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On a problem of Bednarek

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Abstract. We answer a question of Bednarek proposed at the 9th Polish, Slovak and Czech conference in Number Theory.

There are several problems in the literature concerning various arithmetic properties of the digit sum number function, see e.g. [1] and the references given there. In this paper, we deal with a particular problem. Namely, at the 9th Polish, Slovak and Czech Conference on Number Theory, June 11–14, 2012, W. Bednarek (via A. Schinzel) asked the following question.

Question *Is there a positive integer n divisible by $\underbrace{\overline{11\ldots 1}}_{k \text{ times}}$ whose digit sum is less than k?*

Here, we prove that the answer is no in a slightly more general setting. For integers $N \geq 1$ and $b \geq 2$, let $N = \overline{d_m d_{m-1} \ldots d_0}_{(b)}$ be the base b representation of N, where $d_0, \ldots, d_m \in \{0, 1, \ldots, b-1\}$ with $d_m \neq 0$. We have the following result.

Theorem *If $n \geq 2$ is a multiple of $\underbrace{\overline{11\ldots 1}}_{k \text{ times}}{}_{(b)}$, then the sum of its base b digits is greater than or equal to k.*

Proof. We may assume that $k \geq 2$, otherwise there is nothing to prove. Write

$$n = \sum_{i=0}^{m} d_i b^i \,,$$

where $d_0, \ldots, d_m$ are in $\{0, 1, \ldots, b-1\}$ with $d_m \neq 0$. We may also assume that $d_0 \neq 0$. Put $N = (b^k - 1)/(b - 1)$. Then $b^k \equiv 1 \pmod N$. Thus,

$$n \equiv \sum_{j=0}^{k-1} c_j b^j \pmod N \,,$$

where

$$c_j = \sum_{\substack{0 \leq i \leq m \\ i \equiv j \pmod k}} d_i \,.$$

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It is clear that $c_0 + \cdots + c_{k-1}$ is the sum of the digits of n.
For each $\ell \in \{0, 1, \ldots, k-1\}$, put

$$r_{j,\ell} = j + \ell - k \left\lfloor \frac{j+\ell}{k} \right\rfloor ,$$

and consider the integer

$$m_\ell = \sum_{j=0}^{k-1} c_j b^{r_{j,\ell}} .$$

Note that since $b^k \equiv 1 \pmod{N}$, it follows that

$$m_\ell \equiv \sum_{j=0}^{k-1} b^{j+\ell} c_j \pmod{N} \equiv b^\ell n \pmod{N} \equiv 0 \pmod{N},$$

and since $c_j > 0$ for some j, we get that $m_\ell \geq N$. Summing this up for all $\ell \in \{0, 1, \ldots, k-1\}$, we get

$$kN \leq \sum_{\ell=0}^{k-1} \sum_{j=0}^{k-1} b^{r_{j,\ell}} c_j = \sum_{j=0}^{k-1} c_j \sum_{\ell=0}^{k-1} b^{r_{j,\ell}} = N \sum_{j=0}^{k-1} c_j ,$$

so $\sum_{j=0}^{k-1} c_j \geq k$, which is what we wanted to prove. □

Note After this paper was submitted, we learned that Bednarek's question was also asked by Zhi–Wei Sun in [3], who solved the particular case when the modulus b is a prime. We also learned that the main result of this paper was obtained independently by Pan in [2].

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