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Discontinuity of the Fuglede-Kadison determinant on a group von Neumann algebra

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Abstract. We show that in contrast to the case of the operator norm topology on the set of regular operators, the Fuglede-Kadison determinant is not continuous on isomorphisms in the group von Neumann algebra $\mathcal{N}(\mathbb{Z})$ with respect to the strong operator topology. Moreover, in the weak operator topology the determinant is not even continuous on isomorphisms given by multiplication with elements of $\mathbb{Z}[\mathbb{Z}]$. Finally, we define $T \in \mathcal{N}(\mathbb{Z})$ such that for each $\lambda \in \mathbb{R}$ the operator $T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}$ is a self-adjoint weak isomorphism of determinant class but $\lim_{\lambda \to 0} \det(T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}) \neq \det(T)$.

1 Introduction

Fuglede and Kadison [1] introduce their determinant for operators in a finite factor. They prove that, for regular (i.e. invertible) operators, the new determinant shares many algebraic and analytic properties with the usual matrix determinant (which it generalises). That includes continuity with respect to the operator norm. We consider the continuity properties of the generalised Fuglede-Kadison determinant which is used for example by Lück [4, p. 127] to define the topological invariant "L^2-torsion". Let f be an element of a finite von Neumann algebra (N, τ). The (generalised) Fuglede-Kadison determinant of f is

$$\det(f) := \begin{cases} \exp\left(\int_{0+}^{\infty} \ln(\lambda)\, \mathrm{d}F(f)\right), & \text{if } \int_{0+}^{\infty} \ln(\lambda)\, \mathrm{d}F(f) > -\infty, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

In this definition, $F(f)\colon [0,\infty) \to [0,\infty)$ is the spectral density function of f which is defined by $F(f)(\lambda) = \tau(E_{\lambda^2}^{f^*f})$, where $E_{\lambda^2}^{f^*f}$ is a spectral projection of the self--adjoint operator f^*f. The associated measure on the Borel σ-algebra of $\mathbb{R}$ is given by $\mathrm{d}F(f)\,((a,b]) = F(f)(b) - F(f)(a)$ for $a, b \in \mathbb{R}, a < b$. The notation "0+" in 1 means that we omit the possible atom 0 in the domain of integration. The

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omission of that atom is the reason why definition (1) is slightly more general than the original analytic extension of the Fuglede-Kadison determinant to non-regular operators [1, p. 528]. In that extension, all operators with non-zero kernel have determinant zero. In contrast, the generalised Fuglede-Kadison determinant (1) completely ignores the kernel, which leads for example to the odd equation $\det 0 = 1$. However, for injective operators in a finite factor, the original Fuglede-Kadison determinant and its generalisation (1) agree.

Applications

An example of a finite von Neumann algebra is the group von Neumann algebra $\mathcal{N}(G)$ of a discrete group G. It is defined as the set of all operators in $B(l^2(G))$ that commute with the G-action on $l^2(G)$ given by left multiplication. The trace is

$$\begin{aligned} \tau = \operatorname{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) &\to \mathbb{C}\,, \\ T &\mapsto \langle Te, e \rangle_{l^2(G)}\ , \end{aligned} \tag{2}$$

where e is the neutral element of G. In [4, chapter 1], Lück extends that example to the more general theory of morphisms of finite-dimensional Hilbert $\mathcal{N}(G)$-modules. In that context, the Fuglede-Kadison determinant is the main technical ingredient in the definition of L^2-torsion (see [4, chapter 3]).

An important class of operators in $\mathcal{N}(G)$ are those which are given by left multiplication with an element of the integer group ring $\mathbb{Z}G$. Let $(a\colon G \to \mathbb{Z}) \in \mathbb{Z}G$, i.e. $a(g) \neq 0$ for only finitely many $g \in G$. The operator in $\mathcal{N}(G)$ defined by a is

$$\begin{aligned} &A\colon l^2(G) \to l^2(G) \\ &(c_g)_{g\in G} \mapsto \big((Ac)_g\big)_{g\in G}\,, \quad (Ac)_g = \sum_{h\in G} a_h c_{h^{-1}g}\,. \end{aligned}$$

Matrices of such operators are exactly the morphisms of Hilbert $\mathcal{N}(G)$-modules that occur in the study of L^2-invariants of finite free G-CW-complexes. Therefore, determinants of those operators are an important special case of research.

