf{1})} \circ \bar{\phi}$.*

Proof. Suppose the push-forward $\phi_* X$ of any left-invariant vector field X on G is left invariant. By composition with an appropriate left translation, we may assume $\phi(\mathbf{1}) = \mathbf{1}$. Let $A \in \mathfrak{g}$ and $X(g) = gA$ be the corresponding left-invariant vector field. As $\phi_* X$ is left invariant, there exists $A' \in \mathfrak{g}'$ such that

$$(\phi_* X)(\phi(g)) = \phi(g)A'.$$

Thus, as ϕ maps the flow of X to the flow of $\phi_* X$, we have that

$$\phi(g \exp(tA)) = \phi(g) \exp(tA')$$

for all $g \in \mathsf{G}$. Consequently, we find that

$$\phi(g \exp(A)) = \phi(g)\phi(\exp(A))$$

for all $g \in \mathsf{G}$, $A \in \mathfrak{g}$. As any element $g \in \mathsf{G}$ can be written as a finite product

$$g = \exp(A_1) \exp(A_2) \cdots \exp(A_k)$$

where $A_1, \ldots, A_k \in \mathfrak{g}$, it follows that ϕ is a Lie group homomorphism (and hence an isomorphism). The converse is trivial. $\square$

Theorem 1. *Σ and Σ' are S-equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$ for all $u \in U$.*

Proof. Suppose that Σ and Σ' are S-equivalent. There exists a diffeomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $\phi_* \Xi_u = \Xi'_u$ for $u \in U$. Moreover,

$$\phi_*[\Xi_u, \Xi_{\bar{u}}] = [\phi_* \Xi_u, \phi_* \Xi_{\bar{u}}] = [\Xi'_u, \Xi'_{\bar{u}}]$$

for $u, \bar{u} \in U$. The same holds true for higher order brackets, i.e.,

$$\phi_* \big[\cdots [\Xi_{u_1}, \Xi_{u_2}], \ldots, \Xi_{u_k} \big] = \big[\cdots [\Xi'_{u_1}, \Xi'_{u_2}], \ldots, \Xi'_{u_k} \big].$$

As Σ has full rank, it follows that $\{\Xi_u : u \in U\}$ generates $\mathfrak{g}$. Hence, as the Lie bracket of any two left-invariant vector fields is left invariant, it follows that the push-forward $\phi_* X$ of any left-invariant vector field X on G is left invariant. By composition with an appropriate left-translation, we may assume that $\phi(\mathbf{1}) = \mathbf{1}$. Thus $T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$ and, by Lemma 1, ϕ is a Lie group isomorphism.

Conversely, suppose that $\phi : \mathsf{G} \to \mathsf{G}'$ is a Lie group isomorphism as prescribed. Then $\phi \circ L_g = L_{\phi(g)} \circ \phi$ and so

$$T_g\phi \cdot \Xi(g, u) = T_{\mathbf{1}} L_{\phi(g)} \cdot T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\phi(g), u). \qquad \square$$

We now turn to $\mathrm{S_{loc}}$-equivalence. Note that (by left translation) Σ and Σ' are $\mathrm{S_{loc}}$-equivalent at $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if and only if they are $\mathrm{S_{loc}}$-equivalent at $\mathbf{1} \in \mathsf{G}$ and $\mathbf{1} \in \mathsf{G}'$. We give a characterization of $\mathrm{S_{loc}}$-equivalence, analogous to Theorem 1. The result may be proved by “localizing” the argument made in the proof of Theorem 1, or by considering the covering systems on the simply connected universal covering groups (cf. [3]) and applying Theorem 1; the result also follows as a fairly direct consequence of Proposition 1.

Theorem 2. *Σ and Σ' are S_{loc}-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, u)$ for all $u \in U$.*

Remark 1. As left-invariant vector fields are complete, $\mathrm{S_{loc}}$-equivalence implies S-equivalence when the state spaces are simply connected (Proposition 1). This fact can also be readily deduced from Theorems 1 and 2 by use of the following classic result: Let G and G' be connected and simply connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. If $\psi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra isomorphism, then there exists a unique Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi = \psi$ (see, e.g., [10]).

We conclude the section with an example of the classification, under $\mathrm{S_{loc}}$-equivalence, of a class of systems on the three-dimensional Euclidean group.

