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Almost Abelian rings

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Abstract. A ring R is defined to be left almost Abelian if $ae = 0$ implies $aRe = 0$ for $a \in N(R)$ and $e \in E(R)$, where $E(R)$ and $N(R)$ stand respectively for the set of idempotents and the set of nilpotents of R. Some characterizations and properties of such rings are included. It follows that if R is a left almost Abelian ring, then R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is regular. Moreover it is proved that (1) R is an Abelian ring if and only if R is a left almost Abelian left idempotent reflexive ring. (2) R is strongly regular if and only if R is regular and left almost Abelian. (3) A left almost Abelian clean ring is an exchange ring. (4) For a left almost Abelian ring R, it is an exchange $(S, 2)$ ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R.

1 Introduction

Throughout this article, all rings are associative with identity, and all modules are unital. The symbols $J(R)$, $N(R)$, $U(R)$, $E(R)$ will stand respectively for the Jacobson radical, the set of all nilpotent elements, the set of all invertible elements, the set of all idempotent elements of a ring R. For any nonempty subset X of a ring R, $r(X) = r_R(X)$ and $l(X) = l_R(X)$ denote the right annihilator of X and the left annihilator of X, respectively.

The ring R is called left almost Abelian if $ae = 0$ implies $aRe = 0$ for $a \in N(R)$ and $e \in E(R)$, and R is said to be semiabelian [4] if every idempotent of R is either left semicentral or right semicentral. The ring R is called Abelian [1] if every idempotent of R is central. Clearly, Abelian rings are semiabelian and left almost Abelian. Following [4], we know that there exists a semiabelian ring which is not Abelian.

The ring R is called π-regular [1] if for every $a \in R$ there exist $n \geq 1$ and $b \in R$ such that $a^n = a^n b a^n$, and in case of $n = 1$ the ring R is called von Neumann regular. So von Neumann regular rings are π-regular. A ring R is called strongly π-regular if for every $a \in R$ there exist $n \geq 1$ and $b \in R$ such that $a^n = a^{2n} b$,

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and in case of $n = 1$ the ring R is called strongly regular. So strongly regular rings are strongly π-regular. The case when the set $N(R)$ of nilpotent elements of a π-regular ring R is an ideal has been studied by many authors. For examples, in [1], it is shown that if R is an Abelian ring, then R is a π-regular ring if and only if $N(R)$ is an ideal of R and $R/N(R)$ is a strongly regular ring and in [4] it is shown that if R is a semiabelian ring, then R is a π-regular ring if and only if $N(R)$ is an ideal of R and $R/N(R)$ is a strongly regular ring. The goal of this paper is to study the properties of left almost Abelian rings, and to extend some known results on Abelian von Neumann regular rings, π-regular rings, and exchange rings. For instance we prove the following results: if R is a left almost Abelian ring, then R is π-regular if and only if $N(R)$ is an ideal of R and $R/N(R)$ is strongly regular.

2 Characterizations and Properties

It is easy to see that a ring R is Abelian if and only if $ae = 0$ implies $aRe = 0$ for each $a \in R$ and $e \in E(R)$. Motivated by this, we call a ring R left almost Abelian if $ae = 0$ implies $aRe = 0$ for each $a \in N(R)$ and $e \in E(R)$. Clearly, Abelian rings are left almost Abelian. The converse is not true in general. For example, if R is a reduced ring with $E(R) = \{0, 1\}$ then the 2×2 upper triangular matrix ring $UTM_2(R)$ is left almost Abelian but not Abelian.

According to [4], Abelian rings are semiabelian and the converse is not true in general. The following example implies that semiabelian rings need not be left almost Abelian.

Let R be a ring with $E(R) = \{0, 1\}$ and $N(R) \neq 0$. Then

$$E(UTM_2(R)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a, b \in R \right\}.$$

Clearly, $\begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$ is left semicentral and $\begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$ is right semicentral, so $UTM_2(R)$ is semiabelian, but not left almost Abelian. In fact, let $0 \neq a \in N(R)$. Then

$$\begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix} \in N(UTM_2(R)) \qquad \text{and} \qquad \begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 0\,,$$

but

$$\begin{pmatrix} a & -a \\ 0 & 0 \end{pmatrix} UTM_2(R) \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & aR \\ 0 & 0 \end{pmatrix} \neq 0\,.$$

Hence $UTM_2(R)$ is not left almost Abelian.

This example also implies that the upper triangular matrices rings over a left almost Abelian ring need not be left almost Abelian.

Proposition 1.

(1) *The subrings and direct products of left almost Abelian rings are left almost Abelian.*

(2) *Let R be a left almost Abelian ring and $e \in E(R)$. Then*

  (a) $(1 - e)Re \subseteq J(R)\,.$

(b) *If $ReR = R$, then $e = 1$.*

(c) *If M is a maximal left ideal of R and $e \notin M$, then $(1-e)R \subseteq M$.*

(d) *Let M be a maximal left ideal of R and $a \in R$. If $1 - ae \in M$, then $1 - ea \in M$.*

(e) *For any $x \in R$ and $n \geq 1$, $(exe)^n = ex^n e$.*

Proof. (1) is trivial.
(2) (a) For any $a \in R$, write $h = (1-e)a - (1-e)a(1-e)$. Then $h \in N(R)$ and $h(1-e) = 0$. Since R is a left almost Abelian ring, $(1-e)aeR(1-e) = hR(1-e) = 0$. Thus

$$(1-e)ReR(1-e) = \sum_{a\in R}(1-e)aeR(1-e) = 0$$

and so

$$\big((1-e)ReR\big)^2 = 0\,.$$

This implies $(1-e)Re \subseteq J(R)$.

