VOLUME
21/2013
No. 1
ISSN 1804-1388

COMMUNICATIONS IN MATHEMATICS

Available online at
http://cm.osu.cz

UNIVERSITAS OSTRAVIENSIS

Communications in Mathematics 21 (2013) 1–13

Cocalibrated G_2-manifolds with Ricci flat characteristic connection

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Abstract. Any 7-dimensional cocalibrated G_2-manifold admits a unique connection ∇ with skew symmetric torsion (see [8]). We study these manifolds under the additional condition that the ∇-Ricci tensor vanish. In particular we describe their geometry in case of a maximal number of ∇-parallel vector fields.

1 Introduction

Consider a triple (M^n, g, T) consisting of a Riemannian manifold (M^n, g) equipped with a 3-form T. We denote by ∇^g, Ric^g and Scal^g the Levi-Civita connection, the Riemannian Ricci tensor and the scalar curvature. The formula

$$\nabla_X Y := \nabla^g_X Y + \frac{1}{2}\,\mathrm{T}(X, Y, -)$$

defines a metric connection with torsion T. We will denote by Ric^∇ and Scal^∇ its Ricci tensor and scalar curvature respectively. If the Ricci tensor $\mathrm{Ric}^\nabla = 0$ vanishes, then T is a coclosed form, $\delta\mathrm{T} = 0$, and the Riemannian Ricci tensor is completely given by the 3-form T (see [8]),

$$\mathrm{Ric}^g(X, Y) = \frac{1}{4}\sum_{i,j=1}^{n} \mathrm{T}(X, e_i, e_j)\cdot \mathrm{T}(Y, e_i, e_j)\,, \quad \mathrm{Scal}^g = \frac{3}{2}\,\|\mathrm{T}\|^2\,.$$

In particular, the Ricci tensor Ric^g is non-negative, $\mathrm{Ric}^g(X, X) \geq 0$.

Let us introduce the 4-form σ_T depending on T,

$$\sigma_\mathrm{T} = \frac{1}{2}\sum_{i=1}^{n} (e_i \lrcorner \mathrm{T}) \wedge (e_i \lrcorner \mathrm{T})\,.$$

2010 MSC: Primary 53C25, Secondary 81T30
Key words: cocalibrated G_2-manifolds, connections with torsion

If moreover there exists a ∇-parallel spinor field Ψ, then there is an algebraic link between $d\mathrm{T}$, $\nabla\mathrm{T}$ and σ_{T} (see [8]),

$$\big(X \lrcorner\, d\mathrm{T} + 2\,\nabla_X \mathrm{T}\big)\cdot\Psi = 0\,, \quad \big(3\,d\mathrm{T} - 2\sigma_{\mathrm{T}}\big)\cdot\Psi = 0\,.$$

The classification of flat metric connections with skew symmetric torsion has been investigated by Cartan and Schouten in 1926. Complete proofs are known since the beginning of the 70-ties. In [4] one finds a simple proof of this result. Therefore, we are interested in non-flat ($\mathcal{R}^{\nabla} \not\equiv 0$) and ∇-Ricci flat ($\mathrm{Ric}^{\nabla} \equiv 0$) metric connections with skew symmetric torsion $\mathrm{T} \not\equiv 0$.

In this paper we study the 7-dimensional case. Any cocalibrated G_2-manifold admits a unique connection ∇ with skew symmetric torsion and ∇-parallel spinor field Ψ. If this characteristic connection is Ricci flat, then we obtain a solution of the Strominger equations (see [8]),

$$\nabla\Psi = 0\,, \quad \mathrm{Ric}^{\nabla} = 0\,, \quad d*\mathrm{T} = 0\,.$$

If $\mathrm{T} = 0$, M^7 is a Riemannian manifold with holonomy G_2 and $\mathrm{Ric}^g = 0$ follows automatically. The case of $\mathrm{T} \not\equiv 0$ is different. The condition $\mathrm{Ric}^{\nabla} \equiv 0$ is not a consequence of the fact that the holonomy of ∇ is contained in G_2, it is a new condition for the cocalibrated G_2-structure. In this paper we investigate the geometry of the 7-manifolds under consideration. Moreover, we describe all these manifolds with a large number of ∇-parallel vector fields.

2 Examples of Ricci flat connections with skew symmetric torsion

Let us discuss some examples.

Example 1. Any Hermitian manifold admits a unique metric connection ∇ preserving the complex structure and with skew symmetric torsion (see [8]). In [10] the authors constructed on $(k-1)\,(S^2 \times S^4)\,\#\,k\,(S^3 \times S^3)$ a Hermitian structure with vanishing ∇-Ricci tensor, $\mathrm{Ric}^{\nabla} = 0$, for any $k \geq 1$. These examples are toric bundles over special Kähler 4-manifolds.

Example 2. There are 7-dimensional cocalibrated G_2-manifolds (M^7, g, ω^3) with characteristic torsion T such that

$$\nabla\mathrm{T} = 0\,, \quad d\mathrm{T} = 0\,, \quad \delta\mathrm{T} = 0\,, \quad \mathrm{Ric}^{\nabla} = 0\,, \quad \mathfrak{hol}(\nabla) \subset \mathfrak{u}(2) \subset \mathfrak{g}_2\,.$$

The regular G_2-manifolds of this type have been described in [7], Theorem 5.2 (the degenerate case $2a + c = 0$). M^7 is the product $X^4 \times S^3$, where X^4 is a Ricci-flat Kähler manifold and S^3 the round sphere.

