VOLUME
22/2014
No. 1

ISSN 1804-1388 (Print)
ISSN 2336-1298 (Online)

COMMUNICATIONS IN MATHEMATICS

Available online at
http://cm.osu.cz

UNIVERSITAS OSTRAVIENSIS

Communications in Mathematics 22 (2014) 31–48

Stratonovich-Weyl correspondence for the Jacobi group

Benjamin Cahen

Abstract. We construct and study a Stratonovich-Weyl correspondence for the holomorphic representations of the Jacobi group.

1 Introduction

The notion of Stratonovich-Weyl correspondence was introduced in [27] as a generalization of the classical Weyl correspondence [1]. The systematic study of the Stratonovich-Weyl correspondences began with the work of J.M. Gracia-Bondìa, J.C. Vàrilly and their co-workers (see [15], [17], [19] and [20]).

Definition 1. [19] Let G be a Lie group and π a unitary representation of G on a Hilbert space $\mathcal{H}$. Let M be a homogeneous G-space and μ a (suitably normalized) G-invariant measure on M. Then a Stratonovich-Weyl correspondence for the triple (G, π, M) is an isomorphism W from a vector space of operators on $\mathcal{H}$ to a space of (generalized) functions on M satisfying the following properties:

1. W maps the identity operator of $\mathcal{H}$ to the constant function 1;
2. the function $W(A^*)$ is the complex-conjugate of $W(A)$;
3. Covariance: we have $W(\pi(g)\,A\,\pi(g)^{-1})(x) = W(A)(g^{-1}\cdot x)$;
4. Traciality: we have

$$\int_M W(A)(x)W(B)(x)\,d\mu(x) = \mathrm{Tr}(AB).$$

2010 MSC: 22E27, 32M10, 46E22

Key words: Berezin quantization, Berezin transform, quasi-Hermitian Lie group, unitary representation, holomorphic representation, reproducing kernel Hilbert space, Jacobi group, Stratonovich-Weyl correspondence

The basic example is the case when G is the $(2n+1)$-dimensional Heisenberg group H_n acting on $\mathbb{R}^{2n}$ by translations and π is a Schrödinger representation of H_n on $L^2(\mathbb{R}^n)$. In this case, the classical Weyl correspondence gives a Stratonovich-Weyl correspondence for the triple $(H_n, \pi, \mathbb{R}^{2n})$ [18], [19].

Stratonovich-Weyl correspondences were constructed for various Lie group representations, in particular for the massive representations of the Poincaré group [15], [19].

In [13], we constructed and studied a Stratonovich-Weyl correspondence for a quasi-Hermitian Lie group G and a unitary representation π of G which is holomorphically induced from a unitary character of a compactly embedded subgroup K of G (see also our earlier papers [11] and [12]). The construction is based on an idea of [17] consisting in modifying suitably the Berezin correspondence S (see also [2] and [3]). More precisely, recall that S is an isomorphism from the Hilbert space of all Hilbert-Schmidt operators on $\mathcal{H}$ (endowed with the Hilbert-Schmidt norm) onto a space of square integrable functions on a homogeneous complex domain [28]. Then the Stratonovich-Weyl correspondence W is obtained by taking the isometric part in the polar decomposition of S, that is, $W := (SS^*)^{-1/2}S$. Note that $B := SS^*$ is the so-called Berezin transform which have been intensively studied by many authors, see in particular [16], [23], [24], [28], [29].

In [13], we showed that, when the Lie algebra $\mathfrak{g}$ of G is reductive, the mappings B and W can be extended to a class of functions which contains $S(d\pi(X))$ for each $X \in \mathfrak{g}$ and that, for each simple ideal $\mathfrak{s}$ in $\mathfrak{g}$, there exists a constant $c \geq 0$ such that $W(d\pi(X)) = cS(d\pi(X))$ for each $X \in \mathfrak{s}$. However, it seems difficult to obtain the analogous results in the general case.

In the present paper, we aim to study B and W in the particular case of the Jacobi group. The Jacobi group is the semi-direct product of the 3-dimensional real Heisenberg group by the unitary group $SU(1,1)$. This group plays an important role in different areas of Mathematics and Physics as for instance Theory of automorphic forms (Jacobi forms, theta functions), Quantum Optics (squeezed states) and Harmonic Analysis, see [4] and [9]. In particular, the Jacobi group appears as an important example of non-reductive Lie group of Harish-Chandra type [22], [26] and its holomorphic unitary representations were studied by many authors, see [4], [5], [8], [9] and [22]. Moreover, the metaplectic factorization [22] should be used to reduce, in some sense, the general case to the case of some generalized Jacobi group and then the study of the case of the Jacobi group can be considered as a first step towards the general case. Recall that the metaplectic factorization is a method for decomposing a unitary highest weight representation of a quasi-Hermitian Lie group as the tensor product of a unitary highest weight representation of a reductive group and a highest weight representation of some generalized Heisenberg group [22], p. 361.

In this paper, we begin by some generalities on the Jacobi group (Section 2) and its holomorphic representations (Section 3). We introduce the Berezin correspondence S, the Berezin transform B and the Stratonovich-Weyl correspondence W (Section 4). Under some technical assumptions, we extend B to a class of functions which contains $S(d\pi(X))$ for each $X \in \mathfrak{g}$ (Section 5). Finally, we give an explicit expression for $W(d\pi(X))$, $X \in \mathfrak{g}$ (Section 6).

2 The Jacobi group

This section is devoted to generalities on the Jacobi group. The material of this section and of the following section is essentially taken from [18], Chapter 4, [22], Chapters VII and XII, [14] (see also [9]).

Consider the symplectic form ω on $\mathbb{C}^2$ defined by

$$\omega\big((z,w),(z',w')\big)=\frac{i}{2}(zw'-z'w).$$

for $z,w,z',w'\in\mathbb{C}$. The 3-dimensional real Heisenberg group is

$$H:=\big\{\big((z,\bar z),c\big):z\in\mathbb{C},\,c\in\mathbb{R}\big\}$$

endowed with the multiplication

$$\big((z,\bar z),c\big)\cdot\big((z',\bar z'),c'\big)=\Big((z+z',\bar z+\bar z'),c+c'+\tfrac{1}{2}\omega\big((z,\bar z),(z',\bar z')\big)\Big).$$

Then the complexification H^c of H is $H^c:=\{((z,w),c):z,w\in\mathbb{C},c\in\mathbb{C}\}$ and the multiplication of H^c is obtained by replacing $(z,\bar z)$ by (z,w) and $(z',\bar z')$ by (z',w') in the previous formula. We denote by $\mathfrak{h}$ and $\mathfrak{h}^c$ the Lie algebras of H and H^c.

Now consider the group $S:=SU(1,1)$ consisting of all matrices

$$h=\begin{pmatrix} a & b\\ \bar b & \bar a\end{pmatrix},\quad a,b\in\mathbb{C}\,,\quad |a|^2-|b|^2=1\,.$$

The group S acts on H by

$$h\cdot\big((z,\bar z),c\big)=\big(h(z,\bar z),c\big)=(az+b\bar z,\bar a\bar z+\bar b z,c)$$

for h as before, the elements of $\mathbb{C}^2$ being considered as column vectors. Then we can form the semi-direct product $G:=H\rtimes S$ which is called the Jacobi group. The elements of G can be written as $((z,\bar z),c,h)$ where $z\in\mathbb{C}$, $c\in\mathbb{R}$ and $h\in S$. The multiplication of G is thus given by

$$\big((z,\bar z),c,h\big)\cdot\big((z',\bar z'),c',h'\big)=\Big((z,\bar z)+h(z',\bar z'),c+c'+\tfrac{1}{2}\omega\big((z,\bar z),h(z',\bar z')\big),hh'\Big).$$

The complexification of S is $S^c=SL(2,\mathbb{C})$. The complexification G^c of G is then the semi-direct product $G^c=H^c\rtimes SL(2,\mathbb{C})$ and the multiplication of G^c is obtained by replacing $\bar z$ and $\bar z'$ by w and w' in the preceding formula. We denote by $\mathfrak{s}$, $\mathfrak{s}^c$, $\mathfrak{g}$ and $\mathfrak{g}^c$ the Lie algebras of S, S^c, G and G^c. The Lie bracket of $\mathfrak{g}^c$ is given by

$$\Big[\big((z,w),c,A\big),\big((z',w'),c',A'\big)\Big]=\big(A(z',w')-A'(z,w),\omega((z,w),(z',w')),[A,A']\big).$$

Let θ denote conjugation with respect to the real form $\mathfrak{g}$ of $\mathfrak{g}^c$. For $X\in\mathfrak{g}^c$, we set $X^*=-\theta(X)$. We can easily verify that if $X=\big((z,w),c,\left(\begin{smallmatrix} a & b\\ c & -a\end{smallmatrix}\right)\big)\in\mathfrak{g}^c$ then

$$X^*=\left((-\bar w,-\bar z),-\bar c,\begin{pmatrix}\bar a & -\bar c\\ -\bar b & -\bar a\end{pmatrix}\right).$$

Also, we denote by $g \to g^*$ the involutive anti-automorphism of G^c which is obtained by exponentiating $X \to X^*$ to G^c.