(b) is an immediate consequence of (a).

(c) Since $e \notin M$, $Re + M = R$. By (a), $(1-e)Re \subseteq J(R) \subseteq M$, hence

$$(1-e)R = (1-e)Re + (1-e)M \subseteq M\,.$$

(d) Since $1 - ae \in M$, $e \notin M$. By (c), $(1-e)R \subseteq M$. Since $1 - ae = (1-a) + (a - ae)$, $1 - a \in M$, and $1 - ea = (1-a) + ((1-e)a)$ implies $1 - ea \in M$.

(e) Since

$$ex(1-e) \in N(R)\,, \quad ex(1-e)xe \in ((1-e)xe)Re\,,$$

i.e. $ex^2e = e(xe)^2$. Since

$$ex^2e = (exe)^2 + ex(1-e)xe\,, \quad ex^2e = (exe)^2.$$

By induction on n, we obtain $ex^ne = (exe)^n$. □

It is well known that a ring R is Abelian if and only if every idempotent of R is left semicentral and if and only if every idempotent of R is right semicentral. Hence we can construct a left almost Abelian ring which is not semiabelian.

Let R_1 and R_2 be left almost Abelian rings which are not Abelian. Take $e_1 \in R_1$ to be a right semicentral idempotent which is not central and $e_2 \in R_2$ to be a left semicentral idempotent which is not central, then the idempotent (e_1, e_2) is neither right nor left semicentral in $R_1 \oplus R_2$. Hence $R_1 \oplus R_2$ is not semiabelian, while by Proposition 1(1), $R_1 \oplus R_2$ is left almost Abelian.

A ring R is called directly finite if $xy = 1$ implies $yx = 1$ for $x, y \in R$, and R is called left *min-abelian* if for every

$$e \in ME_l(R) = \{e \in E(R) \mid Re \text{ is a minimal left ideal of } R\}\,,$$

e is left semicentral in R. It is well known that Abelian rings are directly finite and left min-abelian.

Corollary 1. *Let R be a left almost Abelian ring. Then*

(1) R is directly finite.

(2) R is left min-abelian.

Proof. (1) Let $ab = 1$, where $a, b \in R$. Set $e = ba$, then $e \in E(R), ae = a$ and $eb = b$. Since R is left almost Abelian, $(1-e)Re \subseteq J(R)$ by Proposition 1(2)(a). So we have $(1-e)a = (1-e)ae \in J(R)$. Therefore, $1-e = (1-e)ab \in J(R)$. This gives $1 = e = ba$, and R is directly finite.

(2) Let $e \in ME_l(R)$. If e is not left semicentral, then there exists $0 \neq a \in R$ such that $ae - eae \neq 0$. Let $h = ae - eae$. Then $eh = 0$, $he = h$ and $0 \neq h \in N(R)$. Since $hR(1-e) \subseteq (1-e)ReR(1-e)$, the equality $hR(1-e) = 0$ follows from the proof of Proposition 1(2)(a). Since $0 \neq Rh \subseteq Re$, $Rh = Re$. Hence $eR(1-e) = 0$, so also $eR = eRe$. Let $e = ch$ for some $c \in R$. Then $h = he = hee = hech = heceh = 0$ what contradicts to $h \neq 0$. Thus e is left semicentral and so R is a left min-abelian ring. □

The following example shows that the converse of Corollary 1 is not true in general.

Let F be a division ring and

$$R = \begin{pmatrix} F & F & F \\ 0 & F & F \\ 0 & 0 & F \end{pmatrix}.$$

For the idempotent $e = e_{11} + e_{33}$ we obtain that

$$eR(1-e)Re = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0,$$

and so R is not left almost Abelian. But by [19, Proposition 2.1] R is left quasi-duo, hence R is left min-abelian by [16, Theorem 1.2].

According to [13], an element e of a ring R is called op-idempotent if $e^2 = -e$. Clearly, an op-idempotent element may not be idempotent. For example, let $R = Z/3Z$. Then $\bar{2} \in R$ is op-idempotent, while it is not idempotent. In [3], Chen called an element $e \in R$ potent if there exists an integer $n \geq 2$ such that $e^n = e$. Clearly, idempotent is potent, while there exists a potent element which is not idempotent. For example, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(Z)$ is a potent element, while it is not idempotent. We denote by $E^o(R)$ and $PE(R)$ the set of all op-idempotent elements and the set of all potent elements of R, respectively. Write

$$P_l(R) = \left\{ k \in R \,\middle|\, {}_R Rk \text{ is projective} \right\}.$$

Clearly, $E(R) \subseteq P_l(R)$. Similarly, we can define $P_r(R)$. Recall that a ring R is left PP (i.e. principally left ideal of R is projective) if ${}_R Ra$ is projective for all $a \in R$. Evidently, R is a left PP ring if and only if $P_l(R) = R$. A ring R is called right GPP if for any $x \in R$, there exists $n \geq 1$ such that $x^n \in P_r(R)$.