Example 3. A suitable deformation of any Sasaki-Einstein manifold yields a metric connection with skew symmetric torsion and vanishing Ricci tensor, see [1].

Next we describe a similar method in order to construct 5-dimensional connections with skew symmetric torsion and vanishing Ricci tensor.

Theorem 1. *Let (Z^4, g, Ω^2) be a 4-dimensional Riemannian manifold equipped with a 2-form Ω^2 such that*

1. $d\Omega^2 = 0\,,\quad d*\Omega^2 = 0$ *and* $\Omega^2 \wedge \Omega^2 = 0$.
2. *The 2-dimensional distributions*
$$E^2 = \big\{X \in TZ^4 : X \lrcorner \Omega^2 = 0\big\}\,,\quad F^2 = \big\{X \in TZ^4 : X \perp E^2\big\}$$
are integrable.
3. *The 2-form is of the form $\Omega^2 = 2a\, f_1 \wedge f_2$, where a is constant and f_1, f_2 is an oriented orthonormal frame in F^2.*
4. *The Riemannian Ricci tensor of Z^4 has two non-negative eigenvalues of multiplicity two,*
$$\mathrm{Ric}^g = 4a^2\,\mathrm{Id} \text{ on } F^2, \quad \mathrm{Ric}^g = 0 \text{ on } E^2.$$
5. *Ω^2 is the curvature form of some $\mathbb{R}^1$- or S^1-connection η.*

Then the principal fibre bundle $\pi\colon N^5 \to Z^4$ defined by Ω^2 admits a Riemannian metric and the torsion form
$$\mathrm{T} = \pi^*(\Omega^2) \wedge \eta$$
yields a metric connection ∇ with the following properties:
$$||\mathrm{T}||^2 = 4a^2, \quad d\mathrm{T} = 0\,, \quad d*\mathrm{T} = 0\,, \quad \mathrm{Ric}^\nabla = 0\,, \quad \nabla\eta = 0\,.$$

Proof. Apply O'Neill's formulas and compute
$$\mathrm{Ric}^g(X,Y) - \frac{1}{4}\sum_{i,j=1}^{5} \mathrm{T}(X,e_i,e_j)\cdot \mathrm{T}(Y,e_i,e_j) = 0\,. \qquad \square$$

Example 4. Let $u = u(x,y)$ be a smooth function of two variables and consider the metric
$$g = e^u\, x\,\big(dx^2 + dy^2\big) + x\, dz^2 + \frac{1}{x}\big(dt + y\, dz\big)^2$$
defined on the set $Z^4 = \{(x,y,t,z) \in \mathbb{R}^4 : x > 0\}$. (Z^4, g) is a Kähler manifold and the Riemannian Ricci tensor has two eigenvalues, namely zero and
$$-\frac{u_{xx} + u_{yy}}{2xe^u},$$
both with multiplicity two (see [5], [11]). If the function u is a solution of the equation
$$-\frac{u_{xx} + u_{yy}}{2xe^u} = 4a^2,$$
Theorem 1 is applicable and we obtain a family of non-flat 5-dimensional examples. Remark that a compact Kähler manifold Z^4 of that type splits into $S^2 \times T^2$, see [6]. The corresponding connection ∇ on the Lie group $N^5 = S^3 \times T^2$ is flat, see [4].

3 Cocalibrated G_2-manifolds with vanishing characteristic Ricci tensor

Consider a cocalibrated G_2-manifold (M^7, g, ω^3),

$$d * \omega^3 = 0 \, , \quad \| \omega^3 \|^2 = 7 \, ,$$

and suppose that the G_2-structure ω^3 is not ∇^g-parallel (i.e. $d\omega^3 \not\equiv 0$). There exists a unique metric connection ∇ with skew symmetric torsion and preserving the G_2-structure ω^3. Its torsion form is given by the formula (see [8]),

$$\mathrm{T} = - * d\omega^3 + \mu \, \omega^3, \quad \mu = \frac{1}{6} \left(d\omega^3 , \, * \omega^3 \right) .$$

The condition $\mathrm{Ric}^{\nabla} = 0$ becomes equivalent to $d\,\mathrm{T} = 0$ and $d * \mathrm{T} = 0$. Indeed, we have:

Theorem 2 ([8, Thm 5.4]). *The following conditions are equivalent:*

1. $\mathrm{Ric}^{\nabla} = 0 \, .$
2. $d\,\mathrm{T} = 0$ *and* $d * \mathrm{T} = 0 \, .$
3. $d\mu = 0$ *and* $d * d\omega^3 - \mu \, d\,\omega^3 = 0 \, .$

Using the G_2-splitting of 3-forms, $\Lambda^3 = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, we know that the characteristic torsion of a cocalibrated G_2-manifold belongs to $\mathrm{T} \in \Lambda^3_1 \oplus \Lambda^3_{27}$. In particular, we obtain

$$\mathrm{T} \wedge \omega^3 = 0 \, .$$

Differentiating the latter equation and using $d\,\mathrm{T} = 0$ one gets

$$\big(* d\,\omega^3 - \mu \, \omega^3 \big) \wedge \omega^3 = 0 \, , \qquad \| \, d\,\omega^3 \|^2 = 6 \, \mu^2 .$$

