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Existence of entropy solutions for degenerate quasilinear elliptic equations in L^1

Albo Carlos Cavalheiro

Abstract. In this article, we prove the existence of entropy solutions for the Dirichlet problem

$$(P) \begin{cases} -\operatorname{div}[\omega(x)\mathcal{A}(x, u, \nabla u)] = f(x) - \operatorname{div}(G), & \text{in } \Omega \\ u(x) = 0, & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded open set of \mathbb{R}^N , $N \geq 2$, $f \in L^1(\Omega)$ and $G/\omega \in [L^{p'}(\Omega, \omega)]^N$.

1 Introduction

The main purpose of this article (see Theorem 2) is to establish the existence of entropy solutions for the Dirichlet problem

$$(P) \begin{cases} -\operatorname{div}[\omega(x)\mathcal{A}(x, u, \nabla u)] = f(x) - \operatorname{div}(G) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set, ω is a weight function (i.e., a locally integrable function on \mathbb{R}^N such that $0 < \omega(x) < \infty$ a.e. $x \in \mathbb{R}^N$), $f \in L^1(\Omega)$, $G = (g_1, \dots, g_N)$ with $G/\omega \in [L^{p'}(\Omega, \omega)]^N$, and the function

$$\mathcal{A}: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$$

satisfies the following conditions:

- (H1) $x \mapsto \mathcal{A}(x, s, \xi)$ is measurable on Ω for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $(s, \xi) \mapsto \mathcal{A}(x, s, \xi)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$ for almost all $x \in \Omega$.
- (H2) $\langle \mathcal{A}(x, s, \xi_1) - \mathcal{A}(x, s, \xi_2), \xi_1 - \xi_2 \rangle > 0$, whenever $\xi_1, \xi_2 \in \mathbb{R}^N$, $\xi_1 \neq \xi_2$ (where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in \mathbb{R}^N).

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(H3) $\langle \mathcal{A}(x, s, \xi), \xi \rangle \geq \lambda |\xi|^p$, with $1 < p < \infty$, and $\lambda > 0$.

(H4) $|\mathcal{A}(x, s, \xi)| \leq K(x) + h_1(x) |s|^{p/p'} + h_2(x) |\xi|^{p/p'}$, where K , h_1 and h_2 are positive functions, with $h_1 \in L^\infty(\Omega)$, $h_2 \in L^\infty(\Omega)$ and $K \in L^{p'}(\Omega, \omega)$ (where $1/p + 1/p' = 1$).

If $f/\omega \in L^{p'}(\Omega, \omega)$ (with $1 < p < \infty$), the problem (P) has been studied in [4], and in this case the problem (P) has a solution $u \in W^{1,p}(\Omega, \omega)$. However, since $L^1(\Omega)$ is not a subspace of $W^{-1,p'}(\Omega, \omega)$ so when we want to consider $f \in L^1(\Omega)$ a different theory is needed.

In [1], a new concept of solution has been introduced for the elliptic equation

$$\begin{cases} -\operatorname{div}[a(x, \nabla u)] = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(when $f \in L^1(\Omega)$) namely entropy solution. In [3] the author studied the degenerate elliptic equation $Lu = f$, where L is a degenerate elliptic operator in divergence form, i.e.,

$$Lu = - \sum_{i,j=1}^n D_j(a_{ij}(x)D_i u),$$

and $f \in L^1(\Omega)$. Note that, in the proof of our main result, many ideas have been adapted from [1] and [3].

For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [5], [6], [8], [9] and [13]).

A class of weights, which is particularly well understood, is the class of A_p weights that was introduced by B. Muckenhoupt in the early 1970s (see [11]).

We propose to solve the problem (P) by approximation with variational solutions: we take $f_n \in C_0^\infty(\Omega)$ such that $f_n \rightarrow f$ in $L^1(\Omega)$, $G_n/\omega \in [L^{p'}(\Omega, \omega)]^N$ such that $G_n/\omega \rightarrow G/\omega$ in $[L^{p'}(\Omega, \omega)]^N$, we find a solution $u_n \in W_0^{1,p}(\Omega, \omega)$ for the problem with right-hand side f_n and G_n , and we will try to pass to the limit as $n \rightarrow \infty$.

2 Definitions and basic results

By a weight we shall mean a locally integrable function ω on \mathbb{R}^N such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^N$. Every weight ω gives rise to a measure on the measurable subsets of \mathbb{R}^N through integration. This measure will be denoted by μ . Thus, $\mu(E) = \int_E \omega(x) dx$ for measurable sets $E \subset \mathbb{R}^N$.