Theorem 1. *The following conditions are equivalent for a ring R:*

(1) R is a left almost Abelian ring;

(2) $ae = 0$ implies $aRe = 0$ for each $a \in N(R)$ and $e \in E^o(R)$;

(3) $ae = 0$ implies $aRe = 0$ for each $a \in N(R)$ and $e \in PE(R)$;

(4) $ak = 0$ implies $aRk = 0$ for each $a \in N(R)$ and $k \in P_l(R)$.

Proof. (1) $\Longleftrightarrow$ (2), (3) $\Longrightarrow$ (1) and (4) $\Longrightarrow$ (1) are trivial.

(1) $\Longrightarrow$ (3) Let $e \in PE(R)$ and $a \in N(R)$ with $ae = 0$. Then there exists $n \geq 2$ such that $e^n = e$. Since $e^{n-1} \in E(R)$ and $ae^{n-1} = 0$, $aRe^{n-1} = 0$ by (1). Thus $aRe = aRe^n = aRe^{n-1}e = 0$.

(1) $\Longrightarrow$ (4) Assume that $a \in N(R)$ and $k \in P_l(R)$ are such that $ak = 0$. Since ${}_RRk$ is projective, there exists $e \in E(R)$ satisfying $l(k) = l(e)$. Hence $ae = 0$, and so $aRe = 0$ by (1). Since $k = ek$, $aRk = aRek = 0$. □

Corollary 2. *Let R be a left PP ring. Then the following conditions are equivalent:*

(1) R is a left almost Abelian ring;

(2) For each $a \in N(R)$ and $b \in R$, $ab = 0$ implies $aRb = 0$;

(3) For each $a \in N(R)$, $r(a)$ is an ideal of R.

A ring R is called left idempotent reflexive if $aRe = 0$ implies $eRa = 0$ for all $a \in R$ and $e \in E(R)$. Clearly, *Abelian* rings are left idempotent reflexive.

Theorem 2. *The following conditions are equivalent for a ring R:*

(1) R is an Abelian ring;

(2) R is an almost Abelian ring and left idempotent reflexive ring;

(3) R is a left idempotent reflexive ring and for any $a, b \in R$ and $e \in E(R)$ we have $eabe = eaebe$.

Proof. (1) $\Longrightarrow$ (2) is trivial.

(2) $\Longrightarrow$ (3) By Proposition 1(2), $ea(1-e)be = 0$ for all $a, b \in R$. Hence $eabe = eaebe$.

(3) $\Longrightarrow$ (1) Let $e \in E(R)$. For any $a \in R$, write $h = ae - eae$. Then

$$hR(1-e) = (1-e)hR(1-e) = (1-e)h(1-e)R(1-e)$$

by (3), so $hR(1-e) = 0$ because $h(1-e) = 0$. Since R is a left idempotent reflexive ring, $(1-e)Rh = 0$, which implies $h = (1-e)h = 0$. Thus $ae = eae$ for all $a \in R$, showing that e is left semicentral. This implies that R is an Abelian ring. □

A ring R is called von Neumann regular if $a \in aRa$ for all $a \in R$ and R is said to be unit-regular if for any $a \in R$, $a = aua$ for some $u \in U(R)$. A ring R is called strongly regular if $a \in a^2R$ for all $a \in R$. Clearly, strongly regular $\Longrightarrow$ unit-regular $\Longrightarrow$ von Neumann regular. Since von Neumann regular rings are semiprime, it follows that von Neumann regular rings are left idempotent reflexive. And it is well known that R is strongly regular if and only if R is von Neumann regular and *Abelian*. In view of Theorem 2, we have the following corollary.

Corollary 3. *The following conditions are equivalent for a ring R:*

(1) R is a strongly regular ring;

(2) R is an unit-regular ring and left almost Abelian ring;

(3) R is a von Neumann regular ring and left almost Abelian ring.

Following [17], a ring R is called left NPP (nil left principally ideal of R is projective) if for any $a \in N(R)$, Ra is projective left R-module. A ring R is said to be reduced if $a^2 = 0$ implies $a = 0$ for each $a \in R$, or equivalently, $N(R) = 0$. Obviously, reduced rings are left NPP, semiprime and *Abelian*. The following theorem gives some new characterizations of reduced rings in terms of left almost Abelian rings and left NPP rings.

Theorem 3. *The following conditions are equivalent for a ring R:*

(1) R is a reduced ring;

(2) R is a left NPP ring, semiprime ring and left almost Abelian ring;

(3) R is a left NPP ring, left idempotent reflexive ring and left almost Abelian ring.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1) By Theorem 2, R is an Abelian ring. Now let $a \in R$ such that $a^2 = 0$. Since R is left NPP, $l(a) = Re, e \in E(R)$. Hence $ea = 0$ and $a = ae$ because $a \in l(a)$. Thus $a = ae = ea = 0$. □

The following theorem is an immediate consequence of Proposition 1(1). We prove this directly.