A different example of application of the determinant is the case of the von Neumann algebra associated to an equivalence relation in a probability space, see [2].

Motivation

The motivation to study the continuity properties of the determinant springs from the desire to understand the behaviour under limits of all constructions that use the determinant, e.g. the L^2-torsion invariant.

The few positive results about the continuity properties of the determinant of morphisms of finite-dimensional Hilbert $\mathcal{N}(G)$-modules [4, p. 129] all consider operator norm convergence and follow essentially from the classical dominated or monotone convergence theorems of Lebesgue and Levi. For example, there is the result that for an injective positive morphism $f\colon U \to U$ in a finite-dimensional Hilbert $\mathcal{N}(G)$-module, we have

$$\lim_{\lambda\to 0^+} \det(f + \lambda \cdot \mathrm{id}_U) = \det(f)\,. \tag{3}$$

Naturally, for such continuity results one expects non-trivial counterexamples when the conditions of no classical convergence theorem for integrals are fulfilled. By non--trivial counterexamples, we mean operators which are as regular as possible. By the latter we shall mean that the kernel and the cokernel are as small as possible, the operator has useful properties such as being self-adjoint, and the operator is of determinant class, i.e. has strictly positive determinant.

For the strong and weak operator topologies, positive results are much harder to obtain since the convergence of operators in those topologies does not imply convergence of the spectral density functions of the operators in any usable sense. We are not aware of any published research in the study of the continuity of the determinant with respect to other topologies than the one induced by the operator norm.

Main results

Our three main results are: The determinant is not continuous on all isomorphisms in $\mathcal{N}(\mathbb{Z})$ with respect to the strong operator topology. In the case of the weak operator topology, the example of discontinuity can be constructed within the class of operators in $\mathcal{N}(\mathbb{Z})$ given by left multiplication with elements of $\mathbb{Z}[\mathbb{Z}]$. Considering the operator norm topology, the Fuglede-Kadison determinant can be discontinuous at $\lambda = 0$ on a line $\{T+\lambda\cdot \mathrm{id}_{l^2(\mathbb{Z})} \mid \lambda \in \mathbb{R}\}$ that consists entirely of weak isomorphisms of determinant class. That is a non-trivial counterexample to (3) in absence of positivity. In all cases the operators are constructed explicitly and the short proofs of their properties suggest how one might construct similar "pathologic" examples in other situations.

Method

The basis for the construction of our examples is the following model for the group von Neumann algebra of the integers. Lück remarks in [4, p. 15] that there is an isometric $\star$-algebra-isomorphism $\mathcal{N}(\mathbb{Z}) \cong L^\infty(S^1)$, where $L^\infty(S^1)$ is identified with the set of pointwise multiplication operators $\{M_g \mid g \in L^\infty(S^1)\} \subset B(L^2(S^1))$ and the involution on $L^\infty(S^1)$ is pointwise complex conjugation. That isomorphism of algebras is induced by an isometry of Hilbert spaces

$$\begin{aligned} l^2(\mathbb{Z}) &\xrightarrow{\cong} L^2(S^1)\,, \\ (a_k)_{k\in\mathbb{Z}} &\longmapsto \Big(z \mapsto \sum_{k\in\mathbb{Z}} a_k z^k\Big), \quad z = \mathrm{e}^{\mathrm{i}\varphi} \in \mathbb{C}\,. \end{aligned} \tag{4}$$

Note that (4) implies that an operator in $\mathcal{N}(\mathbb{Z})$ given by left multiplication with an element $(a_k)_{k\in\mathbb{Z}} \in \mathbb{C}[\mathbb{Z}]$ is identified with the polynomial $\sum_{k\in\mathbb{Z}} a_k z^k$ in $L^\infty(S^1)$. The identification $\mathcal{N}(\mathbb{Z}) \cong L^\infty(S^1)$ allows for simple constructions of concrete morphisms in $\mathcal{N}(\mathbb{Z})$ with prescribed spectral density functions. Moreover, under the identification there is the following simple formula for the determinant [4, p. 128]:

$$\ln \det g = \int_{S^1} \ln|g(z)| \cdot \chi_{\{u\in S^1 \mid g(u)\neq 0\}} \,\mathrm{d}\,\mathrm{vol}_z, \quad g \in L^\infty(S^1) \tag{5}$$

where $\mathrm{d}\,\mathrm{vol}_z$ is the usual "round" measure on S^1, scaled such that $\mathrm{vol}(S^1) = 1$.

