VOLUME
23/2015
No. 1

ISSN 1804-1388 (Print)
ISSN 2336-1298 (Online)

# COMMUNICATIONS IN MATHEMATICS



Available online at
http://cm.osu.cz

UNIVERSITAS OSTRAVIENSIS

Communications in Mathematics 23 (2015) 85–93



# The gap theorems for some extremal submanifolds in a unit sphere

*Xi Guo and Lan Wu*

**Abstract.** Let $M$ be an $n$-dimensional submanifold in the unit sphere $S^{n+p}$, we call $M$ a $k$-extremal submanifold if it is a critical point of the functional $\int_M \rho^{2k}\,\mathrm{d}v$. In this paper, we can study gap phenomenon for these submanifolds.

## 1 Introduction and theorems

Let $x\colon M^n \hookrightarrow S^{n+p}(1)$ be an $n$-dimensional compact submanifold in a unit sphere, and let

- $e_1, \dots, e_n$ be a local orthonormal frame of tangent vector field on $M$,
- $e_{n+1}, \dots, e_{n+p}$ be a local orthonormal frame of normal vector field on $M$,
- $\omega_1, \dots, \omega_n$, $\omega_{n+1}, \dots, \omega_{n+p}$ be its dual coframe field.

Then the second fundamental form and the mean curvature vector of $M$ are

$$A = \sum_{i,j,\alpha} h^\alpha_{ij} \omega_i \otimes \omega_j \otimes e_\alpha, \quad \mathbf{H} = \sum_\alpha H^\alpha e_\alpha = \frac{1}{n} \sum_{i,\alpha} h^\alpha_{ii} e_\alpha. \tag{1}$$

We can define trace-free linear maps $\phi_\alpha\colon T_qM \to T_qM$ by

$$\langle \phi^\alpha X, Y\rangle = \langle A^\alpha X, Y\rangle - \langle X, Y\rangle \langle \mathbf{H}, e_\alpha\rangle,$$

where $q \in M$, $A^\alpha$ is the shape operator of $e_\alpha$,

$$A^\alpha(e_i) = -\sum_j \langle \overline{\nabla}_{e_i} e_\alpha, e_j\rangle e_j = \sum_j h^\alpha_{ij} e_j\,,$$

and we define a bilinear map $\phi\colon T_qM \times T_qM \to T_qM^\perp$ by

*2010 MSC:* 53C40, 53C24
*Key words:* Extremal functional, Mean curvature, Totally umbilical

$$\phi(X,Y)=\sum_{\alpha=n+1}^{n+p}\langle\phi^\alpha X,Y\rangle e_\alpha. \tag{2}$$

It's easy to check that $|\phi|^2=|A|^2-nH^2$, where $H^2=|\mathbf{H}|^2=\sum_\alpha(H^\alpha)^2$, and we denote $\rho=|\phi|$. For any fixed number $k$ with $k\geq 1$, we can define the following functional

$$F_k(x)=\int_M\rho^{2k}\,\mathrm{d}v. \tag{3}$$

When $k=\frac{n}{2}$, it is the Willmore functional. We say $x\colon M\to S^{n+p}$ is a $k$-extremal submanifold if it is a critical point of the functional $F_k(x)$.

It seems very interesting to study the gap phenomenon for submanifolds, and there are some results about compact minimal submanifolds in $S^{n+p}(1)$, such as in [7]. For Willmore submanifolds, H. Li proved:

**Theorem 1.** *[6] Let M be an $n$-dimensional compact Willmore submanifold in $S^{n+p}$, then*

$$\int_M\left[n-\left(2-\frac{1}{p}\right)\rho^2\right]\rho^n\,\mathrm{d}v\leq 0. \tag{4}$$

*In particular, if $\rho^2\leq\frac{n}{2-1/p}$, then either $\rho=0$ and $M$ is a totally umbilical submanifold, or $\rho^2=\frac{n}{2-1/p}$. In the latter case, either $p=1$ and $M$ is a Willmore torus $W_{m,n-m}=S^m\big(\sqrt{\frac{n-m}{n}}\big)\times S^{n-m}\big(\sqrt{\frac{m}{n}}\big)$; or $n=2$, $p=2$ and $M$ is the Veronese surface.*