Definition 1. Let $1 \leq p < \infty$. A weight ω is said to be an A_p -weight, if there is a positive constant $C = C(p, \omega)$ such that for every ball $B \subset \mathbb{R}^N$

$$\begin{aligned} \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\frac{1}{|B|} \int_B \omega^{1/(1-p)}(x) dx \right)^{p-1} &\leq C \quad \text{if } p > 1, \\ \left(\frac{1}{|B|} \int_B \omega(x) dx \right) \left(\operatorname{ess\,sup}_{x \in B} \frac{1}{\omega(x)} \right) &\leq C \quad \text{if } p = 1, \end{aligned}$$

where $|\cdot|$ denotes the N -dimensional Lebesgue measure in \mathbb{R}^N .

If $1 < q \leq p$, then $A_q \subset A_p$ (see [8], [9] or [14] for more information about A_p -weights). As an example of an A_p -weight, the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^N$, is in A_p if and only if, $-N < \alpha < N(p-1)$ (see [12], Chapter IX, Corollary 4.4). If $\varphi \in BMO(\mathbb{R}^N)$ then $\omega(x) = e^{\alpha\varphi(x)} \in A_2$ for some $\alpha > 0$ (see [12]).

Remark 1. If $\omega \in A_p$, $1 < p < \infty$, then

$$\left(\frac{|E|}{|B|}\right)^p \leq C \frac{\mu(E)}{\mu(B)}$$

for all measurable subsets E of B (see 15.5 strong doubling property in [9]). Therefore if $\mu(E) = 0$ then $|E| = 0$. Thus, if $\{u_n\}$ is a sequence of functions defined in B and $u_n \rightarrow u$ μ -a.e. then $u_n \rightarrow u$ a.e..

Definition 2. Let ω be a weight. We shall denote by $L^p(\Omega, \omega)$ ($1 \leq p < \infty$) the Banach space of all measurable functions f defined in Ω for which

$$\|f\|_{L^p(\Omega, \omega)} = \left(\int_{\Omega} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty.$$

We denote $[L^{p'}(\Omega, \omega)]^N = L^{p'}(\Omega, \omega) \times \dots \times L^{p'}(\Omega, \omega)$.

Remark 2. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$ (see [14], Remark 1.2.4). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 3. Let $\Omega \subset \mathbb{R}^N$ a bounded open set, $1 < p < \infty$, k a nonnegative integer and $\omega \in A_p$. We shall denote by $W^{k,p}(\Omega, \omega)$, the weighted Sobolev spaces, the set of all functions $u \in L^p(\Omega, \omega)$ with weak derivatives $D^\alpha u \in L^p(\Omega, \omega)$, $1 \leq |\alpha| \leq k$. The norm in the space $W^{k,p}(\Omega, \omega)$ is defined by

$$\|u\|_{W^{k,p}(\Omega, \omega)} = \left(\int_{\Omega} |u(x)|^p \omega(x) dx + \sum_{1 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u(x)|^p \omega(x) dx \right)^{1/p}. \quad (1)$$

We also define the space $W_0^{k,p}(\Omega, \omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,p}(\Omega, \omega)} = \left(\sum_{1 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u(x)|^p \omega(x) dx \right)^{1/p}.$$

The dual space of $W_0^{1,p}(\Omega, \omega)$ is the space $[W_0^{1,p}(\Omega, \omega)]^* = W^{-1,p'}(\Omega, \omega)$,

$$W^{-1,p'}(\Omega, \omega) = \left\{ T = f - \text{div}(G) : G = (g_1, \dots, g_N), \frac{f}{\omega}, \frac{g_j}{\omega} \in L^{p'}(\Omega, \omega) \right\}.$$

It is evident that a weight function ω which satisfies $0 < C_1 \leq \omega(x) \leq C_2$, for a.e. $x \in \Omega$, gives nothing new (the space $W^{k,p}(\Omega, \omega)$ is then identical with the classical Sobolev space $W^{k,p}(\Omega)$). Consequently, we study all such weight function ω that either vanish in $\Omega \cup \partial\Omega$ or increase to infinity (or both).

We need the following basic result.