At this point we would like to remark that although $\mathcal{N}(\mathbb{Z}) \cong L^\infty(S^1)$ is not a type II factor (because $\mathbb{Z}$ has no infinite conjugacy classes, see [3, Theorem 6.7.5]), $L^\infty(S^1)$ can be embedded into a type II factor as a maximal commutative subalgebra, by the classical group measure space construction. Therefore, operators in $\mathcal{N}(\mathbb{Z})$ can be regarded as elements in a type II factor. Moreover, since all operators involved in our counterexamples are injective, their determinant agrees in fact with the original Fuglede-Kadison determinant from [1], which means that our results apply in particular to the original Fuglede-Kadison determinant.

2 Discontinuity in the Weak Operator Topology

Proposition 1. *There is a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{N}(\mathbb{Z})$ of isomorphisms, given by left multiplication with elements in $\mathbb{Z}[\mathbb{Z}]$, which converges to $\mathrm{id}_{l^2(\mathbb{Z})}$ with respect to the weak operator topology but $\lim_{n\to\infty} \det(A_n) \neq 1 = \det(\mathrm{id}_{l^2(\mathbb{Z})})$.*

Proof. Define A_n to be left multiplication with $({}^n a_k)_{k\in\mathbb{Z}}$, where ${}^n a_0 = 1$, ${}^n a_n = 2$ and ${}^n a_k = 0$ for all $k \in \mathbb{Z}$ other than 0 and n. Under the isometric isomorphism $\mathcal{N}(\mathbb{Z}) \cong L^\infty(S^1)$, A_n corresponds to the polynomial

$$p_n(z) := 1 + 2z^n, \quad z \in S^1 \subset \mathbb{C}\,.$$

The polynomial $1 + 2z^n$ on S^1 is bounded away from zero for each $n \in \mathbb{N}$. Hence A_n is invertible with inverse the operator corresponding to $1/p_n \in L^\infty(S^1)$. In [4, p. 136], Lück proves an example that implies $\det(A_n) = 2$ for all $n \in \mathbb{N}$. From (2) follows immediately that $\det(\mathrm{id}_{l^2(\mathbb{Z})}) = 1$. What is left to show is that $(A_n)_{n\in\mathbb{N}}$ converges to $\mathrm{id}_{l^2(\mathbb{Z})}$ in the weak operator topology as $n \to \infty$. Let $f, g \in C^\infty(S^1)$. Then we have

$$\begin{aligned}\left|\frac{1}{2}\big\langle (p_n - 1)f, g\big\rangle_{L^2(S^1)}\right| &= \left|\int_{S^1} z^n f(z)\,\overline{g(z)}\,\mathrm{d}\,\mathrm{vol}_z\right| \\ &= \left|\int_0^{2\pi} \mathrm{e}^{\mathrm{i}nt} f\big(\exp(\mathrm{i}t)\big)\,\overline{g\big(\exp(\mathrm{i}t)\big)}\,\mathrm{d}t\right| \\ &= \left|\int_0^{2\pi} \frac{1}{\mathrm{i}n}\mathrm{e}^{\mathrm{i}nt}\frac{\mathrm{d}}{\mathrm{d}t}(f\overline{g})\big(\exp(\mathrm{i}t)\big)\,\mathrm{d}t\right| \\ &\le \frac{1}{n}\big\|(fg)'\big\|_\infty \overset{n\to\infty}{\longrightarrow} 0\,. \end{aligned} \tag{6}$$

Now, since $C^\infty(S^1)$ is dense in $L^2(S^1)$ with respect to its Hilbert space norm, we can conclude from (6) that the sequence of operators in $B(L^2(S^1))$ given by pointwise multiplication with p_n converges to $\mathrm{id}_{L^2(S^1)}$ in the weak operator topology as $n \to \infty$. The corresponding claim about the sequence $(A_n)_{n\in\mathbb{N}}$ follows immediately. □

Note that the sequence $(A_n)_{n\in\mathbb{N}}$ from the previous proof does not converge to $\mathrm{id}_{l^2(\mathbb{Z})}$ in the strong operator topology: For each polynomial p_n corresponding to A_n, we have $\|(p_n - 1)f\|_{L^2(S^1)} = 2\,\|f\|_{L^2(S^1)}$ for all $f \in L^2(S^1)$.