And for $k$-extremal submanifolds, Z. Guo and H. Li, the second author proved:

**Theorem 2.** *[1], [9] Let $M$ be an $n$-dimensional compact $k$-extremal submanifold in $S^{n+p}$, $1\leq k<\frac{n}{2}$, then*

$$\int_M\left[n-\left(2-\frac{1}{p}\right)\rho^2\right]\rho^{2k}\,\mathrm{d}v\leq 0. \tag{5}$$

*In particular, if $\rho^2\leq\frac{n}{2-1/p}$, then either $\rho=0$ and $M$ is a totally umbilical submanifold, or $\rho^2=\frac{n}{2-1/p}$. In the latter case, either $p=1,n=2m$ and $M$ is a Clifford torus $C_{m,m}=S^m\left(\sqrt{\frac{1}{2}}\right)\times S^m\left(\sqrt{\frac{1}{2}}\right)$; or $n=2$, $p=2$ and $M$ is the Veronese surface.*

In 2011, H. Xu and D. Yang proved the following pinching theorem for submanifold which is a critical point of the functional $F_1(x)$.

**Theorem 3.** *[8] Let $M$ be an $n$-dimensional compact 1-extremal submanifold in $S^{n+p}$, then there exists an explicit positive constant $A_n$ depending only on $n$ such that if*

$$\left(\int_M\rho^n\,\mathrm{d}v\right)^{\frac{2}{n}}<A_n, \tag{6}$$

$$A_n = \begin{cases} \min\Big\{ \dfrac{n(n-2)^2}{4n(n-1)^2+(n-2)^2}, \\ \qquad \dfrac{(n-2)^2(\frac{n}{2}-n)}{4(\frac{n}{2}-n)(n-1)^2+(n-2)^2} \Big\} C(n)^{-2} \quad (p=1); \\ \dfrac{2}{3}\min\Big\{ \dfrac{n(n-2)^2}{4n(n-1)^2+(n-2)^2}, \\ \qquad \dfrac{(n-2)^2(\frac{n}{2}-n)}{4(\frac{n}{2}-n)(n-1)^2+(n-2)^2} \Big\} C(n)^{-2} \quad (p\geq 2), \end{cases}$$

*then $M$ is a totally umbilical submanifold, where $C(n)$ is a positive constant depending on $n$ which satisfies:*

$$\Big(\int_M f^{\frac{n}{n-1}}\,\mathrm{d}v\Big)^{\frac{n-1}{n}} \leq C(n)\int_M \big(|\nabla f| + (1+H^2)f\big)\,\mathrm{d}v \tag{7}$$

*holds for any $f \in C^1(M)$.*

In this paper, we prove the following theorems for the $k$-extremal submanifold when $1 \leq k < \frac{n}{2}$:

**Theorem 4.** *Let $M$ be an $n$-dimensional compact $k$-extremal submanifold in $S^{n+p}$ $(n \geq 3)$, $1 \leq k < \frac{n}{2}$, then there exists an explicit positive constant $A_{n,k}$ depending only on $n$ and $k$ such that if*

$$\Big(\int_M \rho^n\,\mathrm{d}v\Big)^{\frac{2}{n}} < A_{n,k}, \tag{8}$$

*where*

$$A_{n,k} = \begin{cases} C(n)^{-2}\min\Big\{ \dfrac{n(n-2)^2(2k-1)}{4n(n-1)^2k^2+(2k-1)(n-2)^2}, \\ \qquad \dfrac{(2k-1)(n-2)^2(\frac{n^2}{2k}-n)}{4(\frac{n^2}{2k}-n)(n-1)^2k^2+(2k-1)(n-2)^2} \Big\} \quad (p=1); \\ \dfrac{2}{3}C(n)^{-2}\min\Big\{ \dfrac{n(n-2)^2(2k-1)}{4n(n-1)^2k^2+(2k-1)(n-2)^2}, \\ \qquad \dfrac{(2k-1)(n-2)^2(\frac{n^2}{2k}-n)}{4(\frac{n^2}{2k}-n)(n-1)^2k^2+(2k-1)(n-2)^2} \Big\} \quad (p\geq 2), \end{cases}$$

*then $M$ is a totally umbilical submanifold, where $C(n)$ is the same constant as above.*

