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Almost Abelian rings

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Abstract. A ring $R$ is defined to be left almost Abelian if $ae = 0$ implies $aRe = 0$ for $a \in N(R)$ and $e \in E(R)$, where $E(R)$ and $N(R)$ stand respectively for the set of idempotents and the set of nilpotents of $R$. Some characterizations and properties of such rings are included. It follows that if $R$ is a left almost Abelian ring, then $R$ is $\pi$-regular if and only if $N(R)$ is an ideal of $R$ and $R/N(R)$ is regular. Moreover it is proved that (1) $R$ is an Abelian ring if and only if $R$ is a left almost Abelian left idempotent reflexive ring. (2) $R$ is strongly regular if and only if $R$ is regular and left almost Abelian. (3) A left almost Abelian clean ring is an exchange ring. (4) For a left almost Abelian ring $R$, it is an exchange $(S, 2)$ ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R$.

1 Introduction

Throughout this article, all rings are associative with identity, and all modules are unital. The symbols $J(R)$, $N(R)$, $U(R)$, $E(R)$ will stand respectively for the Jacobson radical, the set of all nilpotent elements, the set of all invertible elements, the set of all idempotent elements of a ring $R$. For any nonempty subset $X$ of a ring $R$, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the right annihilator of $X$ and the left annihilator of $X$, respectively.

The ring $R$ is called left almost Abelian if $ae = 0$ implies $aRe = 0$ for $a \in N(R)$ and $e \in E(R)$, and $R$ is said to be semiabelian if every idempotent of $R$ is either left semicentral or right semicentral. The ring $R$ is called Abelian if every idempotent of $R$ is central. Clearly, Abelian rings are semiabelian and left almost Abelian. Following [4], we know that there exists a semiabelian ring which is not Abelian.

The ring $R$ is called $\pi$-regular if for every $a \in R$ there exist $n \geq 1$ and $b \in R$ such that $a^n = a^n ba^n$, and in case of $n = 1$ the ring $R$ is called von Neumann regular. So von Neumann regular rings are $\pi$-regular. A ring $R$ is called strongly $\pi$-regular if for every $a \in R$ there exist $n \geq 1$ and $b \in R$ such that $a^n = a^{2n}b$.

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and in case of \( n = 1 \) the ring \( R \) is called strongly regular. So strongly regular rings are strongly \( \pi \)-regular. The case when the set \( N(R) \) of nilpotent elements of a \( \pi \)-regular ring \( R \) is an ideal has been studied by many authors. For examples, in [1], it is shown that if \( R \) is an Abelian ring, then \( R \) is a \( \pi \)-regular ring if and only if \( N(R) \) is an ideal of \( R \) and \( R/N(R) \) is a strongly regular ring. The goal of this paper is to study the properties of left almost Abelian rings, and to extend some known results on Abelian von Neumann regular rings, \( \pi \)-regular rings, and exchange rings. For instance we prove the following results: if \( R \) is a left almost Abelian ring, then \( R \) is \( \pi \)-regular if and only if \( N(R) \) is an ideal of \( R \) and \( R/N(R) \) is strongly regular.

2 Characterizations and Properties

It is easy to see that a ring \( R \) is Abelian if and only if \( ae = 0 \) implies \( aRe = 0 \) for each \( a \in R \) and \( e \in E(R) \). Motivated by this, we call a ring \( R \) left almost Abelian if \( ae = 0 \) implies \( aRe = 0 \) for each \( a \in N(R) \) and \( e \in E(R) \). Clearly, Abelian rings are left almost Abelian. The converse is not true in general. For example, if \( R \) is a reduced ring with \( E(R) = \{0, 1\} \) then the \( 2 \times 2 \) upper triangular matrix ring \( UTM_2(R) \) is left almost Abelian but not Abelian.

According to [1], Abelian rings are semiabelian and the converse is not true in general. The following example implies that semiabelian rings need not be left almost Abelian.

Let \( R \) be a ring with \( E(R) = \{0, 1\} \) and \( N(R) \neq 0 \). Then

\[
E(UTM_2(R)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} : a, b \in R \right\}.
\]

Clearly, \( \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \) is left semicentral and \( \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} \) is right semicentral, so \( UTM_2(R) \) is semiabelian, but not left almost Abelian. In fact, let \( 0 \neq a \in N(R) \). Then

\[
\begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix} \in N(UTM_2(R)) \quad \text{and} \quad \begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 0,
\]

but

\[
\begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix} UTM_2(R) \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & aR \\ 0 & 0 \end{pmatrix} \neq 0.
\]

Hence \( UTM_2(R) \) is not left almost Abelian.

This example also implies that the upper triangular matrices rings over a left almost Abelian ring need not be left almost Abelian.

**Proposition 1.**

1. The subrings and direct products of left almost Abelian rings are left almost Abelian.
2. Let \( R \) be a left almost Abelian ring and \( e \in E(R) \). Then
   \[
   (1 - e)Re \subseteq J(R).
   \]
(b) If $ReR = R$, then $e = 1$.

(c) If $M$ is a maximal left ideal of $R$ and $e \notin M$, then $(1 - e)R \subseteq M$.

(d) Let $M$ be a maximal left ideal of $R$ and $a \in R$. If $1 - ae \in M$, then $1 - ea \in M$.

(e) For any $x \in R$ and $n \geq 1$, $(exe)^n = ex^n e$.

Proof. (1) is trivial.

