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Gradient estimates for a nonlinear equation $\Delta_f u + cu^{-\alpha} = 0$ on complete noncompact manifolds

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Abstract. Let (M, g) be a complete noncompact Riemannian manifold. We consider gradient estimates on positive solutions to the following nonlinear equation

$$\Delta_f u + cu^{-\alpha} = 0 \quad \text{in } M,$$

where α, c are two real constants and $\alpha > 0$, f is a smooth real valued function on M and $\Delta_f = \Delta - \nabla f \nabla$. When N is finite and the N-Bakry-Emery Ricci tensor is bounded from below, we obtain a gradient estimate for positive solutions of the above equation. Moreover, under the assumption that ∞-Bakry-Emery Ricci tensor is bounded from below and $|\nabla f|$ is bounded from above, we also obtain a gradient estimate for positive solutions of the above equation. It extends the results of Yang [16].

1 Introduction

Let (M, g) be a complete noncompact n-dimensional Riemannian manifold. For a smooth real-valued function f on M, the drifting Laplacian (see [11], [12]) is defined by $\Delta_f = \Delta - \nabla f \nabla$. There is a naturally associated measure $d\mu = e^{-f} dV$ on M, which makes the operator Δ_f self-adjoint. The N-Bakry-Emery Ricci tensor is defined by

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \nabla^2 f - \frac{1}{N} df \otimes df$$

for $0 \le N \le \infty$ and $N = 0$ if and only if $f = 0$. Here ∇^2 is the Hessian and Ric is the Ricci tensor. In particular, the ∞-Bakry-Emery Ricci tensor is denoted by

$$\mathrm{Ric}_f := \mathrm{Ric}_f^\infty = \mathrm{Ric} + \nabla^2 f$$

with $\mathrm{Ric}_f = \lambda g$ is called a gradient Ricci soliton which is extensively studied in Ricci flow.

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The author in [16] obtained interesting gradient estimates for positive solutions to the following elliptic equation with singular nonlinearity

$$\Delta u + cu^{-\alpha} = 0 \quad \text{in } M, \tag{1}$$

where α, c are two real constants and $\alpha > 0$. For the importance of equation (1), the authors who are interested in it see [5], [8]. In this paper, we consider the following equation

$$\Delta_f u + cu^{-\alpha} = 0 \quad \text{in } M, \tag{2}$$

where f is a smooth real-valued function on M. For some interesting gradient estimates in this direction, for example, we refer to [2], [3], [6], [7], [9], [10], [15]. When N is finite and the N-Bakry-Emery Ricci tensor is bounded from below, we obtain a gradient estimate for positive solutions of the above equation. Moreover, under the assumption that ∞-Bakry-Emery Ricci tensor is bounded from below and $|\nabla f|$ is bounded from above, we also obtain a gradient estimate for positive solutions of the above equation. Main results of this paper are stated as follows:

Theorem 1. Let (M, g) be a complete noncompact n-dimensional Riemannian manifold with N-Bakry-Emery Ricci tensor bounded from below by the constant $-K := -K(2R)$, where $R > 0$ and $K(2R) \geq 0$, in the metric ball $B_p(2R)$ with radius $2R$ around $p \in M$. Let u be a positive solution of (2) with α, c two real constants and $\alpha > 0$. Then

(1) If $c > 0$, we have

$$\begin{aligned} \frac{|\nabla u|^2}{u^2} + cu^{-\alpha-1} \leq & \frac{(n+N)(n+N+2){c_1}^2}{R^2} + \frac{(n+N)[(n+N-1)c_1 + c_2]}{R^2} \\ & + \frac{(n+N)\sqrt{(n+N)K}c_1}{R} + 2(n+N)K. \end{aligned} \tag{3}$$

(2) If $c < 0$, we have

$$\begin{aligned} \frac{|\nabla u|^2}{u^2} + cu^{-\alpha-1} \leq (A + \sqrt{A})|c|(\inf_{B_p(2R)} u)^{-\alpha-1} + \frac{(n+N)[(n+N-1)c_1 + c_2]}{R^2} \\ + \frac{(n+N)c_1^2}{R^2}\Big(n+N+2+\frac{n+N}{2\sqrt{A}}\Big) + \frac{(n+N)\sqrt{(n+N)K}c_1}{R} \\ + \Big(2 + \frac{1}{\sqrt{A}}\Big)(n+N)K, \end{aligned} \tag{4}$$

where $A = (n+N)(\alpha+1)(\alpha+2)$.