**Theorem 5.** *Let $M$ be an $n$-dimensional $(n \geq 3)$ compact $k$-extremal submanifold with flat normal bundle in $S^{n+p}$, $1 \leq k < \frac{n}{2}$. If $\rho^2 \leq n$, then either $\rho = 0$ and $M$ is a totally umbilical submanifold, or $p = 1, n = 2m$ and $M$ is a Clifford torus $C_{m,m} = S^m\Big(\sqrt{\frac{1}{2}}\Big) \times S^m\Big(\sqrt{\frac{1}{2}}\Big)$.*

**Remark 1.** If $k = \frac{n}{2}$, then $A_{n,k} = 0$, so our Theorem 4 is trivial when $k = \frac{n}{2}$. If $k = 1$, $A_{n,1} = A_n$, our Theorem 4 reduces to Xu-Yang's Theorem 3.

## 2 Preliminaries and lemmas

We shall make use of the following convention on the range of indices:

$$1 \le A, B, C \le n+p, \quad 1 \le i, j, k \le n, \quad n+1 \le \alpha, \beta, \gamma \le n+p.$$

We choose a local orthonormal frame field $\{e_1, \ldots, e_n, e_{n+1}, \ldots e_{n+p}\}$ along $M$, with $\{e_i\}_{i=1,2,\ldots,n}$ tangent to $M$ and $\{e_\alpha\}_{\alpha=n+1,n+2,\ldots,n+p}$ normal to $M$. Let $\{\omega_A\}$ be the corresponding dual coframe, and $\{\omega_{AB}\}$ be the connection 1-form on $S^{n+p}$. Restricted on $M$, the curvature tensor, the normal curvature tensor can be given by

$$\mathrm{d}\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{9}$$

$$\mathrm{d}\omega_{\alpha\beta} - \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} = -\frac{1}{2} \sum_{k,l} R^\perp_{\alpha\beta kl} \omega_k \wedge \omega_l. \tag{10}$$

and the mean curvature $\mathbf{H} = \sum_\alpha H^\alpha e_\alpha$, where $H^\alpha = \frac{1}{n} \sum_i h^\alpha_{ii}$.

The covariant derivative of the second fundamental form is given by

$$\sum_k h^\alpha_{ij,k} \omega_k = \mathrm{d}h^\alpha_{ij} + \sum_k h^\alpha_{ki} \omega_{kj} + \sum_k h^\alpha_{kj} \omega_{ki} + \sum_\beta h^\beta_{ij} \omega_{\beta\alpha}, \tag{11}$$

$$\sum_l h^\alpha_{ij,kl} \omega_l = \mathrm{d}h^\alpha_{ij,k} + \sum_l h^\alpha_{lj,k} \omega_{li} + \sum_l h^\alpha_{ij,l} \omega_{lk} + \sum_l h^\alpha_{il,k} \omega_{lj} + \sum_\beta h^\beta_{ij,k} \omega_{\beta\alpha}. \tag{12}$$

In [9], the second author calculated the Euler-Lagrangian equation of $F_k(x)$:

**Lemma 1.** *[9] If $x\colon M \to R^{n+p}(c)$ be an $n$-dimensional submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. Then for $k \ge 1$, $M$ is an extremal submanifold of $F_k(x)$ if and only if for $n+1 \le \alpha \le n+p$,*

$$\begin{aligned} 0 = & -\Delta(\rho^{2k-2}) H^\alpha + 2(n-1) \sum_i (\rho^{2k-2})_{,i} H^\alpha_{,i} \\ & + \sum_{i,j} (\rho^{2k-2})_{,ij} h^\alpha_{ij} + (n-1) \rho^{2k-2} \Delta^\perp H^\alpha \\ & + \rho^{2k-2} \Big[ \sum_{i,j,k,\beta} h^\alpha_{ij} h^\beta_{jk} h^\beta_{ki} - \sum_{i,j,\beta} H^\beta h^\alpha_{ij} h^\beta_{ij} - \frac{n}{2k} \rho^2 H^\alpha \Big]. \end{aligned} \tag{13}$$

Using the above lemma, we can get that:

**Lemma 2.** *If $M$ is an extremal submanifold of $F_k(x)$, then*

$$\begin{aligned} &\int_M \rho^{2k-2} \Big( \Delta H^2 - 2 \sum_{i,j,\alpha} h^\alpha_{ij} H^\alpha_{,ij} \Big) \mathrm{d}v \\ &\qquad = 2 \int_M \rho^{2k-2} |\nabla^\perp \mathbf{H}|^2 \,\mathrm{d}v + 2 \int_M \rho^{2k-2} F \,\mathrm{d}v, \end{aligned} \tag{14}$$

*where $\nabla^\perp$ is the normal connection on $M$, and*

$$F := \sum_{i,j,k,\alpha,\beta} H^\alpha h^\alpha_{ij} h^\beta_{jk} h^\beta_{ji} - \sum_{j,k,\alpha,\beta} H^\alpha H^\beta h^\alpha_{jk} h^\beta_{jk} - \frac{n}{2k} \rho^2 H^2 .$$

*Proof.* Multiplying the equation (13) by $H^\alpha$ and integrating over $M$ we obtain

$$\begin{aligned} 0 = &-\int_M \Delta(\rho^{2k-2})H^2\,\mathrm{d}v + 2(n-1)\int_M \sum_{i,\alpha}(\rho^{2k-2})_{,i}H^\alpha_{,i}H^\alpha\,\mathrm{d}v \\ &+\int_M \sum_{i,j,\alpha}(\rho^{2k-2})_{,ij}h^\alpha_{ij}H^\alpha\,\mathrm{d}v + (n-1)\int_M \sum_{i,\alpha}\rho^{2k-2}H^\alpha_{,ii}H^\alpha\,\mathrm{d}v \\ &+\int_M \rho^{2k-2}F\,\mathrm{d}v, \end{aligned} \tag{15}$$

and integrating by parts, we can get

$$\int_M \sum_{i,\alpha}(\rho^{2k-2})_{,i}H^\alpha_{,i}H^\alpha\,\mathrm{d}v = -\int_M \sum_i \rho^{2k-2}_{,ii}H^2\,\mathrm{d}v - \int_M \sum_{i,\alpha}\rho^{2k-2}_{,i}H^\alpha_{,i}H^\alpha\,\mathrm{d}v,$$

so

$$2\int_M \sum_{i,\alpha}(\rho^{2k-2})_{,i}H^\alpha_{,i}H^\alpha\,\mathrm{d}v = -\int_M \Delta\rho^{2k-2}H^2\,\mathrm{d}v = -\int_M \rho^{2k-2}\Delta H^2\,\mathrm{d}v. \tag{16}$$

Thus we have the following calculations:

$$\begin{aligned} \int_M \sum_{i,j,\alpha}(\rho^{2k-2})_{,ij}h^\alpha_{ij}H^\alpha\,\mathrm{d}v &= -\int_M \sum_{i,j,\alpha}(\rho^{2k-2})_{,i}h^\alpha_{ij,j}H^\alpha\,\mathrm{d}v - \int_M \sum_{i,j,\alpha}(\rho^{2k-2})_{,i}h^\alpha_{ij}H^\alpha_{,j}\,\mathrm{d}v \\ &= -n\int_M \sum_{i,\alpha}(\rho^{2k-2})_{,i}H^\alpha_{,i}H^\alpha\,\mathrm{d}v + \int_M \sum_{i,j,\alpha}\rho^{2k-2}h^\alpha_{ij,i}H^\alpha_{,j}\,\mathrm{d}v \\ &\quad+\int_M \sum_{i,j,\alpha}\rho^{2k-2}h^\alpha_{ij}H^\alpha_{,ji}\,\mathrm{d}v \\ &= \frac{n}{2}\int_M \rho^{2k-2}\Delta H^2\,\mathrm{d}v + n\int_M \rho^{2k-2}|\nabla^\perp\mathbf{H}|^2\,\mathrm{d}v \\ &\quad+\int_M \sum_{i,j,\alpha}\rho^{2k-2}h^\alpha_{ij}H^\alpha_{,ij}\,\mathrm{d}v, \end{aligned} \tag{17}$$

$$\int_M \sum_{i,\alpha}\rho^{2k-2}H^\alpha_{,ii}H^\alpha\,\mathrm{d}v = \frac{1}{2}\int_M \rho^{2k-2}\Delta H^2\,\mathrm{d}v - \int_M \rho^{2k-2}|\nabla^\perp\mathbf{H}|^2\,\mathrm{d}v. \tag{18}$$