Let K be the subgroup of G consisting of all elements $\big((0,0), c, \left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a} \end{smallmatrix}\right)\big)$ where $c \in \mathbb{R}$ and $|a| = 1$. Then the Lie algebra $\mathfrak{k}$ of K is a maximal compactly embedded (Cartan) subalgebra of $\mathfrak{g}$. Let us introduce the linear form ε defined on $\mathfrak{k}$ by $\varepsilon\big((0,0), c, \left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a} \end{smallmatrix}\right)\big) = a$. Then one can easily verify that the roots of $\mathfrak{k}^c$ on $\mathfrak{g}^c$ are $\pm\varepsilon$ and $\pm 2\varepsilon$. In the terminology of [22] p. 234–235, $\pm 2\varepsilon$ are the (non-compact) semisimple roots, $\pm\varepsilon$ are the solvable roots (in that case, there is no compact roots). As in [22] p. 249, we can choose the adapted system of positive roots to be $\{\varepsilon, 2\varepsilon\}$. The root space decomposition of $\mathfrak{g}^c$ is then $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$ where

$$\mathfrak{p}^+ = \mathfrak{g}_\varepsilon \oplus \mathfrak{g}_{2\varepsilon} = \left\{ \left((z,0), 0, \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \right) \,:\, z \in \mathbb{C}, u \in \mathbb{C} \right\}$$

and

$$\mathfrak{p}^- = \mathfrak{g}_{-\varepsilon} \oplus \mathfrak{g}_{-2\varepsilon} = \left\{ \left((0,w), 0, \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix} \right) \,:\, w \in \mathbb{C}, v \in \mathbb{C} \right\}.$$

In the rest of the paper, we denote by $a(z,u)$ the element $\big((z,0), 0, \left(\begin{smallmatrix} 0 & u \\ 0 & 0 \end{smallmatrix}\right)\big)$ of $\mathfrak{p}^+$. Also, we denote by $p_{\mathfrak{p}^+}$, $p_{\mathfrak{k}^c}$ and $p_{\mathfrak{p}^-}$ the projections of $\mathfrak{g}^c$ onto $\mathfrak{p}^+$, $\mathfrak{k}^c$ and $\mathfrak{p}^-$ associated with the above direct decomposition.

Let P^+ and P^- be the analytic subgroups of G^c with Lie algebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$. Then we have

$$P^+ = \left\{ \left((z,0), 0, \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \,:\, z \in \mathbb{C}, u \in \mathbb{C} \right\}$$

and

$$P^- = \left\{ \left((0,w), 0, \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \right) \,:\, w \in \mathbb{C}, v \in \mathbb{C} \right\}.$$

Note that G is a group of the Harish-Chandra type [22], p. 507, that is, the following properties are satisfied:

1. $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$ is a direct sum of vector spaces, $(\mathfrak{p}^+)^* = \mathfrak{p}^-$ and $[\mathfrak{k}^+, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm$;

2. The multiplication map $P^+K^cP^- \to G^c$, $(z,k,y) \to zky$ is a biholomorphic diffeomorphism onto its open image;

3. $G \subset P^+K^cP^-$ and $G \cap K^cP^- = K$.

We denote by $\zeta : P^+K^cP^- \to P^+$, $\kappa : P^+K^cP^- \to K^c$ and $\eta : P^+K^cP^- \to P^-$ the projections onto P^+-, K^c- and P^--components. We can verify that $g = \big((z_0,w_0), c_0, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\big)$ in G^c has a $P^+K^cP^-$-decomposition

$$g = \left((z,0), 0, \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \cdot \left((0,0), \tilde{c}, \begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix} \right) \cdot \left((0,w), 0, \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \right)$$

if and only if $d \neq 0$ and, in this case, we have $z = z_0 - bd^{-1}w_0$, $u = bd^{-1}$, $v = cd^{-1}$, $w = d^{-1}w_0$, $p = d^{-1}$ and $\tilde{c} = c_0 - (1/4)i(z_0 - bd^{-1}w_0)w_0$.

Now we introduce an action (defined almost everywhere) of G^c on $\mathfrak{p}^+$. For $Z\in\mathfrak{p}^+$ and $g\in G^c$ with $g\exp Z\in P^+K^cP^-$, we define the element $g\cdot Z$ of $\mathfrak{p}^+$ by $g\cdot Z:=\log\zeta(g\exp Z)$. From the above formula for the $P^+K^cP^-$-decomposition, we deduce that the action of $g=\big((z_0,w_0),c_0,\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\big)\in G^c$ on $a(z,u)\in\mathfrak{p}^+$ is given by $g\cdot a(z,u)=a(z',u')$ where

$$u'=(au+b)(cu+d)^{-1}$$

and

$$z'=z_0+az-(au+b)(cu+d)^{-1}(w_0+cz)\,.$$

This implies that

$$\mathcal{D}:=G\cdot 0=\{a(z,u)\in\mathfrak{p}^+\;:\;|u|<1\}\cong\mathbb{C}\times\mathbb{D}$$

where $\mathbb{D}$ denotes the unit open disk of $\mathbb{C}$.

We can easily obtain a useful section $Z\to g_Z$ for the action of G on $\mathcal{D}$. Let $Z=a(z,u)\in\mathcal{D}$. Define $g_Z:=((z_0,\bar{z}_0),0,h_0)\in G$ as follows. We set

$$z_0=(1-u\bar{u})^{-1}(z+u\bar{z})\,,\quad a=(1-u\bar{u})^{-1/2},\quad b=(1-u\bar{u})^{-1/2}u$$

and $h_0=\left(\begin{smallmatrix}a&b\\\bar{b}&\bar{a}\end{smallmatrix}\right)$. Then one has $g_Z\cdot 0=Z$.

From this, we can compute the G-invariant measure μ on $\mathcal{D}$. Let $d\mu_L(z,u)$ be the Lebesgue measure on $\mathbb{C}\times\mathbb{D}$ normalized as follows. If $z=x+iy$ and $u=v+iw$ with $x,y,v,w\in\mathbb{R}$ then $d\mu_L(z,u):=dx\,dy\,dv\,dw$. Thus, we easily obtain that $d\mu(Z)=(1-u\bar{u})^{-3}\,d\mu_L(z,u)$. This result can be also deduced from the general formula for the invariant measure, see [22], p. 538.

Now we compute the adjoint and coadjoint actions of G^c. This can be done as follows. We begin by the adjoint action of G^c. Let $g=(v_0,c_0,h_0)\in G^c$ where $v_0\in\mathbb{C}^2$, $c_0\in\mathbb{C}$ and $h_0\in S^c=SL(2,\mathbb{C})$ and $X=(w,c,U)\in\mathfrak{g}^c$ where $w\in\mathbb{C}^2$, $c\in\mathbb{C}$ and $U\in\mathfrak{s}^c$. We set $\exp(tX)=(w(t),c(t),\exp(tU))$. Then, since the derivatives of $w(t)$ and $c(t)$ at $t=0$ are w and c, we find that

$$\begin{aligned}\mathrm{Ad}(g)X&=\frac{d}{dt}(g\exp(tX)g^{-1})|_{t=0}\\&=\big(h_0w-(\mathrm{Ad}(h_0)U)v_0,c+\omega(v_0,h_0w)-\tfrac{1}{2}\omega(v_0,(\mathrm{Ad}(h_0)U)v_0),\mathrm{Ad}(h_0)U\big).\end{aligned}$$

On the other hand, let us denote by $\xi=(u,d,\varphi)$, where $u\in\mathbb{C}^2$, $d\in\mathbb{C}$ and $\varphi\in(\mathfrak{s}^c)^*$, the element of $(\mathfrak{g}^c)^*$ defined by

$$\langle\xi,(w,c,U)\rangle=\omega(u,w)+cd+\langle\varphi,U\rangle.$$

Moreover, for $u,v\in\mathbb{C}^2$, we denote by $v\times u$ the element of $(\mathfrak{s}^c)^*$ defined by

$$\langle v\times u,U\rangle:=\omega(u,Uv)$$

for $U\in\mathfrak{s}^c$.