We compute the length of T,

$$\| \mathrm{T} \|^2 = \| d\omega^3 \|^2 - 2 \, \mu \big(* d\omega^3 , \, \omega^3 \big) + 7 \, \| \omega^3 \|^2 = 6 \, \mu^2 - 12 \, \mu^2 + 7\mu^2 = \mu^2 .$$

Consequently, $\|\mathrm{T}\|^2$ is constant. Moreover, the Riemannian scalar curvature is constant, too,

$$\mathrm{Scal}^g = \frac{3}{2} \, \| \mathrm{T} \|^2 = \frac{3}{2} \, \mu^2 .$$

Since $(\mathrm{T}, \omega^3) = \mu$, we decompose the torsion form into two parts according to the splitting of 3-forms,

$$\mathrm{T} = \mathrm{T}_1 + \mathrm{T}_{27} \, , \qquad \mathrm{T}_1 = \frac{1}{7} \, \mu \, \omega^3, \qquad \mathrm{T}_{27} = - * d\omega^3 + \frac{6}{7} \mu \omega^3 .$$

Corollary 1 ([8, Remark 5.5]). *Let (M^7, g, ω^3) be a compact, cocalibrated G_2-manifold with $\mathrm{Ric}^{\nabla} = 0$ and $\mathrm{T} \neq 0$. Then the third cohomology group is non-trivial,*

$$H^3(M^7; \mathbb{R}) \ \neq 0 \, .$$

Example 5. On the round sphere S^7 there exists a G_2-structure (not cocalibrated) such that $\mathcal{R}^\nabla = 0$ (see [4]). In particular, the Ricci tensor vanishes, $\mathrm{Ric}^\nabla = 0$. The characteristic torsion is coclosed, $\delta \mathrm{T} = 0$, but not closed, $d\,\mathrm{T} \neq 0$.

Remark 1. A cocalibrated G_2-manifold with $\mathrm{Ric}^\nabla = 0$ and $\mathrm{T} \not\equiv 0$ cannot be of pure type Λ^3_1 or Λ^3_{27}. Indeed, if

$$0 = \mathrm{T}_{27} = -*d\omega^3 + \frac{6}{7}\mu\omega^3$$

we differentiate,

$$0 = -d*d\,\omega^3 + \frac{6}{7}\,\mu\,d\,\omega^3$$

and combine the latter formula with equation (3) of Theorem 2. We conclude that $\mu = 0$, $d\omega^3 = 0$ and, finally, $\mathrm{T} = 0$. The second case, i.e. $\mathrm{T}_1 = 0$, implies immediately $\mu = 0$ and $\mathrm{T} = 0$.

There exists a canonical ∇-parallel spinor field Ψ_0 such that

$$\nabla\Psi_0 = 0\,, \qquad \omega^3\cdot\Psi_0 = -\,7\,\Psi_0\,.$$

Since $\Lambda^3_{27}\cdot\Psi_0 = 0$ we obtain

$$\mathrm{T}\cdot\Psi_0 = \mathrm{T}_1\cdot\Psi_0 = -\mu\,\Psi_0\ .$$

The integrability condition for a parallel spinor (see [8]) yields an algebraic restriction for the derivative $\nabla\mathrm{T}$, namely

$$\nabla_X\big(\mathrm{T}\cdot\Psi\big) = \big(\nabla_X\mathrm{T}\big)\cdot\Psi \ = \ 0\ , \quad \sigma_{\mathrm{T}}\cdot\Psi = 0\ , \quad \mathrm{T}^2\cdot\Psi = \|\,\mathrm{T}\,\|^2\,\Psi$$

for any vector $X \in TM^7$ and any ∇-parallel spinor field Ψ. In particular, the characteristic torsion T acts on the space of all ∇-parallel spinors. This condition is not so restrictive. For example, the space of 3-forms $\Sigma^3 \in \Lambda^3_{27}$ killing three spinors has dimension 14, the space killing four spinors has still dimension 9.

4 ∇-parallel vector fields

Via the Riemannian metric we identify vectors with 1-forms. Denote by $\mathcal{P}^\nabla$ the space of all ∇-parallel vector field (1-forms). Any ∇-parallel vector field θ is a Killing field and

$$2\,\nabla^g\theta = d\,\theta = \theta\,\lrcorner\,\mathrm{T}\ , \qquad \nabla^g_\theta\theta = 0\ .$$

holds. This formula together with $d\,\mathrm{T} = 0$ implies that T is preserved by the flow of θ,

$$\mathcal{L}_\theta\mathrm{T} = 0\,.$$

The Riemannian Ricci tensor on θ becomes

$$\mathrm{Ric}^g(\theta,\theta) = \frac{1}{2}\,\|\,d\,\theta\,\|^2\ .$$

The subgroup of G_2 preserving four vectors in $\mathbb{R}^7$ is trivial. The isotropy subgroups of two or three vectors in $\mathbb{R}^7$ coincide and this group is isomorphic to $SU(2) \subset G_2$. Finally, the isotropy subgroup of one vector is isomorphic to $SU(3) \subset G_2$ (see for example [7]). This algebraic observation proves immediately the following

Proposition 1. *If (M^7, g, ω^3) is not ∇-flat, then the possible dimensions of the space $\mathcal{P}^{\nabla}$ are* 0, 1, *or* 3.