Then (15) becomes

$$\begin{aligned} 0 = &-\frac{1}{2}\int_M \rho^{2k-2}\Delta H^2\,\mathrm{d}v + \int_M \rho^{2k-2}|\nabla^\perp\mathbf{H}|^2\,\mathrm{d}v \\ &+\int_M \sum_{i,j,\alpha}\rho^{2k-2}h^\alpha_{ij}H^\alpha_{,ij}\,\mathrm{d}v + \int_M \rho^{2k-2}F\,\mathrm{d}v, \end{aligned} \tag{19}$$

so (14) holds. □

We also need the following inequalities:

**Lemma 3.** *[8] Let $M$ be an $n$-dimensional ($n \geq 3$) compact submanifold in the unit sphere $S^{n+p}$ . Then for any $f \in C^1(M)$, $f \geq 0$, $t > 0$, $f$ satisfies the following inequality*

$$\int_M |\nabla f|^2 \,\mathrm{d}v \geq c_1(n,t)\Big(\int_M f^{\frac{2n}{n-2}} \,\mathrm{d}v\Big)^{\frac{n-2}{n}} - c_2(n,t)\int_M (1+H^2)f^2 \,\mathrm{d}v, \tag{20}$$

*where* $c_1(n,t) = \frac{(n-2)^2}{4C(n)^2(1+t)(n-1)^2}$, $c_2(n,t) = \frac{(n-2)^2}{4t(n-1)^2}$.

**Lemma 4.** *[4] Let $B^1, B^2, \dots, B^m$ be symmetric $(n \times n)$-matrices, Set $S_{\alpha\beta} = \operatorname{tr}(B^\alpha B^\beta)$, $S_\alpha = S_{\alpha\alpha}$, $S = \sum_\alpha S_\alpha$, then*

$$\sum_{\alpha,\beta} |B^\alpha B^\beta - B^\beta B^\alpha|^2 + \sum_{\alpha,\beta} S_{\alpha\beta}^2 \leq \frac{3}{2}\Big(\sum_\alpha S_\alpha\Big)^2, \tag{21}$$

*where* $|B|^2 = \operatorname{tr} B^t B$.

## 3 Proof of the theorems

We also need a Simons' type formula, which can be found in [6]:

**Lemma 5.** *If $x\colon M \to S^{n+m}$ be an $n$-dimensional submanifold, then*

$$\begin{aligned}\frac{1}{2}\Delta\rho^2 = |\nabla A|^2 - n^2|\nabla^\perp \mathbf{H}|^2 + \sum_{i,j,k,\alpha} (h_{ij}^\alpha h_{kk,i}^\alpha)_{,j} \\ + n\sum_{\alpha,\beta,i,j,k} H^\beta \phi_{ij}^\beta \phi_{jk}^\alpha \phi_{ki}^\alpha + n\rho^2 + n^2H^2\rho^2 \\ - \sum_{\alpha,\beta} \sigma_{\alpha\beta}^2 - \sum_{\alpha,\beta,i,j} (R_{\alpha\beta ij}^\perp)^2 - \frac{1}{2}\Delta(nH^2),\end{aligned} \tag{22}$$

*where $\phi$ is the trace-free tensor which defined above, $\sigma_{\alpha\beta} = \sum_{i,j} \phi_{ij}^\alpha \phi_{ij}^\beta$.*

From

$$0 = \int_M \Delta\rho^{2k} \,\mathrm{d}v = 2\int_M \Delta\rho^2\rho^{2k-2} \,\mathrm{d}v + 2\int_M \langle\nabla\rho^2, \nabla\rho^{2k-2}\rangle \,\mathrm{d}v, \tag{23}$$

and (22), we get that

$$\begin{aligned}\frac{1}{2}\int_M \Delta\rho^2\rho^{2k-2} \,\mathrm{d}v = \int_M |\nabla A|^2\rho^{2k-2} \,\mathrm{d}v + n\int_M (\sum_{\alpha,i,j} h_{ij}^\alpha H_{,ij}^\alpha - \frac{1}{2}\Delta H^2)\rho^{2k-2} \,\mathrm{d}v \\ + \int_M E\rho^{2k-2} \,\mathrm{d}v,\end{aligned} \tag{24}$$

where

$$E := n\sum_{\alpha,\beta,i,j,k} H^\beta \phi_{ij}^\beta \phi_{jk}^\alpha \phi_{ki}^\alpha + n\rho^2 + n^2H^2\rho^2 - \sum_{\alpha,\beta} \sigma_{\alpha\beta}^2 - \sum_{\alpha,\beta,i,j} (R_{\alpha\beta ij}^\perp)^2 .$$