Theorem 2. Let (M, g) be a complete noncompact n-dimensional Riemannian manifold and $f \in C^2(M)$ be a function satisfying $|\nabla f| \leq \theta$. Assume that ∞-Bakry-Emery Ricci tensor bounded from below by the constant $-K := -K(2R)$, where $R > 0$ and $K(2R) \geq 0$, in the metric ball $B_p(2R)$ with radius $2R$ around $p \in M$. Let u be a positive solution of (2) with α, c two real constants and $\alpha > 0$. Then

(1) If $c > 0$, we have

$$\frac{|\nabla u|^2}{u^2} + cu^{-\alpha-1} \leq \frac{n[(n+2)c_1^2 + (n-1)c_1 + c_2]}{R^2} + \frac{5nc_1\theta}{R} + 4\theta^2 + \frac{nc_1\sqrt{(n-1)K}}{R} + 2nK. \tag{5}$$

(2) If $c < 0$, we have

$$\frac{|\nabla u|^2}{u^2} + cu^{-\alpha-1} \leq (B+\sqrt{B})|c|(\inf_{B_p(2R)} u)^{-\alpha-1} + \frac{n}{R^2}\Big((2+2n+\frac{n}{\sqrt{B}})c_1^2 + (n-1)c_1 + c_2\Big) + \frac{nc_1\theta}{R} + \Big(1+\frac{1}{2\sqrt{B}}\Big)8\theta^2 + \frac{nc_1\sqrt{(n-1)K}}{R} + \Big(2+\frac{1}{\sqrt{B}}\Big)nK, \tag{6}$$

where $B = n(\alpha+1)(\alpha+2)$.

From (1) in Theorem 1, we obtain the following result immediately:

Corollary 1. *Let (M, g) be a complete noncompact n-dimensional Riemannian manifold with nonnegative N-Bakry-Emery Ricci tensor. Assume that two real constants α, c in (2) are positive. Then the equation (2) does not have a positive smooth solution.*

2 Proof of Theorem 1

Let $h = \log u$. Then one has from (2) that

$$\Delta_f h = \frac{1}{u}\Delta_f u - |\nabla h|^2 = -cu^{-\alpha-1} - |\nabla h|^2.$$

Define $F = cu^{-\alpha-1} + |\nabla h|^2$, then we have $\Delta_f h = -F$. It is well known that for the N-Bakry-Emery Ricci tensor, we have the Bochner formula (see [14]):

$$\begin{aligned}\Delta_f|\nabla h|^2 &\geq \frac{2}{n+N}|\Delta_f h|^2 + 2\langle\nabla h, \nabla(\Delta_f h)\rangle - 2K|\nabla h|^2\\ &= \frac{2}{n+N}F^2 - 2\langle\nabla h, \nabla F\rangle - 2K|\nabla h|^2.\end{aligned}$$

Hence, one gets

$$\begin{aligned}\Delta_f F &= c\Delta_f u^{-\alpha-1} + \Delta_f|\nabla h|^2\\ &\geq c(\alpha+1)(\alpha+2)u^{-\alpha-1}|\nabla h|^2 - c(\alpha+1)u^{-\alpha-2}\Delta_f u\\ &\quad + \frac{2}{n+N}F^2 - 2\langle\nabla h, \nabla F\rangle - 2K|\nabla h|^2.\end{aligned} \tag{7}$$