Theorem 4. *If R is a subdirect product of a family of left almost Abelian rings $\{R_i : i \in I\}$, then R is left almost Abelian.*

Proof. Let $R_i = R/A_i$ where A_i be ideals of R with $\bigcap_{i\in I} A_i = 0$. Let $a \in N(R)$ and $e \in E(R)$ with $ae = 0$. Then $a_i = a + A_i \in N(R_i)$, $e_i = e + A_i \in E(R_i)$ and $(a + A_i)(e + A_i) = 0$ for any $i \in I$. Since each R_i is left almost Abelian, $a_iR_ie_i = 0$ for $i \in I$. This implies $aRe \subseteq A_i$ for all $i \in I$, so we have $aRe \subseteq \bigcap_{i\in I} A_i = 0$. Therefore R is left almost Abelian. □

Recall that a ring R has insertion-of-factors-property (IFP) if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$.

A ring R is called left WIFP (weakly IFP) if for any $a \in N(R)$ and $b \in R$, $ab = 0$ implies $aRb = 0$. By Corollary 2, we know that left PP left almost Abelian rings are left WIFP, and left WIFP rings are left almost Abelian.

Clearly, IFP rings are left WIFP.

Let $Z_2 = Z/2Z$. Then the 2×2 upper triangular matrix ring $R = \begin{pmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{pmatrix}$ is a left almost Abelian and left PP ring, so R is a left WIFP ring. Since R is not an Abelian ring, R is not an IFP ring. Thus there exists a left WIFP ring which is neither Abelian nor IFP.

It is well known that rings whose simple left R-modules are YJ-injective are always semiprime. But in general rings whose simple singular left R-modules are injective (hence also YJ-injective) need not be semiprime.

In [7], it is shown that if R is an IFP ring over which every simple singular left modules are YJ-injective, then R is a reduced weakly regular ring. We can generalize the result as follows.

Theorem 5. *If R is a left WIFP ring whose every simple singular left modules are YJ-injective, then R is a reduced weakly regular ring.*

Proof. First, we show that R is a reduced ring. Let $a^2 = 0$. Suppose that $a \neq 0$. Then there exists a maximal left ideal M containing $r(a)$ because $r(a) \neq R$ and $r(a)$ is a left ideal of R. If M is not essential left ideal of R, then $M = l(e)$ for some $e \in ME_l(R)$. Since $a \in r(a) \subseteq M = l(e)$, $ae = 0$. Hence $e \in r(a) \subseteq M = l(e)$, which is a contradiction. Therefore M must be an essential left ideal of R. Thus R/M is YJ-injective and so any R-homomorphism of Ra into R/M extends to one of R into R/M. Let $f : Ra \longrightarrow R/M$ be defined by $f(ra) = r + M$. Note that f is a well-defined R-homomorphism. Since R/M is YJ-injective, there exists $c \in R$ such that $1 + M = f(a) = ac + M$, but $ac \in r(a) \subseteq M$, which implies $1 \in M$, a contradiction. Hence $a = 0$ and so R is a reduced ring. Therefore R is an IFP ring. By [7, p. 2087–2096], R is also a weakly regular ring. □

Proposition 2. *Let R be a left almost Abelian ring and right GPP ring. Then for each $x \in R$, $x = u + a$, where $u \in P_r(R)$ and $a \in N(R)$.*

Proof. Since R is a right GPP ring, there exists $n \geq 1$ such that $x^n \in P_r(R)$. Clearly, there exists $e \in E(R)$ such that $x^n e = x^n$ and $r(x^n) = r(e)$. Since $xe = (xe)e$ and $r(xe) = r(e)$, $xe \in P_r(R)$ and

$$\big(x(1-e)\big)^{n+1} = x\big((1-e)x(1-e)\big)^n = x(1-e)x^n(1-e)$$

by Proposition 1(2)(e). Hence $x(1-e) \in N(R)$. Let $u = xe$ and $a = x(1-e)$. Then $x = u + a$, $u \in P_r(R)$ and $a \in N(R)$. □

A ring R is called left SF if every simple left R-module is flat, and R is said to be right NFB (nilpotent free Baer ring) if for any $a \in N(R)$, and $b \in R$ with $ab = 0$, there exists $e \in E(R)$ such that $ae = 0$ and $eb = b$. Clearly, right NPP rings are right NFB.