(2) (a) For any $a \in R$, write $h = (1 - e)a - (1 - e)a(1 - e)$. Then $h \in \mathcal{N}(R)$ and $h(1 - e) = 0$. Since $R$ is a left almost Abelian ring, $(1 - e)aeR(1 - e) = hR(1 - e) = 0$. Thus

$$(1 - e)ReR(1 - e) = \sum_{a \in R} (1 - e)aeR(1 - e) = 0$$

and so

$$(1 - e)Re \subseteq J(R).$$

(b) is an immediate consequence of (a).

(c) Since $e \notin M$, $Re + M = R$. By (a), $(1 - e)Re \subseteq J(R) \subseteq M$, hence

$$(1 - e)R = (1 - e)Re + (1 - e)M \subseteq M.$$ 

(d) Since $1 - ae \in M$, $e \notin M$. By (c), $(1 - e)R \subseteq M$. Since $1 - ae = (1 - a) + (a - ae)$, $1 - a \in M$, and $1 - ea = (1 - a) + ((1 - e)a)$ implies $1 - ea \in M$.

(e) Since

$$ex(1 - e) \in N(R), \quad ex(1 - e)xe \in (1 - e)xeRe,$$

i.e. $ex^2 e = e(xe)^2$. Since

$$ex^2 e = (exe)^2 + ex(1 - e)xe, \quad ex^2 e = (exe)^2.$$  

By induction on $n$, we obtain $ex^n e = (exe)^n$. □

It is well known that a ring $R$ is Abelian if and only if every idempotent of $R$ is left semicentral and if and only if every idempotent of $R$ is right semicentral. Hence we can construct a left almost Abelian ring which is not semiabelian.

Let $R_1$ and $R_2$ be left almost Abelian rings which are not Abelian. Take $e_1 \in R_1$ to be a right semicentral idempotent which is not central and $e_2 \in R_2$ to be a left semicentral idempotent which is not central, then the idempotent $(e_1, e_2)$ is neither right nor left semicentral in $R_1 \oplus R_2$. Hence $R_1 \oplus R_2$ is not semiabelian, while by Proposition [1](1), $R_1 \oplus R_2$ is left almost Abelian.

A ring $R$ is called directly finite if $xy = 1$ implies $yx = 1$ for $x, y \in R$, and $R$ is called left min-abelian if for every

$$e \in ME_l(R) = \{e \in E(R) \mid Re \text{ is a minimal left ideal of } R\},$$

e is left semicentral in $R$. It is well known that Abelian rings are directly finite and left min-abelian.
Corollary 1. Let $R$ be a left almost Abelian ring. Then

(1) $R$ is directly finite.

(2) $R$ is left min-abelian.

Proof. (1) Let $ab = 1$, where $a, b \in R$. Set $e = ba$, then $e \in E(R)$, $ae = a$ and $eb = b$. Since $R$ is left almost Abelian, $(1 - e)Re \subseteq J(R)$ by Proposition 1(2)(a). So we have $(1 - e)a = (1 - e)ae \in J(R)$. Therefore, $1 - e = (1 - e)ab \in J(R)$. This gives $1 = e = ba$, and $R$ is directly finite.

(2) Let $e \in ME_l(R)$. If $e$ is not left semicentral, then there exists $0 \neq a \in R$ such that $ae - eae \neq 0$. Let $h = ae - eae$. Then $eh = 0$, $he = h$ and $0 \neq h \in N(R)$. Since $hR(1 - e) \subseteq (1 - e)ReR(1 - e)$, the equality $hR(1 - e) = 0$ follows from the proof of Proposition 1(2)(a). Since $0 \neq Rh \subseteq Re$, $Rh = Re$. Hence $eR(1 - e) = 0$, so also $eR = eRe$. Let $e = ch$ for some $c \in R$. Then $h = he = hec = heceh = 0$ what contradicts to $h \neq 0$. Thus $e$ is left semicentral and so $R$ is a left min-abelian ring. □

The following example shows that the converse of Corollary 1 is not true in general.

Let $F$ be a division ring and

$$R = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}. $$

For the idempotent $e = e_{11} + e_{33}$ we obtain that $eR(1 - e)Re = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0,$

and so $R$ is not left almost Abelian. But by [19, Proposition 2.1] $R$ is left quasi-duo, hence $R$ is left min-abelian by [16, Theorem 1.2].

According to [13], an element $e$ of a ring $R$ is called op-idempotent if $e^2 = -e$. Clearly, an op-idempotent element may not be idempotent. For example, let $R = \mathbb{Z}/3\mathbb{Z}$. Then $2 \in R$ is op-idempotent, while it is not idempotent. In [3], Chen called an element $e \in R$ potent if there exists an integer $n \geq 2$ such that $e^n = e$. Clearly, idempotent is potent, while there exists a potent element which is not idempotent.

For example, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(\mathbb{Z})$ is a potent element, while it is not idempotent.

We denote by $E^o(R)$ and $PE(R)$ the set of all op-idempotent elements and the set of all potent elements of $R$, respectively. Write

$$P_l(R) = \{ k \in R \mid _R Rk \text{ is projective} \}.$$ 

Clearly, $E(R) \subseteq P_l(R)$. Similarly, we can define $P_r(R)$. Recall that a ring $R$ is left PP (i.e. principally left ideal of $R$ is projective) if $_R Ra$ is projective for all $a \in R$. Evidently, $R$ is a left PP ring if and only if $P_l(R) = R$. A ring $R$ is called right GPP if for any $x \in R$, there exists $n \geq 1$ such that $x^n \in P_r(R)$. 
Theorem 1. The following conditions are equivalent for a ring $R$:

1. $R$ is a left almost Abelian ring;
2. $ae = 0$ implies $aRe = 0$ for each $a \in N(R)$ and $e \in E^0(R)$;
3. $ae = 0$ implies $aRe = 0$ for each $a \in N(R)$ and $e \in PE(R)$;
4. $ak = 0$ implies $aRk = 0$ for each $a \in N(R)$ and $k \in P_1(R)$.