Theorem 1. (*The weighted Sobolev inequality*) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let ω be an A_p -weight, $1 < p < \infty$. Then there exist positive constants C_Ω and δ such that for all $f \in C_0^\infty(\Omega)$ and $1 \leq \eta \leq N/(N-1) + \delta$

$$\|f\|_{L^{\eta p}(\Omega, \omega)} \leq C_\Omega \|\nabla f\|_{L^p(\Omega, \omega)}. \quad (2)$$

Proof. See [6], Theorem 1.3. \square

Definition 4. We say that $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$ if $T_k(u) \in W_0^{1,p}(\Omega, \omega)$, for all $k > 0$, where the function $T_k: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k \\ k \operatorname{sign}(s), & \text{if } |s| > k. \end{cases}$$

Remark 3. (i) Note that for given $h > 0$ and $k > 0$ we have

$$T_h(u - T_k(u)) = \begin{cases} 0, & \text{if } |u| \leq k \\ (|u| - k) \operatorname{sign}(u), & \text{if } k < |u| \leq k + h \\ h \operatorname{sign}(u), & \text{if } |u| > k + h. \end{cases}$$

And if $\alpha \in \mathbb{R}$, $\alpha \neq 0$, we have $T_k(\alpha u) = \alpha T_{k/|\alpha|}(u)$.

(ii) If $u \in W_{\text{loc}}^{1,1}(\Omega, \omega)$ then we have

$$\nabla T_k(u) = \chi_{\{|u| < k\}} \nabla u$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^N$.

Definition 5. Let $f \in L^1(\Omega)$, $G/\omega \in [L^{p'}(\Omega, \omega)]^N$ and $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$. We say that u is an entropy solution to problem (P) if

$$\int_{\Omega} \langle \mathcal{A}(x, u, \nabla u), \nabla T_k(u - \varphi) \rangle \omega \, dx = \int_{\Omega} f T_k(u - \varphi) \, dx + \int_{\Omega} \langle G, \nabla T_k(u - \varphi) \rangle \, dx \quad (3)$$

for all $k > 0$ and all $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$.

We recall that the gradient of u which appears in (3) is defined as in Remark 2.8 of [3], that is to say that $\nabla u = \nabla T_k(u)$ on the set where $|u| < k$.

Remark 4. Note that if $u_1, u_2 \in W_0^{1,p}(\Omega, \omega)$ then

$$\varphi = T_k(u_1 + u_2) \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$$

and we have

$$\nabla \varphi = \nabla T_k(u_1 + u_2) = \nabla(u_1 + u_2) \chi_{\{|u_1 + u_2| \leq k\}}.$$

Definition 6. Let $0 < p < \infty$ and let ω be a weight function. We define the weighted Marcinkiewicz space $\mathcal{M}^p(\Omega, \omega)$ as the set of all measurable functions $f: \Omega \rightarrow \mathbb{R}$ such that the function

$$\Gamma_f(k) = \mu(\{x \in \Omega : |f(x)| > k\}), \quad k > 0,$$

satisfies an estimate of the form $\Gamma_f(k) \leq Ck^{-p}$, $0 < C < \infty$.

Remark 5. If $1 \leq q < p$ and $\Omega \subset \mathbb{R}^N$ is a bounded set, we have that

$$L^p(\Omega, \omega) \subset \mathcal{M}^p(\Omega, \omega) \text{ and } \mathcal{M}^p(\Omega, \omega) \subset L^q(\Omega, \omega)$$

(the proof follows the lines of Theorem 2.18.8 in [10]).

Lemma 1. Let $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$ and $\omega \in A_p$, $1 < p < \infty$, be such that

$$\frac{1}{k} \int_{\{|u| < k\}} |\nabla u|^p \omega \, dx \leq M, \tag{4}$$

for every $k > 0$. Then $u \in \mathcal{M}^{p_1}(\Omega, \omega)$, where $p_1 = \eta(p-1)$ (where η is the constant in Theorem 1). More precisely, there exists $C > 0$ such that $\Gamma_u(k) \leq CM^\eta k^{-p_1}$.

Proof. See Lemma 3.3 in [3]. □

Lemma 2. Let $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$, where $\omega \in A_p$, $1 < p < \infty$, be such that

$$\frac{1}{k} \int_{\{|u| < k\}} |\nabla u|^p \omega \, dx \leq M,$$

for every $k > 0$. Then $|\nabla u| \in \mathcal{M}^{p_2}(\Omega, \omega)$, where $p_2 = p p_1 / (p_1 + 1)$ (with η as in Lemma 2 and $p_1 = \eta(p-1)$). More precisely, there exists $C > 0$ such that

$$\Gamma_k(|\nabla u|) \leq CM^{(p_1+\eta)/(p_1+1)} k^{-p_2}.$$

Proof. See Lemma 3.4 in [3]. □

3 Main Result

In this section, we prove the main result of this paper. We need the following result.