Let ξ be a cut-off function such that $\xi(r) = 1$ for $r \leq 1$, $\xi(r) = 0$ for $r \geq 2$, $0 \leq \xi(r) \leq 1$, and

$$0 \geq \xi^{-\frac{1}{2}}(r)\xi^{'}(r) \geq -c_1$$
$$\xi''(r) \geq -c_2$$

for positive constants c_1 and c_2. Denote ϕ by $\rho(x) = d(x, \rho)$ the distance between x and p in M. Let

$$\phi(x) = \xi\left(\frac{\rho(x)}{R}\right).$$

Using an argument of Calabi [1] (see also Cheng and Yau [4]), we can assume without loss of generality that the function ϕ is smooth in $B_p(2R)$. Then, we have

$$\frac{|\nabla\phi|^2}{\phi} \leq \frac{c_1^2}{R^2}. \tag{8}$$

It has been shown by Qian[13] that

$$\Delta_f(\rho^2) \leq (n+N)\Big(1+\sqrt{1+\frac{4K\rho^2}{n+N}}\Big).$$

Hence, we have

$$\begin{aligned}\Delta_f\rho &= \frac{1}{2\rho}[\Delta_f(\rho^2) - 2|\nabla\rho|^2]\\ &\leq \frac{n+N-2}{2\rho} + \frac{n+N}{2\rho}\Big(1+\sqrt{\frac{4K\rho^2}{n+N}}\Big)\\ &= \frac{n+N-1}{\rho} + \sqrt{(n+N)K}.\end{aligned}$$

It follows that

$$\begin{aligned}\Delta_f\phi &= \frac{\xi''(r)|\nabla\rho|^2}{R^2} + \frac{\xi'(r)\Delta_f\rho}{R}\\ &\geq -\frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2}.\end{aligned} \tag{9}$$

Define $G = \phi F$. We may assume that G achieves its maximal value Q at the point $x \in B_p(2R)$ and assume that Q is positive (otherwise the proof is trivial). Then at the point x,

$$0 = \nabla G = \phi\nabla F + F\nabla\phi$$

and $\Delta_f G \leq 0$. Therefore, at the point x, it holds that

$$\begin{aligned}0 \geq \Delta_f G &= \Delta G - \langle\nabla f, \nabla G\rangle\\ &= \phi\Delta_f F + F\Delta_f\phi + 2\langle\nabla\phi, \nabla F\rangle\\ &= \phi\Delta_f F + F\Delta_f\phi - 2F\frac{|\nabla\phi|^2}{\phi}\\ &\geq \frac{2}{n+N}\phi F^2 - 2\phi\langle\nabla h, \nabla F\rangle - 2K\phi|\nabla h|^2\\ &\quad - \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2}F\\ &\quad - \frac{2c_1^2}{R^2}F + c(\alpha+1)(\alpha+2)u^{-\alpha-1}\phi|\nabla h|^2 - c(\alpha+1)u^{-\alpha-2}\phi\Delta_f u,\end{aligned}$$

which shows that

$$
\begin{aligned}
0 \geq\ & \frac{2}{n+N} G^2 + 2G\langle \nabla h, \nabla \phi\rangle - 2K\phi^2|\nabla h|^2 \\
& - \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2} G \\
& - \frac{2c_1^2}{R^2} G + c(\alpha+1)(\alpha+2)u^{-\alpha-1}\phi^2|\nabla h|^2 - c(\alpha+1)u^{-\alpha-2}\phi^2\Delta_f u.
\end{aligned} \tag{10}
$$

Next, we consider the following two cases: (1) $c > 0$; (2) $c < 0$.

(1) When $c > 0$, then we have $F = |\nabla h|^2 + cu^{-\alpha-1} > 0$ and $|\nabla h| < F^{\frac{1}{2}}$. Since

$$
\langle \nabla h, \nabla \phi\rangle \leq |\nabla h||\nabla \phi| \leq \frac{c_1}{R} F^{\frac{1}{2}} \phi^{\frac{1}{2}},
$$

$$
\frac{2c_1}{R} G^{\frac{3}{2}} \leq \frac{(n+N)c_1^2}{R^2} G + \frac{1}{n+N} G^2,
$$

then (10) yields

$$
\begin{aligned}
0 \geq\ & \frac{2}{n+N} G^2 - \frac{2c_1}{R} G^{\frac{3}{2}} - 2K\phi|\nabla h|^2 \\
& - \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2} G \\
& - \frac{2c_1^2}{R^2} G + c(\alpha+1)^2 u^{-\alpha-1}\phi^2|\nabla h|^2 + c(\alpha+1)u^{-\alpha-1}\phi^2 F \\
\geq\ & \frac{1}{n+N} G^2 - \frac{(n+N+2)c_1^2}{R^2} G - 2KG \\
& - \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2} G.
\end{aligned} \tag{11}
$$