3 Discontinuity in the Strong Operator Topology

Proposition 2. *Let $r = (r_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers (e.g. $r_n = \sin(n) + 1$). There is a sequence of isomorphisms $(f_n^r)_{n\in\mathbb{N}} \subset \mathcal{N}(\mathbb{Z})$ such that $f_n^r \to \mathrm{id}_{l^2(\mathbb{Z})}$ in the strong operator topology as $n \to \infty$ and $\det(f_n^r) = \exp(-r_n)$.*

Proof. We use the identification $L^\infty(S^1) \cong \mathcal{N}(\mathbb{Z})$. Let for $n \in \mathbb{N}$ the operator f_n^r correspond to the function $g_n^r \in L^\infty(S^1)$ given by

$$g_n^r\left(\exp(2\pi i t)\right) := \begin{cases} \exp\left(-n \cdot r_n\right), & 0 < t \leq \frac{1}{n}, \\ 1, & \frac{1}{n} < t \leq 1. \end{cases}$$

Then f_n^r is self-adjoint, as g_n^r is real, and invertible with inverse the operator corresponding to the well-defined function $1/g_n^r \in L^\infty(S^1)$.

We prove first that f_n^r converges to $\mathrm{id}_{l^2(\mathbb{Z})}$ in the strong operator topology. This is equivalent to proving that the pointwise multiplication operator $M_{g_n^r} \in B(L^2(S^1))$ converges to $\mathrm{id}_{L^2(S^1)}$.

Let $h \in L^2(S^1)$.

$$\begin{aligned} \left\|M_{g_n^r}(h) - h\right\|_{L^2(S^1)} &= \int_{S^1} \left|g_n^r(z)h(z) - h(z)\right|^2 \mathrm{d}\,\mathrm{vol}_z \\ &= \int_0^1 \left|g_n^r\big(\exp(2\pi i t)\big)h\big(\exp(2\pi i t)\big) - h\big(\exp(2\pi i t)\big)\right|^2 \mathrm{d}t \\ &= \int_0^{1/n} \left|h\big(\exp(2\pi i t)\big)\big(\exp(-n\cdot r_n) - 1\big)\right|^2 \mathrm{d}t \\ &\leq \int_0^{1/n} \left|h\big(\exp(2\pi i t)\big)\right|^2 \mathrm{d}t\,. \end{aligned}$$

The final integral converges to zero as $n \to \infty$ due to σ-additivity of Lebesgue measure. The calculation of the determinant of f_n^r is a very easy task using (5):

$$\begin{aligned} \ln\det(f_n^r) &= \int_{S^1} \ln\big(|g_n^r(z)|\big) \cdot \chi_{\{u\in S^1 \mid g_n^r(u)\neq 0\}} \,\mathrm{d}\,\mathrm{vol}_z \\ &= \int_0^1 \ln\big(g_n^r(\exp(2\pi i t))\big)\mathrm{d}t \\ &= \int_0^{1/n} \ln\big(\exp\left(-n\cdot r_n\right)\big)\mathrm{d}t \\ &= -r_n\,. \end{aligned}$$

□

4 Discontinuity in the Operator Norm Topology

Define $T \in \mathcal{N}(\mathbb{Z})$ as the operator corresponding to $g \in L^\infty(S^1)$, where

$$g\left(\exp(2\pi \mathrm{i} x)\right) := \left(\frac{1}{n} - x\right)^{n(n+1)} - \mathrm{e}^{-\sqrt{n-1}}, \quad x \in \left(\frac{1}{n+1}, \frac{1}{n}\right], \ n \in \mathbb{N}. \tag{7}$$

The shape of the graph of g is illustrated in Figure 1 below. One property of g is that for all $x \in (\frac{1}{n+1}, \frac{1}{n}]$, $n \in \mathbb{N}$, there is a $\delta_n > 0$ such that

$$-\mathrm{e}^{-\sqrt{n}} - \delta_n \geq g\left(\exp(2\pi \mathrm{i} x)\right) \geq -\mathrm{e}^{-\sqrt{n-1}}.$$

The statement in line (4) can be verified using

$$\left|\left(\frac{1}{n} - x\right)^{n(n+1)}\right| \leq \left(\frac{1}{n} - \frac{1}{n+1}\right)^{n(n+1)} = \left(\frac{1}{n(n+1)}\right)^{n(n+1)}$$
$$\leq \frac{1}{2}\left(\mathrm{e}^{-\sqrt{n-1}} - \mathrm{e}^{-\sqrt{n}}\right),$$

where the second inequality is a straightforward check. For example, we can set $\delta_n := \frac{1}{3}(\mathrm{e}^{-\sqrt{n-1}} - \mathrm{e}^{-\sqrt{n}})$. Note that (4) implies that $x \mapsto g\left(\exp(2\pi \mathrm{i} x)\right)$ is a strictly decreasing function since it is strictly decreasing on each interval $(\frac{1}{n+1}, \frac{1}{n}]$.