Lemma 3. Let $\omega \in A_p$, $1 < p < \infty$ and a sequence $\{u_n\}$, $u_n \in W_0^{1,p}(\Omega, \omega)$ satisfies

- (i) $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, \omega)$ and μ -a.e. in Ω ;
- (ii) $\int_{\Omega} \langle \mathcal{A}(x, u_n, \nabla u_n) - \mathcal{A}(x, u_n, \nabla u), \nabla(u_n - u) \rangle \omega \, dx \rightarrow 0$ with $n \rightarrow \infty$.

Then $u_n \rightarrow u$ in $W_0^{1,p}(\Omega, \omega)$.

Proof. The proof of this lemma follows the lines of Lemma 5 in [2]. □

Theorem 2. Let $\omega \in A_p$, $1 < p < \infty$, and $\mathcal{A}(x, s, \xi)$ satisfies the conditions (H1), (H2), (H3) and (H4). Then, there exists an entropy solution u of problem (P). Moreover, $u \in \mathcal{M}^{p_1}(\Omega, \omega)$ and $|\nabla u| \in \mathcal{M}^{p_2}(\Omega, \omega)$, with $p_1 = \eta(p-1)$ and $p_2 = p_1 p / (p_1 + 1)$ (where η is the constant in Theorem 1).

Proof. Considering a sequence $\{f_n\}$, $f_n \in C_0^\infty(\Omega)$, which

$$f_n \rightarrow f \text{ in } L^1(\Omega) \text{ and } \|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)},$$

and a sequence $\{G_n\}$, with $G_n/\omega \in [L^{p'}(\Omega, \omega)]^N$ such that

$$\frac{G_n}{\omega} \rightarrow \frac{G}{\omega} \text{ in } [L^{p'}(\Omega, \omega)]^N \text{ and } \left\| \frac{|G_n|}{\omega} \right\|_{L^{p'}(\Omega, \omega)} \leq \left\| \frac{|G|}{\omega} \right\|_{L^{p'}(\Omega, \omega)}.$$

For each n , there exists a solution $u_n \in W_0^{1,p}(\Omega, \omega)$ of the Dirichlet problem

$$(P_n) \begin{cases} -\operatorname{div}[\omega(x)\mathcal{A}(x, u_n, \nabla u_n)] = f_n(x) - \operatorname{div}(G_n), & \text{in } \Omega \\ u_n(x) = 0, & \text{on } \partial\Omega \end{cases}$$

(by Theorem 1.1 in [4]) that is,

$$\int_{\Omega} \omega \langle \mathcal{A}(x, u_n, \nabla u_n), \nabla \varphi \rangle dx = \int_{\Omega} f_n \varphi dx + \int_{\Omega} \langle G_n, \nabla \varphi \rangle dx, \quad (5)$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega)$. For $\varphi = T_k(u_n)$ we obtain in (5) that

$$\int_{\Omega} \omega \langle \mathcal{A}(x, u_n, \nabla u_n), \nabla T_k(u_n) \rangle dx = \int_{\Omega} f_n T_k(u_n) dx + \int_{\Omega} \langle G_n, \nabla T_k(u_n) \rangle dx. \quad (6)$$

Using (H3) and Remark 3 (ii) we have,

$$\begin{aligned} \int_{\Omega} \omega \langle \mathcal{A}(x, u_n, \nabla u_n), \nabla T_k(u_n) \rangle dx &= \int_{\Omega} \omega \langle \mathcal{A}(x, u_n, \nabla T_k(u_n)), \nabla T_k(u_n) \rangle dx \\ &\geq \lambda \int_{\Omega} |\nabla T_k(u_n)|^p \omega dx. \end{aligned}$$

We also have

$$\left| \int_{\Omega} f_n T_k(u_n) dx \right| \leq \int_{\Omega} |f_n| |T_k(u_n)| dx \leq k \|f_n\|_{L^1(\Omega)} \leq k \|f\|_{L^1(\Omega)},$$

and using Young's inequality there exists a constant $C_1 > 0$ such that

$$\begin{aligned} \left| \int_{\Omega} \langle G_n, \nabla T_k(u_n) \rangle dx \right| &\leq \int_{\Omega} \left| \frac{G_n}{\omega} \right| |\nabla T_k(u_n)| \omega dx \\ &\leq \left(\int_{\Omega} \left| \frac{G_n}{\omega} \right|^{p'} \omega dx \right)^{1/p'} \left(\int_{\Omega} |\nabla T_k(u_n)|^p \omega dx \right)^{1/p} \\ &\leq \frac{\lambda}{2} \int_{\Omega} |\nabla T_k(u_n)|^p \omega dx + C_1 \int_{\Omega} \left| \frac{G_n}{\omega} \right|^{p'} \omega dx \\ &\leq \frac{\lambda}{2} \int_{\Omega} |\nabla T_k(u_n)|^p \omega dx + C_1 \int_{\Omega} \left| \frac{G}{\omega} \right|^{p'} \omega dx. \end{aligned}$$