From (11), we obtain

$$
\begin{aligned}
G \leq\ & \frac{(n+N)(n+N+2)c_1^2}{R^2} + \frac{(n+N)[(n+N-1)c_1 + c_2]}{R^2} \\
& + \frac{(n+N)c_1}{R}\sqrt{(n+N)K} + 2(n+N)K
\end{aligned}
$$

and hence

$$
\begin{aligned}
\sup_{B_p(2R)} F \leq G \leq\ & \frac{(n+N)(n+N+2)c_1^2}{R^2} + \frac{(n+N)[(n+N-1)c_1 + c_2]}{R^2} \\
& + \frac{(n+N)c_1}{R}\sqrt{(n+N)K} + 2(n+N)K.
\end{aligned} \tag{12}
$$

Now (1) of Theorem 1 follows easily from the inequality above.

(2) When $c < 0$, if $F \leq 0$, then the estimate in (2) of Theorem 1 is trivial. Hence we assume $F > 0$. Under the assumption that $F > 0$, one gets $|\nabla h| > F^{\frac{1}{2}}$. Since

$$
2G\langle \nabla h, \nabla \phi\rangle \leq \frac{1}{n+N} G^2 + \frac{(n+N)c_1^2}{R^2} \phi|\nabla h|^2,
$$

then (10) yields

$$\begin{aligned}
0 \geq\ & \frac{1}{n+N}G^2 - \frac{(n+N)c_1^2}{R^2}\phi|\nabla h|^2 - 2K\phi^2|\nabla h|^2 \\
& - \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2}G \\
& - \frac{2c_1^2}{R^2}G + c(\alpha+1)(\alpha+2)(\inf_{B_p(2R)} u)^{-\alpha-1}\phi^2|\nabla h|^2 \\
& + c^2(\alpha+1)(\sup_{B_p(2R)} u)^{-2\alpha-2}\phi^2 \\
\geq\ & \frac{1}{n+N}G^2 - \frac{(n+N)c_1^2}{R^2}\phi F - \frac{(n+N)c_1^2}{R^2}\phi|c|(\inf_{B_p(2R)} u)^{-\alpha-1} \\
& - \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2}G \\
& - \frac{2c_1^2}{R^2}G - J(2R)\phi^2 F - L(2R)\phi^2,
\end{aligned}$$

where

$$J(2R) = 2K - c(\alpha+1)(\alpha+2)(\inf_{B_p(2R)} u)^{-\alpha-1},$$
$$L(2R) = |c|J(2R)(\inf_{B_p(2R)} u)^{-\alpha-1} - c^2(\alpha+1)(\sup_{B_p(2R)} u)^{-2\alpha-2}.$$

This shows that

$$\begin{aligned}
0 \geq\ & \frac{1}{n+N}G^2 \\
& - \Big(\frac{(n+N+2)c_1^2}{R^2} + \frac{(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2}{R^2} + J(2R)\Big)G \\
& - \frac{(n+N)c_1^2}{R^2}|c|(\inf_{B_p(2R)} u)^{-\alpha-1} - L(2R).
\end{aligned}$$

Hence

$$G \leq \frac{b+\sqrt{b^2+4d}}{2} \leq b + \sqrt{d}, \tag{13}$$

where

$$\begin{aligned}
b =\ & (n+N)J(2R) + \frac{(n+N)[(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2]}{R^2} \\
& + \frac{(n+N)(n+N+2)c_1^2}{R^2}, \\
d =\ & (n+N)L(2R) + \frac{(n+N)^2c_1^2}{R^2}|c|(\inf_{B_p(2R)} u)^{-\alpha-1}.
\end{aligned}$$