Proof. (1) $\iff$ (2), (3) $\implies$ (1) and (4) $\implies$ (1) are trivial.

(1) $\implies$ (3) Let $e \in PE(R)$ and $a \in N(R)$ with $ae = 0$. Then there exists $n \geq 2$ such that $e^n = e$. Since $e^{n-1} \in E(R)$ and $ae^{n-1} = 0$, $aRe^{n-1} = 0$ by (1). Thus $aRe = aRe^n = aRe^{n-1}e = 0$.

(1) $\implies$ (4) Assume that $a \in N(R)$ and $k \in P_1(R)$ are such that $ak = 0$. Since $Rk$ is projective, there exists $e \in E(R)$ satisfying $l(k) = l(e)$. Hence $ae = 0$, and so $aRe = 0$ by (1). Since $k = ek$, $aRk = aRek = 0$. $\square$

Corollary 2. Let $R$ be a left PP ring. Then the following conditions are equivalent:

1. $R$ is a left almost Abelian ring;
2. For each $a \in N(R)$ and $b \in R$, $ab = 0$ implies $aRb = 0$;
3. For each $a \in N(R)$, $r(a)$ is an ideal of $R$.

A ring $R$ is called left idempotent reflexive if $aRe = 0$ implies $eRa = 0$ for all $a \in R$ and $e \in E(R)$. Clearly, Abelian rings are left idempotent reflexive.

Theorem 2. The following conditions are equivalent for a ring $R$:

1. $R$ is an Abelian ring;
2. $R$ is an almost Abelian ring and left idempotent reflexive ring;
3. $R$ is a left idempotent reflexive ring and for any $a, b \in R$ and $e \in E(R)$ we have $eabe = eae$.

Proof. (1) $\implies$ (2) is trivial.

(2) $\implies$ (3) By Proposition 1, $ea(1-e)be = 0$ for all $a, b \in R$. Hence $eabe = eae$.

(3) $\implies$ (1) Let $e \in E(R)$. For any $a \in R$, write $h = ae - eae$. Then $hR(1-e) = (1-e)hR(1-e) = (1-e)h(1-e)R(1-e)$ by (3), so $hR(1-e) = 0$ because $h(1-e) = 0$. Since $R$ is a left idempotent reflexive ring, $(1-e)Rh = 0$, which implies $h = (1-e)h = 0$. Thus $ae = eae$ for all $a \in R$, showing that $e$ is left semicentral. This implies that $R$ is an Abelian ring. $\square$

A ring $R$ is called von Neumann regular if $a \in aRa$ for all $a \in R$ and $R$ is said to be unit-regular if for any $a \in R$, $a = auu$ for some $u \in U(R)$. A ring $R$ is called strongly regular if $a \in a^2R$ for all $a \in R$. Clearly, strongly regular $\iff$ unit-regular $\iff$ von Neumann regular. Since von Neumann regular rings are semiprime, it follows that von Neumann regular rings are left idempotent reflexive. And it is well known that $R$ is strongly regular if and only if $R$ is von Neumann regular and Abelian. In view of Theorem 2, we have the following corollary.
Corollary 3. The following conditions are equivalent for a ring $R$:

(1) $R$ is a strongly regular ring;
(2) $R$ is an unit-regular ring and left almost Abelian ring;
(3) $R$ is a von Neumann regular ring and left almost Abelian ring.

Following [17], a ring $R$ is called left NPP (nil left principally ideal of $R$ is projective) if for any $a \in N(R)$, $Ra$ is projective left $R$-module. A ring $R$ is said to be reduced if $a^2 = 0$ implies $a = 0$ for each $a \in R$, or equivalently, $N(R) = 0$. Obviously, reduced rings are left NPP, semiprime and Abelian. The following theorem gives some new characterizations of reduced rings in terms of left almost Abelian rings and left NPP rings.

Theorem 3. The following conditions are equivalent for a ring $R$:

(1) $R$ is a reduced ring;
(2) $R$ is a left NPP ring, semiprime ring and left almost Abelian ring;
(3) $R$ is a left NPP ring, left idempotent reflexive ring and left almost Abelian ring.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) By Theorem 2, $R$ is an Abelian ring. Now let $a \in R$ such that $a^2 = 0$. Since $R$ is left NPP, $l(a) = Re, e \in E(R)$. Hence $ea = 0$ and $a = ae$ because $a \in l(a)$. Thus $a = ae = ea = 0$. □

The following theorem is an immediate consequence of Proposition 1(1). We prove this directly.

Theorem 4. If $R$ is a subdirect product of a family of left almost Abelian rings $\{R_i : i \in I\}$, then $R$ is left almost Abelian.

Proof. Let $R_i = R/A_i$ where $A_i$ be ideals of $R$ with $\bigcap_{i \in I} A_i = 0$. Let $a \in N(R)$ and $e \in E(R)$ with $ae = 0$. Then $a_i = a + A_i \in N(R_i)$, $e_i = e + A_i \in E(R_i)$ and $(a + A_i)(e + A_i) = 0$ for any $i \in I$. Since each $R_i$ is left almost Abelian, $a_i e_i = 0$ for $i \in I$. This implies $aRe \subseteq A_i$ for all $i \in I$, so we have $aRe \subseteq \bigcap_{i \in I} A_i = 0$. Therefore $R$ is left almost Abelian. □

Recall that a ring $R$ has insertion-of-factors-property (IFP) if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$.