Hence in (6) we obtain

$$\lambda \int_{\Omega} |\nabla T_k(u_n)|^p \omega \, dx \leq k \|f\|_{L^1(\Omega)} + \frac{\lambda}{2} \int_{\Omega} |\nabla T_k(u_n)|^p \omega \, dx + C_1 \int_{\Omega} \left| \frac{G}{\omega} \right|^{p'} \omega \, dx,$$

and

$$\begin{aligned} \int_{\Omega} |\nabla T_k(u_n)|^p \omega \, dx &\leq \frac{k}{\lambda} \left(\|f\|_{L^1(\Omega)} + C_1 \left\| \frac{G}{\omega} \right\|_{L^{p'}(\Omega, \omega)}^{p'} \right) \\ &= C_2 k, \quad \text{for all } k > 0. \end{aligned} \tag{7}$$

By Lemma 1 and Lemma 2, we have that the sequence $\{u_n\}$ is bounded in $\mathcal{M}^{p_1}(\Omega, \omega)$ (with $p_1 = \eta(p-1)$) and $\{|\nabla u_n|\}$ is bounded in $\mathcal{M}^{p_2}(\Omega, \omega)$ (with $p_2 = p_1 p / (p_1 + 1)$). Moreover, $\{u_n\}$ is a Cauchy sequence in μ -measure. Consequently, there exists a function u and a subsequence, that we will still denote by $\{u_n\}$, such that

$$u_n \rightarrow u \quad \mu\text{-a.e. in } \Omega, \tag{8}$$

and $u_n \rightarrow u$ a.e. in Ω (by Remark 1).

Using (7) and (8), we have

$$\begin{aligned} T_k(u_n) &\rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega, \omega), \\ T_k(u_n) &\rightarrow T_k(u) \quad \text{strongly in } L^p(\Omega, \omega) \text{ and } \mu\text{-a.e. in } \Omega, \end{aligned} \tag{9}$$

for all $k > 0$. Hence $T_k(u) \in W_0^{1,p}(\Omega, \omega)$.

Furthermore, by the weak lower semicontinuity of the norm $W_0^{1,p}(\Omega, \omega)$, we have that (7) still holds for u , that is,

$$\int_{\Omega} |\nabla T_k(u)|^p \omega \, dx \leq C_2 k.$$

Applying Lemma 1 and Lemma 2, we have that $u \in \mathcal{M}^{p_1}(\Omega, \omega)$ (with $p_1 = \eta(p-1)$) and $|\nabla u| \in \mathcal{M}^{p_2}(\Omega, \omega)$ (with $p_2 = p_1 p / (p_1 + 1)$).

- We need to show that $T_k(u_n) \rightarrow T_k(u)$ strongly in $W_0^{1,p}(\Omega, \omega)$, for all $k > 0$.

Let $h > k$ and applying (5) with function

$$\varphi_n = T_{2k}(u_n - T_h(u_n)) + T_k(u_n) - T_k(u),$$

we get

$$\int_{\Omega} \omega \langle \mathcal{A}(x, u_n, \nabla u_n), \nabla \varphi_n \rangle \, dx = \int_{\Omega} f_n \varphi_n \, dx + \int_{\Omega} \langle G_n, \nabla \varphi_n \rangle \, dx. \tag{10}$$

If we set $M = 4k + h$, we have $\nabla \varphi_n = 0$ for $|u_n| > M$. Hence, since condition (H3) implies that $\mathcal{A}(x, s, 0) = 0$, we can write

$$\int_{\Omega} \omega \langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla \varphi_n \rangle \, dx = \int_{\Omega} f_n \varphi_n \, dx + \int_{\Omega} \langle G_n, \nabla \varphi_n \rangle \, dx. \tag{11}$$

In the left-hand side of (11), we have

$$\begin{aligned}
& \int_{\Omega} \omega \langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx \\
&= \int_{\{|u_n| \leq k\}} \omega \langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx \\
&+ \int_{\{|u_n| > k\}} \omega \langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx
\end{aligned} \tag{12}$$