Let $\xi=(u,d,\varphi)\in(\mathfrak{g}^c)^*$ and $g=(v_0,c_0,h_0)\in G^c$. Then, by using the relation

$$\big\langle\mathrm{Ad}^*(g)\xi,X\big\rangle=\big\langle\xi,\mathrm{Ad}(g^{-1})X\big\rangle$$

for each $X\in\mathfrak{g}^c$, we obtain

$$\mathrm{Ad}^*(g)\xi=\big(h_0u-dv_0,d,\mathrm{Ad}^*(h_0)\varphi+v_0\times(h_0u-\tfrac{d}{2}v_0)\big)$$

By restriction, we also get the formula for the coadjoint action of G.

3 Holomorphic representations

Let us first recall the general method for constructing the holomorphic representations of G, see [22], Chapter XII. We follow the presentation of [13].

Let χ be a unitary character of K. The extension of χ to K^c is also denoted by χ. We set $K_\chi(Z,W) = \chi(\kappa(\exp W^* \exp Z))^{-1}$ for $Z, W \in \mathcal{D}$ and $J_\chi(g,Z) = \chi(\kappa(g \exp Z))$ for $g \in G$ and $Z \in \mathcal{D}$. We consider the Hilbert space $\mathcal{H}_\chi$ of holomorphic functions on $\mathcal{D}$ such that

$$\|f\|_\chi^2 := \int_{\mathcal{D}} |f(Z)|^2 \, K_\chi(Z,Z)^{-1} c_\chi d\mu(Z) < +\infty$$

where the constant c_χ is defined by

$$c_\chi^{-1} = \int_{\mathcal{D}} K_\chi(Z,Z)^{-1} \, d\mu(Z) \, .$$

As we shall see, under some condition on χ, we have that c_χ is well-defined and $\mathcal{H}_\chi \neq (0)$. In that case, $\mathcal{H}_\chi$ contains the polynomials [22], p. 546. Moreover, the formula

$$\big(\pi_\chi(g)f\big)(Z) = J_\chi(g^{-1}, Z) \, f(g^{-1} \cdot Z)$$

defines a unitary representation of G on $\mathcal{H}_\chi$ which is a highest weight representation [22], p. 540.

The space $\mathcal{H}_\chi$ is a reproducing kernel Hilbert space. More precisely, if we set $e_Z(W) := K_\chi(W,Z)$ then we have we have the reproducing property $f(Z) = \langle f, e_Z \rangle_\chi$ for each $f \in \mathcal{H}_\chi$ and each $Z \in \mathcal{D}$ [22], p. 540. Here $\langle \cdot, \cdot \rangle_\chi$ denotes the inner product on $\mathcal{H}_\chi$.

Let's define χ as follows. Fix $\gamma \in \mathbb{R}$ and $m \in \mathbb{Z}$ and for $k = \big((0,0), c, \left(\begin{smallmatrix} a & 0 \\ 0 & \bar{a} \end{smallmatrix}\right)\big) \in K$ set $\chi(k) = e^{i\gamma c} a^{-m}$. Then we have the following result.

Proposition 1. *1. Let $Z = a(z,u) \in \mathcal{D}$ and $W = a(w,v) \in \mathcal{D}$. Then we have*

$$K_\chi(Z,W) = (1 - u\bar{v})^{-m} \exp\left(\frac{\gamma}{4}\left(\frac{2\bar{w}z + \bar{v}z^2 + u\bar{w}^2}{1 - u\bar{v}}\right)\right).$$

2. We have $\mathcal{H}_\chi \neq (0)$ if and only if $m > 3/2$. In this case, we also have $c_\chi = \frac{\gamma}{2\pi^2}(m - \frac{3}{2})$.

Proof.

1. This can be verified by a simple computation based on the $P^+K^cP^-$-decomposition.

2. By [22], Theorem XII.5.6, we have $\mathcal{H}_\chi \neq (0)$ if and only

$$I_\chi := \int_{\mathcal{D}} K_\chi(Z,Z)^{-1} \, d\mu(Z) < \infty \, .$$

Then we have to study the integral I_χ. We begin with the following remark. Let A be a symmetric positive-definite $n \times n$ real matrix. By diagonalizing A we immediately obtain that

$$\int_{\mathbb{R}^n} e^{-Ax \cdot x} \, dx = \pi^{n/2} (\operatorname{Det} A)^{-1/2}.$$

From this, we deduce that

$$\int_{\mathbb{C}} \exp\left(-\frac{\gamma}{4}\left(\frac{2\bar{z}z+\bar{u}z^2+u\bar{z}^2}{1-u\bar{u}}\right)\right) d\mu_L(z) = \frac{2\pi}{\gamma}(1-u\bar{u})^{1/2}$$

for each $u \in \mathbb{D}$. Then, taking into account the above expression for $K_\chi(Z,Z)$, we have

$$\begin{aligned} I_\chi &= \int_{\mathbb{C}\times\mathbb{D}} (1-u\bar{u})^{m-3} \exp\left(-\frac{\gamma}{4}\left(\frac{2\bar{z}z+\bar{u}z^2+u\bar{z}^2}{1-u\bar{u}}\right)\right) d\mu_L(z,u) \\ &= \frac{2\pi}{\gamma}\int_{\mathbb{D}}(1-u\bar{u})^{m-\frac{5}{2}} d\mu_L(u) = \frac{2\pi^2}{\gamma}\int_0^1 (1-t)^{m-\frac{5}{2}} dt\,. \end{aligned}$$

Hence we have $\mathcal{H}_\chi \neq (0)$ if and only if $m > 3/2$ and, in this case, we find that

$$c_\chi^{-1} = I_\chi = \frac{2\pi^2}{\gamma(m-\frac{3}{2})}\,. \qquad \square$$

Let $g = \big((z_0,\bar{z}_0), c_0, \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a}\end{smallmatrix}\right)\big) \in G$ and $Z = a(z,u) \in \mathcal{D}$. Then, setting

$$c := c_0 + \frac{i}{4}(\bar{b}z_0 - a\bar{z}_0)z - \frac{i}{4}\Big(z_0 + az - \frac{au+b}{\bar{b}u+\bar{a}}(\bar{z}_0+\bar{b}z)\Big)(\bar{z}_0+\bar{b}z),$$

one can verify that $J_\chi(g,Z) = e^{i\gamma c}(\bar{b}u+\bar{a})^m$. This gives an explicit but rather complicated expression for $(\pi_\chi(g)f)(Z)$.

Now we compute the derived representation $d\pi_\chi$.

Proposition 2. *For each $X \in \mathfrak{g}^c$, $Z = a(z,u) \in \mathcal{D}$ and $f \in \mathcal{H}_\chi$, we have*

$$\big(d\pi_\chi(X)f\big)(Z) = d\chi(p_{\mathfrak{k}^c}(e^{-\operatorname{ad} Z}X))\,f(Z) - (df)_Z\big(p_{\mathfrak{p}^+}(e^{-\operatorname{ad} Z}X)\big).$$

More precisely

1. *If $X = \big((w,0),0,\left(\begin{smallmatrix} 0 & v \\ 0 & 0\end{smallmatrix}\right)\big) \in \mathfrak{p}^+$ then we have $(d\pi_\chi(X)f)(Z) = -df_Z(X)$ hence*
$$d\pi_\chi(X)f = -w\partial_z f - v\partial_u f;$$
2. *If $X = \big((0,0),c,\left(\begin{smallmatrix} a & 0 \\ 0 & -a\end{smallmatrix}\right)\big) \in \mathfrak{k}^c$ then we have*
$$(d\pi_\chi(X)f)(Z) = d\chi(X)f(Z) + df_Z([Z,X])$$
hence
$$d\pi_\chi(X)f = (i\gamma c - ma)f - a(z\partial_z f + 2u\partial_u f)\,;$$
3. *If $X = \big((0,w),0,\left(\begin{smallmatrix} 0 & 0 \\ v & 0\end{smallmatrix}\right)\big) \in \mathfrak{p}^-$ then we have*
$$\begin{aligned}\big(d\pi_\chi(X)f\big)(Z) &= (d\chi\circ p_{\mathfrak{k}^c})\Big(-[Z,X] + \frac{1}{2}\big[Z,[Z,X]\big]\Big)f(Z) \\ &\quad - (df_Z\circ p_{\mathfrak{p}^+})\Big(-[Z,X] + \frac{1}{2}\big[Z,[Z,X]\big]\Big)\end{aligned}$$
hence
$$d\pi_\chi(X)f = \Big(\frac{\gamma}{4}z(2w+vz) + muv\Big)f + u(w+vz)\partial_z f + u^2 v\partial_u f\,.$$