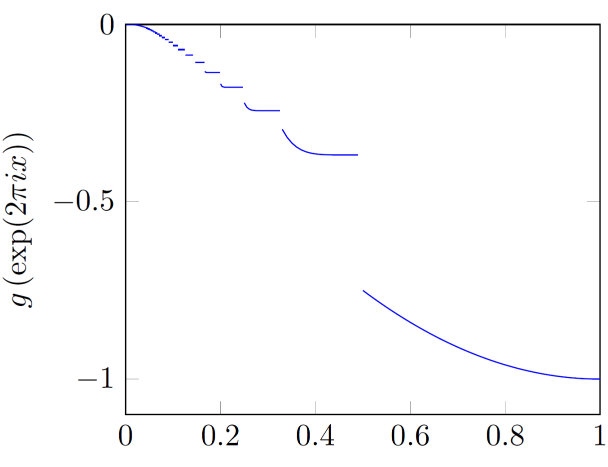

Figure 1: Qualitative picture of the graph of g. The slope of the $\left(\frac{1}{n} - x\right)^{n(n+1)}$-segments is strongly exaggerated.

4.1 Verification of the properties of T

Note that for $\lambda \in \mathbb{R}$, the operator $T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}$ corresponds to $g + \lambda \cdot 1$, where 1 is the "constant 1" function on S^1.

Proposition 3. *For each $\lambda \in \mathbb{R}$ the operator $T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}$ is a weak isomorphism.*

Proof. As T is self-adjoint, the claim is equivalent to the claim that $T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}$ is injective, i.e. for $\lambda \in \mathbb{R}$ the zero locus of $g + \lambda \cdot 1$ is a null set in S^1. Since $x \mapsto g(\exp(2\pi \mathrm{i} x))$ is strictly decreasing, $g + \lambda \cdot 1$ can have at most one zero. $\square$

Proposition 4. *For each $\lambda \in \mathbb{R}$ the operator $T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}$ is of determinant class.*

Proof. To simplify notation, set $\gamma_\lambda(x) := g(\exp(2\pi \mathrm{i} x)) + \lambda$ for $x \in (0, 1]$.

Case $\lambda = 0$: We use equation (5).

$$\begin{aligned}
\ln \det(T) &= \int_{S^1} \ln(|g(z)|) \cdot \chi_{\{u \in S^1 \,|\, g(u) \neq 0\}} \,\mathrm{d}\,\mathrm{vol}_z = \sum_{n \in \mathbb{N}} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \ln |\gamma_\lambda(x)| \,\mathrm{d}x \\
&\overset{(4)}{\geq} \sum_{n \in \mathbb{N}} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \ln\left(\mathrm{e}^{-\sqrt{n}}\right) \mathrm{d}x \\
&= \sum_{n \in \mathbb{N}} \left(\frac{1}{n} - \frac{1}{n+1}\right) \cdot (-\sqrt{n}) = \sum_{n \in \mathbb{N}} \frac{-1}{\sqrt{n}(n+1)} \\
&> -\infty\,.
\end{aligned}$$

Case $\lambda = \mathrm{e}^{-\sqrt{m-1}}$, $m \in \mathbb{N}$: Again, we use equation (5).

$$\begin{aligned}
\ln \det\left(T + \lambda_m \cdot \mathrm{id}_{l^2(\mathbb{Z})}\right) &= \sum_{n \in \mathbb{N}} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \ln |\gamma_\lambda(x)| \,\mathrm{d}x \\
&\geq \int_{\frac{1}{m+1}}^{\frac{1}{m}} \ln |\gamma_\lambda(x)| \,\mathrm{d}x + \int_0^1 \ln \min\{\delta_m, \delta_{m-1}\} \,\mathrm{d}x \\
&= \int_{\frac{1}{m+1}}^{\frac{1}{m}} \ln\left(\left(\frac{1}{m} - x\right)^{m(m+1)}\right) \mathrm{d}x + \ln \min\{\delta_m, \delta_{m-1}\} \\
&= \ln\left(\frac{1}{m(m+1)}\right) - 1 + \ln\left(\min\{\delta_m, \delta_{m-1}\}\right) \\
&> -\infty\,. \qquad (8)
\end{aligned}$$