A ring $R$ is called left WIFP (weakly IFP) if for any $a \in N(R)$ and $b \in R$, $ab = 0$ implies $aRb = 0$. By Corollary 2, we know that left PP left almost Abelian rings are left WIFP, and left WIFP rings are left almost Abelian.

Clearly, IFP rings are left WIFP.

Let $Z_2 = Z/2Z$. Then the $2 \times 2$ upper triangular matrix ring $R = \begin{pmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{pmatrix}$ is a left almost Abelian and left PP ring, so $R$ is a left WIFP ring. Since $R$ is not
an Abelian ring, \( R \) is not an IFP ring. Thus there exists a left WIFP ring which is neither Abelian nor IFP.

It is well known that rings whose simple left \( R \)-modules are YJ-injective are always semiprime. But in general rings whose simple singular left \( R \)-modules are injective (hence also YJ-injective) need not be semiprime.

In [7], it is shown that if \( R \) is an IFP ring over which every simple singular left modules are YJ-injective, then \( R \) is a reduced weakly regular ring. We can generalize the result as follows.

**Theorem 5.** If \( R \) is a left WIFP ring whose every simple singular left modules are YJ-injective, then \( R \) is a reduced weakly regular ring.

**Proof.** First, we show that \( R \) is a reduced ring. Let \( a^2 = 0 \). Suppose that \( a \neq 0 \). Then there exists a maximal left ideal \( M \) containing \( r(a) \) because \( r(a) \neq R \) and \( r(a) \) is a left ideal of \( R \). If \( M \) is not essential left ideal of \( R \), then \( M = l(e) \) for some \( e \in ME_1(R) \). Since \( a \in r(a) \subseteq M = l(e) \), \( ae = 0 \). Hence \( e \in r(a) \subseteq M = l(e) \), which is a contradiction. Therefore \( M \) must be an essential left ideal of \( R \). Thus \( R/M \) is YJ-injective and so any \( R \)-homomorphism of \( Ra \) into \( R/M \) extends to one of \( R \) into \( R/M \). Let \( f : Ra \to R/M \) be defined by \( f(ra) = r + M \). Note that \( f \) is a well-defined \( R \)-homomorphism. Since \( R/M \) is YJ-injective, there exists \( c \in R \) such that \( 1 + M = f(a) = ac + M \), but \( ac \in r(a) \subseteq M \), which implies \( 1 \in M \), a contradiction. Hence \( a = 0 \) and so \( R \) is a reduced ring. Therefore \( R \) is an IFP ring. By [7] p. 2087–2096, \( R \) is also a weakly regular ring. \( \square \)

**Proposition 2.** Let \( R \) be a left almost Abelian ring and right GPP ring. Then for each \( x \in R \), \( x = u + a \), where \( u \in P_r(R) \) and \( a \in N(R) \).

**Proof.** Since \( R \) is a right GPP ring, there exists \( n \geq 1 \) such that \( x^n \in P_r(R) \). Clearly, there exists \( e \in E(R) \) such that \( x^n e = x^n \) and \( r(x^n) = r(e) \). Since \( xe = (xe)e \) and \( r(xe) = r(e) \), \( xe \in P_r(R) \) and

\[
(x(1-e))^{n+1} = x((1-e)x(1-e))^n = x(1-e)x^n(1-e)
\]

by Proposition [7] 2(e). Hence \( x(1-e) \in N(R) \). Let \( u = xe \) and \( a = x(1-e) \). Then \( x = u + a \), \( u \in P_r(R) \) and \( a \in N(R) \). \( \square \)

A ring \( R \) is called left SF if every simple left \( R \)-module is flat, and \( R \) is said to be right NFB (nilpotent free Baer ring) if for any \( a \in N(R) \), and \( b \in R \) with \( ab = 0 \), there exists \( e \in E(R) \) such that \( ae = 0 \) and \( eb = b \). Clearly, right NPP rings are right NFB.

**Proposition 3.** Let \( R \) be a left SF ring. If \( R \) is a left almost Abelian right NFB ring, then \( R \) is a strongly regular ring.

**Proof.** It is well known that reduced left SF rings are strongly regular. We claim that \( R \) is reduced. In fact, if \( a^2 = 0 \), then \( Ra + r(aR) = R \). If not, then there exists maximal left ideal \( M \) of \( R \) containing \( Ra + r(aR) \). Since \( R \) is a left SF ring, \( R/M \) is flat as a left \( R \)-module. Since \( a \in Ra \subseteq M \), \( a = ab \) for some \( b \in M \). Since
$R$ is a right NFB ring, there exists $e \in E(R)$ such that $ae = 0$ and $e(1-b) = 1-b$. Since $R$ is a left almost Abelian ring, $aRe = 0$. Hence $aR(1-b) = aRe(1-b) = 0$, which implies $1-b \in r(aR) \subseteq M$. This is a contradiction. Hence $Ra + r(aR) = R$.