4.1 The case of three ∇-parallel vector fields

We discuss the case that there are three orthonormal and ∇-parallel 1-forms θ_1, θ_2, θ_3. Then $\omega^3(\theta_1, \theta_2, -)$ is ∇-parallel, too. If it does not coincide with θ_3, then we have at least four ∇-parallel 1-forms, i.e. the G_2-connection ∇ is flat. Under our assumption $\mathcal{R}^{\nabla} \not\equiv 0$ we conclude that

$$\omega^3(\theta_1, \theta_2, -) = \theta_3 \, , \qquad \omega^3(\theta_1, \theta_2, \theta_3) = 1 \, .$$

The holonomy of the connection ∇ is contained in $\mathfrak{su}(2) \subset \mathfrak{g}_2$. Moreover, the spinors

$$\Psi_0 \, , \quad \Psi_1 := \theta_1 \cdot \Psi_0 \, , \qquad \Psi_2 := \theta_2 \cdot \Psi_0 \, , \qquad \Psi_3 := \theta_3 \cdot \Psi_0$$

are all ∇-parallel spinors. The torsion form T acts as a symmetric endomorphism on the space $\mathrm{Lin}(\Psi_0, \Psi_1, \Psi_2, \Psi_3)$ and $\mathrm{T} \cdot \Psi_0 = -\,\mu\,\Psi_0$. Consequently, T acts on the 3-dimensional space $\mathrm{Lin}(\Psi_1, \Psi_2, \Psi_3)$ and $\mathrm{T}^2 = \|\mathrm{T}\|^2 \cdot \mathrm{Id} = \mu^2 \cdot \mathrm{Id}$. We decompose the torsion form into

$$\mathrm{T} = \mathrm{T}_1 + \mathrm{T}_{27} = \frac{1}{7}\,\mu\,\omega^3 + \mathrm{T}_{27}$$

and we use the known action of ω^3 on spinors:

$$\omega^3 \cdot \Psi_0 = -7\Psi_0 \, , \qquad \omega^3 \cdot \Psi_i = \Psi_i \, , \; i = 1, 2, 3 \, , \qquad \mathrm{T}_{27} \cdot \Psi_0 = 0 \, .$$

Finally, $\mathrm{T}_{27} \in \Lambda^3_{27}$ preserves the space $\mathrm{Lin}(\Psi_1, \Psi_2, \Psi_3)$ and

$$\mathrm{T}^2_{27} + \frac{2}{7}\,\mu\,\mathrm{T}_{27} = \frac{48}{49}\,\mu^2.$$

Without loss of generality we may assume that Ψ_1, Ψ_2, Ψ_3 are eigenspinors of T_{27},

$$\mathrm{T}_{27} \cdot \Psi_i = m_i\,\Psi_i \, , \quad m_i^2 + \frac{2}{7}\,m_i\,\mu = \frac{48}{49}\,\mu^2, \qquad i = 1, 2, 3 \, .$$

We fix an orthonormal basis $e_1, \ldots, e_7$ such that

$$\omega^3 = e_{127} + e_{135} - e_{146} - e_{236} - e_{245} + e_{347} + e_{567}$$

and $\theta_1 = e_1, \theta_2 = e_2, \theta_3 = e_7$. This is possible, since we already have $\omega^3(\theta_1, \theta_2, \theta_3) = 1$. Let

$$\mathrm{T}_{27} = \sum_{i<j<k} t_{ijk}\, e_{ijk}$$

be the 3-form T_{27} and introduce the following numbers:

$$a := t_{236} + t_{245} \, , \qquad b := t_{347} + t_{567} \, , \qquad c := t_{235} - t_{246} \, .$$

A purely algebraic computation yields the following

Lemma 1. *The space of all 3-forms $T_{27} \in \Lambda^3_{27}$ such that $T_{27} \cdot \Psi_i = m_i \Psi_i$, $i = 1, 2, 3$ is an affine space of dimension 9. A parameterization is given by*

$$\begin{aligned} T_{27} = & \Big(-\frac{m_1}{2} - b\Big) e_{127} - t_{156}\, e_{134} + \Big(\frac{m_1}{2} + t_{146} + a\Big) e_{135} \\ & - t_{145}\, e_{136} + t_{145}\, e_{145} + t_{146}\, e_{146} + t_{156}\, e_{156} - t_{256}\, e_{234} \\ & + t_{235}\, e_{235} + t_{236}\, e_{236} + t_{245}\, e_{245} + t_{246}\, e_{246} + t_{256}\, e_{256} + t_{347}\, e_{347} \\ & + t_{467}\, e_{357} - t_{457}\, e_{367} + t_{457}\, e_{457} + t_{467}\, e_{467} + t_{567}\, e_{567}\,. \end{aligned}$$

and

$$m_1 + 2a + 2b = m_2\,, \quad -2a + 2b = m_3\,, \quad c = 0\,.$$

Corollary 2. *For $X \perp \mathrm{Lin}(\theta_1, \theta_2, \theta_3)$ we have*

$$T(\theta_i, \theta_j, X) = 0\,, \quad T = (\theta_1 \lrcorner T) \wedge \theta_1 + (\theta_2 \lrcorner T) \wedge \theta_2 + (\theta_3 \lrcorner T) \wedge \theta_3\,.$$

We solve the linear system with respect to a and b:

$$a = -\frac{1}{4}\big(m_1 - m_2 + m_3\big)\,, \quad b = \frac{1}{4}\big(-m_1 + m_2 + m_3\big)\,.$$

In particular,

$$m_1 + 2b = \frac{1}{2}\big(m_1 + m_2 + m_3\big)\,.$$

We are interested in the value

$$T(\theta_1, \theta_2, \theta_3) = \frac{1}{7}\mu - \frac{m_1}{2} - b = \frac{1}{7}\mu - \frac{1}{4}\big(m_1 + m_2 + m_3\big)\,.$$

We have 8 possibilities, namely

$$m_i = \frac{6}{7}\mu \qquad \text{or} \qquad m_i = -\frac{8}{7}\mu\,.$$

Therefore

$$T(\theta_1, \theta_2, \theta_3) = 0\,, \quad \pm\frac{1}{2}\mu \quad \text{or} \quad \mu\,.$$

We summarize the result.