Proof. Taking into account the fact that $\mathfrak{p}^+$ is abelian, the first formula for $d\pi_\chi(X)$ follows from [13], Proposition 3.3 (see also [22], Proposition XII.2.1). The other formulas follows from the first formula by routine computations which are mainly based on the following fact. For each $X \in \mathfrak{p}^-$ and $Z \in \mathfrak{p}^+$ we have $[Z,[Z,[Z,X]]]=0$ hence we get

$$e^{-\operatorname{ad} Z}(X) = X - [Z,X] + \frac{1}{2}[Z,[Z,X]]\,. \qquad \square$$

From this proposition we deduce the following result which will be needed later.

Proposition 3. *Let $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$. Then the operator $d\pi_\chi(X_1X_2\cdots X_q)$ is a sum of terms of the form $P_{r,s}(z,u)\partial_z^r\partial_u^s$ where $r+s \le q$ and $P_{r,s}$ is a polynomial of degree $\le 2q$.*

Proof. This result is proved by induction on q by using Proposition 2. $\square$

4 Berezin correspondence and Stratonovich-Weyl correspondence

We first review some general facts about the Berezin correspondence, the Berezin transform and the Stratonovich-Weyl correspondence.

The Berezin correspondence S_χ is defined as follows. Consider an operator (not necessarily bounded) A on $\mathcal{H}_\chi$ whose domain contains e_Z for each $Z \in \mathcal{D}$. Then the Berezin symbol of A is the function $S_\chi(A)$ defined on $\mathcal{D}$ by

$$S_\chi(A)(Z) := \frac{\langle A\, e_Z\,,\, e_Z\rangle_\chi}{\langle e_Z\,,\, e_Z\rangle_\chi}.$$

We can verify that each operator is determined by its Berezin symbol and that if an operator A has adjoint A^* then we have $S_\chi(A^*) = \overline{S_\chi(A)}$ [6], [7]. Moreover, for each operator A on $\mathcal{H}_\chi$ whose domain contains the coherent states e_Z for each $Z \in \mathcal{D}$ and each $g \in G$, the domain of $\pi_\chi(g^{-1})A\pi_\chi(g)$ also contains e_Z for each $Z \in \mathcal{D}$ and we have

$$S_\chi\big(\pi_\chi(g)^{-1}A\pi_\chi(g)\big)(Z) = S_\chi(A)(g\cdot Z),$$

that is, S_χ is G-equivariant, see [13] where we have also proved the following result.

Proposition 4. *1. For $g \in G$ and $Z \in \mathcal{D}$, we have*

$$S_\chi\big(\pi_\chi(g)\big)(Z) = \chi\big(\kappa(\exp Z^* g^{-1}\exp Z)^{-1}\kappa(\exp Z^*\exp Z)\big).$$

2. For $X \in \mathfrak{g}^c$ and $Z \in \mathcal{D}$, we have

$$S_\chi\big(d\pi_\chi(X)\big)(Z) = d\chi\big(p_{\mathfrak{k}^c}(\operatorname{Ad}(\zeta(\exp Z^*\exp Z)^{-1}\exp Z^*)\,X\big).$$

Consider the linear form ξ on $\mathfrak{g}^c$ defined by $\xi = -\operatorname{id}\chi$ on $\mathfrak{k}^c$ and $\xi = 0$ on $\mathfrak{p}^\pm$. Then we have $\xi(\mathfrak{g}) \subset \mathbb{R}$ and the restriction ξ_χ of ξ to $\mathfrak{g}$ is an element of $\mathfrak{g}^*$. More precisely, with the notation of Section 2 we have $\xi_\chi = (0,\gamma,\varphi_m)$ where $\varphi_m \in \mathfrak{s}^*$ is defined by $\varphi_m\left(\begin{smallmatrix} a & b \\ c & -a\end{smallmatrix}\right) = ima$.

We denote by $\mathcal{O}(\xi_\chi)$ the orbit of ξ_χ in $\mathfrak{g}^*$ for the coadjoint action of G. This orbit is said to be associated with π_χ by the Kostant-Kirillov method of orbits, see [13], [21]. Moreover, we have the following result.

Proposition 5. *[13]*

1. *For each $Z\in\mathcal{D}$, let*
$$\Psi_\chi(Z):=\mathrm{Ad}^*\big(\exp(-Z^*)\,\zeta(\exp Z^*\exp Z)\big)\,\xi_\chi\,.$$
Then, for each $X\in\mathfrak{g}^c$ and each $Z\in\mathcal{D}$, we have
$$S\big(d\pi_\chi(X)\big)(Z)=i\big\langle\Psi_\chi(Z),X\big\rangle\,.$$
2. *For each $g\in G$ and each $Z\in\mathcal{D}$, we have $\Psi_\chi(g\cdot Z)=\mathrm{Ad}^*(g)\,\Psi_\chi(Z)$.*
3. *The map Ψ_χ is a diffeomorphism from $\mathcal{D}$ onto $\mathcal{O}(\xi_\chi)$.*

We aim to make the expression of Ψ_χ more explicit. To this goal, we introduce the following notation. For $\varphi\in\mathfrak{s}^*$, let $\alpha(\varphi)$ the unique element of $\mathfrak{s}$ such that $\langle\varphi,X\rangle=\mathrm{Tr}(\alpha(\varphi)X)$ for each $X\in\mathfrak{s}$. In particular, one has $\alpha(\varphi_m)=\frac{m}{2}\left(\begin{smallmatrix} i & 0\\ 0 & -i\end{smallmatrix}\right)$. Moreover, for $u=(x,\bar{x})\in\mathbb{C}^2$ and $v=(y,\bar{y})\in\mathbb{C}^2$ we have
$$\alpha(v\times u)=\frac{i}{4}\begin{pmatrix}-x\bar{y}-\bar{x}y & 2xy\\ -2\bar{x}\bar{y} & x\bar{y}+\bar{x}y\end{pmatrix}.$$
Note also that α intertwines Ad^* and Ad. Then, by a direct calculation we obtain the following result.

Proposition 6. *Let $Z=a(z,u)\in\mathcal{D}$. Let us denote by $\varphi(z,u)$ the element of $\mathfrak{s}^*$ defined by*
$$\begin{aligned}\alpha\big(\varphi(z,u)\big)=&\frac{mi}{2}(1-u\bar{u})^{-1}\begin{pmatrix}1+u\bar{u} & -2u\\ 2\bar{u} & -(1+u\bar{u})\end{pmatrix}\\ &-\frac{\gamma i}{4}(1-u\bar{u})^{-2}\begin{pmatrix}-|z+u\bar{z}|^2 & (z+u\bar{z})^2\\ -(\bar{z}+\bar{u}z)^2 & |z+u\bar{z}|^2\end{pmatrix}.\end{aligned}$$
Then we have
$$\Psi_\chi(Z)=\big(-\gamma(1-u\bar{u})^{-1}(z+u\bar{z},\bar{z}+\bar{u}z),\gamma,\varphi(z,u)\big)\,.$$

Let us recall the construction of the Stratonovich-Weyl correspondence [12], [13], [17]. Denote by $\mathcal{L}_2(\mathcal{H}_\chi)$ the space of all Hilbert-Schmidt operators on $\mathcal{H}_\chi$ and by μ_χ the G-invariant measure on $\mathcal{D}$ defined by $d\mu_\chi(Z)=c_\chi d\mu(Z)$. Then the map S_χ is a bounded operator from $\mathcal{L}_2(\mathcal{H}_\chi)$ into $L^2(\mathcal{D},\mu_\chi)$ which is one-to-one and has dense range [25], [28].