Proposition 3. *Let R be a left SF ring. If R is a left almost Abelian right NFB ring, then R is a strongly regular ring.*

Proof. It is well known that reduced left SF rings are strongly regular. We claim that R is reduced. In fact, if $a^2 = 0$, then $Ra + r(aR) = R$. If not, then there exists maximal left ideal M of R containing $Ra + r(aR)$. Since R is a left SF ring, R/M is flat as a left R-module. Since $a \in Ra \subseteq M$, $a = ab$ for some $b \in M$. Since

R is a right NFB ring, there exists $e \in E(R)$ such that $ae = 0$ and $e(1-b) = 1-b$. Since R is a left almost Abelian ring, $aRe = 0$. Hence $aR(1-b) = aRe(1-b) = 0$, which implies $1-b \in r(aR) \subseteq M$. This is a contradiction. Hence $Ra + r(aR) = R$. Let $1 = ca + x$, where $c \in R$ and $x \in r(aR)$. Therefore, $a = aca + ax = aca$. Since $a(1-ca) = 0$ and $1-ca \in E(R)$, $aR(1-ca) = 0$. Hence $ac(1-ca) = 0$, this gives $ac = acca$ and $a = aca = accaa = 0$. □

Corollary 4. *The following conditions are equivalent for a ring R:*

(1) R is a strongly regular ring;

(2) R is a left SF ring, left almost Abelian ring and right NFB ring;

(3) R is a left SF ring, left almost Abelian ring and right NPP ring;

(4) R is a left SF ring, left almost Abelian ring and right PP ring.

Let R be a ring and M a bimodule over R. The trivial extension of R and M is $R \propto M = \{(a,x)|a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by $(a,x)(b,y) = (ab, ay + xb)$. Clearly $R \propto M$ is a ring and $0 \propto M = \{(0,x)|x \in M\}$ is a nonzero nilpotent ideal of $R \propto M$.

Let R be a ring, M a bimodule over R. Write

$$T(R,M) = \left\{ \begin{pmatrix} c & x \\ 0 & c \end{pmatrix} \;\middle|\; c \in R, x \in M \right\},$$

then $T(R,M)$ is a ring and $T(R,M) \cong R \propto M$.

Let R be a ring and $R[x]$ denote the ring of polynomials over R. Clearly, $R[x]/(x^2) \cong R \propto R$.

A right R-module M is called normal if $me = 0$ implies $mRe = 0$ for each $m \in M$ and $e \in E(R)$. Clearly, every right module over an Abelian ring is normal.

Proposition 4. *Let M be a (R,R)-bimodule. Then $T(R,M)$ is a left almost Abelian ring if and only if R is a left almost Abelian ring and M is a right normal R-module.*

Proof. Assume that $T(R,M)$ is a left almost Abelian ring. Then R is a left almost Abelian ring by Proposition 1. Let $m \in M$ and $e \in E(R)$ satisfy $me = 0$. Then

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} = 0 .$$

Since $T(R,M)$ is left almost Abelian,

$$\begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} = 0$$

for each $r \in R$. Therefore $mre = 0$ for each $r \in R$, that is, $mRe = 0$, and M is a right normal R-module.

Conversely, assume that R is left almost Abelian and M is a right normal R-module. Let

$$A = \begin{pmatrix} a & x \\ 0 & a \end{pmatrix} \in N\big(T(R,M)\big)$$

and

$$E = \begin{pmatrix} e & y \\ 0 & e \end{pmatrix} \in E\big(T(R,M)\big)$$

satisfy $AE = 0$. Then $a \in N(R)$, $e \in E(R)$ and we have the following equations:

$$ey + ye = y\,, \tag{1}$$

$$ae = 0\,, \tag{2}$$

$$ay + xe = 0\,. \tag{3}$$

Since R is almost Abelian, $aRe = 0$ by (2). Hence, by (1), we have

$$ay = aey + aye = 0\,. \tag{4}$$

Thus (3) implies

$$xe = (ay + xe) - ay = 0\,. \tag{5}$$

Since M is right normal R-module, $xRe = 0$.

Now, for each $B = \begin{pmatrix} b & z \\ 0 & b \end{pmatrix} \in E(T(R,M))$, we have

$$ABE = \begin{pmatrix} abe & aby + aze + xbe \\ 0 & abe \end{pmatrix}\,. \tag{6}$$

Since $abe, aze, abye \in aRe$, $abe = abye = aze = 0$. Similarly $aby = abey + abye$ implies $aby = 0$ and $xbe \in xRe$ implies $xbe = 0$.

Thus $ABE = 0$, and this gives $AT(R,M)E = 0$. Hence $T(R,M)$ is a left almost Abelian ring. □

Corollary 5. *Let M be an (R,R)-bimodule. Then $R \propto M$ is a left almost Abelian ring if and only if R is a left almost Abelian ring and M is a right normal R-module.*

Let R be a left almost Abelian ring and I an ideal of R. If $I \subseteq N(R)$, then I is right normal as right R-module. Hence by Proposition 4 and Corollary 5, we have the following corollary.

Corollary 6. *Let I be an ideal of R and $I \subseteq N(R)$. Then the following conditions are equivalent:*

(1) R is a left almost Abelian ring;

(2) $T(R,I)$ is a left almost Abelian ring;

(3) $R \propto I$ is a left almost Abelian ring.

It is well known that a ring R is Abelian if and only if for each $e, g \in E(R)$, $ge = 0$ implies $gRe = 0$. Hence, a ring R is Abelian if and only if every right R-module is normal and if and only if R_R is normal. Thus, by Proposition 4, we have the following corollary.