Example 1. Any two-input inhomogeneous control affine system on the Euclidean group $\mathsf{SE}(2)$ is $\mathrm{S_{loc}}$-equivalent to exactly one of the following systems

$$\begin{aligned}
\Sigma_{1,\alpha\beta\boldsymbol{\gamma}} :&\quad \alpha E_3 + u_1(E_1 + \gamma_1 E_2) + u_2(\beta E_2)\\
\Sigma_{2,\alpha\beta\boldsymbol{\gamma}} :&\quad \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(\alpha E_3) + u_2 E_2\\
\Sigma_{3,\alpha\beta\boldsymbol{\gamma}} :&\quad \beta E_1 + \gamma_1 E_2 + \gamma_2 E_3 + u_1(E_2 + \gamma_3 E_3) + u_2(\alpha E_3).
\end{aligned}$$

Here $\alpha > 0$, $\beta \neq 0$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}$, with different values of these parameters yielding distinct (non-equivalent) class representatives. (The standard basis elements E_1, E_2, E_3 of the Lie algebra $\mathfrak{se}(2)$ have commutator relations given by $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, and $[E_1, E_2] = 0$.) For a classification, under $\mathrm{S_{loc}}$-equivalence, of full-rank left-invariant control affine systems on $\mathsf{SE}(2)$, see [1].

Indeed, the group of automorphisms of $\mathfrak{se}(2)$ is

$$\mathsf{Aut}(\mathfrak{se}(2)) = \left\{ \begin{bmatrix} x & y & v \\ -\varsigma y & \varsigma x & w \\ 0 & 0 & \varsigma \end{bmatrix} \,:\, x, y, v, w \in \mathbb{R},\, x^2 + y^2 \neq 0,\, \varsigma = \pm 1 \right\}.$$

Let $\Sigma = (\mathsf{SE}(2), \Xi)$,

$$\Xi(\mathbf{1}, u) = \sum_{i=1}^{3} a_i E_i + u_1 \sum_{i=1}^{3} b_i E_i + u_2 \sum_{i=1}^{3} c_i E_i,$$

or in *matrix form*

$$\Sigma : \quad \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right].$$

It is then straightforward to show that there exists an automorphism $\psi \in \mathsf{Aut}(\mathfrak{se}(2))$ such that

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & \gamma_1 & \beta \\ \alpha & 0 & 0 \end{array}\right] \quad \text{if} \quad b_3 = 0,\ c_3 = 0$$

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} \beta & 0 & 0 \\ \gamma_1 & 0 & 1 \\ \gamma_2 & \alpha & 0 \end{array}\right] \quad \text{if} \quad b_3 \neq 0,\ c_3 = 0$$

or

$$\psi \cdot \left[\begin{array}{c|cc} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{array}\right] = \left[\begin{array}{c|cc} \beta & 0 & 0 \\ \gamma_1 & 1 & 0 \\ \gamma_2 & \gamma_3 & \alpha \end{array}\right] \quad \text{if} \quad c_3 \neq 0.$$

Thus Σ is $\mathrm{S_{loc}}$-equivalent to $\Sigma_{1,\alpha\beta\boldsymbol{\gamma}}$, $\Sigma_{2,\alpha\beta\boldsymbol{\gamma}}$, or $\Sigma_{3,\alpha\beta\boldsymbol{\gamma}}$. It is a simple matter to verify that these class representatives are non-equivalent.

4 Detached feedback equivalence

We specialize feedback equivalence in the context of left-invariant control systems by requiring that the feedback transformations are G-invariant. Let $\Sigma = (\mathsf{G}, \Xi)$ and $\Sigma' = (\mathsf{G}', \Xi')$ be left-invariant control systems. Then Σ and Σ' are called *locally detached feedback equivalent* (shortly $\mathrm{DF_{loc}}$-equivalent) at points $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if there exist open neighbourhoods N and N' of a and a', respectively, and a diffeomorphism

$$\Phi : N \times U \to N' \times U', \quad (g, u) \mapsto (\phi(g), \bar{\varphi}(u))$$

such that $\phi(a) = a'$ and $T_g\phi \cdot \Xi(g, u) = \Xi'(\phi(g), \bar{\varphi}(u))$ for $g \in N$ and $u \in U$. Systems Σ and Σ' are called *detached feedback equivalent* (shortly DF-equivalent) if this happens globally (i.e., $N = \mathsf{G}$ and $N' = \mathsf{G}'$).