Let $1 = ca + x$, where $c \in R$ and $x \in r(aR)$. Therefore, $a = aca + ax = aca$. Since $a(1-ca) = 0$ and $1-ca \in E(R)$, $aR(1-ca) = 0$. Hence $ac(1-ca) = 0$, this gives $ac = acca$ and $a = aca = accaa = 0$. □

**Corollary 4.** The following conditions are equivalent for a ring $R$:

(1) $R$ is a strongly regular ring;
(2) $R$ is a left SF ring, left almost Abelian ring and right NFB ring;
(3) $R$ is a left SF ring, left almost Abelian ring and right NPP ring;
(4) $R$ is a left SF ring, left almost Abelian ring and right PP ring.

Let $R$ be a ring and $M$ a bimodule over $R$. The trivial extension of $R$ and $M$ is $R \propto M = \{(a,x) | a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by $(a, x)(b, y) = (ab, ay + xb)$. Clearly $R \propto M$ is a ring and $0 \propto M = \{(0, x) | x \in M\}$ is a nonzero nilpotent ideal of $R \propto M$.

Let $R$ be a ring, $M$ a bimodule over $R$. Write

$$T(R, M) = \left\{ \begin{pmatrix} c & x \\ 0 & c \end{pmatrix} \bigg| c \in R, x \in M \right\} ,$$

then $T(R, M)$ is a ring and $T(R, M) \cong R \propto M$.

Let $R$ be a ring and $R[x]$ denote the ring of polynomials over $R$. Clearly, $R[x]/(x^2) \cong R \propto R$.

A right $R$-module $M$ is called normal if $me = 0$ implies $mRe = 0$ for each $m \in M$ and $e \in E(R)$. Clearly, every right module over an Abelian ring is normal.

**Proposition 4.** Let $M$ be a $(R, R)$-bimodule. Then $T(R, M)$ is a left almost Abelian ring if and only if $R$ is a left almost Abelian ring and $M$ is a right normal $R$-module.

*Proof.* Assume that $T(R, M)$ is a left almost Abelian ring. Then $R$ is a left almost Abelian ring by Proposition 1. Let $m \in M$ and $e \in E(R)$ satisfy $me = 0$. Then

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} = 0 .$$

Since $T(R, M)$ is left almost Abelian,

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} = 0$$

for each $r \in R$. Therefore $mre = 0$ for each $r \in R$, that is, $mRe = 0$, and $M$ is a right normal $R$-module.
Conversely, assume that $R$ is left almost Abelian and $M$ is a right normal $R$-module. Let

$$A = \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} \in N(T(R, M))$$

and

$$E = \begin{pmatrix} e & y \\ 0 & e \end{pmatrix} \in E(T(R, M))$$

satisfy $AE = 0$. Then $a \in N(R)$, $e \in E(R)$ and we have the following equations:

$$ey + ye = y, \quad (1)$$
$$ae = 0, \quad (2)$$
$$ay + xe = 0. \quad (3)$$

Since $R$ is almost Abelian, $aRe = 0$ by [2]. Hence, by (1), we have

$$ay = aey + ay = 0. \quad (4)$$

Thus (3) implies

$$xe = (ay + xe) - ay = 0. \quad (5)$$

Since $M$ is right normal $R$-module, $xRe = 0$.

Now, for each $B = \begin{pmatrix} b & z \\ 0 & b \end{pmatrix} \in E(T(R, M))$, we have

$$ABE = \begin{pmatrix} abe & aby + aze + xbe \\ 0 & abe \end{pmatrix}. \quad (6)$$

Since $abe, aze, abye \in aRe$, $abe = abye = aze = 0$. Similarly $aby = abey + abye$ implies $aby = 0$ and $xbe \in xRe$ implies $xbe = 0$.

Thus $ABE = 0$, and this gives $AT(R, M)E = 0$. Hence $T(R, M)$ is a left almost Abelian ring. \hfill \Box

**Corollary 5.** Let $M$ be an $(R, R)$-bimodule. Then $R \bowtie M$ is a left almost Abelian ring if and only if $R$ is a left almost Abelian ring and $M$ is a right normal $R$-module.

Let $R$ be a left almost Abelian ring and $I$ an ideal of $R$. If $I \subseteq N(R)$, then $I$ is right normal as right $R$-module. Hence by Proposition 4 and Corollary 5 we have the following corollary.

**Corollary 6.** Let $I$ be an ideal of $R$ and $I \subseteq N(R)$. Then the following conditions are equivalent:

1. $R$ is a left almost Abelian ring;
2. $T(R, I)$ is a left almost Abelian ring;
3. $R \bowtie I$ is a left almost Abelian ring.
It is well known that a ring $R$ is Abelian if and only if for each $e, g \in E(R)$, $ge = 0$ implies $gRe = 0$. Hence, a ring $R$ is Abelian if and only if every right $R$-module is normal and if and only if $R_R$ is normal. Thus, by Proposition 4, we have the following corollary.

**Corollary 7.** Let $R$ be a ring. Then the following conditions are equivalent:

1. $R$ is an Abelian ring;
2. $T(R, R)$ is a left almost Abelian ring;
3. $R \times R$ is a left almost Abelian ring;
4. $R[x]/(x^2)$ is a left almost Abelian ring.

### 3 Almost Abelian $\pi$-regular rings

For convenience, we list the following notions which appeared in the first section of this paper. Let $R$ be a ring and $a \in R$. Then $a$ is called $\pi$-regular, if there exist $n \geq 1$ and $b \in R$ such that $a^n = a^n ba^n$. If $n = 1$, $a$ is called von Neumann regular. Further $a$ is said to be strongly $\pi$-regular, if $a^n = a^{n+1}b$, and if $n = 1$, $a$ is called strongly regular. A ring $R$ is called von Neumann regular, strongly regular, $\pi$-regular and strongly $\pi$-regular, if every element of $R$ is von Neumann regular, strongly regular, $\pi$-regular and strongly $\pi$-regular, respectively. For convenience, we list some known facts which are necessary for the study of $\pi$-regularity of rings.