Corollary 7. *Let R be a ring. Then the following conditions are equivalent:*

(1) R is an Abelian ring;

(2) $T(R, R)$ is a left almost Abelian ring;

(3) $R \propto R$ is a left almost Abelian ring;

(4) $R[x]/(x^2)$ is a left almost Abelian ring.

3 Almost Abelian π-regular rings

For convenience, we list the following notions which appeared in the first section of this paper. Let R be a ring and $a \in R$. Then a is called π-regular, if there exist $n \geq 1$ and $b \in R$ such that $a^n = a^n b a^n$. If $n = 1$, a is called von Neumann regular. Further a is said to be strongly π-regular, if $a^n = a^{n+1}b$, and if $n = 1$, a is called strongly regular. A ring R is called von Neumann regular, strongly regular, π-regular and strongly π-regular, if every element of R is von Neumann regular, strongly regular, π-regular and strongly π-regular, respectively. For convenience, we list some known facts which are necessary for the study of π-regularity of rings.

Lemma 1. *[11, Theorem 23.2] The following conditions are equivalent for a ring R.*

(1) R is strongly π-regular.

(2) Every prime factor ring of R is strongly π-regular.

(3) $R/P(R)$ is strongly π-regular.

Proposition 5. *Let R be a left almost Abelian ring and $x \in R$. Then:*

(1) If x is von Neumann regular, then x is strongly regular.

(2) If x is π-regular, then there exists an $e \in E(R)$ such that ex is von Neumann regular and $(1-e)x \in N(R)$.

(3) R is π-regular if and only if R is strongly π-regular.

Proof. (1) Let $x = xyx$ for some $y \in R$. Write $e = yx$. Then $e^2 = e \in R$ and $x = xe$. By Proposition 1(2),

$$e = eee = eyxe = eyexe = eyex = ey^2x^2$$

so, we have $x = xe = xy^2x^2$. Similarly, we can show that $x = x^2y^2x$. Therefore x is strongly regular.

(2) By hypothesis, there exists a positive integer n such that x^n is regular. By (1), x^n is strongly regular. By [10], $x^n = x^n u x^n$ and $x^n u = u x^n$ for some $u \in U(R)$. Let $e = x^n u$. Then $e \in E(R)$, $x^n = ex^n$ and $x^n = ev$, where $v = u^{-1}$. Since

$$(ex)(x^{n-1}u)(ex) = ex^n uex = evuex = ex\,,$$

ex is von Neumann regular. On the other hand, by Proposition 1(2),

$$((1-e)x)^n(1-e) = (1-e)x^n(1-e) = (1-e)ev(1-e) = 0\,,$$

so, we have $((1-e)x)^{n+1} = 0$. Hence $(1-e)x \in N(R)$.

(3) follows from (1). □

The module ${}_RM$ has the finite exchange property if for every module ${}_RA$ and any two decompositions $A = M' \oplus N = \oplus_{i\in I} A_i$ with $M' \cong M$ and I finite set, there exist submodules $A'_i \subseteq A_i$ such that $A = M' \oplus (\oplus_{i\in I} A'_i)$.

Warfield [15] called a ring R an exchange ring if ${}_RR$ has the finite exchange property and showed that this definition is left-right symmetric. Nicholson [9] showed that R is an exchange ring if and only if idempotents can be lifted modulo every left (equivalently, right) ideal of R.

Theorem 6. *Let R be a left almost Abelian exchange ring. Then R/P is a local ring for every prime ideal of R.*

Proof. According to [14, Theorem 1], an exchange ring with only two idempotents is a local ring. Since R is an exchange ring, idempotents can be lifted modulo P. For any idempotent element g of R/P, there exists idempotent e of R such that $e + P = g$. Since R is a left almost Abelian, $eR(1-e)Re = 0$ by Proposition 1(2). Hence $gR/P(\bar{1}-g)R/Pg = 0$. Since R/P is a prime ring, $g = 0$ or $g = \bar{1}$, therefore R/P only has two idempotents. Since R/P is an exchange ring, R/P is a local ring. □

Corollary 8. *Let R be a left almost Abelian exchange ring. Then R/P is a division ring for every left (resp., right) primitive ideal of R.*

It is easy to show that if R is an exchange ring with $J(R) = 0$, then R is reduced if and only if R is left almost Abelian. Combining this fact with Theorem 3 and [8, Theorem 4.6], we have the following lemma.

Lemma 2. *If R is an exchange ring, then the following conditions are equivalent.*

(1) $R/J(R)$ is reduced.

(2) $R/J(R)$ is Abelian.

(3) $R/J(R)$ is left almost Abelian.

(4) R is quasi-duo.

(5) R is left quasi-duo.

Theorem 7. *Let R be an exchange ring, then the following conditions are equivalent.*

(1) $N(R) \subseteq J(R)$.

(2) $R/J(R)$ is a left almost Abelian ring.

If $J(R)$ is also nil, then the above conditions are equivalent to any of the following.

(3) $N(R)$ is a left ideal of R.

(4) $N(R)$ is a right ideal of R.

(5) R is an NI ring (i.e. the set of all nilpotent elements forms an ideal of R).