The Berezin transform is the operator on $L^2(\mathcal{D},\mu_\chi)$ defined by $B_\chi:=S_\chi S_\chi^*$. We can easily verify that we have
$$B_\chi F(Z)=\int_{\mathcal{D}}F(W)\,\frac{|\langle e_Z,e_W\rangle|_\chi^2}{\langle e_Z,e_Z\rangle_\chi\langle e_W,e_W\rangle_\chi}\,d\mu_\chi(W)\tag{1}$$
(see [6], [28], [29] for instance).

Let ρ be the left-regular representation of G on $L^2(\mathcal{D},\mu_\chi)$. As a consequence of the equivariance property for S_χ, we see that B_χ commute with $\rho(g)$ for each $g\in G$.

Let us consider the polar decomposition of S_χ:

$$S_\chi=(S_\chi S_\chi^*)^{1/2}W_\chi=B_\chi^{1/2}W_\chi$$

where $W_\chi:=B_\chi^{-1/2}S_\chi$ is a unitary operator from $\mathcal{L}_2(\mathcal{H}_\chi)$ onto $L^2(\mathcal{D},\mu_\chi)$. We immediately obtain the following proposition. Note that, by 2. of Proposition 5, the measure $\mu_0:=(\Psi_\chi^{-1})^*(\mu_\chi)$ is a G-invariant measure on $\mathcal{O}(\xi_\chi)$.

Proposition 7. *1) The map $W_\chi\colon\mathcal{L}_2(\mathcal{H}_\chi)\to L^2(\mathcal{D},\mu_\chi)$ is a Stratonovich-Weyl correspondence for the triple $(G,\pi_\chi,\mathcal{D})$, that is, we have*

1. $W_\chi(A^*)=\overline{W_\chi(A)}$;
2. $W_\chi(\pi_\chi(g)\,A\,\pi_\chi(g)^{-1})(Z)=W_\chi(A)(g^{-1}\cdot Z)$;
3. *W_χ is unitary.*

2) Similarly, the map $\mathcal{W}_\chi\colon\mathcal{L}_2(\mathcal{H}_\chi)\to L^2(\mathcal{O}(\xi_\chi),\mu_0)$ defined by

$$\mathcal{W}_\chi(A)=W_\chi(A)\circ\Psi_\chi^{-1}$$

is a Stratonovich-Weyl correspondence for the triple $(G,\pi_\chi,\mathcal{O}(\xi_\chi))$.

Here we have relaxed the requirement of Definition 1 that W_χ maps the identity operator I to the constant function 1 which is not adapted to the present setting (here I is not Hilbert-Schmidt). However, this requirement should hold in some generalized sense, see for instance [19].

5 Extension of the Berezin transform

In this section, we show how to extend the Berezin transform to a class of functions which contains $S_\chi(d\pi_\chi(X))$ for each $X\in\mathfrak{g}^c$, in order to define and study $W_\chi(d\pi_\chi(X))$. This question was investigated in [13] but only in the case of a reductive Lie group. Here we adapt the method of [13], Section 6 to the case of the Jacobi group.

We use the following notation. If L is a Lie group and X is an element of the Lie algebra of L then we denote by X^+ the corresponding right invariant vector field on L, that is, $X^+(h)=\frac{d}{dt}(\exp tX)h|_{t=0}$ for $h\in L$. By differentiating the multiplication map from $P^+\times K^c\times P^-$ onto $P^+K^cP^-$, we can easily prove the following result.

Lemma 1. *[10], [13] Let $X\in\mathfrak{g}^c$ and $g=z\,k\,y$ where $z\in P^+$, $k\in K^c$ and $y\in P^-$. We have*

1. $d\zeta_g\big(X^+(g)\big)=\big(\mathrm{Ad}(z)\,p_{\mathfrak{p}^+}(\mathrm{Ad}(z^{-1})\,X)\big)^+(z)$.
2. $d\kappa_g\big(X^+(g)\big)=\big(p_{\mathfrak{k}^c}(\mathrm{Ad}(z^{-1})\,X)\big)^+(k)$.

3. $d\eta_g\big(X^+(g)\big) = \big(\mathrm{Ad}(k^{-1})\, p_{\mathfrak{p}^-}(\mathrm{Ad}(z^{-1})\, X)\big)^+(y)$.

For Z, $W \in \mathcal{D}$, we set $l_Z(W) := \log \eta(\exp Z^* \exp W) \in \mathfrak{p}^-$.

Lemma 2. *1. For each Z, $W \in \mathcal{D}$ and $V \in \mathfrak{p}^+$, we have*

$$\frac{d}{dt}\, e_Z(W+tV)\big|_{t=0} = -e_Z(W)\,(d\chi\circ p_{\mathfrak{k}^c})\Big([l_Z(W),V]+\frac{1}{2}\big[l_Z(W),[l_Z(W),V]\big]\Big).$$

2. For each Z, $W \in \mathcal{D}$ and $V \in \mathfrak{p}^+$, we have

$$\frac{d}{dt}\, l_Z(W+tV)\big|_{t=0} = p_{\mathfrak{p}^-}\Big([l_Z(W),V]+\frac{1}{2}\big[l_Z(W),[l_Z(W),V]\big]\Big).$$

3. Let r and s be non-negative integers such that $r+s\leq q$. Let $Z\in\mathcal{D}$. Then the function $(\partial_z^r\partial_u^s e_Z)(W)$ is of the form $e_Z(W)Q(l_Z(W))$ where Q is a polynomial on $\mathfrak{p}^-$ of degree $\leq 2q$.

4. Let r and s be non-negative integers such that $r+s\leq q$. Then $S_\chi(\partial_z^r\partial_u^s)$ is of the form $(1-u\bar{u})^{-2q}P(z,\bar{z},u,\bar{u})$ where P is a polynomial whose degree in $(z,\bar{z})$ is $\leq 2q$.

5. For each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, the function $S_\chi(d\pi_\chi(X_1X_2\cdots X_q))(Z)$ is of the form $(1-u\bar{u})^{-2q}P(z,\bar{z},u,\bar{u})$ where P is a polynomial whose degree in $(z,\bar{z})$ is $\leq 4q$.

Proof. Following the same lines as in the proof of Lemma 4.1 in [12], we use 2. of Lemma 1. For Z, $W\in\mathcal{D}$ and $V\in\mathfrak{p}^+$, we have

$$\begin{aligned}
\frac{d}{dt}\, e_Z(W+tV)\big|_{t=0} &= \frac{d}{dt}\,\chi^{-1}\left(\kappa(\exp Z^*\exp W\exp tV)\right)\big|_{t=0}\\
&= d\chi^{-1}_{\kappa(\exp Z^*\exp W)} d\kappa_{\exp Z^*\exp W}\left(\big(\mathrm{Ad}(\exp Z^*\exp W)\,V\big)^+(\exp Z^*\exp W)\right)\\
&= -\chi^{-1}\big(\kappa(\exp Z^*\exp W)\big)\, d\chi\Big(p_{\mathfrak{k}^c}\Big(\mathrm{Ad}\big(\kappa(\exp Z^*\exp W)\eta(\exp Z^*\exp W)\big)V\Big)\Big)\\
&= -e_Z(W)(d\chi\circ p_{\mathfrak{k}^c})\Big(\mathrm{Ad}\big(\eta(\exp Z^*\exp W)\big)V\Big).
\end{aligned}$$

But here we have $[\mathfrak{p}^-,[\mathfrak{p}^-,[\mathfrak{p}^-,\mathfrak{p}^+]]]=(0)$. This implies that

$$p_{\mathfrak{k}^c}\Big(\mathrm{Ad}\big(\eta(\exp Z^*\exp W)\big)V\Big) = p_{\mathfrak{k}^c}\Big([l_Z(W),V]+\frac{1}{2}\big[l_Z(W),[l_Z(W),V]\big]\Big).$$

Hence Statement 1. is proved. Similarly, we prove 2. From 1. and 2., we deduce 3. by induction on q.