**Lemma 1.** [11, Theorem 23.2] The following conditions are equivalent for a ring $R$.

1. $R$ is strongly $\pi$-regular.
2. Every prime factor ring of $R$ is strongly $\pi$-regular.
3. $R/P(R)$ is strongly $\pi$-regular.

**Proposition 5.** Let $R$ be a left almost Abelian ring and $x \in R$. Then:

1. If $x$ is von Neumann regular, then $x$ is strongly regular.
2. If $x$ is $\pi$-regular, then there exists an $e \in E(R)$ such that $ex$ is von Neumann regular and $(1 - e)x \in N(R)$.
3. $R$ is $\pi$-regular if and only if $R$ is strongly $\pi$-regular.

**Proof.**

1. Let $x = yx$ for some $y \in R$. Write $e = yx$. Then $e^2 = e \in R$ and $x = xe$. By Proposition [12],

$$e = eee = eyxe = eyexe = eyex = ey^2x^2$$

so, we have $x = xe = xy^2x^2$. Similarly, we can show that $x = x^2y^2x$. Therefore $x$ is strongly regular.

2. By hypothesis, there exists a positive integer $n$ such that $x^n$ is regular. By (1), $x^n$ is strongly regular. By [10], $x^n = x^n ux^n$ and $x^n u = ux^n$ for some $u \in U(R)$. Let $e = x^n u$. Then $e \in E(R)$, $x^n = ex^n$ and $x^n = ev$, where $v = u^{-1}$. Since

$$(ex)(x^{n-1}u)(ex) = ex^n uex = evuex = ex,$$
ex is von Neumann regular. On the other hand, by Proposition 1(2),

\[((1 - e)x)^n(1 - e) = (1 - e)x^n(1 - e) = (1 - e)ev(1 - e) = 0\],

so, we have \((1 - e)x^{n+1} = 0\). Hence \((1 - e)x \in N(R)\).

(3) follows from (1). □

The module \(R M\) has the finite exchange property if for every module \(R A\) and any two decompositions \(A = M' \oplus N = \oplus_{i \in I} A_i\) with \(M' \cong M\) and \(I\) finite set, there exist submodules \(A'_i \subseteq A_i\) such that \(A = M' \oplus (\oplus_{i \in I} A'_i)\).

Warfield [15] called a ring \(R\) an exchange ring if \(R R\) has the finite exchange property and showed that this definition is left-right symmetric. Nicholson [9] showed that \(R\) is an exchange ring if and only if idempotents can be lifted modulo every left (equivalently, right) ideal of \(R\).

**Theorem 6.** Let \(R\) be a left almost Abelian exchange ring. Then \(R/P\) is a local ring for every prime ideal of \(R\).

**Proof.** According to [14, Theorem 1], an exchange ring with only two idempotents is a local ring. Since \(R\) is an exchange ring, idempotents can be lifted modulo \(P\). For any idempotent element \(g\) of \(R/P\), there exists idempotent \(e\) of \(R\) such that \(e + P = g\). Since \(R\) is a left almost Abelian, \(eR(1 - e)Re = 0\) by Proposition 1(2). Hence \(gR/P(1 - g)R/Pg = 0\). Since \(R/P\) is a prime ring, \(g = 0\) or \(g = 1\), therefore \(R/P\) only has two idempotents. Since \(R/P\) is an exchange ring, \(R/P\) is a local ring. □

**Corollary 8.** Let \(R\) be a left almost Abelian exchange ring. Then \(R/P\) is a division ring for every left (resp., right) primitive ideal of \(R\).

It is easy to show that if \(R\) is an exchange ring with \(J(R) = 0\), then \(R\) is reduced if and only if \(R\) is left almost Abelian. Combining this fact with Theorem 3 and [8, Theorem 4.6], we have the following lemma.

**Lemma 2.** If \(R\) is an exchange ring, then the following conditions are equivalent.

1. \(R/J(R)\) is reduced.
2. \(R/J(R)\) is Abelian.
3. \(R/J(R)\) is left almost Abelian.
4. \(R\) is quasi-duo.
5. \(R\) is left quasi-duo.

**Theorem 7.** Let \(R\) be an exchange ring, then the following conditions are equivalent.

1. \(N(R) \subseteq J(R)\).
2. \(R/J(R)\) is a left almost Abelian ring.
If \( J(R) \) is also nil, then the above conditions are equivalent to any of the following.

3) \( N(R) \) is a left ideal of \( R \).
4) \( N(R) \) is a right ideal of \( R \).
5) \( R \) is an NI ring (i.e. the set of all nilpotent elements forms an ideal of \( R \)).

Proof. (1) \( \implies \) (2) Because \( R \) is an exchange ring there exists \( e \in E(R) \) such that \( e + J(R) = i \) for any \( i \in E(R/J(R)) \). On the other hand, for any \( a \in R \), \( ae - eae \in N(R) \), so, we have \( ae - eae \in J(R) \) by (1). This shows that \( i \) is left semicentral in \( R/J(R) \), hence \( R/J(R) \) is left almost Abelian.

(2) \( \implies \) (1) By Lemma 2, \( R/J(R) \) is reduced, therefore \( N(R/J(R)) = 0 \), so, we have \( N(R) \subseteq J(R) \).