(a) If $|u_n| \leq k$. Since $h > k$, if $|u_n| \leq k < h$, then $T_h(u_n) = T_k(u_n) = u_n$. Hence,

$$u_n - T_h(u_n) + T_k(u_n) - T_k(u) = u_n - T_k(u).$$

We also have that $|u_n - u| \leq 2k$. Then, since $\nabla T_M(u_n) = \nabla T_k(u_n)$ (because $|u_n| \leq k < M$),

$$\begin{aligned}
& \int_{\{|u_n| \leq k\}} \omega \langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx \\
&= \int_{\{|u_n| \leq k\}} \omega \langle \mathcal{A}(x, T_k(u_n), \nabla T_k(u_n)), \nabla(T_k(u_n) - T_k(u)) \rangle dx \\
&= \int_{\Omega} \omega \langle \mathcal{A}(x, T_k(u_n), \nabla T_k(u_n)), \nabla(T_k(u_n) - T_k(u)) \rangle dx.
\end{aligned}$$

(b) If $|u_n| > k$. Since u_n , $T_k(u_n)$ and $T_k(u)$ are in $W_0^{1,p}(\Omega, \omega)$, if

$$|u_n - T_h(u_n) + T_k(u_n) - T_k(u)| \leq 2k,$$

we obtain

$$\begin{aligned}
\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) &= \nabla(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \\
&= \nabla u_n - \nabla T_h(u_n) + \nabla T_k(u_n) - \nabla T_k(u) \\
&= \nabla u_n - \nabla T_h(u_n) - \nabla T_k(u)
\end{aligned}$$

(because $\nabla T_k(u_n) = 0$ if $|u_n| > k$). There are two possible cases as follows:

(i) If $k < |u_n| < h$, we have $\nabla T_h(u_n) = \nabla u_n$. Then

$$\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = -\nabla T_k(u).$$

(ii) If $h < |u_n| \leq M$, we have that $\nabla T_h(u_n) = 0$. Then

$$\nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) = \nabla u_n - \nabla T_k(u) = \nabla T_M(u_n) - \nabla T_k(u).$$

Since $\langle \mathcal{A}(x, s, \xi), \xi \rangle \geq \lambda |\xi|^p \geq 0$, in both cases we obtain

$$\begin{aligned}
& \langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle \\
&\geq -\langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_k(u) \rangle \\
&\geq -|\mathcal{A}(x, T_M(x), \nabla T_M(x))| |\nabla T_k(u)|.
\end{aligned}$$

Therefore we obtain in (12)

$$\begin{aligned}
 & \int_{\Omega} \omega \langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx \\
 &= \int_{\{|u_n| \leq k\}} \omega \langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx \\
 & \quad + \int_{\{|u_n| > k\}} \omega \langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx \\
 & \geq \int_{\Omega} \omega \langle \mathcal{A}(x, T_k(u_n), \nabla T_k(u_n)), \nabla(T_k(u_n) - T_k(u)) \rangle dx \\
 & \quad - \int_{\{|u_n| > k\}} \omega |\mathcal{A}(x, T_M(u_n), \nabla T_M(u_n))| |\nabla T_k(u)| dx.
 \end{aligned}$$

Hence, in (11) we obtain

$$\begin{aligned}
 & \int_{\Omega} \omega \langle \mathcal{A}(x, T_k(u_n), \nabla T_k(u_n)) - \mathcal{A}(x, T_k(u), \nabla T_k(u)), \nabla(T_k(u_n) - T_k(u)) \rangle dx \\
 & \leq \int_{\{|u_n| > k\}} \omega |\mathcal{A}(x, T_M(u_n), \nabla T_M(u_n))| |\nabla T_k(u)| dx \\
 & \quad + \int_{\Omega} f_n T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) dx \\
 & \quad + \int_{\Omega} \langle G_n, \nabla T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u)) \rangle dx \\
 & \quad - \int_{\Omega} \omega \langle \mathcal{A}(x, T_k(u_n), \nabla T_k(u)), \nabla(T_k(u_n) - T_k(u)) \rangle dx. \tag{13}
 \end{aligned}$$

Considering the test function $\psi_n = T_{2k}(u_n - T_h(u_n))$ in (5), we have

$$\int_{\Omega} \omega \langle \mathcal{A}(x, u_n, \nabla u_n), \nabla \psi_n \rangle dx = \int_{\Omega} f_n \psi_n dx + \int_{\Omega} \langle G_n, \nabla \psi_n \rangle dx,$$

and by (7) we obtain

$$\int_{\Omega} |\nabla T_{2k}(u_n - T_h(u_n))|^p \omega dx \leq C_2(2k + 1), \text{ for all } k \geq 1.$$

Now using that $T_{2k}(u_n - T_h(u_n)) \rightharpoonup T_{2k}(u - T_h(u))$ weakly in $W_0^{1,p}(\Omega, \omega)$ (by (9) and Remark 3 (i)), we have

$$\int_{\Omega} |\nabla T_{2k}(u - T_h(u))|^p \omega dx \leq C_2(2k + 1). \tag{14}$$