We firstly characterize DF-equivalence. (The argument is very similar to the one used in the proof of Theorem 1.)

Theorem 3. *Σ and Σ' are DF-equivalent if and only if there exists a Lie group isomorphism $\phi : \mathsf{G} \to \mathsf{G}'$ such that $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$.*

Proof. Suppose that Σ and Σ' are DF-equivalent. There exists diffeomorphisms $\phi : \mathsf{G} \to \mathsf{G}'$ and $\varphi : U \to U'$ such that $\phi_* \Xi_u = \Xi'_{\varphi(u)}$ for $u \in U$. Moreover,

$$\phi_*[\Xi_u, \Xi_{\bar{u}}] = [\phi_* \Xi_u, \phi_* \Xi_{\bar{u}}] = [\Xi'_{\varphi(u)}, \Xi'_{\varphi(\bar{u})}]$$

for $u, \bar{u} \in U$ and similarly for higher order brackets. Therefore it follows that the push-forward $\phi_* X$ of any left-invariant vector field X on G is left invariant. By composition with an appropriate left-translation, we may assume that $\phi(\mathbf{1}) = \mathbf{1}$. Thus

$$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u))$$

and so $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$. Also, by Lemma 1, ϕ is a Lie group isomorphism.

Conversely, suppose that $\phi : \mathsf{G} \to \mathsf{G}'$ is a Lie group isomorphism as prescribed. As $T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$, there exists a unique diffeomorphism $\phi : U \to U'$ such that

$$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\mathbf{1}, \varphi(u)).$$

Hence, as $\phi \circ L_g = L_{\phi(g)} \circ \phi$, it follows that

$$T_g\phi \cdot \Xi(g, u) = T_{\mathbf{1}}\phi(g) \cdot T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, u) = \Xi'(\phi(g), \varphi(u)). \qquad \square$$

Remark 2. Systems Σ and Σ' are F-equivalent if there exists a *diffeomorphism* $\phi : \mathsf{G} \to \mathsf{G}'$ such that (the push-forward) $\phi_* \mathcal{F} = \mathcal{F}'$. Here $g \mapsto \mathcal{F}(g) = g\Gamma$ is the field of admissible velocities. The specialization to DF-equivalence corresponds to the existence of a *Lie group isomorphism* ϕ such that $\phi_* \mathcal{F} = \mathcal{F}'$. Thus F-equivalence is weaker than DF-equivalence. For example, suppose $\Gamma = \mathfrak{g}$, $\Gamma' = \mathfrak{g}'$, and G is diffeomorphic to G'. Then Σ and Σ' are F-equivalent. However, Σ and Σ' will be DF-equivalent only if G and G' are, in addition, isomorphic as Lie groups.

We now proceed to DF_{loc}-equivalence. We point out that systems Σ and Σ' are DF_{loc}-equivalent at $a \in \mathsf{G}$ and $a' \in \mathsf{G}'$ if and only if they are DF_{loc}-equivalent at $\mathbf{1} \in \mathsf{G}$ and $\mathbf{1} \in \mathsf{G}'$. We give a characterization of DF_{loc}-equivalence, analogous to Theorem 3. As with S-equivalence, the result may be proved by "localizing" the argument made in the proof of Theorem 3, or by considering the covering systems on the simply connected universal covering groups (cf. [3]) and applying Theorem 3. Alternatively, one can make use of the fact that any DF_{loc}-equivalence (resp. DF-equivalence) transformation decomposes into a S_{loc}-equivalence (resp. S-equivalence) transformation and a reparametrization (by which we mean a transformation of the form $\Xi'(g, u) = \Xi(g, \varphi(u))$).

Theorem 4. *Σ and Σ' are DF_{loc}-equivalent if and only if there exists a Lie algebra isomorphism $\psi : \mathfrak{g} \to \mathfrak{g}'$ such that $\psi \cdot \Gamma = \Gamma'$.*

Remark 3. As with S-equivalence, we have that DF_{loc}-equivalence implies DF--equivalence when the state spaces are simply connected (cf. Remark 1).