Now we assume that \( J(R) \) is nil, then \( J(R) \subseteq N(R) \).

By (1), \( N(R) = J(R) \) is an ideal, so \( R \) is an NI ring. Thus (1) \( \implies \) (5).

(5) \( \implies \) (4) \( \implies \) (1) and (5) \( \implies \) (3) \( \implies \) (1) are trivial. □

It is known that \( \pi \)-regular rings are exchange and the Jacobson radical of \( \pi \)-regular ring is nil. Hence Theorem 7 implies that for a \( \pi \)-regular ring \( R \), \( R \) is an NI ring if and only if \( R/J(R) \) is a left almost Abelian ring.

The following corollary generalizes [1, Theorem 2].

**Corollary 9.** Let \( R \) be a left almost Abelian \( \pi \)-regular ring. Then \( N(R) = J(R) \), so \( R \) is an NI ring.

Proof. It is an immediate consequence of Theorem 7 and Proposition 1(2)(b). □

In terms of Corollary 9, we have the following theorem, which generalizes [1, Theorem 3].

**Theorem 8.** Let \( R \) be a left almost Abelian ring. Then \( R \) is \( \pi \)-regular if and only \( N(R) \) is an ideal of \( R \) and \( R/N(R) \) is von Neumann regular. In this case \( R \) is strongly \( \pi \)-regular.

Proof. (\( \implies \)) Suppose that \( R \) is \( \pi \)-regular. By Corollary 9, \( R \) is an NI ring and \( N(R) = J(R) \). Therefore \( R/N(R) \) is a reduced \( \pi \)-regular ring, so, \( R/N(R) \) is strongly regular.

(\( \iff \)) Assume that \( N(R) \) is an ideal of \( R \) and \( \tilde{R} = R/N(R) \) is a von Neumann regular ring. Then \( R/N(R) \) is strongly regular because \( R/N(R) \) is a reduced ring. To prove that \( R \) is \( \pi \)-regular, it is sufficient to prove (Lemma 1) that \( R/P \) is strongly \( \pi \)-regular for every prime ideal \( P \) of \( R \). If \( x \in R \), then \( \bar{x} = x + J(R) \in \tilde{R} \) is unit regular. So we have \( \bar{x} = e\bar{u} = \bar{u}e \) with \( e \in E(R) \) and \( u \in U(R) \) because idempotents and units of \( \tilde{R} \) can be lifted modulo \( N(R) \). Hence

\[
x = eu + a = ue + b, \quad \text{where} \ a, b \in N(R),
\]

which implies

\[
ex = e(u + a) \quad \text{and} \quad xe = (u + b)e,
\]
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and

\[(1 - e)x = x - ex = (1 - e)a \in N(R),
\]

\[x(1 - e) = x - xe = b(1 - e) \in N(R).\]

So there exists a positive integer \(n\) such that \([((1-e)x)^n = x(1-e)^n = 0\). If \(e \in P\), then \(x^n \in P\) and \(\hat{x} = x + P \in N(R/P)\), so \(\hat{x}\) is strongly \(\pi\)-regular in \(R/P\). If \(e \notin P\), then since \(R\) is left almost Abelian, \(eR(1-e)Re = 0 \subseteq P\) and \(1 - e \in P\), which gives \(\hat{e} = \hat{1}\) in \(R/P\). This implies \(\hat{x} = \hat{e}\hat{x} = e(\hat{u} + a) = \hat{u} + a\) in \(R/P\). Hence \(\hat{x}\) is a unit and so it is a strongly \(\pi\)-regular element in \(R/P\), and the proof is completed.

**Corollary 10.** Suppose \(R\) is left almost Abelian \(\pi\)-regular and let \(P\) be a prime ideal of \(R\), then:

1. Every element of \(R/P\) is either nilpotent or unit.
2. If \(N(R) \subseteq P\), then \(R/P\) is a division ring.
3. If \(P\) is left or right primitive ideal of \(R\), then \(R/P\) is a division ring.

Hence \(R\) is strongly \(\pi\)-regular with \(J(R) = N(R)\).

**Corollary 11.** Let \(R\) be a left almost Abelian \(\pi\)-regular ring. If \(R\) is indecomposable, then \(R\) is local and \(N(R) = J(R)\).

**Proof.** By Theorem \([8, \text{Theorem 4.6}]\), \(N(R) = J(R)\). Let \(x \in R\). If \(x \notin J(R)\), then \(x \notin N(R)\). Since \(R\) is \(\pi\)-regular, there exists \(n \geq 1\) and \(y \in R\) such that \(x^n = x^n y x^n\). Set \(e = y x^n\). Then \(e^2 = e\) and \(x^n = x^n e\). Since \(R\) is indecomposable, either \(e = 0\) or \(e = 1\). Since \(x \notin N(R)\), \(e \neq 0\). Hence \(e = 1\), that is \(y x^n = 1\). By Corollary \([1]\), \(R\) is directly finite, and \(x\) is invertible. This shows that \(R\) is a local ring.

In \([8, \text{Theorem 4.6}]\), it is proved that for a ring \(R\), if \(R/J(R)\) is an exchange ring, then \(R\) is left quasi-duo if and only if \(R/J(R)\) is Abelian.

**Theorem 9.** Let \(R\) be a left almost Abelian exchange ring. Then \(R\) is a left and right quasi-duo ring.

**Proof.** Since \(R\) is a left almost Abelian exchange ring, \(R/J(R)\) is Abelian exchange by the proof of Corollary \([9]\). By Lemma \([2]\), \(R/J(R)\) is reduced, and by \([8, \text{Theorem 4.6}]\), \(R\) is left and right quasi-duo.