Theorem 3. *Let (M^7, g, ω^3) be a cocalibrated G_2-manifold and ∇ its characteristic connection. Suppose that $\mathrm{Ric}^\nabla = 0$, $\|T\|^2 = \mu^2 > 0$ and $\mathcal{R}^\nabla \not\equiv 0$. If $\theta_1, \theta_2, \theta_3$ are three orthonormal and ∇-parallel vector fields, then*

1. *$\omega^3(\theta_1, \theta_2, \theta_3) = 1$.*
2. *$T(\theta_1, \theta_2, \theta_3)$ is constant and has only four possible values: $0, \pm\mu/2, \mu$.*
3. *$T(\theta_i, \theta_j, X) = 0$ for $X \perp \mathrm{Lin}(\theta_1, \theta_2, \theta_3)$.*

In particular

$$\begin{aligned}\mathrm{T} &= (\theta_1 \lrcorner \mathrm{T}) \wedge \theta_1 + (\theta_2 \lrcorner \mathrm{T}) \wedge \theta_2 + (\theta_3 \lrcorner \mathrm{T}) \wedge \theta_3 \\ &= d\theta_1 \wedge \theta_1 + d\theta_2 \wedge \theta_2 + d\theta_3 \wedge \theta_3 \,.\end{aligned}$$

and

$$[\theta_1, \theta_2] = -\mathrm{T}(\theta_1, \theta_2, \theta_3)\, \theta_3$$

is proportional to θ_3*. The* 3*-dimensional space* $\mathrm{Lin}(\theta_1, \theta_2, \theta_3)$ *is closed with respect to the Lie bracket and is a Lie subalgebra of the Killing vector fields. This algebra is either commutative or isomorphic to* $\mathfrak{so}(3)$*.*

Remark 2. Since we do not assume that the torsion form T is ∇-parallel, it is not obvious by general arguments that $[\theta_1, \theta_2] = -\mathrm{T}(\theta_1, \theta_2)$ is again ∇-parallel.

We can classify the case of $\mathrm{T}(\theta_1, \theta_2, \theta_3) = \mu$ immediately. Indeed, we have then $\|\mathrm{T}\|^2 \geq \mu^2$. On the other hand, we know that $\|\mathrm{T}\|^2 = \mu^2$ holds. It follows that

$$\mathrm{T} = \mu\, \theta_1 \wedge \theta_2 \wedge \theta_3 \quad \text{and} \quad \nabla \mathrm{T} = 0\,.$$

Cocalibrated G_2-structures with characteristic holonomy $\mathfrak{su}(2)$ and a characteristic torsion of the given type have been classified at the end of our paper [7]. We apply this result and obtain

Theorem 4. *Let* (M^7, g, ω^3) *be a complete, cocalibrated* G_2*-manifold and* ∇ *its characteristic connection. Suppose that* $\mathrm{Ric}^{\nabla} = 0$*. If* $\theta_1, \theta_2, \theta_3$ *are three orthonormal and* ∇*-parallel vector fields and* $\mathrm{T}(\theta_1, \theta_2, \theta_3) = \mu$*, then the universal covering of* M^7 *is isometric to the product* $X^4 \times S^3$*, where* X^4 *is a complete anti-self dual and Ricci flat Riemannian manifold.*

If $\mathrm{T}(\theta_1, \theta_2, \theta_3) = 0$ the 3-dimensional abelian Lie group acts on M^7 locally free as a group of isometries and preserves the torsion form T. Moreover, we obtain the 2-forms $d\theta_i = \theta_i \lrcorner \mathrm{T}$ and

$$\mathcal{L}_{\theta_i}(\theta_j \lrcorner \mathrm{T}) = 0\,, \qquad \theta_i \lrcorner \theta_j \lrcorner \mathrm{T} = 0\,.$$

We will investigate the special case, where two of these 2-forms vanish, later.

Remark 3. We do not have any results in case of $|\mathrm{T}(\theta_1, \theta_2, \theta_3)| = \mu/2$.

4.2 Special ∇-parallel vector fields

There are special ∇-parallel vector fields (1-forms), namely

$$\mathcal{SP}^{\nabla} := \big\{\theta : \nabla^g \theta = 0 \text{ and } \theta \lrcorner \mathrm{T} = 0\big\} \subset \mathcal{P}^{\nabla}.$$

A consequence of the formula in Theorem 3 is the following

Corollary 3. *If* $\mathrm{T} \not\equiv 0$ *and* $\mathcal{R}^{\nabla} \not\equiv 0$*, then* $\dim(\mathcal{SP}^{\nabla}) \leq 2$*.*

Proposition 2. *If* $\theta \in \mathcal{SP}^{\nabla}$ *is special* ∇*-parallel, then*

$$\nabla^g_\theta\, \omega^3 = 0\,, \qquad d(\theta \lrcorner \omega^3) = \theta \lrcorner d\,\omega^3, \qquad \mathcal{L}_\theta(\theta \lrcorner \omega^3) = 0\,.$$