We have (by Remark 3 (i) and (ii) and (14))

$$\begin{aligned}
\int_{\Omega} |G| |\nabla T_{2k}(u - T_h(u))| \, dx &= \int_{\{h < |u| < 2k+h\}} |G| |\nabla u| \, dx \\
&\leq \left(\int_{\{|u| \geq h\}} |G/\omega|^{p'} \omega \, dx \right)^{1/p'} \left(\int_{\{h < |u| < 2k+h\}} |\nabla u|^p \omega \, dx \right)^{1/p} \\
&= \left(\int_{\{|u| \geq h\}} |G/\omega|^{p'} \omega \, dx \right)^{1/p'} \left(\int_{\Omega} |\nabla T_{2k}(u - T_h(u))|^p \omega \, dx \right)^{1/p} \\
&= C_3 \left(\int_{\{|u| \geq h\}} |G/\omega|^{p'} \omega \, dx \right)^{1/p'},
\end{aligned}$$

where C_3 depends on k but not on h . Therefore we have

$$\lim_{h \rightarrow \infty} \int_{\Omega} \langle G, \nabla T_{2k}(u - T_h(u)) \rangle \, dx = 0.$$

We also have (by Theorem 1)

$$\begin{aligned}
\int_{\Omega} |T_{2k}(u - T_h(u))|^p \omega \, dx &\leq C_{\Omega} \int_{\Omega} |\nabla T_{2k}(u - T_h(u))|^p \omega \, dx \\
&\leq C_{\Omega} C_2(2k+1).
\end{aligned}$$

Moreover, by Lebesgue's theorem, we obtain

$$\lim_{h \rightarrow \infty} \int_{\Omega} f T_{2k}(u - T_h(u)) \, dx = 0.$$

We can fix a positive real number h_{ε} sufficiently large to have

$$\int_{\Omega} f T_{2k}(u - T_{h_{\varepsilon}}(u)) \, dx + \int_{\Omega} \langle G, \nabla T_{2k}(u - T_{h_{\varepsilon}}(u)) \rangle \, dx \leq \varepsilon. \quad (15)$$

Considering $h = h_{\varepsilon}$ in (13) (and $M = M_{\varepsilon} = 4k + h_{\varepsilon}$), by (H4) and (7), we have

$$\begin{aligned}
&\int_{\Omega} |\mathcal{A}(x, T_M(u_n), \nabla T_M(u_n))|^{p'} \omega \, dx \\
&\leq \int_{\Omega} \left(K(x) + h_1(x) |T_M(u_n)|^{p/p'} + h_2(x) |\nabla T_M(u_n)|^{p/p'} \right)^{p'} \omega \, dx \\
&\leq C \left[\int_{\Omega} K^{p'}(x) \omega \, dx + \int_{\Omega} h_1^{p'}(x) |T_M(u_n)|^p \omega \, dx \right. \\
&\quad \left. + \int_{\Omega} h_2^{p'}(x) |\nabla T_M(u_n)|^p \omega \, dx \right] \\
&\leq C \left(\|K\|_{L^{p'}(\Omega, \omega)}^{p'} + \|h_1\|_{L^{\infty}(\Omega)}^{p'} \int_{\Omega} |T_M(u_n)|^p \omega \, dx \right. \\
&\quad \left. + \|h_2\|_{L^{\infty}(\Omega)}^{p'} \int_{\Omega} |\nabla T_M(u_n)|^p \omega \, dx \right) \\
&\leq C \left(\|K\|_{L^{p'}(\Omega, \omega)}^{p'} + \|h_1\|_{L^{\infty}(\Omega)}^{p'} M^p \mu(\Omega) + \|h_2\|_{L^{\infty}(\Omega)}^{p'} M C_2 \right),
\end{aligned}$$

that is, $|\mathcal{A}(x, T_M(u_n), \nabla T_M(u_n))|$ is bounded in $L^{p'}(\Omega, \omega)$.

Moreover, $\chi_{\{|u_n|>k\}}|\nabla T_k(u)| \rightarrow 0$ in $L^p(\Omega, \omega)$ as $n \rightarrow \infty$. Therefore,

$$\lim_{n \rightarrow \infty} \int_{\{|u_n|>k\}} |\mathcal{A}(x, T_M(u_n), \nabla T_M(u_n))| |\nabla T_k(u)| \omega \, dx = 0. \quad (16)$$

Furthermore, we have that

$$T_{2k}(u_n - T_h(u_n)) + T_k(u_n) - T_k(u) \rightharpoonup T_{2k}(u - T_h(u)),$$

weakly in $W_0^{1,p}(\Omega, \omega)$, as $n \rightarrow \infty$.