Proof. (1) $\Longrightarrow$ (2) Because R is an exchange ring there exists $e \in E(R)$ such that $e + J(R) = i$ for any $i \in E(R/J(R))$. On the other hand, for any $a \in R$, $ae - eae \in N(R)$, so, we have $ae - eae \in J(R)$ by (1). This shows that i is left semicentral in $R/J(R)$, hence $R/J(R)$ is left almost Abelian.

(2) $\Longrightarrow$ (1) By Lemma 2, $R/J(R)$ is reduced, therefore $N(R/J(R)) = 0$, so, we have $N(R) \subseteq J(R)$.

Now we assume that $J(R)$ is nil, then $J(R) \subseteq N(R)$.

By (1), $N(R) = J(R)$ is an ideal, so R is an NI ring. Thus (1) $\Longrightarrow$ (5).

(5) $\Longrightarrow$ (4) $\Longrightarrow$ (1) and (5) $\Longrightarrow$ (3) $\Longrightarrow$ (1) are trivial. □

It is known that π-regular rings are exchange and the Jacobson radical of π-regular ring is nil. Hence Theorem 7 implies that for a π-regular ring R, R is an NI ring if and only if $R/J(R)$ is a left almost Abelian ring.

The following corollary generalizes [1, Theorem 2].

Corollary 9. *Let R be a left almost Abelian π-regular ring. Then $N(R) = J(R)$, so R is an NI ring.*

Proof. It is an immediate consequence of Theorem 7 and Proposition 1(2)(b). □

In terms of Corollary 9, we have the following theorem, which generalizes [1, Theorem 3].

Theorem 8. *Let R be a left almost Abelian ring. Then R is π-regular if and only $N(R)$ is an ideal of R and $R/N(R)$ is von Neumann regular. In this case R is strongly π-regular.*

Proof. ($\Longrightarrow$) Suppose that R is π-regular. By Corollary 9, R is an NI ring and $N(R) = J(R)$. Therefore $R/N(R)$ is a reduced π-regular ring, so, $R/N(R)$ is strongly regular.

($\Longleftarrow$) Assume that $N(R)$ is an ideal of R and $\bar{R} = R/N(R)$ is a von Neumann regular ring. Then $R/N(R)$ is strongly regular because $R/N(R)$ is a reduced ring. To prove that R is π-regular, it is sufficient to prove (Lemma 1) that R/P is strongly π-regular for every prime ideal P of R. If $x \in R$, then $\bar{x} = x + J(R) \in \bar{R}$ is unit regular. So we have $\bar{x} = \bar{e}\bar{u} = \bar{u}\bar{e}$ with $e \in E(R)$ and $u \in U(R)$ because idempotents and units of $\bar{R}$ can be lifted modulo $N(R)$. Hence

$$x = eu + a = ue + b\,, \quad \text{where } a, b \in N(R)\,,$$

which implies

$$ex = e(u + a) \quad \text{and} \quad xe = (u + b)e\,,$$

and

$$(1-e)x = x - ex = (1-e)a \in N(R)\,,$$
$$x(1-e) = x - xe = b(1-e) \in N(R)\,.$$

So there exists a positive integer n such that $[(1-e)x]^n = [x(1-e)]^n = 0$. If $e \in P$, then $x^n \in P$ and $\hat{x} = x+P \in N(R/P)$, so $\hat{x}$ is strongly π-regular in R/P. If $e \notin P$, then since R is left almost Abelian, $eR(1-e)Re = 0 \subseteq P$ and $1-e \in P$, which gives $\hat{e} = \hat{1}$ in R/P. This implies $\hat{x} = \hat{e}\hat{x} = e\widehat{(u+a)} = \widehat{u+a}$ in R/P. Hence $\hat{x}$ is a unit and so it is a strongly $\pi-$regular element in R/P, and the proof is completed. □

Corollary 10. *Suppose R is left almost Abelian π-regular and let P be a prime ideal of R, then:*

(1) Every element of R/P is either nilpotent or unit.

(2) If $N(R) \subseteq P$, then R/P is a division ring.

(3) If P is left or right primitive ideal of R, then R/P is a division ring.

Hence R is strongly π-regular with $J(R) = N(R)$.

Corollary 11. *Let R be a left almost Abelian π-regular ring. If R is indecomposable, then R is local and $N(R) = J(R)$.*

Proof. By Theorem 8, $N(R) = J(R)$. Let $x \in R$. If $x \notin J(R)$, then $x \notin N(R)$. Since R is π-regular, there exists $n \geq 1$ and $y \in R$ such that $x^n = x^n y x^n$. Set $e = yx^n$. Then $e^2 = e$ and $x^n = x^n e$. Since R is indecomposable, either $e = 0$ or $e = 1$. Since $x \notin N(R)$, $e \neq 0$. Hence $e = 1$, that is $yx^n = 1$. By Corollary 1, R is directly finite, and x is invertible. This shows that R is a local ring. □

In [8, Theorem 4.6], it is proved that for a ring R, if $R/J(R)$ is an exchange ring, then R is left quasi-duo if and only if $R/J(R)$ is Abelian.