Now, let $Z=a(z,u)\in\mathcal{D}$. Then we have

$$l_Z(Z) = \Big(\big(0,-(1-u\bar{u})^{-1}(\bar{z}+\bar{u}z)\big),0,\left(\begin{smallmatrix}0 & 0\\ -(1-u\bar{u})^{-1}\bar{u} & 0\end{smallmatrix}\right)\Big)$$

and, by 3., $S_\chi(\partial_z^r\partial_u^s)(Z)=Q(l_Z(Z))$ is a linear combination of terms of the form

$$\big((1-u\bar{u})^{-1}(\bar{z}+\bar{u}z)\big)^i\big((1-u\bar{u})^{-1}\bar{u}\big)^j$$

for $i+j\leq 2q$. This gives 4. Finally, 5. follows from 4. and Proposition 3. $\square$

We are now in position to extend the Berezin transform. The following proposition is analogous to Proposition 6.3 of [13] (see also Proposition 4.1 of [12]).

Proposition 8. *If $m > 2q + \frac{3}{2}$ then for each $X_1, X_2, \ldots, X_q \in \mathfrak{g}^c$, the Berezin transform of $S_\chi\big(d\pi_\chi(X_1X_2\cdots X_q)\big)$ is well-defined.*

Proof. First recall the equality $e_{g\cdot Z} = \overline{\chi(\kappa(g\exp Z))}\pi_\chi(g)e_Z$ for each $Z\in\mathcal{D}$ and each $g\in G$ [13]. Then, by performing the change of variables $W \to g_Z\cdot W$ in equation (1), we obtain

$$(B_\chi F)(Z) = \int_{\mathcal{D}} F(g_Z\cdot W)\langle e_W, e_W\rangle_\chi^{-1}\, d\mu_\chi(W).$$

Now we take $F = S_\chi\big(d\pi_\chi(X_1X_2\cdots X_q)\big)$. Fixing $Z\in\mathcal{D}$, we set $Y_k := \mathrm{Ad}(g_Z^{-1})X_k$ for $k = 1, 2, \ldots, q$. Then, by using the G-invariance of S_χ, we get

$$F(g_Z\cdot W) = S_\chi(d\pi_\chi(Y_1Y_2\cdots Y_q))(W)$$

for each $W\in\mathcal{D}$. Thus, by 1. of Proposition 1 and 5. of Lemma 2, we see that the proposition will be established if we prove that under the condition that $m > 2q+\frac{3}{2}$ the integral

$$I := \int_{\mathbb{C}\times\mathbb{D}} P(z,\bar z,u,\bar u)(1-u\bar u)^{-2q+m-3}\exp\left(-\frac{\gamma}{4}\left(\frac{2\bar z z + \bar u z^2 + u\bar z^2}{1-u\bar u}\right)\right) d\mu_L(z,u)$$

converges for each polynomial $P(z,\bar z,u,\bar u)$ whose degree in $(z,\bar z)$ is $\leq 2q$.

We set $z = x+iy$ with $x,y\in\mathbb{R}$ and $u = a+ib$ with $a,b\in\mathbb{R}$. Then we have

$$2z\bar z + \bar u z^2 + u\bar z^2 = 2\big((1+a)x^2 + (1-a)y^2 + 2bxy\big).$$

For $u\neq 0$, this quadratic form can be reduced by means of an orthonormal change of variables of the form $z\to vz$ ($|v| = 1$). Under this change of variables, the integral I becomes

$$I = \int_{\mathbb{C}\times\mathbb{D}} P(vz,\overline{vz},u,\bar u)(1-u\bar u)^{-2q+m-3} \times \exp\left(-\frac{\gamma}{2}\left(\frac{(1+|u|)x^2 + (1-|u|)y^2}{1-u\bar u}\right)\right) d\mu_L(z,u)\,.$$

Now, we make the last change of variables

$$x' = (1-|u|)^{-1/2}x,\quad y' = (1+|u|)^{-1/2}y$$

and we obtain

$$I = \int_{\mathbb{C}\times\mathbb{D}} P(v(x'\sqrt{1-|u|} + iy'\sqrt{1+|u|}), \bar v(x'\sqrt{1-|u|} - iy'\sqrt{1+|u|}), u, \bar u) \times (1-u\bar u)^{-2q+m-\frac{5}{2}} e^{-\frac{\gamma}{2}|z'|^2}\, d\mu_L(z',u).$$

Finally we see that, under the condition that $m > 2q+\frac{3}{2}$, we have

$$\int_{\mathbb{D}} (1-u\bar u)^{-2q+m-\frac{5}{2}}\, d\mu_L(u) < +\infty$$

hence I converges. This ends the proof. $\square$

6 Stratonovich-Weyl symbols of derived representation operators

In this section, we assume that $m > \frac{7}{2}$. Then, by Proposition 8, the Berezin transform of $S_\chi(d\pi_\chi(X))$ is well-defined for each $X \in \mathfrak{g}^c$. As we shall see, this fact can be used to define $W_\chi(d\pi_\chi(X))$ for $X \in \mathfrak{g}^c$. The first step is to introduce a vector space of functions on $\mathcal{D}$ which is stable under B_χ and contains $S_\chi(d\pi_\chi(X))$ for each $X \in \mathfrak{g}^c$.

Note that, for each $X \in \mathfrak{g}^c$ and $Z \in \mathcal{D}$, we have $S_\chi(d\pi_\chi(X))(Z) = i\xi(\mathrm{Ad}(g_Z^{-1})X)$ by Proposition 5. Then, we introduce the space $\mathcal{S}$ generated by the functions $Z \to \phi_0(\mathrm{Ad}(g_Z^{-1})X)$ where $X \in \mathfrak{g}^c$ and ϕ_0 is an element of $(\mathfrak{g}^c)^*$ which is $\mathrm{Ad}^*(K)$-invariant. The following lemma can be easily verified, see [14].

Lemma 3. *Let φ_0 denote the element of $(\mathfrak{s}^c)^*$ defined by $\langle \varphi_0, \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)\rangle = a$. Then the elements of $(\mathfrak{g}^c)^*$ which are fixed by K are the elements of the form $(0, d, \lambda\varphi_0)$ where $d, \lambda \in \mathbb{C}$.*

Then we have the following result.

Proposition 9. *Let $\phi\colon \mathcal{D} \times \mathfrak{g}^c \to \mathbb{C}$ be a function such that*
(i) For each $Z \in \mathcal{D}$, the map $X \to \phi(Z, X)$ is linear;
(ii) For each $X \in \mathfrak{g}^c$, $g \in G$ and $Z \in \mathcal{D}$, we have $\phi(g \cdot Z, X) = \phi(Z, \mathrm{Ad}(g^{-1})X)$.
Then

1. *The element ϕ_0 of $(\mathfrak{g}^c)^*$ defined by $\phi_0(X) := \phi(0, X)$ is fixed by K;*
2. *For each $X \in \mathfrak{g}^c$ and $Z \in \mathcal{D}$, we have*

$$\begin{aligned}\phi(Z, X) &= \phi_0(\mathrm{Ad}(g_Z^{-1})X) = \phi_0\left(\mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X\right) \\ &= (\phi_0 \circ p_{\mathfrak{k}^c})\left(\mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X\right).\end{aligned}$$

In particular, for each $X \in \mathfrak{g}^c$, the function $\phi(\cdot, X)$ is in $\mathcal{S}$.

3. *For each $X \in \mathfrak{g}^c$, the Berezin transform of the function $\phi(\cdot, X)$ is well-defined and there exists a function $\psi\colon \mathcal{D} \times \mathfrak{g}^c \to \mathbb{C}$ satisfying (i) and (ii) such that, for each $X \in \mathfrak{g}^c$, one has $\psi(\cdot, X) = B_\chi(\phi(\cdot, X))$.*

Proof. 1. This follows from (ii).

2. By (ii) again, we have $\phi(Z, X) = \phi_0(\mathrm{Ad}(g_Z^{-1})X)$ for each $X \in \mathfrak{g}^c$ and $Z \in \mathcal{D}$. Now, by [14], there exists $k_Z \in K$ such that $g_Z = \exp(-Z^*)\zeta(\exp Z^* \exp Z)k_Z^{-1}$. Then we have

$$\begin{aligned}\phi(Z, X) &= \phi_0\left(\mathrm{Ad}(k_Z\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X\right) \\ &= \phi_0\left(\mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X\right)\end{aligned}$$

and, since $\phi_0|_{\mathfrak{p}^\pm} = 0$ by Lemma 3, we also have

$$\phi(Z, X) = (\phi_0 \circ p_{\mathfrak{k}^c})\left(\mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1} \exp Z^*)X\right).$$

3. We prove the first assertion by the same method as in the proof of Proposition 8. The second assertion immediately follows from the fact that B_χ commutes to the $\rho(g)$, $g \in G$ (see Section 4). $\square$

Now we aim to compute the Berezin transform of a function $\phi(\cdot, X)$ as before. We need the following lemma.