Case $\lambda = \mathrm{e}^{-\sqrt{m-1}} + (\frac{1}{m} - \frac{1}{m+1})^{m(m+1)}$, $m \in \mathbb{N}$: The point $(\frac{1}{m+1}, 0)$ is a limit point of the graph of γ_λ. We have $|\gamma_\lambda(x)| \geq \delta_m$ for all $x \in (\frac{1}{m+2}, \frac{1}{m+1}]$. Note that $\gamma_\lambda|_{(\frac{1}{m+1}, \frac{1}{m}]}$ is a polynomial whose derivative has a right limit for $x \to \frac{1}{m+1}^+$ which is strictly greater than zero. If d_m is that limit, we can find $\varepsilon > 0$ such that $|\gamma_\lambda(x)| \geq \frac{d_m}{2} |x - \frac{1}{m+1}|$ for all $x \in [\frac{1}{m+1} - \varepsilon, \frac{1}{m+1} + \varepsilon]$. On $(0, 1] \backslash [\frac{1}{m+1} - \varepsilon, \frac{1}{m+1} + \varepsilon]$,

γ_λ is bounded away from zero by some bound $\delta > 0$. We can estimate using (5):

$$\begin{aligned}
\ln\det\left(T + \lambda \cdot \mathrm{id}_{l^2(\mathbb{Z})}\right) &= \sum_{n\in\mathbb{N}} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \ln|\gamma_\lambda(x)| \,\mathrm{d}x \\
&\geq \int_{\frac{1}{m+1}-\varepsilon}^{\frac{1}{m+1}+\varepsilon} \ln\left|\frac{1}{2}d_m\left(x - \frac{1}{m+1}\right)\right| \mathrm{d}x + \int_0^1 \ln\delta \,\mathrm{d}x \\
&= 2\varepsilon\left(\ln\left(\frac{1}{2}d_m\right) + \ln(\varepsilon) - 1\right) + \ln\delta \\
&> -\infty\,.
\end{aligned}$$

Case $\mathrm{e}^{-\sqrt{m-1}} < \lambda < \mathrm{e}^{-\sqrt{m-1}} + \left(\frac{1}{m} - \frac{1}{m+1}\right)^{m(m+1)}$, $m \in \mathbb{N}$: The graph of γ_λ cuts the x-axis at some $x_0 \in (\frac{1}{m+1}, \frac{1}{m})$. We can proceed as in the previous case.

For other $\lambda \in \mathbb{R}$: The function γ_λ is bounded away from 0 so the case is trivial. □

Proposition 5. *There is a sequence* $(\lambda_m)_{m\in\mathbb{N}} \subset (0,1]$ *converging to zero such that* $\det\left(T + \lambda_m \cdot \mathrm{id}_{l^2(\mathbb{Z})}\right) < \frac{1}{m(m+1)}$. *So* $\lim_{m\to\infty}\det\left(T + \lambda_m \cdot \mathrm{id}_{l^2(\mathbb{Z})}\right) = 0 \neq \det(T)$.

Proof. Set $\lambda_m := \mathrm{e}^{-\sqrt{m-1}}$. Similarly as in the previous proof, use (5):

$$\begin{aligned}
\ln\det\left(T + \lambda_m \cdot \mathrm{id}_{l^2(\mathbb{Z})}\right) &= \sum_{n\in\mathbb{N}} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \ln\left|g\left(\exp(2\pi\mathrm{i}x)\right) + \lambda_m\right| \mathrm{d}x \\
&\leq \int_{\frac{1}{m+1}}^{\frac{1}{m}} \ln\left|g\left(\exp(2\pi\mathrm{i}x)\right) + \lambda_m\right| \mathrm{d}x && (9) \\
&= \ln\left(\frac{1}{m(m+1)}\right) - 1\,. && (10)
\end{aligned}$$

In line (9) we used that the summands in the previous line are non-positive. In line (10) we used the estimate (8). □

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