Using (14) and (23),

$$\begin{aligned}0 = &\int_M (|\nabla A|^2 - n|\nabla^\perp \mathbf{H}|^2)\rho^{2k-2}\,\mathrm{d}v \\ &+ \int_M (E - nF)\rho^{2k-2}\,\mathrm{d}v + (2k-2)\int_M |\nabla\rho|^2\rho^{2k-2}\,\mathrm{d}v,\end{aligned} \tag{25}$$

from Lemma 2.1 in [8] we know that

$$|\nabla A|^2 - n|\nabla^\perp \mathbf{H}|^2 = \sum_{\alpha,i,j,k}(\phi^\alpha_{ij,k})^2 \geq |\nabla\rho|^2. \tag{26}$$

By a direct computation, we have that

$$E - nF = n\rho^2 + \frac{n^2}{2k}\rho^2 H^2 - n\sum_{\alpha,\beta,i,j} H^\alpha H^\beta \phi^\alpha_{ij}\phi^\beta_{ij} - \sum_{\alpha,\beta}\sigma^2_{\alpha\beta} - \sum_{\alpha,\beta,i,j}(R^\perp_{\alpha\beta ij})^2, \tag{27}$$

for

$$\sum_{\alpha,\beta,i,j} H^\alpha H^\beta \phi^\alpha_{ij}\phi^\beta_{ij} = \sum_{i,j}\Big(\sum_\alpha H^\alpha\phi^\alpha_{ij}\Big)^2 \leq \bigg(\sum_{i,j}\Big(\sum_\alpha \phi^\alpha_{ij}\Big)^2\bigg)\bigg(\Big(\sum_\alpha H^\alpha\Big)^2\bigg) = \rho^2 H^2, \tag{28}$$

then

$$\begin{aligned}0 \geq &\frac{2k-1}{k^2}\int_M |\nabla\rho^k|^2\,\mathrm{d}v \\ &+ \int_M \Big[n\rho^2 + \Big(\frac{n^2}{2k} - n\Big)H^2\rho^2 - \sum_{\alpha,\beta}\sigma^2_{\alpha\beta} - \sum_{\alpha,\beta,i,j}\big(R^\perp_{\alpha\beta ij}\big)^2\Big]\rho^{2k-2}\,\mathrm{d}v.\end{aligned} \tag{29}$$

*Proof.* (Theorem 4) From Lemma 4,

$$E - nF \geq n\rho^2 + (\frac{n^2}{2k} - n)\rho^2 H^2 - \eta\rho^4, \tag{30}$$

where $\eta = \min(\frac{3}{2}, 2 - \frac{1}{p})$.

From (25), (26) and (30), we know that the following inequality holds,

$$\frac{2k-1}{k^2}\int_M |\nabla\rho^k|^2\,\mathrm{d}v + \int_M \Big[n + \Big(\frac{n^2}{2k} - n\Big)H^2 - \eta\rho^2\Big]\rho^{2k}\,\mathrm{d}v \leq 0, \tag{31}$$

and with Lemma 3 and (31), we can get:

$$\begin{aligned}0 \geq &\frac{2k-1}{k^2}c_1(n,t)\Big(\int_M \rho^{\frac{2n}{n-2}k}\,\mathrm{d}v\Big)^{\frac{n-2}{n}} + \Big(n - \frac{2k-1}{k^2}c_2(n,t)\Big)\Big(\int_M \rho^{2k}\,\mathrm{d}v\Big) \\ &+ \Big(\frac{n^2}{2k} - n - \frac{2k-1}{k^2}c_2(n,t)\Big)\Big(\int_M H^2\rho^{2k}\,\mathrm{d}v\Big) - \eta\int_M \rho^{2k+2}\,\mathrm{d}v.\end{aligned} \tag{32}$$