Lemma 4. *For each $u \in \mathbb{D}$, we have*

$$I_1(u) = \int_{\mathbb{C}} \exp\left(-\frac{\gamma}{4}\left(2\bar{z}z + \bar{u}z^2 + u\bar{z}^2\right)\right) d\mu_L(z) = \frac{2\pi}{\gamma}(1-u\bar{u})^{-1/2};$$
$$I_2(u) = \int_{\mathbb{C}} z^2 \exp\left(-\frac{\gamma}{4}\left(2\bar{z}z + \bar{u}z^2 + u\bar{z}^2\right)\right) d\mu_L(z) = -\frac{4\pi}{\gamma^2}u(1-u\bar{u})^{-3/2};$$
$$I_3(u) = \int_{\mathbb{C}} \bar{z}^2 \exp\left(-\frac{\gamma}{4}\left(2\bar{z}z + \bar{u}z^2 + u\bar{z}^2\right)\right) d\mu_L(z) = -\frac{4\pi}{\gamma^2}\bar{u}(1-u\bar{u})^{-3/2};$$
$$I_4(u) = \int_{\mathbb{C}} z\bar{z} \exp\left(-\frac{\gamma}{4}\left(2\bar{z}z + \bar{u}z^2 + u\bar{z}^2\right)\right) d\mu_L(z) = \frac{4\pi}{\gamma^2}(1-u\bar{u})^{-3/2}.$$

Proof. The integral $I_1(u)$ has been computed in the proof of Proposition 1. By taking the derivative of $I_1(u)$ with respect to u, we obtain the desired expression of $I_2(u)$ and, by conjugation, the expression of $I_3(u)$. Finally, integrating by parts, we have

$$I_1(u) = -\int_{\mathbb{C}} z\partial_z\left(\exp\left(-\frac{\gamma}{4}\left(2\bar{z}z + \bar{u}z^2 + u\bar{z}^2\right)\right)\right) d\mu_L(z) = \frac{\gamma}{2}\big(I_4(u) + \bar{u}I_2(u)\big)$$

and we get the expression of $I_4(u)$. □

We introduce the following basis of $\mathfrak{g}^c$:

$$X_1 = \big((1,0),0,0\big); \qquad Y_1 = \big((0,0),0,\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\big); \qquad H_1 = \big((0,0),1,0\big);$$
$$X_2 = \big((0,1),0,0\big); \qquad Y_2 = \big((0,0),0,\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)\big); \qquad H_2 = \big((0,0),0,\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\big).$$

Also, we denote by ϕ^1 and ϕ^2 the elements of $\mathcal{S}$ defined by $\phi^1_0 = (0,1,0)$ and $\phi^2_0 = (0,0,\varphi_0)$.

Proposition 10. *Let $\mu := \frac{1}{\gamma}i\frac{2m-3}{2m-5}$ and $\nu := \frac{2m-1}{2m-5}$. Let $\phi \in \mathcal{S}$ be defined by $\phi_0 = (0,d,\lambda\varphi_0)$ with $d,\lambda \in \mathbb{C}$. Let $\psi \in \mathcal{S}$ such that $\psi(\cdot,X) = B_\chi(\phi(\cdot,X))$ for each $X \in \mathfrak{g}^c$. Then we have $\psi_0 = (0,d,d\mu + \lambda\nu)$.*

Proof. We have just to compute the Berezin transforms $\psi^1(\cdot,X)$ and $\psi^2(\cdot,X)$ of $\phi^1(\cdot,X)$ and $\phi^2(\cdot,X)$.

We can write $\psi^1_0 = (0,d_1,\lambda_1\varphi_0)$ with $d_1,\lambda_1 \in \mathbb{C}$. For each $Z \in \mathcal{D}$, we have $\mathrm{Ad}(g_Z^{-1})H_1 = H_1$ hence $\phi^1(Z,H_1) = \phi^1_0(\mathrm{Ad}(g_Z^{-1})H_1) = 1$. This implies that

$$\psi^1_0(H_1) = \int_{\mathcal{D}} \langle e_Z, e_Z\rangle_\chi^{-1} d\mu_\chi(Z) = 1.$$

Consequently, we find $d_1 = 1$. On the other hand, we can verify that, for each $Z = a(z,u) \in \mathcal{D}$, we have

$$\phi^1(Z,H_2) = \phi^1_0\big(\mathrm{Ad}(g_Z^{-1})H_2\big) = \frac{i}{2}(1-u\bar{u})^{-2}\big(\bar{u}z^2 + (1+u\bar{u})z\bar{z} + u\bar{z}^2\big).$$

Then we get

$$\begin{aligned}\psi_0^1(H_2) &= \frac{i}{2}c_\chi \int_{\mathbb{C}\times\mathbb{D}} (1-u\bar{u})^{m-5}\big(\bar{u}z^2 + (1+u\bar{u})z\bar{z} + u\bar{z}^2\big) \\ &\quad \times \exp\left(-\frac{\gamma}{4}\left(\frac{2\bar{z}z + \bar{u}z^2 + u\bar{z}^2}{1-u\bar{u}}\right)\right) d\mu_L(z,u) \\ &= \frac{i}{2}c_\chi \int_{\mathbb{C}\times\mathbb{D}} (1-u\bar{u})^{m-3}\big(\bar{u}z^2 + (1+u\bar{u})z\bar{z} + u\bar{z}^2\big) \\ &\quad \times \exp\left(-\frac{\gamma}{4}\left(2\bar{z}z + \bar{u}z^2 + u\bar{z}^2\right)\right) d\mu_L(z,u)\end{aligned}$$

and, using Lemma 4, we obtain

$$\psi_0^1(H_2) = \frac{i}{2}c_\chi \frac{4\pi}{\gamma^2}\int_{\mathbb{D}} (1-u\bar{u})^{m-\frac{7}{2}} d\mu_L(u) = \frac{i}{2}c_\chi \frac{4\pi^2}{\gamma^2}\frac{1}{m-\frac{5}{2}}\,.$$

Thus, taking into account the value of c_χ (see Proposition 1), we can conclude that $\psi_0^1(H_2) = \mu$ hence $\lambda_1 = \mu$.

Similarly, we write $\psi_0^2 = (0, d_2, \lambda_2\varphi_0)$ with $d_2, \lambda_2 \in \mathbb{C}$ and, since we have

$$\phi^2(Z, H_1) = \phi_0^2(\mathrm{Ad}(g_Z^{-1})H_1) = 0$$

and

$$\phi^2(Z, H_2) = \phi_0^2(\mathrm{Ad}(g_Z^{-1})H_2) = (1-u\bar{u})^{-1}(1+u\bar{u})$$

we obtain $d_2 = \psi_0^2(H_1) = 0$ and $\lambda_2 = \psi_0^2(H_2) = \nu$. □

Now we show that $B_\chi \colon \mathcal{S} \to \mathcal{S}$ can be diagonalized and has positive eigenvalues.

Lemma 5. *The functions* $\phi^1(\cdot, X_1)$, $\phi^1(\cdot, X_2)$, $\phi^1(\cdot, H_1)$, $\phi^1(\cdot, Y_1)$,$\phi^1(\cdot, Y_2)$, $\phi^1(\cdot, H_2)$, $\phi^2(\cdot, Y_1)$, $\phi^2(\cdot, Y_2)$, $\phi^2(\cdot, H_2)$ *form a basis* $\mathcal{B}_0$ *of* $\mathcal{S}$.

Proof. Note that $\phi^2(\cdot, X_1) = \phi^2(\cdot, X_2) = \phi^2(\cdot, H_1) = 0$. Moreover, we can easily verify that the equality $\phi^1(\cdot, X) = \phi^2(\cdot, Y)$ where $X \in \mathfrak{g}^c$ and $Y \in \mathrm{Span}_{\mathbb{C}}\{Y_1, Y_2, H_2\}$ implies $X = Y = 0$. The result follows. □

Let us introduce the functions

$$\begin{aligned} s_1 &:= (1-\nu)\phi^1(\cdot, Y_1) + \mu\phi^2(\cdot, Y_1)\,; \\ s_2 &:= (1-\nu)\phi^1(\cdot, Y_2) + \mu\phi^2(\cdot, Y_2)\,; \\ s_3 &:= (1-\nu)\phi^1(\cdot, H_2) + \mu\phi^2(\cdot, H_2)\,. \end{aligned}$$

By combining the previous lemma with Proposition 10 we immediately obtain the following result.