Let $m = (\inf_{B_p(2R)} u)^{-\alpha-1}$, $M = (\sup_{B_p(2R)} u)^{-\alpha-1}$. We have

$$\begin{aligned}\sqrt{d} &= \sqrt{(n+N)c^2(\alpha+1)[(\alpha+2)m^2 - M^2] + [\frac{(n+N)c_1^2}{R^2}|c| + 2(n+N)|c|K]m} \\ &\leq \sqrt{(n+N)c^2(\alpha+1)(\alpha+2)m^2 + [\frac{(n+N)c_1^2}{R^2}|c| + 2(n+N)|c|K]m} \\ &\leq \sqrt{(n+N)(\alpha+1)(\alpha+2)}|c|m + \frac{\frac{(n+N)c_1^2}{R^2} + 2(n+N)K}{2\sqrt{(n+N)(\alpha+1)(\alpha+2)}}.\end{aligned}$$

It follows from (13) that

$$\begin{aligned}G &\leq 2(n+N)K + A|c|m + \frac{(n+N)[(n+N-1+\sqrt{(n+N)K}R)c_1 + c_2]}{R^2} \\ &\quad + \frac{(n+N)(n+N+2)c_1^2}{R^2} + \sqrt{A}|c|m + \frac{\frac{(n+N)^2c_1^2}{R^2} + 2(n+N)K}{2\sqrt{A}} \\ &= (A+\sqrt{A})|c|m + \frac{(n+N)[(n+N-1)c_1 + c_2]}{R^2} \\ &\quad + \frac{(n+N)c_1^2}{R^2}\left(n+N+2+\frac{n+N}{2\sqrt{A}}\right) \\ &\quad + \frac{(n+N)\sqrt{(n+N)K}c_1}{R} + \left(2+\frac{1}{\sqrt{A}}\right)(n+N)K,\end{aligned} \tag{14}$$

where

$$A = (n+N)(\alpha+1)(\alpha+2).$$

Therefore, we obtain (2) of Theorem 1. □

3 Proof of Theorem 2

Let $h = \log u$. Then we have

$$\Delta_f h = -cu^{-\alpha-1} - |\nabla h|^2.$$

Denote by $F = cu^{-\alpha-1} + |\nabla h|^2$, then we have $\Delta_f h = -F$. Applying the Bochner formula to h, we get (see [14]):

$$\Delta_f |\nabla h|^2 = 2|D^2 h|^2 + 2\langle \nabla h, \nabla(\Delta_f h)\rangle + 2\mathrm{Ric}_f(\nabla h, \nabla h). \tag{15}$$

Since

$$\begin{aligned}|D^2 h|^2 &\geq \frac{1}{n}(\Delta h)^2 \\ &= \frac{1}{n}[F - \langle \nabla h, \nabla f\rangle]^2 \\ &\geq \frac{1}{n}F^2 - \frac{2}{n}F\langle \nabla h, \nabla f\rangle,\end{aligned}$$

then we derive from (15)

$$\Delta_f |\nabla h|^2 \geq \frac{2}{n}F^2 - \frac{4}{n}F\langle \nabla h, \nabla f\rangle - 2\langle \nabla h, \nabla F\rangle - 2K|\nabla h|^2. \tag{16}$$

Thus we have

$$\begin{aligned}\Delta_f F &= c\Delta_f u^{-\alpha-1} + \Delta_f |\nabla h|^2 \\ &\geq c(\alpha+1)(\alpha+2)u^{-\alpha-1}|\nabla h|^2 - c(\alpha+1)u^{-\alpha-2}\Delta_f u \\ &\quad + \frac{2}{n}F^2 - \frac{4}{n}F\langle \nabla h, \nabla f\rangle - 2\langle \nabla h, \nabla F\rangle - 2K|\nabla h|^2.\end{aligned} \tag{17}$$