We revisit the class of systems considered in Example 1 and, in contrast, now classify these systems up to DF_{loc}-equivalence. We also give an example of the classification of a class of systems on another three-dimensional Lie group, namely the pseudo-orthogonal group.

Example 2. Any two-input inhomogeneous control affine system on $\mathsf{SE}(2)$ is $\mathrm{DF}_{\mathrm{loc}}$-equivalent to exactly one of the following systems (see [6])

$$\begin{aligned}\Sigma_1 : &\quad E_1 + u_1 E_2 + u_2 E_3\\ \Sigma_{2,\alpha} : &\quad \alpha E_3 + u_1 E_1 + u_2 E_2.\end{aligned}$$

Here $\alpha > 0$ parametrizes a family of class representatives, each different value corresponding to a distinct non-equivalent representative.

Indeed, let $\Sigma = (\mathsf{SE}(2), \Xi)$ be an inhomogeneous system with trace

$$\Gamma = \sum_{i=1}^{3} a_i E_i + \left\langle \sum_{i=1}^{3} b_i E_i, \sum_{i=1}^{3} c_i E_i \right\rangle.$$

If $c_3 \neq 0$ or $b_3 \neq 0$, then

$$\Gamma = a_1' E_1 + a_2' E_2 + \langle b_1' E_1 + b_2' E_2, c_1' E_1 + c_2' E_2 + E_3 \rangle.$$

Now either $b_1' \neq 0$ or $b_2' \neq 0$, and so

$$\begin{bmatrix} b_2' & -b_1' \\ b_1' & b_2' \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_2' \\ a_1' \end{bmatrix}$$

has a unique solution (with $v_2 \neq 0$). Therefore

$$\psi = \begin{bmatrix} v_2 b_2' & v_2 b_1' & c_1' \\ -v_2 b_1' & v_2 b_2' & c_2' \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that

$$\psi \cdot \Gamma_1 = \psi \cdot (E_1 + \langle E_2, E_3 \rangle) = \Gamma.$$

Thus Σ is $\mathrm{DF}_{\mathrm{loc}}$-equivalent to Σ_1. On the other hand, suppose $b_3 = 0$ and $c_3 = 0$. Then $\Gamma = a_3 E_3 + \langle E_1, E_2 \rangle$. Hence $\psi = \mathrm{diag}(1, 1, \mathrm{sgn}(a_3))$ is an automorphism such that $\psi \cdot \Gamma = \alpha E_3 + \langle E_1, E_2 \rangle$ with $\alpha > 0$. Thus Σ is $\mathrm{DF}_{\mathrm{loc}}$-equivalent to $\Sigma_{2,\alpha}$. As the subspace $\langle E_1, E_2 \rangle$ is invariant (under automorphisms), Σ_1 and $\Sigma_{2,\alpha}$ cannot be $\mathrm{DF}_{\mathrm{loc}}$-equivalent. It is easy to show that $\Sigma_{2,\alpha}$ a $\Sigma_{2,\alpha'}$ are $\mathrm{DF}_{\mathrm{loc}}$-equivalent only if $\alpha = \alpha'$.

Example 3. Any two-input homogeneous control affine system on the pseudo-orthogonal group $\mathsf{SO}(2,1)$ is $\mathrm{DF}_{\mathrm{loc}}$-equivalent to exactly one of the following systems (see [4])

$$\begin{aligned}\Sigma_1 : &\quad \Xi_1(\mathbf{1}, u) = u_1 E_1 + u_2 E_2\\ \Sigma_2 : &\quad \Xi_2(\mathbf{1}, u) = u_1 E_2 + u_2 E_3.\end{aligned}$$

(Here the commutator relations are given by $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$, and $[E_1, E_2] = -E_3$.)

Indeed, the group of automorphisms of $\mathfrak{so}(2,1)$ is

$$\mathsf{Aut}(\mathfrak{so}(2,1)) = \mathsf{SO}(2,1) = \left\{g \in \mathbb{R}^{3\times 3} : g^\top J g = J,\ \det g = 1\right\}.$$

Here $J = \operatorname{diag}(1,1,-1)$ and each automorphism ψ is identified with its corresponding matrix g. The Lorentzian product $\odot$ on $\mathfrak{so}(2,1)$ is given by

$$A \odot B = a_1 b_1 + a_2 b_2 - a_3 b_3.$$

(Here $A = \sum_{i=1}^{3} a_i E_i$ and $B = \sum_{i=1}^{3} b_i E_i$.) Any automorphism ψ preserves $\odot$, i.e., $(\psi \cdot A) \odot (\psi \cdot B) = A \odot B$.