Hence, by (9), (15) and (16), passing to the limit in (13), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} \langle \mathcal{A}(x, T_k(u_n), \nabla T_k(u_n)) - \mathcal{A}(x, T_k(u_n), \nabla T_k(u)), \nabla(T_k(u_n) - T_k(u)) \rangle \omega \, dx \\ \leq \int_{\Omega} f T_{2k}(u - T_{h_\varepsilon}(u)) \, dx + \int_{\Omega} \langle G, \nabla T_{2k}(u - T_{h_\varepsilon}(u)) \rangle \, dx \leq \varepsilon, \end{aligned}$$

for all $\varepsilon > 0$, that is,

$$\int_{\Omega} \langle \mathcal{A}(x, T_k(u_n), \nabla T_k(u_n)) - \mathcal{A}(x, T_k(u_n), \nabla T_k(u)), \nabla(T_k(u_n) - T_k(u)) \rangle \omega \rightarrow 0,$$

as $n \rightarrow \infty$. Applying Lemma 3 we get

$$T_k(u_n) \rightarrow T_k(u) \quad (17)$$

strongly in $W_0^{1,p}(\Omega, \omega)$ for every $k > 0$. This convergence implies that, for every fixed $k > 0$

$$\mathcal{A}(x, T_k(u_n), \nabla T_k(u_n)) \rightarrow \mathcal{A}(x, T_k(u), \nabla T_k(u)) \quad (18)$$

in $(L^{p'}(\Omega, \omega))^N = L^{p'}(\Omega, \omega) \times \cdots \times L^{p'}(\Omega, \omega)$.

• Finally, we need to show that u is an entropy solution to Dirichlet problem (P). Let us take $\psi_n = T_k(u_n - \varphi)$ as test function in (5), with $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$. We obtain,

$$\int_{\Omega} \omega \langle \mathcal{A}(x, u_n, \nabla u_n), \nabla \psi_n \rangle \, dx = \int_{\Omega} f_n \psi_n \, dx + \int_{\Omega} \langle G_n, \nabla \psi_n \rangle \, dx. \quad (19)$$

If $M = k + \|\varphi\|_{L^\infty(\Omega)}$ and $n > M$, we have

$$\begin{aligned} \int_{\Omega} \omega \langle \mathcal{A}(x, u_n, \nabla u_n), \nabla T_k(u_n - \varphi) \rangle \, dx \\ = \int_{\Omega} \omega \langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_k(u_n - \varphi) \rangle \, dx. \end{aligned}$$

Hence, in (19) we obtain

$$\begin{aligned} \int_{\Omega} \omega \langle \mathcal{A}(x, T_M(u_n), \nabla T_M(u_n)), \nabla T_k(u_n - \varphi) \rangle dx \\ = \int_{\Omega} f_n T_k(u_n - \varphi) dx + \int_{\Omega} \langle G, \nabla T_k(u_n - \varphi) \rangle dx. \end{aligned} \quad (20)$$

Therefore, by (9) and (18), passing to the limit as $n \rightarrow \infty$ in (20), we obtain

$$\int_{\Omega} \omega \langle \mathcal{A}(x, u, \nabla u), \nabla T_k(u - \varphi) \rangle dx = \int_{\Omega} f T_k(u - \varphi) dx + \int_{\Omega} \langle G, \nabla T_k(u - \varphi) \rangle dx$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$ and for each $k > 0$.

Therefore u is an entropy solution of problem (P). \square

Example 1. Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, the weight function

$$\begin{aligned} \omega(x, y) &= (x^2 + y^2)^{-1/2} \quad (\omega \in A_3), \\ f(x, y) &= \frac{\cos(xy)}{(x^2 + y^2)^{1/3}}, \\ G(x, y) &= ((x^2 + y^2) \sin(xy), (x^2 + y^2)^{-1/3} \cos(xy)) \end{aligned}$$

and $\mathcal{A}: \Omega \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\mathcal{A}((x, y), s, \xi) = |\xi| \xi$. By Theorem 2, the problem

$$(P) \begin{cases} -\operatorname{div}[(x^2 + y^2)^{-1/2} \mathcal{A}(x, u, \nabla u)] = \frac{\cos(xy)}{(x^2 + y^2)^{1/3}} - \operatorname{div}(G(x, y)), & \text{in } \Omega \\ u(x, y) = 0, & \text{on } \partial\Omega \end{cases}$$

has an entropy solution.

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