Let ξ be a cut-off function such that $\xi(r) = 1$ for $r \leq 1$, $\xi(r) = 0$ for $r \geq 2$, $0 \leq \xi(r) \leq 1$, and

$$0 \geq \xi^{-\frac{1}{2}}(r)\xi^{'}(r) \geq -c_1$$

$$\xi''(r) \geq -c_2$$

for positive constants c_1 and c_2. Denote ϕ by $\rho(x) = d(x, \rho)$ the distance between x and p in M. Let

$$\phi(x) = \xi\left(\frac{\rho(x)}{R}\right).$$

Using an argument of Calabi [1] (see also Cheng and Yau [4]), we can assume without loss of generality that the function ϕ is smooth in $B_{2R}(p)$. Then, we have

$$\frac{|\nabla \phi|^2}{\phi} \leq \frac{c_1^2}{R^2}. \tag{18}$$

Since $\mathrm{Ric}_f \geq -K$ and $|\nabla f| \leq \theta$, we have from the Theorem 1.1 in [14]:

$$\begin{aligned}\Delta_f \rho &\leq \sqrt{(n-1)K}\coth\Big(\sqrt{\frac{K}{n-1}}\rho\Big) + \theta \\ &\leq (n-1)\Big(\frac{1}{\rho} + \sqrt{\frac{K}{n-1}}\Big) + \theta.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\Delta_f \phi &= \frac{\xi''(r)|\nabla \rho|^2}{R^2} + \frac{\xi'(r)\Delta_f \rho}{R} \\ &\geq -\frac{(n-1+\sqrt{(n-1)K}R + \theta R)c_1 + c_2}{R^2}.\end{aligned} \tag{19}$$

Define $G = \phi F$. We assume that G achieves its maximal value Q at the point $x \in B_p(2R)$ and assume that Q is positive (otherwise the proof is trivial). Then at the point x,

$$0 = \nabla G = \phi \nabla F + F\nabla \phi$$

and $\Delta_f G \leq 0$. This shows that

$$\nabla F = -\frac{F}{\phi}\nabla \phi.$$

Therefore, at the point x, it holds that

$$\begin{aligned}
0 &\geq \Delta_f G = \phi \Delta_f F + F \Delta_f \phi + 2\langle \nabla \phi, \nabla F \rangle \\
&= \phi \Delta_f F + F \Delta_f \phi - 2F \frac{|\nabla \phi|^2}{\phi} \\
&\geq \frac{2}{n}\phi F^2 - \frac{4}{n}\phi F \langle \nabla h, \nabla f \rangle - 2\phi \langle \nabla h, \nabla F \rangle - 2K\phi |\nabla h|^2 \\
&\quad - \frac{(n-1+\sqrt{(n-1)K}R + \theta R)c_1 + c_2}{R^2} F - \frac{2c_1^2}{R^2} F \\
&\quad + c(\alpha+1)(\alpha+2)u^{-\alpha-1}\phi |\nabla h|^2 - c(\alpha+1)u^{-\alpha-2}\phi \Delta_f u,
\end{aligned}$$

which means that

$$\begin{aligned}
0 \geq{} & \frac{2}{n}G^2 - \frac{4}{n}\phi G \langle \nabla h, \nabla f \rangle + 2G\langle \nabla h, \nabla \phi \rangle - 2K\phi^2 |\nabla h|^2 \\
& - \frac{2c_1^2 + (n-1)c_1 + c_2}{R^2} G - \frac{(\sqrt{(n-1)K} + \theta)c_1}{R} G \\
& + c(\alpha+1)(\alpha+2)u^{-\alpha-1}\phi^2 |\nabla h|^2 - c(\alpha+1)u^{-\alpha-2}\phi^2 \Delta_f u.
\end{aligned} \tag{20}$$

Next, we consider two cases: (1) $c > 0$; (2)$c < 0$.

(1) When $c > 0$, we have $F = |\nabla h|^2 + cu^{-\alpha-1} > 0$ and $|\nabla h| < F^{\frac{1}{2}}$. Since

$$|\langle \nabla h, \nabla \phi \rangle| \leq |\nabla h||\nabla \phi| \leq \frac{c_1}{R} F^{\frac{1}{2}} \phi^{\frac{1}{2}},$$

$$|\langle \nabla h, \nabla f \rangle| \leq |\nabla h||\nabla f| \leq F^{\frac{1}{2}} |\nabla f|,$$