Theorem 9. *Let R be a left almost Abelian exchange ring. Then R is a left and right quasi-duo ring.*

Proof. Since R is a left almost Abelian exchange ring, $R/J(R)$ is Abelian exchange by the proof of Corollary 9. By Lemma 2, $R/J(R)$ is reduced, and by [8, Theorem 4.6], R is left and right quasi-duo. □

Combining Theorem 9 with Lemma 2 and [8, Corollary 4.7], we have the following corollary.

Corollary 12. *Let R be a left almost Abelian π-regular ring, then $R/J(R)$ is a duo ring and R is a quasi-duo ring.*

Proposition 6. *Let R be a π-regular ring such that $N(R)$ form a one-sided ideal of R. Then R is quasi-duo.*

Proof. We claim that $R/J(R)$ is reduced. To see this, let $x \in R$ be such that $x^2 \in J(R)$. Since $J(R)$ is nil, $(x^2)^m = 0$ for some $m \geq 1$. Therefore $x \in N(R)$. Since $N(R)$ is a one-sided ideal of R, $N(R) \subseteq J(R)$ and so, we have $x \in J(R)$. Having shown that $R/J(R)$ is reduced, R is quasi-duo by [8, Theorem 4.6]. □

Recall that a ring R is semi-π-regular if $R/J(R)$ is π-regular and idempotents can be lifted modulo $J(R)$. Combining Theorem 8 with Theorem 2, we have the following corollary.

Corollary 13. *Let R be a left almost Abelian semi-π-regular ring, then $R/J(R)$ is a strongly regular ring.*

We end this section with the following example which gives a non-Abelian left almost Abelian π-regular ring.

Let F be a division ring and $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Clearly, R is a left almost Abelian π-regular ring. But R is not Abelian.

4 Applications

Following [9], a ring R is called clean if every element of R is a sum of a unit and an idempotent. Clean rings are always exchange rings, and the converse is true if R is Abelian.

Proposition 7. *Let R be a left almost Abelian ring. Then R is clean if and only if R is exchange.*

Proof. One direction is trivial.

For the other direction, let R be an exchange ring, then $R/J(R)$ is exchange and idempotents can be lifted modulo $J(R)$. By Proposition 1 (2)(b), $R/J(R)$ is Abelian. Therefore $R/J(R)$ is clean by [9], so, by [2, Proposition 7], R is a clean ring. □

In [5], it is shown that if R is a unit regular ring in which 2 is invertible, then every element in R is a sum of two units. The ring R is called an $(S, 2)$ ring [6] if every element in R is a sum of at least two units of R. In [1, Theorem 6] it is proved that if R is an Abelian π-regular ring, then R is an $(S, 2)$ ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R. We can generalize this result to left almost Abelian rings, however, we need the following lemma.

Lemma 3.

(1) R is an $(S, 2)$ ring if and only if $R/J(R)$ is an $(S, 2)$ ring.

(2) $\mathbb{Z}/2\mathbb{Z}$ is a homomorphic image of R if and only if $\mathbb{Z}/2\mathbb{Z}$ is a homomorphic image of $R/J(R)$.

Theorem 10. *Let R be a left almost Abelian π-regular ring. Then R is an $(S, 2)$ ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of R.*

Proof. Since R is a left almost Abelian π-regular ring, $R/J(R)$ is strongly regular by Theorem 8 and Corollary 10. Hence $R/J(R)$ is Abelian π-regular. By [1, Theorem 6], $R/J(R)$ is an $(S, 2)$ ring if and only if $\mathbb{Z}/2\mathbb{Z}$ is not a homomorphic image of $R/J(R)$. Then Lemma 3 finishes the proof. □

In light of Theorem 10, we have the following corollaries:

Corollary 14. *Let R be a left almost Abelian π-regular ring such that $2 = 1 + 1 \in U(R)$. Then R is an $(S, 2)$ ring.*

Corollary 15. *Let R be a left almost Abelian π-regular ring. Then R is an $(S, 2)$ ring if and only if for some $d \in U(R)$, $1 + d \in U(R)$.*

Recall that a ring R is said to have stable range 1 [12] if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right invertible. Clearly, R has stable range 1 if and only if $R/J(R)$ has stable range 1. In [19, Theorem 6], it is showed that exchange rings with all idempotents central have stable range 1. We now generalize this result as follows.

Theorem 11. *Left almost Abelian exchange rings have stable range* 1*.*

Proof. Let R be a left almost Abelian exchange ring. Then $R/J(R)$ is exchange with all idempotents central, so, by [19, Theorem 6], $R/J(R)$ has stable range 1. Therefore R has stable range 1. □

In [18], the ring R is said to satisfy the unit 1-stable condition if for any $a, b, c \in R$ with $ab + c = 1$, there exists $u \in U(R)$ such that $au + c \in U(R)$. It is easy to prove that R satisfies the unit 1-stable condition if and only if $R/J(R)$ satisfies the unit 1-stable condition.

Proposition 8. *Let R be a left almost Abelian exchange ring, then the following conditions are equivalent:*

(1) R is an $(S, 2)$ ring.

(2) R satisfies the unit 1*-stable condition.*

(3) Every factor ring R_1 of R is an $(S, 2)$ ring.

(4) $\mathbb{Z}_2$ is not a homomorphic image of R.

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