Let Σ be a system with trace $\Gamma = \langle A, B\rangle$. The *sign* $\sigma(\Gamma)$ of Γ is given by

$$\sigma(\Gamma) = \operatorname{sgn}\left(\begin{vmatrix} A \odot A & A \odot B \\ A \odot B & B \odot B \end{vmatrix}\right).$$

($\sigma(\Gamma)$ does not depend on the parametrization.) As $\odot$ is preserved by automorphisms, it follows that $\sigma(\psi \cdot \Gamma) = \sigma(\Gamma)$. A straightforward computation shows that if $\sigma(\Gamma) = 0$, then Σ does not have full rank.

Suppose $\sigma(\Gamma) = -1$. Then we may assume that $a_3 \neq 0$ or $b_3 \neq 0$. Hence

$$\Gamma = \left\langle a_1' E_1 + a_2' E_2 + E_3, r\sin\theta E_1 + r\cos\theta E_2\right\rangle.$$

Thus

$$\psi = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an automorphism such that $\psi \cdot \Gamma = \langle a_1'' E_1 + E_3, E_2\rangle$. Now, as $\sigma(\psi \cdot \Gamma) = -1$, we have $(a_1'')^2 - 1 < 0$ and so $\psi \cdot \Gamma = \langle \sinh\vartheta E_1 + \cosh\vartheta E_3, E_2\rangle$. Therefore

$$\psi' = \begin{bmatrix} \cosh\vartheta & 0 & -\sinh\vartheta \\ 0 & 1 & 0 \\ -\sinh\vartheta & 0 & \cosh\vartheta \end{bmatrix}$$

is an automorphism such that $\psi' \cdot \psi \cdot \Gamma = \langle E_3, E_2\rangle$. Thus Σ is $\mathrm{DF}_{\mathrm{loc}}$-equivalent to Σ_1.

If $\sigma(\Gamma) = 1$, then a similar argument shows that there exists an automorphism ψ such that $\psi \cdot \Gamma = \langle E_1, E_2\rangle$ (and so Σ is $\mathrm{DF}_{\mathrm{loc}}$-equivalent to Σ_2). Lastly, Σ_1 and Σ_2 are non-equivalent systems, as $\sigma(\Gamma_1) = 1$ and $\sigma(\Gamma_2) = -1$.

5 Conclusion

In recent decades, attention has been drawn to invariant control systems evolving on (matrix) Lie groups of low dimension. We believe that this paper facilitates the structuring and comparison of such systems. We summarize the results in a table (see the next page).

Equivalence	Characterization	
S-equiv	$T_{\mathbf{1}}\phi \cdot \Xi(\mathbf{1}, \cdot) = \Xi'(\mathbf{1}, \cdot)$	$\phi : \mathsf{G} \to \mathsf{G}'$ Lie group isomorphism
DF-equiv	$T_{\mathbf{1}}\phi \cdot \Gamma = \Gamma'$	
$\mathrm{S}_{\mathrm{loc}}$-equiv	$\psi \cdot \Xi(\mathbf{1}, \cdot) = \Xi'(\mathbf{1}, \cdot)$	$\psi : \mathfrak{g} \to \mathfrak{g}'$ Lie algebra isomorphism
$\mathrm{DF}_{\mathrm{loc}}$-equiv	$\psi \cdot \Gamma = \Gamma'$	

The (four) characterizations of equivalences provide efficient means to classify various distinguished subclasses of left-invariant control systems. For instance, if one considers the problem of classifying under $\mathrm{DF}_{\mathrm{loc}}$-equivalence, one may restrict to systems with a fixed Lie algebra $\mathfrak{g}$. Σ and Σ' are then $\mathrm{DF}_{\mathrm{loc}}$-equivalent if and only if their traces Γ and Γ' are equivalent under the relation

$$\Gamma \sim \Gamma' \iff \exists \psi \in \mathsf{Aut}(\mathsf{G}),\ \psi \cdot \Gamma = \Gamma'.$$

This reduces the problem of classifying control affine systems (under $\mathrm{DF}_{\mathrm{loc}}$-equivalence) to that of classifying affine subspaces of $\mathfrak{g}$. In the case of control affine systems evolving on three-dimensional Lie groups, a full classification under $\mathrm{DF}_{\mathrm{loc}}$-equivalence has been obtained in [4], [5], [6].