then from (20) we obtain

$$\begin{aligned}
0 \geq{} & \frac{2}{n}G^2 - \frac{4}{n}|\nabla f| G^{\frac{3}{2}} - \frac{2c_1}{R}G^{\frac{3}{2}} - 2K\phi |\nabla h|^2 - \frac{2c_1^2 + (n-1)c_1 + c_2}{R^2} G \\
& - \frac{(\sqrt{(n-1)K} + \theta)c_1}{R} G + c(\alpha+1)^2 u^{-\alpha-1}\phi^2 |\nabla h|^2 \\
& + c(\alpha+1)u^{-\alpha-1}\phi^2 F \\
\geq{} & \frac{2}{n}G^2 - \frac{4}{n}|\nabla f| G^{\frac{3}{2}} - \frac{2c_1}{R}G^{\frac{3}{2}} - 2KG - \frac{2c_1^2 + (n-1)c_1 + c_2}{R^2} G \\
& - \frac{(\sqrt{(n-1)K} + \theta)c_1}{R} G.
\end{aligned} \tag{21}$$

Using the Schwarz inequality, one has

$$\begin{aligned}
\Big(\frac{4}{n}|\nabla f| + \frac{2c_1}{R}\Big) G^{\frac{3}{2}} &\leq n\Big(\frac{2}{n}|\nabla f| + \frac{c_1}{R}\Big)^2 G + \frac{1}{n}G^2 \\
&= \Big(\frac{4}{n}|\nabla f|^2 + \frac{4c_1}{R}|\nabla f| + \frac{nc_1^2}{R^2}\Big) G + \frac{1}{n}G^2.
\end{aligned} \tag{22}$$

Inserting (22) into (21) yields

$$\begin{aligned}
0 \geq{} & \frac{1}{n}G^2 - \Big(\frac{4}{n}|\nabla f|^2 + \frac{4c_1}{R}|\nabla f|\Big) G - 2KG \\
& - \frac{(n+2)c_1^2 + (n-1)c_1 + c_2}{R^2} G - \frac{(\sqrt{(n-1)K} + \theta)c_1}{R} G.
\end{aligned}$$

Hence

$$G \le \frac{n[(n+2)c_1^2+(n-1)c_1+c_2]}{R^2} + \frac{5nc_1\theta}{R} + 4\theta^2 + \frac{nc_1\sqrt{(n-1)K}}{R} + 2nK, \quad (23)$$

and

$$\sup_{B_p(2R)} F \le G \le \frac{n[(n+2)c_1^2+(n-1)c_1+c_2]}{R^2} + \frac{5nc_1\theta}{R} + 4\theta^2 + \frac{nc_1\sqrt{(n-1)K}}{R} + 2nK.$$

We complete the proof of (1) in Theorem 2.

(2) When $c < 0$, if $F \le 0$, then the estimate in (2) of Theorem 2 is trivial. Hence we assume $F > 0$ and hence $|\nabla h| > F^{\frac{1}{2}}$. Noticing

$$2G\langle \nabla h, \nabla \phi\rangle \le 2\frac{c_1}{R}G\phi^{\frac{1}{2}}|\nabla h| \le \frac{1}{2n}G^2 + \frac{2nc_1^2}{R^2}\phi|\nabla h|^2,$$
$$\frac{4}{n}\phi G\langle \nabla h, \nabla f\rangle \le \frac{4}{n}\phi G|\nabla h||\nabla f| \le \frac{1}{2n}G^2 + \frac{8}{n}|\nabla f|^2\phi^2|\nabla h|^2,$$

we have from (20)

$$\begin{aligned}
0 \ge{}& \frac{1}{n}G^2 - \frac{8}{n}|\nabla f|^2\phi^2|\nabla h|^2 - \frac{2nc_1^2}{R^2}\phi|\nabla h|^2 - 2K\phi^2|\nabla h|^2 - \frac{2c_1^2+(n-1)c_1+c_2}{R^2}G \\
&- \frac{(\sqrt{(n-1)K}+\theta)c_1}{R}G + c(\alpha+1)(\alpha+2)(\inf_{B_p(2R)} u)^{-\alpha-1}\phi^2|\nabla h|^2 \\
&+ c^2(\alpha+1)(\sup_{B_p(2R)} u)^{-2\alpha-2}\phi^2 \\
\ge{}& \frac{1}{n}G^2 - \Big(\frac{8}{n}|\nabla f|^2 + \frac{2nc_1^2}{R^2}\Big)\phi F - \Big(\frac{8}{n}|\nabla f|^2 + \frac{2nc_1^2}{R^2}\Big)\phi|c|(\inf_{B_p(2R)} u)^{-\alpha-1} \\
&- \frac{2c_1^2+(n-1)c_1+c_2}{R^2}G - \frac{(\sqrt{(n-1)K}+\theta)c_1}{R}G - J(2R)\phi^2F - L(2R)\phi^2,
\end{aligned}$$