Lemma 6. *The functions* $\phi^1(\cdot, X_1)$, $\phi^1(\cdot, X_2)$, $\phi^1(\cdot, H_1)$, s_1, s_2, s_3, $\phi^2(\cdot, Y_1)$, $\phi^2(\cdot, Y_2)$, $\phi^2(\cdot, H_2)$ *form a basis of* $\mathcal{S}$ *consisting in eigenvectors of* B_χ. *With respect* *to this basis,* $B_\chi|_{\mathcal{S}}$ *has matrix* $\mathrm{Diag}(1,1,1,1,1,1,\nu,\nu,\nu)$. *Moreover, the matrix of* $B_\chi|_{\mathcal{S}}$ *with respect to* $\mathcal{B}_0$ *is the* 9×9 *matrix*

$$\begin{pmatrix} I_3 & O & O \\ O & I_3 & O \\ O & \mu I_3 & \nu I_3 \end{pmatrix}$$

where I_3 *denotes the* 3×3 *identity matrix.*

Recall that for each $X\in\mathfrak{g}^c$, we have $S_\chi(d\pi_\chi(X))\in\mathcal{S}$. Then we see that the expression $W_\chi(d\pi_\chi(X))=B_\chi^{-1/2}(S_\chi(d\pi_\chi(X)))$ is well-defined. More precisely, we have the following result.

Proposition 11. *For each* $X\in\mathrm{Span}_{\mathbb{C}}\{X_1,X_2,H_1\}$, *we have*

$$W_\chi\bigl(d\pi_\chi(X)\bigr)=S_\chi\bigl(d\pi_\chi(X)\bigr)\,.$$

For each $X\in\mathrm{Span}_{\mathbb{C}}\{Y_1,Y_2,H_2\}$, *we have*

$$W_\chi\bigl(d\pi_\chi(X)\bigr)=S_\chi\bigl(d\pi_\chi(X)\bigr)+(1-\nu^{-1/2})\left(\frac{i\gamma\mu}{1-\nu}+m\right)\phi^2(\cdot,X)\,.$$

Proof. Recall that for each $X\in\mathfrak{g}^c$ we have

$$S_\chi(d\pi_\chi(X))(Z)=d\chi(\mathrm{Ad}(g_Z^{-1})X)=i\gamma\phi^1(Z,X)-m\phi^2(Z,X).$$

From Lemma 6, we deduce that the matrix of $B_\chi^{-1/2}$ with respect to $\mathcal{B}_0$ is

$$\begin{pmatrix} I_3 & O & O \\ O & I_3 & O \\ O & -\frac{\mu\nu^{-1/2}}{1+\nu^{1/2}}I_3 & \nu^{-1/2}I_3 \end{pmatrix}.$$

The result then follows. ∎

Acknowledgements

I thank the referee for a careful reading of the manuscript and many valuable comments.

References

[1] S.T. Ali, M. Englis: Quantization methods: a guide for physicists and analysts. *Rev. Math. Phys.* 17 (4) (2005) 391–490.

[2] J. Arazy, H. Upmeier: *Invariant symbolic calculi and eigenvalues of invariant operators on symmetric domains.* Function spaces, interpolation theory and related topics (Lund, 2000) 151–211. De Gruyter, Berlin (2002).

[3] J. Arazy, H. Upmeier: Weyl Calculus for Complex and Real Symmetric Domains, Harmonic analysis on complex homogeneous domains and Lie groups (Rome, 2001). *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 13 (3–4) (2002) 165–181.

[4] S. Berceanu: A holomorphic representation of the Jacobi algebra. *Rev. Math. Phys.* 18 (2006) 163–199.

[5] S. Berceanu, A. Gheorghe: On the geometry of Siegel-Jacobi domains. *Int. J. Geom. Methods Mod. Phys.* 8 (2011) 1783–1798.

[6] F.A. Berezin: Quantization. *Math. USSR Izv.* 8 (5) (1974) 1109–1165.

[7] F.A. Berezin: Quantization in complex symmetric domains. *Math. USSR Izv.* 9 (2) (1975) 341–379.

[8] R. Berndt, S. Böcherer: Jacobi forms and discrete series representations of the Jacobi group. *Math. Z.* 204 (1990) 13–44.

[9] R. Berndt, R. Schmidt: *Elements of the representation theory of the Jacobi group,* Progress in Mathematics 163. Birkhäuser Verlag, Basel (1998).

[10] B. Cahen: Berezin quantization for discrete series. *Beiträge Algebra Geom.* 51 (2010) 301–311.

[11] B. Cahen: Stratonovich-Weyl correspondence for compact semisimple Lie groups. *Rend. Circ. Mat. Palermo* 59 (2010) 331–354.

[12] B. Cahen: Stratonovich-Weyl correspondence for discrete series representations. *Arch. Math. (Brno)* 47 (2011) 41–58.

[13] B. Cahen: Berezin Quantization and Holomorphic Representations. *Rend. Sem. Mat. Univ. Padova* 129 (2013) 277–297.

[14] B. Cahen: Global Parametrization of Scalar Holomorphic Coadjoint Orbits of a Quasi-Hermitian Lie Group. *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica* 52 (2013) 35–48.

[15] J.F. Cariñena, J.M. Gracia-Bondìa, J.C. Vàrilly: Relativistic quantum kinematics in the Moyal representation. *J. Phys. A: Math. Gen.* 23 (1990) 901–933.

[16] M. Davidson, G. Òlafsson, G. Zhang: Laplace and Segal-Bargmann transforms on Hermitian symmetric spaces and orthogonal polynomials. *J. Funct. Anal.* 204 (2003) 157–195.

[17] H. Figueroa, J.M. Gracia-Bondìa, J.C. Vàrilly: Moyal quantization with compact symmetry groups and noncommutative analysis. *J. Math. Phys.* 31 (1990) 2664–2671.

[18] B. Folland: *Harmonic Analysis in Phase Space.* Princeton Univ. Press (1989).

[19] J.M. Gracia-Bondìa: *Generalized Moyal quantization on homogeneous symplectic spaces,* Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990), 93–114, Contemp. Math., 134. Amer. Math. Soc., Providence, RI (1992).

[20] J.M. Gracia-Bondìa, J.C. Vàrilly: The Moyal Representation for Spin. *Ann. Physics* 190 (1989) 107–148.

[21] A.A. Kirillov: *Lectures on the Orbit Method,* Graduate Studies in Mathematics, Vol. 64. American Mathematical Society, Providence, Rhode Island (2004).

[22] K-H. Neeb: *Holomorphy and Convexity in Lie Theory,* de Gruyter Expositions in Mathematics, Vol. 28. Walter de Gruyter, Berlin, New-York (2000).

[23] T. Nomura: Berezin Transforms and Group representations. *J. Lie Theory* 8 (1998) 433–440.

[24] B. Ørsted, G. Zhang: Weyl Quantization and Tensor Products of Fock and Bergman Spaces. *Indiana Univ. Math. J.* 43 (2) (1994) 551–583.

[25] J. Peetre, G. Zhang: A weighted Plancherel formula III. The case of a hyperbolic matrix ball. *Collect. Math.* 43 (1992) 273–301.

[26] I. Satake: *Algebraic structures of symmetric domains.* Iwanami Sho-ten, Tokyo and Princeton Univ. Press, Princeton, NJ (1971).

[27] R.L. Stratonovich: On distributions in representation space. *Soviet Physics. JETP* 4 (1957) 891–898.

[28] A. Unterberger, H. Upmeier: Berezin transform and invariant differential operators. *Commun. Math. Phys.* 164 (3) (1994) 563–597.

[29] G. Zhang: Berezin transform on compact Hermitian symmetric spaces. *Manuscripta Math.* 97 (1998) 371–388.

Author's address:
Université de Lorraine, Site de Metz, UFR-MIM, Département de mathématiques, Bâtiment A, Ile du Saulcy, CS 50128, F-57045, Metz cedex 01, France.

E-mail: benjamin.cahen@univ-lorraine.fr

Received: 2 December, 2013
Accepted for publication: 2 March, 2014
Communicated by: Ilka Agricola