Combining Theorem \([9]\) with Lemma \([2]\) and \([8, \text{Corollary 4.7}]\), we have the following corollary.

**Corollary 12.** Let \(R\) be a left almost Abelian \(\pi\)-regular ring, then \(R/J(R)\) is a duo ring and \(R\) is a quasi-duo ring.

**Proposition 6.** Let \(R\) be a \(\pi\)-regular ring such that \(N(R)\) form a one-sided ideal of \(R\). Then \(R\) is quasi-duo.
Proof. We claim that $R/J(R)$ is reduced. To see this, let $x \in R$ be such that $x^2 \in J(R)$. Since $J(R)$ is nil, $(x^2)^m = 0$ for some $m \geq 1$. Therefore $x \in N(R)$. Since $N(R)$ is a one-sided ideal of $R$, $N(R) \subseteq J(R)$ and so, we have $x \in J(R)$. Having shown that $R/J(R)$ is reduced, $R$ is quasi-duo by [8, Theorem 4.6]. □

Recall that a ring $R$ is semi-$\pi$-regular if $R/J(R)$ is $\pi$-regular and idempotents can be lifted modulo $J(R)$. Combining Theorem 8 with Theorem 2, we have the following corollary.

**Corollary 13.** Let $R$ be a left almost Abelian semi-$\pi$-regular ring, then $R/J(R)$ is a strongly regular ring.

We end this section with the following example which gives a non-Abelian left almost Abelian $\pi$-regular ring.

Let $F$ be a division ring and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Clearly, $R$ is a left almost Abelian $\pi$-regular ring. But $R$ is not Abelian.

4 Applications

Following [9], a ring $R$ is called clean if every element of $R$ is a sum of a unit and an idempotent. Clean rings are always exchange rings, and the converse is true if $R$ is Abelian.

**Proposition 7.** Let $R$ be a left almost Abelian ring. Then $R$ is clean if and only if $R$ is exchange.

*Proof.** One direction is trivial.

For the other direction, let $R$ be an exchange ring, then $R/J(R)$ is exchange and idempotents can be lifted modulo $J(R)$. By Proposition 1 (2)(b), $R/J(R)$ is Abelian. Therefore $R/J(R)$ is clean by [9], so, by [2] Proposition 7, $R$ is a clean ring. □

In [5], it is shown that if $R$ is a unit regular ring in which 2 is invertible, then every element in $R$ is a sum of two units. The ring $R$ is called an $(S, 2)$ ring [6] if every element in $R$ is a sum of at least two units of $R$. In [1, Theorem 6] it is proved that if $R$ is an Abelian $\pi$-regular ring, then $R$ is an $(S, 2)$ ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R$. We can generalize this result to left almost Abelian rings, however, we need the following lemma.

**Lemma 3.**

(1) $R$ is an $(S, 2)$ ring if and only if $R/J(R)$ is an $(S, 2)$ ring.

(2) $\mathbb{Z}/2\mathbb{Z}$ is a homomorphic image of $R$ if and only if $\mathbb{Z}/2\mathbb{Z}$ is a homomorphic image of $R/J(R)$.

**Theorem 10.** Let $R$ be a left almost Abelian $\pi$-regular ring. Then $R$ is an $(S, 2)$ ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R$. 

Proof. Since $R$ is a left almost Abelian $\pi$-regular ring, $R/J(R)$ is strongly regular by Theorem 8 and Corollary 10. Hence $R/J(R)$ is Abelian $\pi$-regular. By Theorem 6, $R/J(R)$ is an $(S,2)$ ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R/J(R)$. Then Lemma 3 finishes the proof. □

In light of Theorem 10, we have the following corollaries:

**Corollary 14.** Let $R$ be a left almost Abelian $\pi$-regular ring such that $2 = 1 + 1 \in U(R)$. Then $R$ is an $(S,2)$ ring.

**Corollary 15.** Let $R$ be a left almost Abelian $\pi$-regular ring. Then $R$ is an $(S,2)$ ring if and only if for some $d \in U(R)$, $1 + d \in U(R)$.

Recall that a ring $R$ is said to have stable range 1 [12] if for any $a,b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right invertible. Clearly, $R$ has stable range 1 if and only if $R/J(R)$ has stable range 1. In Theorem 6, it is showed that exchange rings with all idempotents central have stable range 1. We now generalize this result as follows.

**Theorem 11.** Left almost Abelian exchange rings have stable range 1.

**Proof.** Let $R$ be a left almost Abelian exchange ring. Then $R/J(R)$ is exchange with all idempotents central, so, by Theorem 6, $R/J(R)$ has stable range 1. Therefore $R$ has stable range 1. □

In [18], the ring $R$ is said to satisfy the unit 1-stable condition if for any $a,b,c \in R$ with $ab + c = 1$, there exists $u \in U(R)$ such that $au + c \in U(R)$. It is easy to prove that $R$ satisfies the unit 1-stable condition if and only if $R/J(R)$ satisfies the unit 1-stable condition.

**Proposition 8.** Let $R$ be a left almost Abelian exchange ring, then the following conditions are equivalent:

1. $R$ is an $(S,2)$ ring.
2. $R$ satisfies the unit 1-stable condition.
3. Every factor ring $R_1$ of $R$ is an $(S,2)$ ring.
4. $\mathbb{Z}_2$ is not a homomorphic image of $R$.

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