where

$$J(2R) = 2K - c(\alpha+1)(\alpha+2)(\inf_{B_p(2R)} u)^{-\alpha-1},$$
$$L(2R) = |c|J(2R)(\inf_{B_p(2R)} u)^{-\alpha-1} - c^2(\alpha+1)(\sup_{B_p(2R)} u)^{-2\alpha-2}.$$

This shows that

$$\begin{aligned}
0 \ge{}& \frac{1}{n}G^2 \\
&- \Big(\frac{8}{n}|\nabla f|^2 + \frac{(2n+2)c_1^2+(n-1)c_1+c_2}{R^2} + \frac{(\sqrt{(n-1)K}+\theta)c_1}{R} + J(2R)\Big)G \\
&- \Big(\frac{8}{n}|\nabla f|^2 + \frac{2nc_1^2}{R^2}\Big)|c|(\inf_{B_p(2R)} u)^{-\alpha-1} - L(2R).
\end{aligned}$$

Hence one has

$$G \le \frac{b + \sqrt{b^2 + 4d}}{2} \le b + \sqrt{d}, \tag{24}$$

where

$$b = nJ(2R) + 8|\nabla f|^2 + \frac{n[(2n+2)c_1^2 + (n-1)c_1 + c_2]}{R^2} + \frac{nc_1(\sqrt{(n-1)K} + \theta)}{R},$$

$$d = nL(2R) + \Big(8|\nabla f|^2 + \frac{2n^2c_1^2}{R^2}\Big)|c|(\inf_{B_p(2R)} u)^{-\alpha-1}.$$

Let $m = (\inf_{B_p(2R)} u)^{-\alpha-1}$, $M = (\sup_{B_p(2R)} u)^{-\alpha-1}$. We have

$$\begin{aligned}\sqrt{d} &= \sqrt{nc^2(\alpha+1)[(\alpha+2)m^2 - M^2] + (2nK + 8|\nabla f|^2 + \frac{2n^2c_1^2}{R^2})|c|m}\\ &\le \sqrt{nc^2(\alpha+1)(\alpha+2)m^2 + (2nK + 8|\nabla f|^2 + \frac{2n^2c_1^2}{R^2})|c|m}\\ &\le \sqrt{n(\alpha+1)(\alpha+2)}|c|m + \frac{nK + 4|\nabla f|^2 + \frac{n^2c_1^2}{R^2}}{\sqrt{n(\alpha+1)(\alpha+2)}}.\end{aligned}$$

It follows from (24) and $|\nabla f| \le \theta$ that

$$\begin{aligned}G &\le 2nK + B|c|m + 8\theta^2 + \frac{n[(2n+2)c_1^2 + (n-1)c_1 + c_2]}{R^2}\\ &\quad + \frac{nc_1(\sqrt{(n-1)K} + \theta)}{R} + \sqrt{B}|c|m + \frac{nK + 4\theta^2 + \frac{n^2c_1^2}{R^2}}{\sqrt{B}}\\ &= (B + \sqrt{B})|c|m + \frac{n}{R^2}\Big((2 + 2n + \frac{n}{\sqrt{B}})c_1^2 + (n-1)c_1 + c_2\Big) + \frac{nc_1\theta}{R}\\ &\quad + \Big(1 + \frac{1}{2\sqrt{B}}\Big)8\theta^2 + \frac{nc_1\sqrt{(n-1)K}}{R} + \Big(2 + \frac{1}{\sqrt{B}}\Big)nK,\end{aligned}$$

where

$$B = n(\alpha+1)(\alpha+2).$$

The proof of (2) in Theorem 2 is completed finally. □

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