Proof. Since $\theta \lrcorner \mathrm{T} = 0$ we get

$$\nabla_\theta S = \nabla^g_\theta S + \frac{1}{2}\,\rho_*(\theta \lrcorner \mathrm{T})(S) = \nabla^g_\theta S$$

for any tensor S. Here ρ_* denotes action of $\mathfrak{so}(7)$ in the corresponding tensor representation. In particular,

$$\nabla^g_\theta\, \omega^3 = 0\,.$$

Since θ is ∇^g-parallel, we have $\nabla^g(\theta\lrcorner\omega^3) = \theta\lrcorner\nabla^g\omega^3$. Using an orthonormal frame with $\theta = e_7$ we compute the differential

$$\begin{aligned} d\,(\theta \lrcorner \omega^3) &= \sum_{i=1}^{7} \nabla^g_{e_i}(\theta \lrcorner \omega^3) \wedge e_i = \sum_{i=1}^{6} (\theta \lrcorner \nabla^g_{e_i}\omega^3) \wedge e_i \ + \ 0 = \sum_{i=1}^{6} \theta \lrcorner (\nabla^g_{e_i}\omega^3 \wedge e_i) \\ &= \sum_{i=1}^{6} \theta \lrcorner (\nabla^g_{e_i}\omega^3 \wedge e_i) \ + \ \theta \lrcorner (\nabla^g_\theta \omega^3 \wedge \theta) = \theta \lrcorner d\omega^3. \end{aligned}$$

Finally, $\mathcal{L}_\theta(\theta \lrcorner \omega^3) = \theta \lrcorner d(\theta \lrcorner \omega^3) = \theta \lrcorner \theta \lrcorner d\,\omega^3 = 0$. □

Theorem 5. *Let (M^7, g, ω^3) be a compact, cocalibrated* G_2*-manifold and ∇ its characteristic connection. Suppose that* $\mathrm{Ric}^\nabla = 0$, $\|\mathrm{T}\|^2 = \mu^2 > 0$ *and* $\mathcal{R}^\nabla \not\equiv 0$. *Then the space of harmonic* 1- *forms coincides with* $\mathcal{SP}^\nabla$,

$$H^1(M^7; \mathbb{R}) = \big\{\theta : \Delta^g\theta = 0\big\} = \mathcal{SP}^\nabla.$$

In particular, the second Betti number is bounded, $b_2(M^7) \le 2$.

Proof. The result follows directly from the Weitzenboeck formula for 1-forms and the link between Ric^g and the torsion form T,

$$\begin{aligned} 0 &= \int_{M^7} g(\Delta^g\theta, \theta) = \int_{M^7} \|\nabla^g\theta\|^2 + \int_{M^7} \mathrm{Ric}^g(\theta,\theta) \\ &= \int_{M^7} \|\nabla^g\theta\|^2 + \frac{1}{2}\int_{M^7} \|\theta \lrcorner \mathrm{T}\|^2. \end{aligned}$$

□

4.3 The case of two special ∇-parallel vector fields

Suppose that there exist two special ∇-parallel vector fields θ_1, θ_2,

$$\nabla^g\ \theta_1 = \nabla^g\theta_2 = 0\,, \qquad \theta_1 \lrcorner \mathrm{T} = \theta_2 \lrcorner \mathrm{T} = 0\,.$$

Then $\omega^3(\theta_1, \theta_2, -) = \theta_3$ is the third ∇-parallel (non-special) vector field and we have

$$\mathrm{T}(\theta_1,\theta_2,\theta_3) = 0\,, \qquad [\theta_1,\theta_2] = [\theta_1,\theta_3] = [\theta_2,\theta_3] = 0\,.$$

The conditions $\theta_1 \lrcorner \mathrm{T} = \theta_2 \lrcorner \mathrm{T} = 0$ restrict the algebraic type of the torsion form. In fact, Theorem 3 yields that the possible torsion forms depend on two parameters only. Indeed, there are two possibilities. The first case:

$$a = \frac{2}{7}\,\mu\,, \quad b = \frac{5}{7}\,\mu\,, \quad m_1 = -\frac{8}{7}\,\mu\,, \quad m_2 = m_3 = \frac{6}{7}\,\mu\,.$$

The second case:

$$a = \frac{2}{7}\,\mu\,, \quad b = -\,\frac{2}{7}\,\mu\,, \quad m_1 = \frac{6}{7}\,\mu\,, \quad m_2 = \frac{6}{7}\,\mu\,, \quad m_3 = -\frac{8}{7}\,\mu\,.$$

Introducing a new notation for the frame

$$f_1 := e_3\,, \quad f_2 := e_4\,, \quad f_3 := e_5\,, \quad f_4 := e_6\,, \quad f_5 := e_7$$

we obtain the following formula for the torsion form:

$$\begin{aligned} \mathrm{T} &= (t_{125} + \mu/7) f_{125} + t_{245}(f_{135} + f_{245}) \\ &\quad + t_{235}(-f_{145} + f_{235}) + (t_{345} + \mu/7) f_{345}\,, \\ b &= t_{125} + t_{345} = \frac{5}{7}\,\mu \quad \text{or} \quad -\,\frac{2}{7}\,\mu\,, \\ \mu^2 &= \|\mathrm{T}\|^2 = \Big(t_{125} + \frac{\mu}{7}\Big)^2 + \Big(t_{345} + \frac{\mu}{7}\Big)^2 + 2\,t_{245}^2 + 2\,t_{235}^2\,. \end{aligned}$$