Using the Hölder's inequality, we have

$$
\begin{aligned}
0 \geq & \left[\frac{2k-1}{k^2} c_1(n,t) - \eta \Big(\int_M \rho^n \, \mathrm{d}v\Big)^{\frac{2}{n}}\right] \Big(\int_M \rho^{\frac{2n}{n-2}k} \, \mathrm{d}v\Big)^{\frac{n-2}{n}} \\
& + \left(n - \frac{2k-1}{k^2} c_2(n,t)\right) \Big(\int_M \rho^{2k} \, \mathrm{d}v\Big) \\
& + \left[\frac{n^2}{2k} - n - \frac{2k-1}{k^2} c_2(n,t)\right] \Big(\int_M H^2 \rho^{2k} \, \mathrm{d}v\Big),
\end{aligned}
$$

let $t = \frac{(n-2)^2(2k-1)}{4(n-1)^2k^2} \max\left(\frac{2k}{n^2-2kn}, \frac{1}{n}\right)$, then Theorem 4 follows. □

*Proof.* (Theorem 5) If $M$ has normal flat bundle, then (29) become

$$
\begin{aligned}
0 \geq & \frac{2k-1}{k^2} \int_M |\nabla \rho^k|^2 \, \mathrm{d}v \\
& + \int_M \left[n\rho^2 + \Big(\frac{n^2}{2k} - n\Big) H^2 \rho^2 - \sum_{\alpha,\beta} \sigma_{\alpha\beta}^2\right] \rho^{2k-2} \, \mathrm{d}v \\
\geq & \int_M \left[n\rho^2 + \Big(\frac{n^2}{2k} - n\Big) H^2 \rho^2 - \rho^4\right] \rho^{2k-2} \, \mathrm{d}v \\
\geq & \int_M (n - \rho^2) \rho^{2k} \, \mathrm{d}v. \qquad (33)
\end{aligned}
$$

So if $\rho \leq n$, then either $\rho = 0$ and $M$ is a totally umbilical submanifold, or $\rho^2 = n$, for $k < \frac{n}{2}$, from (33), we know that $H = 0$, with the Theorem 3 in [3], we know that $M$ lies in a $(n+1)$-dimensional unit sphere, so the Theorem 5 follows from the Theorem 2. □

## Acknowledgements

The authors thank the referee for useful comments, and they would like to thank Professor Haizhong Li for his useful advice. The first author was supported by NSFC (Grant No. 11426097).

## References

[1] Z. Guo, H. Li: A variational problem for submanifolds in a sphere. *Monatsh. Math.* 152 (2007) 295–302.

[2] D. Hoffman, J. Spruck: Sobolev and isoperimetric inequalities for Riemannian submanifolds. *Comm. Pure. Appl. Math.* 27 (1974) 715–727.

[3] K. Kenmotsu: Some remarks on minimal submanifolds. *Tohoku. Math. J.* 22 (1970) 240–248.

[4] A.-M. Li, J.-M. Li.: An intrinsic rigidity theorem for minimal submanifolds in a sphere. *Arch. Math.* 58 (1992) 582–594.

[5] H. Li.: Willmore hypersurfaces in a sphere. *Asian. J. Math.* 5 (2001) 365–378.

[6] H. Li.: Willmore submanifolds in a sphere. *Math. Res. Letters* 9 (2002) 771–790.

[7] J. Simons.: Minimal varieties in Riemannian manifolds. *Ann. of Math.* 88 (1968) 62–105.

[8] H.-W. Xu, D. Yang.: The gap phenomenon for extremal submanifolds in a Sphere. *Differential Geom and its Applications* 29 (2011) 26–34.

[9] L. Wu.: A class of variational problems for submanifolds in a space form. *Houston J. Math.* 35 (2009) 435–450.

*Authors' address*
School of Mathematics and statistics, Hubei University, Wuhan 430062, P. R. China, Department of Mathematics, Renmin University of China, Beijing 100872, P. R. China

*E-mail:* guoxi@hubu.edu.cn, wulan@ruc.edu.cn

*Received:* 24th February, 2015
*Accepted for publication:* 10th March, 2015
*Communicated by:* Haizhong Li