Acknowledgements

This research was supported in part by the European Union's Seventh Framework Programme (FP7/2007–2013, grant no. 317721). R. Biggs would also like to acknowledge the financial support of the Claude Leon Foundation.

References

[1] R.M. Adams, R. Biggs, C.C. Remsing: Equivalence of control systems on the Euclidean group SE(2). *Control Cybernet.* 41 (2012) 513–524.

[2] A.A. Agrachev, Y.L. Sachkov: *Control Theory from the Geometric Viewpoint.* Springer Science & Business Media (2004).

[3] R. Biggs, C.C. Remsing: A category of control systems. *An. Şt. Univ. Ovidius Constanţa* 20 (2012) 355–368.

[4] R. Biggs, C.C. Remsing: Control affine systems on semisimple three-dimensional Lie groups. *An. Şt. Univ. "A.I. Cuza" Iaşi. Ser. Mat.* 59 (2013) 399–414.

[5] R. Biggs, C.C. Remsing: Control affine systems on solvable three-dimensional Lie groups, I. *Arch. Math. (Brno)* 49 (2013) 187–197.

[6] R. Biggs, C.C. Remsing: Control affine systems on solvable three-dimensional Lie groups, II. *Note Mat.* 33 (2013) 19–31.

[7] R.W. Brockett: System theory on group manifolds and coset spaces. *SIAM J. Control* 10 (1972) 265–284.

[8] V.I. Elkin: Affine control systems: their equivalence, classification, quotient systems, and subsystems. *J. Math. Sci.* 88 (1998) 675–721.

[9] R.B. Gardner, W.F. Shadwick: Feedback equivalence of control systems. *Systems Control Lett.* 8 (1987) 463–465.

[10] V.V. Gorbatsevich, A.L. Onishchik, E.B. Vinberg: *Foundations of Lie Theory and Lie Transformation Groups*. Springer Science & Business Media (1997).

[11] B. Jakubczyk: Equivalence and invariants of nonlinear control systems. *Nonlinear Controllability and Optimal Control* (1990) 177–218. In: H.J. Sussmann (ed.)

[12] B. Jakubczyk: Critical Hamiltonians and feedback invariants. *Geometry of Feedback and Optimal Control* (1998) 219–256. In: B. Jakubczyk, W. Respondek (eds.)

[13] B. Jakubczyk, W. Respondek: On linearization of control systems. *Bull. Acad. Polon. Sci. Ser. Sci. Math.* 28 (1980) 517–522.

[14] V. Jurdjevic: *Geometric Control Theory*. Cambridge University Press (1997).

[15] V. Jurdjevic, H.J. Sussmann: Control systems on Lie groups. *J. Diff. Equations* 12 (1972) 313–329.

[16] A.J. Krener: On the equivalence of control systems and the linearization of nonlinear systems. *SIAM J. Control* 11 (1973) 670–676.

[17] C.C. Remsing: Optimal control and Hamilton-Poisson formalism. *Int. J. Pure Appl. Math.* 59 (2010) 11–17.

[18] W. Respondek, I.A. Tall: Feedback equivalence of nonlinear control systems: a survey on formal approach. *Chaos in Automatic Control* (2006) 137–262. In: W. Perruquetti, J.-P. Barbot (eds.)

[19] Y.L. Sachkov: Control theory on Lie groups. *J. Math. Sci.* 156 (2009) 381–439.

[20] H.J. Sussmann: An extension of a theorem of Nagano on transitive Lie algebras. *Proc. Amer. Math. Soc.* 45 (1974) 349–356.

[21] H.J. Sussmann: Lie brackets, real analyticity and geometric control. *Differential Geometric Control Theory* (1983) 1–116. In: R.W. Brockett, R.S. Millman, H.J. Sussmann (eds.)

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Received: 19 November, 2014
Accepted for publication: 23 November, 2015
Communicated by: Anthony Bloch