If M^7 is complete, its universal covering splits into $N^5 \times \mathbb{R}^2$ and the torsion T as well as the form $\theta_3 = e_7 = f_5$ are forms on N^5. This follows form $\mathcal{L}_{\theta_i}\mathrm{T} = 0$, $\mathcal{L}_{\theta_i}\theta_3 = 0$ for $i = 1, 2$. We reduced the dimension. $(N^5, g, \nabla, \mathrm{T}, \theta_3)$ is a 5-dimensional Riemannian manifold equipped with a torsion form T as well as a metric connection ∇ such that

$$d * \mathrm{T} = 0\,, \quad d\,\mathrm{T} = 0\,, \quad ||\mathrm{T}||^2 = 0\,, \quad \mathrm{Ric}^{\nabla} = 0\,,$$

$$\mathcal{R}^{\nabla} \not\equiv 0\,, \quad \mathfrak{hol}(\nabla) \subset \mathfrak{su}(2) \subset \mathfrak{g}_2$$

hold. θ_3 is ∇-parallel on N^5,

$$\nabla\theta_3 = 0\,, \quad d\,\theta_3 = \theta_3 \lrcorner\, \mathrm{T}\,, \quad \mathrm{T} = \theta_3 \wedge d\,\theta_3\,, \quad 0 = d\,\mathrm{T} = d\,\theta_3 \wedge d\,\theta_3\,.$$

Consider the case of $b = -2\mu/7$. Then

$$t_{125} + \frac{\mu}{7} = -t_{345} - \frac{\mu}{7}$$

and we obtain

$$*\mathrm{T} = -\theta_3 \lrcorner\, \mathrm{T} = -d\theta_3\,, \quad *d\theta_3 = -\mathrm{T} = -d\theta_3 \wedge \theta_3\,.$$

We multiply the latter equation by $d\theta_3$:

$$\|d\theta_3\|^2 = d\theta_3 \wedge *d\theta_3 = -\theta_3 \wedge d\theta_3 \wedge d\theta_3 = 0\,.$$

Consequently, $b = -2\,\mu/7$ implies that the torsion form vanishes, $\mathrm{T} = 0$, i.e. the second case is impossible.

We observe that there are three ∇-parallel 2-forms on N^5, namely,

$$\Omega_i^2 := \theta_i \lrcorner \big(\omega^3 - \theta_1 \wedge \theta_2 \wedge \theta_3\big)\,.$$

Consequently, $\mathfrak{hol}(\nabla) \subset \mathfrak{su}(2)$. We can express these forms in our local frame,

$$\Omega^2_1 = f_{13} - f_{24}\,,$$
$$\Omega^2_2 = -\,f_{14} - f_{23}\,,$$
$$\Omega^2_3 = f_{12} + f_{34}\,.$$

Remark that

$$\big(\theta_3 \lrcorner \mathrm{T}\,,\,\Omega^2_1\big) = \big(\theta_3 \lrcorner \mathrm{T}\,,\,\Omega^2_2\big) = 0\,, \quad \big(\theta_3 \lrcorner \mathrm{T}\,,\,\Omega^2_3\big) = b + \frac{2}{7}\,\mu = \mu$$

holds.

Theorem 6. *The kernel of* T

$$E^2 := \big\{X \in TN^5 : X \lrcorner \mathrm{T} = 0\big\}$$

is a 2*-dimensional subbundle of* TN^5*. The tangent bundle splits into two subbundles of dimension* 2 *and* 3*, respectively,*

$$TN^5 = E^2 \oplus (E^2)^{\perp}.$$

θ_3 *belongs to* $(E^2)^{\perp}$ *and the torsion form is given by*

$$\mathrm{T} = \mu\, f^*_1 \wedge f^*_2 \wedge \theta_3\,,$$

where f^*_1, f^*_2, θ_3 *is an orthonormal basis in* $(E^2)^{\perp}$*. Both subbundles are involutive and* N^5 *splits locally (but the* 2*- und* 3*-dimensional leaves are not totally geodesic).*

Proof. We compute the determinant of the skew symmetric endomorphism $\theta_3 \lrcorner \mathrm{T}$ on the space of all vectors being orthogonal to θ_3,

$$\mathrm{Det}(\theta_3 \lrcorner \mathrm{T}) = \frac{1}{4}\Big(-\,b^2 - \frac{4}{7}\,b\,\mu + \frac{45}{49}\,\mu^2\Big)^2 = 0\,.$$

This proves that the dimension of E^2 equals two. Let $f^*_1, f^*_2, f^*_3, f^*_4, f^*_5 = \theta_3$ be an orthonormal frame such that

$$\mathrm{Lin}(f^*_1, f^*_2, f^*_5) = \big(E^2\big)^{\perp}, \quad \mathrm{Lin}(f^*_3, f^*_4) = E^2.$$

Since μ is constant and $d\,\mathrm{T} = d * \mathrm{T} = 0$ we have

$$d\big(f^*_1 \wedge f^*_2 \wedge f^*_5\big) = 0\,, \quad d\big(f^*_3 \wedge f^*_4\big) = 0\,.$$

We differentiate the equations $f^*_3 \wedge f^*_3 \wedge f^*_4 = 0$, $f^*_4 \wedge f^*_3 \wedge f^*_4 = 0$,

$$0 = df^*_3 \wedge (f^*_3 \wedge f^*_4) - f^*_3 \wedge d(f^*_3 \wedge f^*_4) = df^*_3 \wedge (f^*_3 \wedge f^*_4)$$
$$0 = df^*_4 \wedge (f^*_3 \wedge f^*_4) - f^*_4 \wedge d(f^*_3 \wedge f^*_4) = df^*_4 \wedge (f^*_3 \wedge f^*_4)\,.$$

By the Frobenius Theorem, the bundle $(E^2)^{\perp}$ is involutive. Similarly we have

$$df^*_1 \wedge (f^*_1 \wedge f^*_2 \wedge f^*_5) = df^*_2 \wedge (f^*_1 \wedge f^*_2 \wedge f^*_5) = df^*_5 \wedge (f^*_1 \wedge f^*_2 \wedge f^*_5) = 0$$

and the bundle E^2 is involutive. □

This splitting is not ∇-parallel ($\nabla \mathrm{T} \neq 0$), but the flow of θ_3 preserves the splitting ($\mathcal{L}_{\theta_3}\mathrm{T} = 0$). The Ricci tensor preserves the splitting, too. Indeed, it depends only on T and we compute easily:

Theorem 7. *The Ricci tensor* Ric^g *preserves the splitting of the tangent bundle and*

$$\mathrm{Ric}^g_{|E^2} = 0\,, \qquad \mathrm{Ric}^g_{|(E^2)^\perp} = \frac{1}{2}\mu^2\,\mathrm{Id}\,.$$

In particular, the Ricci tensor of (N^5, g) *has constant eigenvalues, and these are* 0 *and* $\mu^2/2 > 0$.

The 2-form $d\theta_3$ is invariant under the flow of θ_3,

$$\mathcal{L}_{\theta_3}\big(d\theta_3\big) = 0 \quad \text{and} \quad d\theta_3 \wedge d\theta_3 = 0\,.$$

If the orbit space $Z^4 := N^5/\theta_3$ is smooth, its tangent bundle splits into two involutive 2-dimensional subbundles. $d\theta_3$ defines a 2-form on Z^4 satisfying all the conditions of Theorem 1. However, we have an additional condition for $(N^5, g, \nabla, \mathrm{T}, \theta_3)$, namely the holonomy of ∇ should be contained in $\mathfrak{su}(2) \subset \mathfrak{g}_2$ and the holonomy representation is in $\mathbb{C}^2 \subset \mathbb{R}^5$. This is equivalent to the condition that there are three ∇-parallel 2-forms $\Omega^2_1, \Omega^2_2, \Omega^2_3$. The 2-form Ω^2_3 plays a special role on N^5. Indeed, it projects down to a Kähler form on Z^4.

Proposition 3.

$$\nabla\,\Omega^2_3 = 0\,, \quad d\,\Omega^2_3 = 0\,, \quad \mathcal{L}_{\theta_3}\Omega^2_3 = 0\,.$$

In particular, if Z^4 *is smooth, then* $\Omega^2_3 \in \Lambda^2_+(Z^4)$ *defines a* ∇^g*-parallel, self-dual 2-form on* Z^4.

Proof. Using the frame $f_1, \ldots, f_5$ one easily computes the formula

$$\Omega^2_3 = \frac{1}{\mu}\,\big(*\mathrm{T} + d\,\theta_3\big) = \frac{1}{\mu}\,\big(*\mathrm{T} + \theta_3 \lrcorner\, \mathrm{T}\big)\,.$$

Since $d*\mathrm{T} = 0$ we obtain $d\,\Omega^2_3 = 0$. Moreover, $\mathcal{L}_{\theta_3}\mathrm{T} = 0$, and

$$\mathcal{L}_{\theta_3}\Omega^2_3 = \frac{1}{\mu}\,\mathcal{L}_{\theta_3}(d\theta_3) \;=\; \frac{1}{\mu}\big(\theta_3 \lrcorner\, (\theta_3 \lrcorner\, \mathrm{T})\big) = 0\,. \qquad \square$$

A similar algebraic computation yields the following formulas.

Proposition 4.

$$d\,\Omega^2_1 = \mu\,\Omega^2_2 \wedge \theta_3\,, \qquad d\,\Omega^2_2 = -\,\mu\,\Omega^2_1 \wedge \theta_3\,,$$

$$\mathcal{L}_{\theta_3}\Omega^2_1 = \mu\,\Omega^2_2\,, \qquad \mathcal{L}_{\theta_3}\Omega^2_2 = -\,\mu\,\Omega^2_1\,.$$

Proof. Since the 2-forms are ∇-parallel, we can compute the derivatives using the formula (see [2])

$$d\,\Omega^2 = \sum_{j=1}^{5}(f_j \lrcorner\, \Omega^2) \wedge (f_j \lrcorner\, \mathrm{T})\,. \qquad \square$$

Remark 4. In the frame $f_1^*, \ldots, f_5^*$ we have $\Omega^2_3 = f_1^* \wedge f_2^* + f_3^* \wedge f_4^*$, too. In particular, Ω^2_3 is completely defined by T and θ_3. If Z^4 is smooth and compact, then $Z^4 = S^2 \times T^2$, see [6], and the connection ∇ on $M^7 = N^5 \times \mathbb{R}^2 = S^3 \times T^2 \times \mathbb{R}^2$ becomes flat.

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Received: 15 March, 2013
Accepted for publication: 29 March, 2013
Communicated by: Ilka Agricola